# Cardinal Invariants $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$
# Sh643

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Abstract. This paper studies cardinal invariants $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$, the natural generalizations of the invariants $\mathfrak{b}$ and $\mathfrak{t}$ to a regular cardinal $\kappa$.

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## §0 Introduction

Cardinal invariants $\mathfrak{b}$ and $\mathfrak{t}$ were introduced by Rothberger [Ro39], [Ro48]. They are cardinals between $\aleph_1$ and $2^{\aleph_0}$ and have been extensively studied over the years. The survey paper [Bsxx] contains much information about these two invariants as well as many other cardinal invariants of the continuum.

The goal of this paper is to study the natural generalizations of $\mathfrak{b}$ and $\mathfrak{t}$ to higher regular cardinals, namely $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$ respectively, where $\kappa$ is a regular cardinal. The results presented here are that the relationship $\mathfrak{t} \leq \mathfrak{b}$ (shown by Rothberger [Ro48]) also holds for $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$ and that, assuming, e.g., that $\kappa = \kappa^{<\kappa} \geq \beth_\omega$, if $\kappa \leq \mu < \mathfrak{t}_\kappa$ then $2^\kappa = 2^\mu$. These results are then used as constraints in the forcing construction of model in which $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$ can take on essentially any preassigned regular value.

The cardinal $\mathfrak{b}_\kappa$ was studied in [CuSh 541] where it was shown that the value of $\mathfrak{b}_\kappa$ does not have any influence on the value of $2^\mu$ for $\kappa \leq \mu < \mathfrak{b}_\kappa$ even if $GCH$ is assumed to hold below $\kappa$. However, the same does not hold for $\mathfrak{t}_\kappa$ as it is shown in §2.

In an earlier version, a wrong "improvement" of §1 was used and we thank the referee for detecting this.

## §1 Conventions and elementary facts

*1.1 Notation.* 1) For cardinals $\lambda$ and $\kappa$ let $[\kappa]^\lambda = \{X \subseteq \kappa :\mid X \mid = \lambda\}$ and ${}^\lambda\kappa$ is the set of functions from $\lambda$ to $\kappa$. The symbol $\kappa^\lambda$ is used to denote the cardinality of the set $\{f : f : \lambda \to \kappa\}$.
2) For $A, B \in [\kappa]^\kappa$ let $A \subseteq^* B$ iff $|A \backslash B| < \kappa$ and $A \subset^* B$ iff $|A \backslash B| < \kappa \wedge |B \backslash A| = \kappa$. Let "$A$ is an almost subset of $B$" mean $A \subseteq^* B$. For $f, g \in {}^\kappa\kappa$ let $f <^* g$ iff $\exists\beta < \kappa\forall\alpha > \beta(f(\alpha) < g(\alpha))$. Then $\mathscr{F} \subseteq {}^\kappa\kappa$ is unbounded (or $\leq^*$-unbounded) in $({}^\kappa\kappa, <^*)$ mean that $\forall f \in \kappa^\kappa \exists g \in \mathscr{F}(g \not<^* f)$.
3) For a filter $D$ on a set $A$ let $A =$ Dom$(D)$.

**1.2 Definition.** For regular cardinal $\kappa$ let

$$\mathfrak{b}_\kappa = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \kappa^\kappa \text{ and } \mathscr{F} \text{ is } <^* \text{-unbounded in } \kappa^\kappa\},$$

$$\begin{aligned} \mathfrak{t}_\kappa = \min\{|\mathscr{T}| :& \mathscr{T} \subseteq [\kappa]^\kappa, |\mathscr{T}| \geq \kappa, \mathscr{T} \text{ is well ordered by } \subset^*, \\ & (\forall C \in [\kappa]^\kappa)(|\kappa \setminus C| = \kappa \Rightarrow \exists A \in \mathscr{T}(|A \setminus C| = \kappa)) \\ & \mathscr{F} \text{ is with no } \subseteq^* \text{-last element} \\ & \text{and has no } \subseteq^* \text{-unbounded subset of cardinality } < \kappa\}. \end{aligned}$$

In this notation $\mathfrak{b} = \mathfrak{b}_\omega$ and $\mathfrak{t} = \mathfrak{t}_\omega$.

An equivalent formulation of $\mathfrak{t}_\kappa$ is obtained if $\supseteq^*$ is used instead of $\subseteq^*$. Standard arguments show that

*1.3 Fact.* For any regular cardinal $\kappa, \kappa^+ \leq \mathfrak{b}_\kappa, \mathfrak{t}_\kappa \leq 2^\kappa$ and in the definition of $\mathfrak{b}_\kappa, \mathscr{F}$ may be assumed to be well ordered by $<^*$ and consisting only of strictly increasing functions. Thus, both $\mathfrak{b}_\kappa$ and $\mathfrak{t}_\kappa$ are regular cardinals.

**1.4 Fact.** $\mathfrak{t}_\kappa \leq \mathfrak{b}_\kappa$.

*Proof.* On the case $\kappa = \omega$ see, e.g., [Bsxx]. So assume $\kappa > \omega$ and by way of contradiction assume $\mathfrak{b}_\kappa < \mathfrak{t}_\kappa$. Let $\{f_\alpha : \alpha < \mathfrak{b}_\kappa\} \subseteq {}^\kappa\kappa$ be $<^*$-unbounded in $({}^\kappa\kappa, <^*)$ and without loss of generality by 1.3 such that $\alpha < \beta \to f_\alpha <^* f_\beta$. For each $\alpha < \mathfrak{b}_\kappa$ let $C_\alpha = \{\xi < \kappa : \forall\zeta < \xi(f_\alpha(\zeta) < \xi)\}$. Then each $C_\alpha$ is closed unbounded in $\kappa$ and $\alpha \leq \beta \to C_\beta \subseteq^* C_\alpha$. Since we are assuming that $\mathfrak{b}_\kappa < \mathfrak{t}_\kappa$, there is $A \in [\kappa]^\kappa$ such that $\forall \alpha < \mathfrak{b}_\kappa (A \subset^* C_\alpha)$. Let $f : \kappa \to A$ be such that $\forall \xi < \kappa (\xi < f(\xi))$. Fix $\alpha < \mathfrak{b}_\kappa$ and let $i_\alpha$ be such that $A \setminus i_\alpha \subseteq C_\alpha \setminus i_\alpha$. Let $\xi \in \kappa \setminus i_\alpha$ and $\zeta = \min(C_\alpha \setminus (\xi + 1))$ and note that $f_\alpha(\xi) < \zeta$ by the definition of $C_\alpha$. However, $A \setminus \xi \subseteq C_\alpha \setminus \xi$, and $\xi < f(\xi)$, so $\zeta \leq f(\xi)$. In other words, $(\forall \xi \in \kappa \setminus i_\alpha)(f_\alpha(\xi) < f(\xi))$; hence $f$ is a $<^*$-bound for $\{f_\alpha : \alpha < \mathfrak{b}_\kappa\}$. This is a contradiction and the lemma is proved. $\square_{1.4}$

## §2 Combinatorics

The goal of this section is to show that if, e.g., $\kappa^{<\kappa} = \kappa \geq \beth_\omega$ then $2^\mu = 2^\kappa$ for any $\mu$ with $\kappa \leq \mu < \mathfrak{t}_\kappa$. Naturally we start from the scheme of the proof of $\omega \leq \mu < \mathfrak{t} \to 2^\mu = 2^\omega$, namely to use $\mu < \mathfrak{t}_\kappa$ to construct a binary tree in $(\mathscr{P}(\kappa), \subset^*)$ of height $\mu$. However, when $\kappa$ is uncountable a difficulty arises in the construction at limit stages of cofinality less than $\kappa$, a case which does not occur when $\kappa = \aleph_0$. The difficulty comes from the fact that the intersection of a $\subset^*$-decreasing sequence in $[\kappa]^\kappa$ of limit length less than $\kappa$ may be empty. To deal with this difficulty, a notion of a closed subset of $\kappa$ with respect to a certain parameter is introduced next.
From where comes the condition $\kappa \geq \beth_\omega$? From using a result from pcf theory [Sh 460].

**2.1 Main Lemma.** *If $\kappa$ is regular <u>then</u> for every $\mu < \mathfrak{t}_\kappa$ we have $2^\mu = 2^\kappa$ provided that at least one of the following conditions holds:*

$CND_1$: $\kappa \geq \beth_\omega$ and $\kappa = \kappa^{<\kappa}$.

$CND_2$: diamond on $\kappa$ or at least $Dl_\kappa$ (see below, see more e.g., in [Sh 460]).

$CND_3$: there is a sequence $\bar{D} = \langle D_{\delta,\gamma} : \delta \text{ limit} < \kappa \text{ and } \gamma < \gamma_\delta \rangle$ such that:

- $(a)$ $\gamma_\delta < \kappa$ (for each limit $\delta < \kappa$)
- $(b)$ $D_{\delta,\gamma}$ is a filter on $\delta$ to which all cobounded subsets of $\delta$ belongs
- $(c)$ for every unbounded subset of $A$ of $\kappa$, for stationarily many ordinals $\delta < \kappa$ we have: for some $\gamma < \gamma_\delta, A \cap \delta \in D_{\delta,\gamma}$
- $(d)$ moreover, if $\tau < \kappa$ is regular and $A_i$ an unbounded subset of $\kappa$ for $i < \kappa$ and for $i < j < \tau, A_j$ is an almost subset of $A_i$ (i.e., $A_j \subseteq^* A_i$) <u>then</u> for stationarily many $\delta < \kappa$ some $\gamma < \gamma_\delta$ satisfies: for every $i < \tau$ we have $A_i \cap \delta \in D_{\delta,\gamma}$.

**2.2 Definition.** Let $D\ell_\kappa$ mean: $\kappa$ is regular uncountable and there is a sequence $\langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$ such that:

- $(a)$ $\mathscr{P}_\alpha$ is a family of subsets of $\alpha$
- $(b)$ $\mathscr{P}_\alpha$ has cardinality $< \lambda$
- $(c)$ for every $A \subseteq \lambda$ the set $\{\delta < \lambda : A \cap \delta \in \mathscr{P}_\delta\}$ is a stationary subset of $\lambda$.

*Proof.* By [Sh 460], $CND_1$ implies $CND_2$.

Easily $CND_2$ implies $CND_3$, so we shall assume the latter. Let $E$ be a club of $\kappa$ such that each member is a limit ordinal and $\delta < \alpha \in E$ implies $(\delta + \gamma_\delta + \omega) < \alpha$; here $\delta$ always denotes a limit ordinal. The proof is preceded by a definition and some facts.
Below $A, B$ denote subsets of $\kappa$, a fix regular uncountable cardinal; unbounded means unbounded in $\kappa$.

**2.3 Definition.** 1) A subset $A$ of $\kappa$ is called $(E, \bar{D})$-closed when: for every $\delta \in E$, if $A \cap \delta \in D_{\delta,\gamma}$ then $\delta + \gamma \in A$.
2) The atomic $(E, \bar{D})$-closure atcl$(A)$ of $A$, a subset of $\kappa$ is $A \cup \{\delta + \gamma : \delta \in E, \gamma < \gamma_\delta$ and $A \cap \delta \in D_{\delta,\gamma}\}$.
3) We define $\mathrm{cl}^\alpha(A)$ for $A$ a subset of $\kappa$ and $\alpha$ an ordinal, by induction on $\alpha$ : $\mathrm{cl}^\alpha(A) = A \cup \cup\{\mathrm{atcl}(\mathrm{cl}^\beta(A)) : \beta < \alpha\}$.
4) We define the $(E, \bar{D})$-closure of $A$, cl$(A)$ as $\mathrm{cl}^\alpha(A)$ for every large $\alpha$ large enough; (see 2.4(3)).

*2.4 Fact.* 1) $\kappa$ is $(E, \bar{D})$-closed, unbounded in $\kappa$.
2) For $\alpha < \beta$ we have $A \subseteq \mathrm{cl}^\alpha(A) \subseteq \mathrm{cl}^\beta(A)$.
3) $\mathrm{cl}^\alpha(A) \subseteq \mathrm{cl}^\kappa(A) = \mathrm{cl}^\beta(A)$ if $\alpha < \kappa \leq \beta$.
4) If $\delta \in E$ and $A \subseteq \kappa$ <u>then</u> $\mathrm{cl}(A \cap (\delta + \gamma_\delta)) = \mathrm{cl}(A) \cap (\delta + \gamma_\delta)$.

*2.5 Fact.* For $A$ a subset of $\kappa$

(a) cl$(A)$ is a $(E, \bar{D})$-closed set
(b) cl$(A)$ is the minimal $(E, \bar{D})$-closed set which includes $A$
(c) cl$(A)$ is bounded (in $\kappa$) <u>iff</u> $A$ is bounded (in fact if $A$ is a subset of $\delta \in E$ then cl$(A)$ is a subset of $\delta + \gamma_\delta$).

*2.6 Fact.* If $A$ is $(E, \bar{D})$-closed and unbounded, and $B$ is an unbounded, almost a subset of $A$ <u>then</u> cl$(B)$ is an unbounded, $(E, \bar{D})$-closed almost subset of $A$.

*Proof.* By 2.4(4).

*2.7 Fact.* cl$(A)$ is the increasing union of $\mathrm{cl}(A \cap \alpha)$ for $\alpha < \kappa$.

*2.8 Fact.* If $A$ is $(E, \bar{D})$-closed and unbounded (subset of $\kappa$) <u>then</u> we can find two disjoint $(E, \bar{D})$-closed unbounded subsets of it.

*Proof.* We choose by induction on $i < \kappa$, ordinals $\alpha_i, \beta_i$ such that:

$(*)$ $\alpha_i, \beta_i$ are distinct members of $A$ and larger than the supremum of the $(E, \bar{D})$-closure of $\{\alpha_i, \beta_j : j < i\}$.

There is no problem to do it and $\mathrm{cl}(\{\alpha_i : i < \kappa\})$, $\mathrm{cl}(\{\beta_i : i < \kappa\}$ are two sets as required. $\square_{2.8}$

*2.9 Fact.* If $A_i$ is a $(E, \bar{D})$-closed unbounded subset of $\kappa$ for $i < \tau, \tau$ a regular cardinal $< \kappa$ and for $i < j < \tau$ the set $A_j$ is an almost subset of $A_i$ then their intersection is a $(E, \bar{D})$-closed unbounded subset of $\kappa$.

*Proof of 1.2.* By the last demand in $CND_3$, i.e., clause (d). $\square_{2.9}$

Continuation of the proof of 2.1: Now let $\mu < \mathfrak{t}_\kappa$. We choose by induction on $\zeta \le \mu$ for every sequence $\eta$ of zeroes and ones of length $\zeta$, a set $A_\eta$ such that:

$(A)$ $A_\eta$ is a subset of $B$ of cardinality $\kappa$

$(B)$ $A_\eta$ is $(E, \bar{D})$-closed

$(C)$ if $\rho$ is an initial segment of $\eta$ then $A_\eta \backslash A_\rho$ has cardinality $< \kappa$

$(D)$ if $\rho \in {}^\varepsilon 2, \varepsilon < \zeta$ then $A_{\rho\hat{\ }<0>}, A_{\rho\hat{\ }<1>}$ are disjoint.

If we succeed, clearly $\{A_\rho : \rho$ a sequence of zeroes and ones of length $\mu\}$ is a family of $2^\mu$ pairwise almost disjoint subsets of $\kappa$, so $2^\mu \le 2^\kappa$ thus finishing.
In stage $\zeta = 0$ use fact 2.4(1).
In limit stages of cofinality $\ge \kappa$, we use the hypothesis $\mu < \mathfrak{t}_\kappa$ to get an unbounded $A^0_\rho$, almost included in each $A_{\rho \restriction \varepsilon}$ for $\epsilon < \zeta$. Let $A_\rho$ be $\mathrm{cl}(A^0_\rho)$, it is $(E, \bar{D})$-closed (by fact 2.5, clause (a)) is unbounded (by fact 2.5, clause (c)) and is almost a subset of $A_{\rho \restriction \epsilon}$ for each $\epsilon < \zeta$ by Fact 2.6.
In limit stages $\zeta$ of cofinality $< \kappa$, we choose an increasing sequence $\langle \epsilon_i : i < \mathrm{cf}(\zeta) \rangle$ of ordinals $< \zeta$ converging to $\zeta$. We let $A^0_\rho = \cap\{A_{\rho \restriction \epsilon_i} : i < \mathrm{cf}(\zeta)\}$. By the Fact 2.9, $A^0_\rho$ has cardinality $\kappa$. Let $A_\rho$ be the $(E, \bar{D})$-closure of $A^0_\rho$, now $A_\rho$ is an unbounded $(E, \bar{D})$-closed subset of $\kappa$ (see Fact 2.5, clauses (c), (a)), it is almost subset of each $A_{\rho \restriction \varepsilon_i}$ by Fact 2.6, hence is as required.
Lastly, for successor stages use Fact 2.8. $\square_{2.1}$

*2.10 Remark.* In Lemma 2.1 it suffices to assume the following variant.
$CND_4$: we can find a set $\mathscr{D}$ such that:

$(a)$ $\mathscr{D}$ is a family of $\le \kappa$ filters

($b$) each filter $D$ from $\mathscr{D}$, is a filter on some $\alpha = \alpha[D] < \kappa$

($c$) if $A \in [\kappa]^\kappa$ then for some $D \in \mathscr{D}$ we have[1] $A \cap \alpha[D] \in D$

($d$) if $\theta = \operatorname{cf}(\theta) < \kappa$ and $A_i \in [\kappa]^\kappa$ for $i < \kappa$ and $i < j < \theta \Rightarrow A_j \subseteq A_i$ then for some $D \in \mathscr{D}$ we have $j < \theta \Rightarrow A_j \in D$

($e$) for each $\beta < \kappa$, the set $\{D : \beta \cap \alpha[D] \in D\}$ has cardinality $< \kappa$.

Why? First note that for some $\bar{D}$

($*$) $\bar{D} = \langle D_i : i \in S^* \rangle$ list $\mathscr{D}$ with no repetitions where $\alpha[D_i] \le i$ and $S^* \subseteq \kappa$ is unbounded.
[Why? Note that $|\mathscr{D}| \le \kappa$ by clause (a) but if $|\mathscr{D}| < \kappa$ then $\alpha^* = \cup\{\alpha[D] : D \in \mathscr{D}\}$ is $< \kappa$ so $A =: (\alpha, \kappa) \in [\kappa]^\kappa$ but by clause (c) there is $D \in \mathscr{D}$, such that $A \cap \alpha[D] \in D$ easy contradiction. So together $|\mathscr{D}| = \kappa$, let $\langle D^0_i : i < \kappa \rangle$ list $\mathscr{D}$, and let us define $\zeta(i) < \kappa$ strictly increasing such that $\alpha[D^0_i] < \zeta(i)$. Now let $S^* = \{\zeta(i) : i < \lambda\}, D_{\zeta(i)} = D^0_i$.]

Also let $E$ be $\{\delta < \lambda : \delta = \sup(S^* \cap \delta), \delta$ limit ordinal and for no $\alpha < \delta \le \zeta \in S^*$ do we have $\alpha \cap \operatorname{Dom}(D_\zeta) \in D_\zeta\}$, so $E$ is a club of $\lambda$, and continue as above.

[1] we can let $\mathscr{D}$ be over $\operatorname{Dom}(D) \subseteq \alpha$, no real difference

## §3 Forcing

For regular $\kappa$ we know that $\kappa < \mathfrak{t}_\kappa \leq \mathfrak{b}_\kappa$, both regular, so we may wonder are there additional restrictions.

We use the previous section by which if $\diamondsuit_\kappa, 2^\kappa < 2^\lambda$ then $\mathfrak{t} \leq \lambda$, so making $\mathfrak{b}_k$ larger than some such $\lambda$ guarantees this.

Let $\kappa$ be a regular uncountable cardinal and let $\lambda$, $\mu$, $\theta$ be cardinals such that $\kappa < \lambda \leq \mu \leq \theta$ with $\lambda$, $\mu$ regular and $\mathrm{cf}(\theta) \geq \lambda$. This section deals with the construction of a model for $\mathfrak{t}_\kappa = \lambda$, $\mathfrak{b}_\kappa = \mu$ and $2^\kappa = \theta$. The idea behind the construction is as follows: Start with a countable transitive model (c.t.m.) $N$ for $ZFC + GCH$. Expand $N$ to a model $M$ by forcing with the standard partial order for adding $\theta^+$ many subsets of $\lambda$ (see below). Then

$$M \models \text{“}\forall \xi < \lambda (2^\xi = \xi^+ \wedge 2^\lambda = \theta^+)\text{”}.$$

In $M$, perform an iterated forcing construction with $< \kappa$-supports of length $\theta \cdot \mu$ (ordinal product) with $\kappa$-closed and $\kappa^+$-c.c. partial orders as follows: At stages which are not of the form $\theta \cdot \xi$ $(\xi < \mu)$ towers in $(\mathscr{P}(\kappa), \subset^*)$ of height $\eta$ are destroyed for $\kappa < \eta < \lambda$. At stages of the form $\theta \cdot \xi$ a function from $\kappa$ to $\kappa$ is added to eventually dominate all the functions from $\kappa$ to $\kappa$ constructed by that stage. The bookkeeping is arranged in such a way that by the end of the construction all towers of height $\eta$ for $\kappa < \eta < \lambda$ are considered so that in the final model $\mathfrak{t}_\kappa \geq \lambda$. However, in the final model

$$\forall \xi ((\xi < \kappa \to 2^\xi = \xi^+) \wedge (\kappa \leq \xi < \lambda \to 2^\xi = \theta)) \wedge 2^\lambda = \theta^+$$

so that, by the previous section, $\mathfrak{t}_\kappa = \lambda$. By virtue of adding dominating functions at stages of the form $\theta \cdot \xi$, the final model has a scale in $(\kappa^\kappa, <^*)$ of order type $\mu$ so that $\mathfrak{b}_\kappa = \mu$.

The rest of this section deals with the details of the construction. In showing that the final model has the desired properties it is important to know that cardinals are not collapsed. A standard way of proving this is to show that the final partial order obtained by the iteration is $\kappa$-closed and has the $\kappa^+$-cc this follows (see [Sh 80]) but a self-contained proof is given. And to show that the final partial order has the two properties, the names for the partial orders used in the iteration must be carefully selected. The discussion here will be analogous to the discussion in the final section of [Ku83] which deals with countable support iterations. Also many proofs are omited here since they are analogous to the proofs of the corresponding facts in [Ku83].

**3.1 Definition.** Let $\mathbb{P}$ be a partial order and $\pi$ a $\mathbb{P}$-name for a partial order. $\pi$ is full for $< \kappa$-sequences iff whenever $\alpha < \kappa$, $p \in \mathbb{P}$, $\rho_\xi \in \mathrm{dom}(\pi)$ $(\xi < \alpha)$ and for each $\xi < \zeta < \alpha$

$$p \Vdash \text{“}\rho_\zeta, \rho_\xi \in \pi \wedge \rho_\zeta \leq \rho_\xi\text{”}$$

then there is a $\sigma \in \mathrm{dom}(\pi)$ such that $p \Vdash$ “$\sigma \in \pi$” and $p \Vdash$ “$\sigma \leq \rho_\xi$” for all $\xi < \alpha$.

The reason for using names which are full for $< \kappa$-sequences is because of the following

**3.2 Lemma.** *Let $M$ be a c.t.m. for ZFC and in $M$ let*

$$\langle \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle, \langle \pi_\xi : \xi < \alpha \rangle \rangle$$

*be a $< \kappa$-support iterated forcing construction and suppose that for each $\xi$, the $\mathbb{P}_\xi$-name $\pi_\xi$ is full for $< \kappa$-sequences. Then $\mathbb{P}_\alpha$ is $\kappa$-closed in $M$.*

The next few paragraphs show how to select names for partial orders in the construction so that they are full for $< \kappa$-sequences. First consider the partial order which destroys a tower in $(\mathscr{P}(\kappa), \subset^*)$. Let $\epsilon$ be a regular cardinal with $\kappa < \epsilon < \lambda$ and $a = \langle a_\xi : \xi < \epsilon \rangle$ a tower in $(\mathscr{P}(\kappa), \subset^*)$. In the following subsets of $\kappa$ are identified with their characteristic functions.

**3.3 Definition.** $\mathbb{T}_a = \{(s, x) : s$ is a function $\wedge$ $\mathrm{dom}(s) \in \kappa \wedge$ $\mathrm{ran}(s) \subseteq 2 \wedge x \in [\epsilon]^{<\kappa}\}$ with $(s_2, x_2) \leq (s_1, x_1)$ iff

(1) $s_1 \subseteq s_2 \wedge x_1 \subseteq x_2$,

(2) $\forall \xi \in x_1 \forall \eta \in$ $\mathrm{dom}(s_2) \setminus$ $\mathrm{dom}(s_1)(a_\xi(\eta) \leq s_2(\eta))$.

Then $\mathbb{T}_a$ is a partial order and it is $\kappa$-closed and $\kappa^+$-c.c. (assuming $\kappa^{<\kappa} = \kappa$). Let $G$ be $\mathbb{T}_a$-generic over $M$ and $b = \cup\{s : \exists x((s, x) \in G)\}$. Since $G$ intersects suitably chosen dense subsets of $\mathbb{T}_a$ in $M$, then $b \subseteq \kappa$, $|b| = |\kappa \setminus b| = \kappa$ and $\forall \xi < \epsilon(a_\xi \subseteq^* b)$ so that $a$ ceases to be a tower in $M[G]$.

Since the $< \kappa$-support iteration is sensitive to the particular names used for the partial orders, a suitable name for $\mathbb{T}_a$ is formulated next.

**3.4 Definition.** Assume that $\mathbb{P} \in M$, ($\mathbb{P}$ is $\kappa$-closed) and

$$\mathbf{1} \Vdash \text{“}\tau \text{ is an } \check{\epsilon}\text{-tower in } (\mathscr{P}(\kappa), \subset^*)\text{”}.$$

A standard name for $\mathbb{T}_\tau$ is $\langle \sigma, \leq_\sigma, \mathbf{1}_\sigma \rangle$, where

$$\begin{aligned}\sigma = \{\langle \mathrm{op}(\check{s}, \rho), \mathbf{1}_{\mathbb{P}} \rangle : & s \text{ is a function } \wedge \ \mathrm{dom}(s) \in \kappa \wedge \ \mathrm{ran}(s) \subseteq 2 \wedge \\ & \mathbf{1} \Vdash \text{“}\rho \subseteq \tau \wedge |\rho| < \kappa\text{”} \wedge \rho \text{ is a nice name for a subset of } \tau\}\end{aligned}$$

and $\mathbf{1}_\sigma = \ \mathrm{op}(\check{0}, \check{0})$.

Here *op* is the invariant name for the ordered pair and $\rho$ is a nice name for a subset of $\tau$ if

$$\rho = \cup\{\{\pi\} \times A_\pi : \pi \in \ \mathrm{dom}(\tau)\}$$

and each $A_\pi$ is an antichain in $\mathbb{P}$. It is irrelevant what type of name we use for $\leq_\sigma$ as long as it is forced by $\mathbf{1}_{\mathbb{P}}$ to be the correct partial order on $\mathbb{T}_\tau$.

In $M$, let $\mathbb{P}$, $\tau$, and $\sigma$ be as in the definition above. Let $G$ be $\mathbb{P}$-generic over $M$ and $a = \tau_G$. Then in $M[G]$, $\sigma_G = \mathbb{T}_a$. In addition, $\sigma$ is full for $< \kappa$-sequences.

The dominating function partial order is considered next. Let $F \subseteq \kappa^\kappa$. In the final construction $F$ will be equal to $\kappa^\kappa$, but for the general discussion $F$ is any subset of $\kappa^\kappa$.

**3.5 Definition.** $\mathbb{D}_F = \{(s, x) : s$ is a function $\wedge$ $\mathrm{dom}(s) \in \kappa \wedge$ $\mathrm{ran}(s) \subseteq \kappa \wedge x \in [F]^{<\kappa}\}$ where $(s_2, x_2) \leq (s_1, x_1)$ if

(1) $s_1 \subseteq s_2 \wedge x_1 \subseteq x_2$,

(2) $\forall f \in x_1 \forall \alpha \in \mathrm{dom}(s_2) \setminus \mathrm{dom}(s_1)(f(\alpha) < s_2(\alpha))$.

Then $\mathbb{D}_F$ is a partial order and is $\kappa$-closed and $\kappa^+$-c.c. (assuming $\kappa^{<\kappa} = \kappa$). Let $G$ be $\mathbb{D}_F$-generic over $M$ and $g = \cup\{s : \exists x((s, x) \in G)\}$. Then since $G$ intersects suitably chosen dense subsets of $\mathbb{D}_F$ in $M, g$ is a function from $\kappa$ to $\kappa$ which eventually dominates every function in $F$, i.e., $\forall f \in F(f <^* g)$.

**3.6 Definition.** Assume that $\mathbb{P} \in M$, $(\mathbb{P}$ is $\kappa$-closed$)^M$, and $\mathbf{1} \Vdash$ "$\varphi \subseteq \kappa^\kappa$". The standard $\mathbb{P}$-name for $\mathbb{D}_\varphi$ is $\langle \psi, \leq_\psi, \mathbf{1}_\psi \rangle$, where

$$\begin{aligned}\psi = \{\langle op(\check{s}, \phi), \mathbf{1}_{\mathbb{P}} \rangle : & s \text{ is a function } \wedge \text{ dom}(s) \in \kappa \wedge \text{ ran}(s) \subseteq \kappa \wedge \\ & \mathbf{1}_{\mathbb{P}} \Vdash \text{"}\phi \subseteq \varphi \wedge |\phi| < \kappa\text{"} \wedge \phi \\ & \text{is a nice name for a subset of } \varphi\}\end{aligned}$$

and $\mathbf{1}_\psi = op(\check{0}, \check{0})$.

The choice of the $\mathbb{P}$-name $\leq_\psi$ is, once again, irrelevant as long as it is forced by $\mathbf{1}_{\mathbb{P}}$ to be the correct partial order on $\mathbb{D}_\varphi$.

In $M$, let $\mathbb{P}$, $\varphi$, $\psi$, be as above. Let $G$ be $\mathbb{P}$-generic over $M$ and $F = \varphi_G$. Then, in $M[G]$, $\psi_G = \mathbb{D}_F$. In addition, $\psi$ is full for $<\kappa$-sequences. The use of full names for $< \kappa$-sequences will guarantee, as indicated earlier, that the iteration is $\kappa$-closed. The use of standard names will imply that the iteration also satisfies the $\kappa^+$-cc so that all the cardinals are preserved in the final model.

Now follows the main result of this section.

**3.7 Theorem.** *Let $N$ be a c.t.m. for $ZFC + GCH$ and, in $N$, let $\kappa < \lambda \leq \mu \leq \theta$ be cardinals such that $\kappa, \lambda, \mu$ are regular and* $\mathrm{cf}(\theta) \geq \lambda$. *Then there is a cardinal preserving extension $M[G]$ of $N$ such that*

$$M[G] \models \text{"}\mathfrak{t}_\kappa = \lambda \wedge \mathfrak{b}_\kappa = \mu \wedge 2^\kappa = \theta\text{"}.$$

*Proof.* Let $\alpha$, $\beta$ be cardinals with $\alpha$ regular, $\alpha < \beta$, and $\mathrm{cf}(\beta) > \alpha$. Then $\mathbb{F}n(\beta \times \alpha, 2, \alpha)$ is the standard partial order for adding $\beta$-many subsets of $\alpha$ (see [Ku83]). It is $\alpha$-closed and $\alpha^+$-c.c. (assuming $\alpha^{<\alpha} = \alpha$), so it preserves cardinals.

Let $N$ be a c.t.m. for $ZFC + GCH$. In $N$, let $\kappa < \lambda \leq \mu \leq \theta$ be cardinals such that $\kappa$, $\lambda$, $\mu$ are regular and $\mathrm{cf}(\theta) > \kappa$. The goal is to produce an extension of $N$ in which $\mathfrak{t}_\kappa = \lambda$, $\mathfrak{b}_\kappa = \mu$ and $2^\kappa = \theta$. Let $H$ be $\mathbb{F}n(\theta^+ \times \lambda, 2, \lambda)$-generic over $N$ and let $N[H] = M$. Then

$$M \models \text{"}ZFC + \forall \xi < \lambda (2^\xi = \xi^+) + 2^\lambda = \theta^{+}\text{"}$$

$\kappa$, $\lambda$, $\mu$ are still regular and all the cardinals are preserved. Now, in $M$, perform an iterated forcing construction of length $\theta \cdot \mu$ (ordinal product) with $< \kappa$-supports, i.e., build an iterated forcing construction

$$\langle\langle \mathbb{P}_\xi : \xi \le \theta\cdot\mu\rangle, \langle \pi_\xi : \xi < \theta\cdot\mu\rangle\rangle$$

with supports of size less than $\kappa$ each $\mathbb{P}_\xi$ having cardinality $\le \lambda$.

Given $\mathbb{P}_\xi$, if $\xi$ is not of the form $\theta\cdot\xi$, list all the $\mathbb{P}_\xi$-names for towers in $(\mathscr{P}(\kappa), \subset^*)$ of size $\eta$ for all $\kappa < \eta < \lambda$; for example, let $\langle \sigma^\xi_\gamma : \gamma < \theta\rangle$ enumerate all $\mathbb{P}_\xi$-names $\sigma$ such that for some $\eta$, with $\kappa < \eta < \lambda$, $\sigma$ is a nice $\mathbb{P}_\xi$-name for a subset of $(\eta \times \kappa)$ with the property that there is a name $\tau^\xi_\gamma$ such that

$$\begin{aligned}\mathbf{1} \Vdash \text{“}\tau^\xi_\gamma =& \{x \subseteq \kappa : \exists \zeta < \eta (x) = \{\nu : (\zeta,\nu) \in \sigma^\xi_\gamma\} \\ & \text{is a tower in } (\mathscr{P}(\kappa), \subset^*) \text{ of size } \eta\}\text{”}.\end{aligned}$$

Let $\Theta = (\theta\cdot\mu) \setminus \{\theta\cdot\xi : \xi < \mu\}$ and let $f : \Theta \to (\theta\cdot\mu)\times\theta$ be a bookkeeping function such that $f$ is onto and $\forall \xi,\beta,\gamma (f(\xi) = (\beta,\gamma) \to \beta < \xi)$. If $f(\xi) = (\beta,\gamma)$, let $\tau_\xi$ be a $\mathbb{P}_\xi$-name for the same object for which $\tau^\beta_\gamma$ is a $\mathbb{P}_\beta$-name. Let $\pi_\xi$ be the standard $\mathbb{P}_\xi$-name for $\mathbb{T}_{\tau_\xi}$. And if $\xi$ is of the form $\theta\cdot\zeta$, let $\varphi_\xi$ be a $\mathbb{P}_\xi$-name for $\kappa^\kappa$ and let $\pi_\xi$ be the standard $\mathbb{P}_\xi$-name for $\mathbb{D}_{\varphi_\xi}$. This finishes the iteration.

By Lemma 9 $\mathbb{P}_{\theta\cdot\mu}$ is $\kappa$-closed in $M$. In fact, $\mathbb{P}_{\theta\cdot\mu}$ has the property that each decreasing sequence of length $< \kappa$ has a greatest lower bound so that the set $\mathbb{P}'$ of elements $p \in \mathbb{P}_{\theta\cdot\mu}$ with the property that the first coordinate of $p(\gamma)$, for $\gamma \in \operatorname{dom}(p)$, is a real object and not just a $\mathbb{P}_\gamma$-name, is dense in $\mathbb{P}_{\theta\cdot\mu}$. Therefore, to show that $\mathbb{P}_{\theta\cdot\mu}$ also has the $\kappa^+$-cc in $M$ it suffices to show that $\mathbb{P}'$ has the $\kappa^+$-cc in $M$. So, in $M$, let $p^\gamma \in \mathbb{P}'$ for $\gamma < \kappa^+$. By $\kappa^{<\kappa} = \kappa$, the $\Delta$-system lemma (see Theorem II,1.6 in [Ku83]) implies that there is an $X \in [\kappa^+]^{\kappa^+}$ such that $\{\ \text{support}(p^\gamma) : \gamma < \kappa^+\}$ for a $\Delta$-system with root $r$. Let $p^\gamma = \langle \rho^\gamma_\xi : \xi < \theta\cdot\mu\rangle$, and let $\rho^\gamma_\xi = op(\check{s}^\gamma_\xi, \sigma^\gamma_\xi)$. By $\kappa^{<\kappa} = \kappa$, there is a $Y \in [X]^{\kappa^+}$ such that for all $\xi \in r$, the $s^\gamma_\xi$ for $\gamma \in Y$ are all the same; say $s^\gamma_\xi = s_\xi$ for $\xi \in r$ and $\gamma \in Y$. But then the $p^\gamma$ for $\gamma \in Y$ are pairwise compatible; to see this observe that if $\gamma, \delta \in Y$, then $p^\gamma$, $p^\delta$ have as a common extension $\langle p_\xi : \xi < \theta\cdot\mu\rangle$, where $\rho_\xi$ is

(*a*) $\rho^\gamma_\xi$ if $\xi \notin\ \text{support}(p^\delta)$,

(*b*) $\rho^\delta_\xi$ if $\xi \notin\ \text{support}(p^\gamma)$,

(*c*) $op(\check{s}_\xi, \sigma_\xi)$ if $\xi \in r$,

where $\sigma_\xi$ is a nice name which satisfies $\mathbf{1}_\xi \Vdash$ “$\sigma_\xi = \sigma^\gamma_\xi \cup \sigma^\delta_\xi$”. So $\mathbb{P}_{\theta\cdot\mu}$ has the $\kappa^+$-c.c. and together with being $\kappa$-closed preserves all the cardinal numbers. Let $G$ be $\mathbb{P}_{\theta\cdot\mu}$-generic over $M$. Since at each stage of the form $\theta\cdot\xi$, a function from $\kappa$ to $\kappa$ is added which eventually dominates all the functions in $\kappa^\kappa$ constructed by that stage, it follows that, in $M[G]$, there is a scale in $(\kappa^\kappa, <^*)$ of order type $\mu$

so that $\mathfrak{b}_\kappa = \mu$. In addition, since at each stage of the iteration a new element to $\kappa^\kappa$ or $\mathscr{P}(\kappa)$ is added, it follows that $M[G] \models$ "$2^\kappa = |\theta \cdot \mu| = \theta$". Finally, $M[G]$ contains no towers in $(\mathscr{P}(\kappa), \subseteq^*)$ of order type $\eta$ for $\kappa < \eta < \lambda$ since by the bookkeeping device all such towers are considered and eventually destroyed at some stage of the iteration, so that $\mathfrak{t}_\kappa \geq \lambda$. However, $M[G] \models$ "$\forall \xi (\kappa \leq \xi < \lambda \rightarrow 2^\xi = \theta)$" and $M[G] \models$ "$2^\lambda = \theta^+$" since $M \models$ "$2^\lambda = \theta^+$" and clearly $\diamondsuit_\kappa$ holds (e.g., without loss of generality $M[G] \models \diamondsuit_\kappa$ and $\kappa$-closed forcing preserve it so that by the previous section $\mathfrak{t}_\kappa = \lambda$. This finishes the proof of this theorem. $\square$

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