REGULAR SUBALGEBRAS OF COMPLETE BOOLEAN ALGEBRAS

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Abstract. It is proved that the following conditions are equivalent:

(a) there exists a complete, atomless, σ–centered Boolean algebra, which does not contain any regular, atomless, countable subalgebra,

(b) there exists a nowhere dense ultrafilter on ω.

Therefore the existence of such algebras is undecidable in ZFC. In "forcing language" condition (a) says that there exists a non–trivial σ–centered forcing not adding Cohen reals

A subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{A}$ is called regular whenever for every $X \subseteq \mathbb{B}$, $\sup_{\mathbb{B}} X = \mathbf{1}$ implies $\sup_{\mathbb{A}} X = \mathbf{1}$; see e.g. Heindorf and Shapiro [6]. Clearly, every dense subalgebra is regular. Although every complete Boolean algebra contains a free Boolean algebra of the same size (see the Balcar-Franek Theorem; [2]), not always such an embedding is regular. For instance, if $\mathbb{B}$ is a measure algebra, then it contains a free subalgebra of the same cardinality as $\mathbb{B}$, but $\mathbb{B}$ cannot contain any infinite free Boolean algebra as a regular subalgebra. Indeed, measure algebras are weakly σ-distributive but free Boolean algebras are not, and a regular subalgebra of a weakly σ-distributive one is again σ-distributive. Thus $\mathbb{B}$ does not contain any free Boolean algebra. On the other hand, measure algebras are not σ-centered. So, a natural question arises whether there exists a σ-centered, complete, atomless Boolean algebra $\mathbb{B}$ without regular free subalgebras. Since countable atomless Boolean algebras are free and every free Boolean algebra contains a countable regular free subalgebra, it is enough to ask whether $\mathbb{B}$ contains a countable regular subalgebra. In the paper we prove that such an algebra exists iff there exists a nowhere dense ultrafilter.

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Definition 1 (Baumgartner [3])**.** *A filter D on ω is called nowhere dense if for every function f from ω to the Cantor set ${}^{\omega}2$ there exists a set $A \in D$ such that $f(A)$ is nowhere dense in ${}^{\omega}2$.*

In the sequel we will rather interested in nowhere dense ultrafilters. Observe that every P-ultrafilter (i.e. every P-point in ω^*) is a nowhere dense ultrafilter.

Theorem 1. *There exists an atomless, complete, σ-centered Boolean algebra without any countable atomless regular subalgebras iff there exists a nowhere dense ultrafilter.*

By a recent result of Saharon Shelah [7] there exists a model of ZFC in which there are no nowhere dense ultrafilters. So it is consistent with ZFC that there are no atomless, complete, σ-centered Boolean algebras without any countable regular subalgebras.

In the first part of the paper, forcing methods are used to show that nowhere dense ultrafilters exist whenever there exists a σ-centered forcing $\mathbb{P}$ such that above every element of $\mathbb{P}$ there are two incompatible ones and $\mathbb{P}$ does not add any Cohen real. The forcing constructed here uses some ideas from Gitik and Shelah [5]. They have shown that if $\mathbb{P}$ is a σ-centered forcing notion, $\{A_n : n < \omega\}$ are subsets of $\mathbb{P}$ witnessing this, and both $\mathbb{P}$ and A_n's are Borel, then $\mathbb{P}$ adds a Cohen real. On the other hand it is known that a forcing $\mathbb{P}$ adds a Cohen real iff the complete Boolean algebra $\mathbb{B} = RO(\mathbb{P})$ contains an element u such that the reduced Boolean algebra $\mathbb{B}|u$ has a regular infinite free Boolean subalgebra. Thus, to prove the Theorem 1 we need to show in particular the following:

Theorem 2. *If there exists a σ-centered forcing $\mathbb{P}$ such that above every element of $\mathbb{P}$ there are two incompatible ones and $\mathbb{P}$ does not add any Cohen real then there exists a nowhere dense ultrafilter on ω.*

We shall proceed with the proof by some definitions and a lemma.

Definition 2. (a) *A forcing $\mathbb{P}$ is called σ-centered if $\mathbb{P} = \bigcup\{A_n : n < \omega\}$ where each A_n is directed, i. e., for every $p, q \in A_n$ there exists $r \in A_n$ such that $p \leqslant r$ and $q \leqslant r$.*

(b) *A forcing $\mathbb{P}$ adds a Cohen real if there exists a $\mathbb{P}$–name $\underline{\underline{r}} \in {}^{\omega} 2$ such that for every open dense set $\mathcal{D} \subset {}^{\omega}2$ we have $\Vdash_{\mathbb{P}}$ "$\underline{\underline{r}} \in \mathcal{D}^*$", where $\mathcal{D}^*$ denotes the encoding of $\mathcal{D}$ in the Boolean universe.*

Remarks .

(a) The order of forcing in this notation is inverse of the one in the Boolean algebra.

(b) We can just assume that there is a member p of $\mathbb{P}$ such that if q is above p then there are r_1 and r_2 above q which are incompatible in $\mathbb{P}$.

Definition 3. *A set $X \subseteq {}^{\omega>}2$ is somewhere dense if there exists an $\eta \in {}^{\omega>}2$ such that for every $\nu \in {}^{\omega>}2$ there is $\varrho \in X$ with $\eta^\frown\nu \trianglelefteq \varrho$, where $\eta^\frown\nu$ stands for the concatenation of η and ν and the relation $\trianglelefteq$ means that ϱ is an extension of the sequence $\eta^\frown\nu$.*

Lemma . *A filter D on ω is not nowhere dense iff it is a so-called well behaved filter, i.e., there is a function $f\colon \omega \to {}^{\omega>}2$ such that for every $B \in D$ the range of f restricted to B is somewhere-dense.*

Proof. Suppose $f\colon \omega \to {}^{\omega}2$ be such that for every $B \in D$ the image of B is not nowhere dense. Without loss of generality we can assume that the range of f is dense in itself. Since every closed and dense in itself subset of the Cantor cube ${}^{\omega}2$ is homeomorphic to the whole ${}^{\omega}2$ we can assume also that the range of f is dense in ${}^{\omega}2$. Moreover, since it is countable it can be identified with a subset of the set ${}^{\omega>}2$ of all rational points of the Cantor set. Thus without loss of generality we can assume that f maps ω into ${}^{\omega>}2$. On the other hand a set $X \subseteq {}^{\omega>}2$ is nowhere dense whenever for every $\eta \in {}^{\omega>}2$ there exists some $\nu \in {}^{\omega>}2$ such that the set of all sequences extending $\eta^\frown\nu$ is disjoint from X. Therefore, since the image of B under f is not nowhere dense in ${}^{\omega>}2$, it can be identified with a somewhere dense subset of ${}^{\omega>}2$. This in fact completes the proof of the lemma. □

Remark . If D is a filter on ω and $\mathcal{P}(\omega)/\mathrm{D}$ is infinite then D is not nowhere dense. Indeed, if $\langle A_n\colon n < \omega\rangle$ is a partition of ω such that $\omega \setminus A_n \notin D$ for all $n < \omega$ and $\langle e_n\colon n < \omega\rangle$ list the set ${}^{\omega>}2$ then the map $f\colon \omega \to {}^{\omega>}2$ defined by the formula

$$f(e) = e_n \quad \text{iff} \quad e \in A_n$$

witnes "D is well behaved".

Proof. [of Theorem 2] Assume that there are no nowhere dense ultrafilters. Further assume that $\mathbb{P}$ is a forcing in which above each element there are two incompatible ones and $\mathbb{P} = \bigcup\{A_n\colon n < \omega\}$ where each A_n is directed. We start with the following known fact which we prove here for the sake of completeness: □

Fact (0)**.** *Every forcing $\mathbb{Q}$ with Knaster condition such that above every element of $\mathbb{Q}$ there are two incompatible ones, adds a real.*

In fact, by assumption, forcing with $\mathbb{Q}$ adds a new subset to $\mathbb{Q}$, hence a new subset to some ordinal. In the set

$$\mathcal{K} = \{(\alpha, p, \underset{\sim}{\tau})\colon p \in \mathbb{Q}, \alpha \text{ an ordinal and } \underset{\sim}{\tau} \text{ a } \mathbb{Q}-\text{name of a subset of } \alpha \text{ such that } p \Vdash \text{“}\underset{\sim}{\tau} \notin V\text{”}\}$$

we choose $(\alpha, p, \underset{\sim}{\tau})$ with α being minimal. So necessarily α is a cardinal and

$$p \Vdash \text{“the tree } ({}^{\alpha>}2, \trianglelefteq) \text{ has a new } \alpha\text{-branch in } V^{\mathbb{Q}}\text{”}$$

So, as $\mathbb{Q}$ satisfies the Knaster condition (which follows from σ-centered), necessarily $\mathrm{cf}(\alpha) = \aleph_0$ and letting $\alpha = \bigcup_{n<\omega} \alpha_n$, where $\alpha_n < \alpha_{n+1}$ for some countable $w \subseteq {}^{\alpha>}2$ we get

$$p \Vdash \text{“}(\forall n < \omega)(\underset{\sim}{\tau} \restriction \alpha_n \in w)\text{”},$$

so $p \Vdash$ “we add a new subset to $w, |w| = \aleph_0$”.

We have shown that $I = \{p \in \mathbb{Q}\colon p \Vdash \text{“}\underset{=}{r} \in {}^{\omega}2 \text{ is new ” for some } \mathbb{Q}-\text{name } \underset{=}{r}\}$ is a dense subset of $\mathbb{Q}$. So let $\{p_i\colon i < \omega\} \subseteq I$ be a maximal antichain and let $\underset{=}{r}_i$ be such that $p_i \Vdash$ “$\underset{=}{r}_i$ is new ”. By density of I we can define the $\mathbb{Q}$-name $\underset{=}{r}$ as follows: $\underset{=}{r} = \underset{=}{r}_i$ if $p_i \in G_{\mathbb{Q}}$. This completes the proof of Fact (0).

Now we fix a $\mathbb{P}$-name of a new real $\underset{=}{r} \in {}^{\omega}2$ added by $\mathbb{P}$. For every $p \in \mathbb{P}$ we set

$$T_p = \{\eta \in {}^{\omega>}2\colon \neg(p \Vdash \neg(\text{“}\eta \trianglelefteq \underset{=}{r}\text{”}))\},$$

i.e., $\eta \in T_p$ iff there exists $q \in \mathbb{P}$ such that $p \leqslant q$ and $q \Vdash$ “$\eta = \underset{=}{r} \restriction \lg\eta$”, where $\lg \eta$ denotes the length of the sequence η.

Fact (1). *For every $p \in \mathbb{P}$, T_p is a subtree of ${}^{\omega>}2$, i.e $\eta \trianglelefteq \nu$ and $\nu \in T_p$ implies $\eta \in T_p$ and $\langle\rangle \in T_p$, where $\langle\rangle$ denotes the empty sequence.*

Indeed, if $\eta \trianglelefteq \nu$ and $\nu = \underset{=}{r} \restriction \lg \nu$, then $\eta = \underset{=}{r} \restriction \lg \eta$.

Fact (2). *The tree T_p has no maximal elements.*

To prove the Fact (2) we fix $\eta \in T_p$. Then there is $q \in \mathbb{P}$ such that $p \leqslant q$ and

$$q \Vdash \text{“}\underset{=}{r} \restriction \lg \eta = \eta\text{”}.$$

Let $k = \lg(\eta)$, so $I = \{r \in \mathbb{P}\colon r \text{ forces a value to } \underset{=}{r} \restriction (k+1)\}$ is a dense and open subset of $\mathbb{P}$, hence there is $q' \in \mathbb{P}$ such that $q \leqslant q'$ and q' forces a value to $\underset{=}{r} \restriction (k+1)$, say ϑ. So q' also forces $\underset{=}{r} \restriction k = \vartheta \restriction k$, but $q \leqslant q'$ and $q \Vdash$ “$\underset{=}{r} \restriction k = \eta$ hence $\vartheta \restriction k = \eta$”. As q' witnesses $\vartheta \in T_p$ and $\vartheta \in {}^{k+1}2$ and $\eta \in {}^{k}2$, $\eta \trianglelefteq \vartheta$, this completes the proof of Fact (2).

Fact (3). *The set* $\lim T_p$ *of all* ω*-branches through* T_p *is closed, i.e., if* $\eta \in {}^{\omega}2 \setminus \lim T_p$ *then there exists* $\nu \in {}^{\omega>}2$ *such that* $\nu \trianglelefteq \eta$ *and the set of all* ω*-branches extending* ν *is disjoint from* $\lim T_p$.

Indeed, if $\eta \in {}^{\omega}2 \setminus \lim T_p$ then there exists $n \in \omega$ such that $n \leqslant m < \omega$ implies $\eta \restriction m \notin T_p$. By Fact 1 it is clear that every ω-branch extending $\nu = \eta \restriction n$ does not belong to T_p, which proves the Fact 3.

Now let us observe that the family

$$\{T_p \colon p \in A_n\}$$

is directed under inclusion, i.e. if $p, q \in A_n$ and $r \in \mathbb{P}$ is such that $p \leqslant r$ and $q \leqslant r$ then

$$T_r \subseteq T_p \cap T_q.$$

Indeed, if $\eta \in {}^{\omega>}2$ and there exists $s \geqslant r$ such that $s \Vdash$ "$\eta = \underset{\sim}{r} \restriction \lg \eta$" then of course $s \geqslant p$ and $s \geqslant q$ and thus η belongs to T_p and T_q.

So by compactness of ${}^{\omega}2$ and Facts 1-3 we get the following:

Fact (4). *The set*

$$T_n = \bigcap \{T_p \colon p \in A_n\}$$

is a subtree of ${}^{\omega>}2$ *and the set of* ω*-branches of* T_n *is non-empty.*

Now we make a choice:

$$\eta^*_n \quad \text{is an } \omega \text{ - branch of } T_n. \tag{1}$$

Subsequently for every $n < \omega$ and every $p \in A_n$ we define

$$B^n_p = \{k < \omega \colon (\exists q \in \mathbb{P})(p \leqslant q \wedge q \Vdash \text{"}\underset{\sim}{r} \restriction k = \eta^*_n \restriction k \ \ \& \ \ \underset{\sim}{r}(k) \neq \eta^*_n(k)\text{"})\}$$

We have the following:

Fact (5). *For every* $n < \omega$ *and every* $p \in A_n$ *the set* B^n_p *is infinite.*

Indeed, since $p \in A_n$ and T_n is a subtree of T_p, η^*_n is an ω-branch of T_p. Let us fix $m < \omega$. Then, by the definition of T_p, there exists $r \in \mathbb{P}$ such that $r \geqslant p$ and

$$r \Vdash \text{"}\eta^*_n \restriction m = \underset{\sim}{r} \restriction m\text{"}.$$

On the other hand

$$\Vdash_{\mathbb{P}} \text{"}\underset{\sim}{r} \neq \eta^*_n\text{"},$$

because $\underset{\sim}{r}$ is a new real. Thus for some $q \in \mathbb{P}$, $q \geqslant r$ and $k < \omega$ we get

$$q \Vdash \text{"}\underset{\sim}{r} \restriction k \neq \eta^*_n \restriction k\text{"}.$$

We can assume that k is minimal with such a property. Since $r \leqslant q$, it must be $k > m$. But $q \geqslant p$ and thus, by minimality of k, we have $k - 1 \in B^n_p$, which proves the Fact 5.

Now we establish for every $n<\omega$ the following definition:

$$\mathcal{D}^0_n=\{B\subseteq\omega\colon(\exists p\in A_n)(|B^n_p\setminus B|<\omega)\}.$$

Fact (6). *For every $n<\omega$, $\mathcal{D}^0_n$ is a filter.*

Indeed, let $B_1,B_2\in\mathcal{D}^0_n$. Then there exist $p_1,p_2\in A_n$ such that both $B^n_{p_1}\setminus B_1$ and $B^n_{p_2}\setminus B_2$ are finite. Since A_n is directed we can choose $r\in A_n$ such that $p_1\leqslant r$ and $p_2\leqslant r$. On the other hand, from the definition of B^n_p it easily follows that

$$p\leqslant q\quad\text{implies}\quad B^n_q\subseteq B^n_p.$$

Thus $B^n_r\subseteq B^n_{p_1}\cap B^n_{p_2}$ and therefore

$$B^n_r\setminus(B_1\cap B_2)\subseteq(B^n_{p_1}\setminus B_1)\cup(B^n_{p_2}\setminus B_2)$$

is finite. Clearly, every superset of an element of $\mathcal{D}^0_n$ also belongs to $\mathcal{D}^0_n$ and, by the Fact 5, $\mathcal{D}^0_n$ does not contain the empty set, which completes the proof of Fact 6.

Now by Fact 5 and Fact 6, we can make the following choice: for $n<\omega$

$$\mathcal{D}_n \text{ is a non-principal ultrafilter containing } \mathcal{D}^0_n \tag{2}$$

By our hypothesis the ultrafilters $\mathcal{D}_n$ are not nowhere dense and so by Lemma for every $n<\omega$ we can choose a function $f_n\colon\omega\to{}^{\omega>}2$ such that

$$(\forall B\in\mathcal{D}_n)(\exists u\in{}^{\omega>}2)(\forall\nu\in{}^{\omega>}2)(\exists k\in B)(u^\frown\nu\trianglelefteq f_n(k)). \tag{3}$$

Without loss of generality we may assume that the empty sequence does not belong to the range of f_n.

Now we have to come back to the sequence $\{\eta^*_n\colon n<\omega\}$ of ω-branches of the trees T_n. Since it can happen that the sequence is not one-to-one we consider the set

$$Y=\{n<\omega\colon\eta^*_n\notin\{\eta^*_m\colon m<n\}\}.$$

Then for $n,m\in Y$ we have $\eta^*_n\neq\eta^*_m$ whenever $n\neq m$.

In the sequel we shall need the following:

Claim . *If $\langle\eta_n\colon n<\omega\rangle\subseteq{}^{\omega}2$ is a sequence of distinct ω-branches of a tree $T\subseteq{}^{\omega>}2$ there exists an increasing sequence $\langle e_n\colon n<\omega\rangle\subseteq\omega$ such that for all $n<m<\omega$ we have*

$$\{\eta_n\restriction l\colon e_n<l<\omega\}\cap\{\eta_m\restriction l\colon e_m<l<\omega\}=\emptyset. \tag{$*$}$$

To prove the claim observe that $\eta_n \restriction l \neq \eta_m \restriction l$ and $k > l$ implies $\eta_n \restriction k \neq \eta_m \restriction k$. Now assume that $e_0, \ldots, e_n$ are defined so that the condition $(*)$ holds true. Since $\eta_{n+1} \notin \{\eta_0, \ldots, \eta_n\}$ there exists $k < \omega$ such that $\eta_0 \restriction k, \ldots, \eta_n \restriction k, \eta_{n+1} \restriction k$ are pairwise different. We can assume that $k > e_n$ and e_{n+1} to be the first such k. This completes the proof of the claim.

Now using the claim we can choose an increasing sequence $\langle e_n \colon n < \omega \rangle \subseteq \omega$ in such a way that, letting

$$C_n = \{\eta^*_n \restriction l \colon e_n \leqslant l < \omega\},$$

the sequence $\langle C_n \colon n \in Y \rangle$ consists of pairwise disjoint sets, and so that we have

$$\eta^*_n = \eta^*_m \Leftrightarrow e_n = e_m \Leftrightarrow C_n = C_m.$$

Finally, for $\eta \in {}^{\omega}2$ we define

$$u(\eta) = \{n \in Y : (\exists l < \omega)(\eta \restriction l = \eta^*_n \restriction l \wedge (\forall m < n)(\eta \restriction l \neq \eta^*_m \restriction l))\},$$

$n_k(\eta) =$ the k–th member of $u(\eta)$,

$$m_k(\eta) = \min\{m < \omega : e_{n_k(\eta)} < m \wedge \eta \restriction (m+1) \ntrianglelefteq \eta^*_{n_k(\eta)}\},$$

i.e. $m_k(\eta)$ is the smallest $m > e_{n_k(\eta)}$ such that

$$\eta \restriction (m+1) \neq \eta^*_{n_k(\eta)} \restriction (m+1).$$

By definition of $m_k(\eta)$, we have

$$e_{n_k(\eta)} < m_k(\eta).$$

Clearly we also have

(i) $u(\eta)$ is well-defined,
(ii) $n_k(\eta)$ is well–defined if $k < |u(\eta)|$,
(iii) $m_k(\eta)$ is well–defined if $k < |u(\eta)|$ and $\eta \neq \eta^*_{n_k}$.

Now we can define a function $\tau : {}^{\omega}2 \setminus \{\eta^*_n : n < \omega\} \to {}^{\omega\geq}2$ by the formula:

$$\tau(\eta) = f_{n_0(\eta)}(m_0(\eta))^\frown f_{n_1(\eta)}(m_1(\eta))^\frown \cdots,$$

where, for $n < \omega$, f_n is the function from the condition (3). From the formula it follows easily that $\tau(\eta) \in {}^{\omega\geq}2$ and it is well defined if $\eta \notin \{\eta^*_n : n < \omega\}$ and moreover $\tau(\eta)$ is infinite whenever $u(\eta)$ is infinite, as $\langle\rangle \notin$ Range (f_n).

To complete the proof of the theorem it remains to show:

Fact (7). $\Vdash_{\mathbb{P}}$ *"$\tau(\underset{\sim}{r})$ is Cohen over V".*

To prove this fact we fix an open dense set $I \subseteq {}^{\omega>}2$ and a $p \in \mathbb{P}$ and we show that there is a $q \in \mathbb{P}$ with $p \leq q$ such that $q \Vdash_{\mathbb{P}}$ "$\tau(\underline{r}) \in [I]$", where $[I]$ is the name of $\{\eta \in {}^{\omega}2 : t \trianglelefteq \eta$ for some $t \in I\}$ in the generic extension. Let $n < \omega$ be such that $p \in A_n$ and let $n^\otimes = \min\{m < \omega : \eta^*_m = \eta^*_n\}$. Clearly $n^\otimes \leq n$ and $n^\otimes \in Y$. Then $u(\eta^*_n)$ is well defined and $n^\otimes \in u(\eta^*_n)$; in fact $n^\otimes$ is the last member of $u(\eta^*_n)$. Let $k = |u(\eta^*_n)| - 1$, so $n_k(\eta^*_n) = n^\otimes$. Also $m_i(\eta^*_n)$ is well defined and finite for $i < k$. Then we set

$$\nu^\otimes = f_{n_0(\eta^*_n)}(m_0(\eta^*_n))^\frown \cdots {}^\frown f_{n_{k-1}(\eta^*_n)}(m_{k-1}(\eta^*_n)),$$

so if $k = 0$, i.e., if $u(\eta^*_n)$ is a singleton, then $\nu^\otimes$ is the empty sequence.

Clearly $\nu^\otimes \in {}^{\omega>}2$. Also we have

$$p \nVdash_{\mathbb{P}} \text{"}\underline{r} \restriction (e_n + 1) \ntrianglelefteq \eta^*_n\text{"}.$$

Hence

$$p \nVdash_{\mathbb{P}} \text{"}\neg\varphi\text{"},$$

where φ is the formula asserting $u(\eta^*_n)$ is an initial segment of $u(\underline{r})$. Note that φ implies $(\forall i < k)(n_i(\underline{r}) = n_i(\eta^*_n)) \wedge m_i(\underline{r}) = m_i(\eta^*_n)$. Since $p \Vdash_{\mathbb{P}}$ "$\underline{r} \neq \eta^*_{n^\otimes}$", it follows that

$$p \Vdash_{\mathbb{P}} \text{"}\varphi \rightarrow m_k(\underline{r}) \text{ is well–defined"}.$$

Let

$$Z = \{\varrho \in {}^{\omega>}2 : p \nVdash_{\mathbb{P}} \text{"}\neg(\varphi \wedge f_{n_k(\underline{r})}(m_k(\underline{r})) = \varrho)\text{"}\}.$$

It is enough to show that Z is a somewhere dense subset of ${}^{\omega>}2$. [Suppose that Z is a somewhere dense subset of ${}^{\omega>}2$. Then there is $\varrho_0 \in {}^{\omega>}2$ such that for any $\nu \in {}^{\omega>}2$ there is $\varrho \in Z$ with $\varrho_0{}^\frown\nu \trianglelefteq \varrho$. Let $\tilde{\varrho}_0 = \nu^{\otimes\frown}\varrho_0$ and let $\nu \in {}^{\omega>}2$ be such that $\tilde{\varrho}_0{}^\frown\nu \in I$. Then there is $\varrho \in Z$ such that $\tilde{\varrho}_0{}^\frown\nu \trianglelefteq \varrho$. Let $q \geq p$ be such that $q \Vdash_{\mathbb{P}}$ "$\varphi \wedge f_{n_k}(\underline{r}) = \varrho$". Then $q \Vdash_{\mathbb{P}}$ "$\tilde{\varrho}_0{}^\frown\nu \trianglelefteq \tau(\underline{r})$". And hence we can conclude that $q \Vdash_{\mathbb{P}}$ "$\tau(\underline{r}) \in [I]$".]

Now, we have

$$p \nVdash_{\mathbb{P}} \text{"}\neg(n_k(\underline{r}) = n^\otimes \vee \neg\varphi)\text{"}.$$

Hence

$$Z = \{\varrho \in {}^{\omega>}2 : p \nVdash_{\mathbb{P}} \text{"}\neg(f_{n^\otimes}(m_k(\underline{r})) = \varrho \wedge \varphi)\text{"}\}.$$

Thus, by the choice of $f_{n^\otimes}$, it is enough to prove:

$$B_0 = \{m < \omega : p \nVdash_{\mathbb{P}} \text{"}m_k(\underline{r}) \neq m \vee \neg\varphi\text{"}\} \in \mathcal{D}_{n^\otimes}.$$

[Suppose that $B_0 \in \mathcal{D}_{n^\otimes}$. Then, by (3), there is $\varrho \in {}^{\omega>}2$ such that $(\forall \nu \in {}^{\omega>}2)(\exists k \in B_0)(\varrho{}^\frown\nu \trianglelefteq f_{n^\otimes}(k))$.]

We have $\mathcal{D}_{n^\otimes} = \mathcal{D}_n$. Hence it is enough to show $B_0 \in \mathcal{D}_n$. By definition of $m_k(\underline{r})$ and since $\varphi \rightarrow n_k(\underline{r}) = n^\otimes$, this is equivalent to:

$$\{m < \omega : p \nVdash_{\mathbb{P}} \text{"}\underline{r} \restriction m \neq \eta^*_{n^\otimes} \restriction m \vee \underline{r}(m+1) = \eta^*_{n^\otimes}(m+1) \vee \neg\varphi\text{"}\} \in \mathcal{D}_n.$$

But $\eta^*_{n\otimes} = \eta^*_n$ and $p \in A_n$. Hence, by definition of $\mathcal{D}^0_n$, the set above does belong to $\mathcal{D}^0_n \subseteq \mathcal{D}_n$. □

Finally we prove that the converse to Theorem 2 is also true, i. e., we shall show that whenever there exists a nowhere dense ultrafilter there exists a σ-centered forcing $\mathbb{P}$ with the property that above each element there are two incompatible ones and moreover $\mathbb{P}$ does not add a Cohen real. To prove this fact we shall use some topological methods, but we can also write it using forcing.

Recall, a subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{A}$ is *regular whenever* $\sup_{\mathbb{A}} X = 1$ *for every* $X \subseteq \mathbb{B}$ *such that* $\sup_{\mathbb{B}} X = 1$. *The subalgebra* $\mathbb{B}$ *is regular iff the corresponding map of the Stone spaces is semi-open, i. e., the image of every non-empty clopen set has non-empty interior. Using nowhere dense ultrafilters we construct a dense in itself, separable, extremally disconnected compact space (= Stone space of an atomless,* σ*-centered, complete Boolean algebra) which has no semi-open continuous maps onto the Cantor set.*

We use a topology on the set ${}^{\omega>}\omega = \bigcup\{{}^n\omega\colon n < \omega\}$. *If* $s \in {}^{\omega>}\omega$ *is a sequence of length* n *and* $k \in \omega$, *then* $s^\frown k$ *denotes the sequence of length* $n+1$ *extending* s *in such a way that the* n*-th term is* k. *For a set* $A \subseteq \omega$ *we set* $s^\frown A = \{s^\frown k\colon k \in A\}$. *For a given ultrafilter* $p \subseteq \mathcal{P}(\omega)$ *we consider a topology* $\mathcal{T}_p$ *on* ${}^{\omega>}\omega$ *given by the formula:*

$$U \in \mathcal{T}_p \text{ iff for every } s \in U \text{ there exists } A \in p \text{ such that } s^\frown A \subseteq U.$$

The set ${}^{\omega>}\omega$ *equipped with the topology* $\mathcal{T}_p$ *we denote* G_p. *The space* G_p *is known to be Hausdorff and extremally disconnected; see e. g. Dow, Gubbi and Szymanski, (*[4]*). Hence the Čech-Stone extension* βG_p *is extremally disconnected, compact, separable, and dense in itself.*

Under a much stronger assumption that there exists a P–point the next theorem was proved by A. Blass [1].

Theorem 3. *If there exists a nowhere dense ultrafilter then there exists a* σ*-centered forcing* $\mathbb{P}$ *such that above every element of* $\mathbb{P}$ *there are two incompatible ones and* $\mathbb{P}$ *does not add any Cohen real.*

Proof. By virtue of a theorem of Silver, it is enough to show that there exists a σ-centered, complete, atomless Boolean algebra $\mathbb{B}$ such that $\mathbb{B}$ does not contain any regular free subalgebra. For this goal we shall use the topological space G_p described above. It remains to show that whenever p is a nowhere dense ultrafilter and $f\colon \beta G_p \to {}^\omega\{0,1\}$ is continuous, then there exists a non-empty clopen set $H \subseteq \beta G_p$ such that $\operatorname{int} f(H) = \emptyset$.

First of all we notice that since p is a nowhere dense ultrafilter, for every $s \in {}^{\omega>}\omega$ there exists $A_s \in p$ such that

$$\operatorname{int}\operatorname{cl} f(s^\frown A_s) = \emptyset. \tag{4}$$

In the sequel L_n will denote the set of all sequences of length n, i. e., L_n is the n-th level of the tree ${}^{\omega>}\omega$. In particular, $L_0 = \{s_0\}$ is the empty sequence. By induction we define a sequence of sets $\{U_n \colon n < \omega\}$ such that $U_n \subseteq L_n$ for every $n < \omega$ and, moreover

$$\operatorname{int}\operatorname{cl} f(U_n) = \emptyset, \tag{5}$$

$$\text{for every } s \in U_n \text{ there exists } A \in p \text{ such that } s^\frown A \subseteq U_{n+1}. \tag{6}$$

We set $U_0 = \{s_0\}$ and $U_1 = s_0{}^\frown A_{s_0}$. Assume U_n is defined, say $U_n = \{s_k \colon k < \omega\}$. Then by continuity of f and the condition (4) we can choose $A_k \in p$ in such a way that $\operatorname{int}\operatorname{cl} f(s_k{}^\frown A_k) = \emptyset$ and moreover, the diameter of $\operatorname{cl} f(s_k{}^\frown A_k)$ is not greater than $\frac{1}{k}$. Clearly, s_k is an accumulation point of $s_k{}^\frown A_k$, because $A_k \in p$. Hence, for every $k < \omega$ we get

$$\operatorname{cl} f(s_k{}^\frown A_k) \cap \operatorname{cl} f(U_n) \neq \emptyset.$$

Therefore, since diameters of the sets $\operatorname{cl} f(s_k{}^\frown A_k)$ tend to zero, the set of accumulation points of the set $\bigcup\{\operatorname{cl} f(s_k{}^\frown A_k) \colon k < \omega\}$ is contained in $\operatorname{cl} f(U_n)$. Indeed, every ε-neighbourhood of the set $\operatorname{cl} f(U_n)$ has to contain all but finitely many sets of the form $\operatorname{cl} f(s_k{}^\frown A_k)$. So the set $\operatorname{cl} f(U_n) \cup \bigcup\{\operatorname{cl} f(s_k{}^\frown A_k) \colon k < \omega\}$ is closed. It is also nowhere dense as it is a countable union of nowhere dense sets and is closed. Now we set

$$U_{n+1} = \bigcup\{s_k{}^\frown A_k \colon k < \omega\}$$

and observe that

$$\operatorname{cl} f(U_{n+1}) \subseteq \operatorname{cl} f(U_n) \cup \bigcup\{\operatorname{cl} f(s_k{}^\frown A_k) \colon k < \omega\}.$$

Thus the set $f(U_{n+1})$ is nowhere dense, which completes the construction of U_n's.

By the condition (5) , there exists a dense set

$$\{x_n \colon n < \omega\} \subseteq {}^{\omega}\{0,1\} \setminus \bigcup\{\operatorname{cl} f(U_n) \colon n < \omega\}.$$

In particular, for every $n, k < \omega$ we have

$$f^{-1}(\{x_n\}) \cap \operatorname{cl} U_k = \emptyset,$$

where "cl" denotes here the closure in βG_p. Now, for every $n < \omega$ we choose a clopen set $V_n \subseteq \beta G_p$ such that

$$f^{-1}(\{x_n\}) \subseteq V_n \subseteq \beta G_p \setminus (\operatorname{cl} U_0 \cup \cdots \cup U_n). \tag{7}$$

By induction we construct a sequence $\{W_n : n < \omega\}$ such that the following conditions hold:

$$W_n \subseteq U_n \text{ for } n < \omega \text{ and } W_0 = U_0 \tag{8}$$

for every $s \in W_n$ there exists $B_s \in p$ such that

$$s ^\frown B_s \subseteq U \setminus (V_0 \cup \cdots \cup V_n), \tag{9}$$

$$W_{n+1} = \bigcup \{s ^\frown B_s : s \in W_n\}. \tag{10}$$

Assume the sets $W_0, \ldots, W_n$ are defined in such a way that (8), (9) and (10) are satisfied. Then we have in particular

$$W_n \subseteq U_n \setminus (V_0 \cup \cdots \cup V_{n-1});$$

by the condition (7) we also have

$$U_n \subseteq \beta G_p \setminus V_n.$$

Hence we get $W_n \subseteq U_n \setminus (V_0 \cup \cdots \cup V_n)$. Since the set $U_n \setminus (V_0 \cup \cdots \cup V_n)$ is open, for every $s \in W_n$ we can choose $B_s \in p$ such that $s ^\frown B_s \subseteq U_n \setminus (V_0 \cup \cdots \cup V_n)$. Then it is enough to set $W_{n+1} = \bigcup \{s ^\frown B_s : s \in W_n\}$.

Clearly the set $W = \bigcup \{W_n : n < \omega\}$ is open in G_p and $W \cap V_n = \emptyset$ for every $n < \omega$. Indeed, if $m > n$, then $W_m \cap V_n = \emptyset$ by the conditions (9) and (10), whereas for $m \leqslant n$, $W_m \cap V_n = \emptyset$ because $W_m \subseteq U_m$ and $U_m \cap V_n = \emptyset$ by the condition (7). Since V_n is a clopen set in βG_p we also have

$$\operatorname{cl} W \cap V_n = \emptyset$$

for every $n < \omega$. Since βG_p is extremally disconnected, $\operatorname{cl} W$ is clopen subset of βG_p and, by the last equality and condition (7) we get

$$f(\operatorname{cl} W) \cap \{x_n : n < \omega\} = \emptyset.$$

Therefore $f(\operatorname{cl} W)$ is nowhere dense, because $\{x_n : n < \omega\}$ is dense in ${}^{\omega}\{0, 1\}$, which completes the proof. □

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