# On inverse $\gamma$-systems and the number of $L_{\infty\lambda}$-equivalent, non-isomorphic models for $\lambda$ singular

Saharon Shelah * Pauli Väisänen †

October 6, 2003

**Abstract**

Suppose $\lambda$ is a singular cardinal of uncountable cofinality $\kappa$. For a model $\mathcal{M}$ of cardinality $\lambda$, let $\mathrm{No}(\mathcal{M})$ denote the number of isomorphism types of models $\mathcal{N}$ of cardinality $\lambda$ which are $L_{\infty\lambda}$-equivalent to $\mathcal{M}$. In [She85] Shelah considered inverse $\kappa$-systems $\mathcal{A}$ of abelian groups and their certain kind of quotient limits $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$. In particular Shelah proved in [She85, Fact 3.10] that for every cardinal $\mu$ there exists an inverse $\kappa$-system $\mathcal{A}$ such that $\mathcal{A}$ consists of abelian groups having cardinality at most $\mu^\kappa$ and $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big) = \mu$. Later in [She86, Theorem 3.3] Shelah showed a strict connection between inverse $\kappa$-systems and possible values of No (under the assumption that $\theta^\kappa < \lambda$ for every $\theta < \lambda$): if $\mathcal{A}$ is an inverse $\kappa$-system of abelian groups having cardinality $< \lambda$, then there is a model $\mathcal{M}$ such that $\mathrm{card}(\mathcal{M}) = \lambda$ and $\mathrm{No}(\mathcal{M}) = \mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big)$. The following was an immediate consequence (when $\theta^\kappa < \lambda$ for every $\theta < \lambda$): for every nonzero $\mu < \lambda$ or $\mu = \lambda^\kappa$ there is a model $\mathcal{M}_\mu$ of cardinality $\lambda$ with $\mathrm{No}(\mathcal{M}_\mu) = \mu$. In this paper we show: for every nonzero $\mu \leq \lambda^\kappa$ there is an inverse $\kappa$-system $\mathcal{A}$ of abelian groups having cardinality $< \lambda$ such that $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big) = \mu$ (under the assumptions $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for all $\theta < \lambda$ when $\mu > \lambda$), with the obvious new consequence concerning the possible value of No. Specifically, the case $\mathrm{No}(\mathcal{M}) = \lambda$ is possible when $\theta^\kappa < \lambda$ for every $\theta < \lambda$.[1]

## 1 Introduction

Suppose $\lambda$ is a cardinal. For a model $\mathcal{M}$ we let $\mathrm{card}(\mathcal{M})$ denote the cardinality of the universe of $\mathcal{M}$. When $\mathcal{M}$ and $\mathcal{N}$ are models of the same vocabulary and they satisfy the same sentences of the infinitary language $L_{\infty\lambda}$, we write $\mathcal{M} \equiv_{\infty\lambda} \mathcal{N}$. For any model $\mathcal{M}$ of cardinality $\lambda$ we define $\mathrm{No}(\mathcal{M})$ to be the cardinality of the set

$$\big\{\mathcal{N}/\cong \;\mid\; \mathrm{card}(\mathcal{N}) = \lambda \text{ and } \mathcal{N} \equiv_{\infty\lambda} \mathcal{M}\big\},$$

where $\mathcal{N}/\cong$ is the equivalence class of $\mathcal{N}$ under the isomorphism relation. Our principal purpose is to study the possible values of $\mathrm{No}(\mathcal{M})$ for models $\mathcal{M}$ of singular cardinality with uncountable cofinality.

When $\mathcal{M}$ is countable, $\mathrm{No}(\mathcal{M}) = 1$ by [Sco65]. This result extends to structures of cardinality $\lambda$ when $\lambda$ is a singular cardinal of countable cofinality [Cha68].

*Thanks to GIF for its support of this research and also to University of Helsinki for funding a visit of the first author to Helsinki in August 1996. Pub. No. 644.

†The second author wishes to thank Tapani Hyttinen under whose supervision he did his share of the paper.

[1] 1991 Mathematics Subject Classification: primary 03C55; secondary 03C75. Key words: number of models, infinitary logic, inverse $\gamma$-system.

If $V = L$, $\lambda$ is an uncountable regular cardinal which is not weakly compact, and $\mathcal{M}$ is a model of cardinality $\lambda$, then $\mathrm{No}(\mathcal{M})$ has either the value 1 or $2^\lambda$. For $\lambda = \aleph_1$ this result was first proved in [Pal77a]. Later in [She81] Shelah extended this result to all other regular non-weakly compact cardinals. The possibility $\mathrm{No}(\mathcal{M}) = \aleph_0$ is consistent with ZFC + GCH in case $\lambda = \aleph_1$, as remarked in [She81]. The values $\mathrm{No}(\mathcal{M}) \in \omega \setminus \{0, 1\}$ are proved to be consistent with ZFC + GCH in the forthcoming paper of the authors [SV97] (number 646 in Shelah's publications).

The case $\mathcal{M}$ has cardinality of a weakly compact cardinal is dealt with in [She82] by Shelah. The result is that for $\kappa$ weakly compact there is for every $1 \leq \mu \leq \kappa$ a model $\mathcal{M}_\mu$ such that $\mathrm{No}(\mathcal{M}_\mu) = \mu$. There is in preparation by the authors a paper where the question for $\kappa$ weakly compact is revisited.

The case $\mathcal{M}$ is of singular cardinality $\lambda$ with uncountable cofinality $\kappa$ was first treated in [She85], where the relations of $\mathcal{M}$ have infinitely many places. Later in [She86] Shelah improved the result by showing that if $\theta^\kappa < \lambda$ for every $\theta < \lambda$ and $0 < \mu < \lambda$ then $\mathrm{No}(\mathcal{M}) = \mu$ is possible for a model $\mathcal{M}$ having cardinality $\lambda$ and relations of finitely many places only. The main idea in those papers was to transform the problem of possible values of $\mathrm{No}(\mathcal{M})$ into a question concerning possible cardinalities of "quotient limit" $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ of an inverse system $\mathcal{A}$ of groups [She86, Theorem 3.3]:

**Theorem 1** ($\lambda$ *cardinal with* $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$)*If* $\theta^\kappa < \lambda$ *for every* $\theta < \lambda$ *and* $\mathcal{A}$ *is an inverse* $\kappa$*-system of abelian groups having cardinality* $< \lambda$*, then there is a model* $\mathcal{M}$ *of cardinality* $\lambda$ (*with relations having finitely many places only*) *such that* $\mathrm{No}(\mathcal{M}) = \mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big)$.

Actually the groups in [She86, Theorem 3.3] are not limited to be abelian. However, abelian groups suffice for the present purposes.

The recent paper fills a gap left open since the paper [She86]. We present a uniform way to construct inverse $\kappa$-system of abelian groups having a quotient limit of desired cardinality. The most important new case is that the cardinality of a quotient limit can be $\lambda$ for some inverse system (in other cases, where the result below can be applied, the Singular Cardinal Hypothesis fails). The result of this paper is:

**Theorem 2** ($\lambda$ *cardinal with* $\lambda > \mathrm{cf}(\lambda) = \kappa > \aleph_0$)*For every nonzero* $\mu \leq \lambda$ *there is an inverse* $\kappa$*-system* $\mathcal{A} = \langle G_i, h_{i,j} \mid i < j < \kappa \rangle$ *of abelian groups satisfying that* $\mathrm{card}(G_i) < \lambda$ *for every* $i < \kappa$ *and* $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})\big) = \mu$*. The same conclusion holds also for the values* $\lambda < \mu \leq \lambda^\kappa$ *under the assumption that* $2^\kappa < \lambda$ *and* $\theta^{<\kappa} < \lambda$ *for every* $\theta < \lambda$*.*

So the general method used here to find new possibilities for the values of $\mathrm{No}(\mathcal{M})$ is the same as in [She86]. As an immediate consequence of the last theorem we get:

**Theorem 3** *Suppose* $\lambda$ *is a singular cardinal of uncountable cofinality* $\kappa$*. For each nonzero* $\mu \leq \lambda^\kappa$ *there is a model* $\mathcal{M}$ (*with relations having finitely many places only*) *satisfying* $\mathrm{card}(\mathcal{M}) = \lambda$ *and* $\mathrm{No}(\mathcal{M}) = \mu$*, provided that* $\theta^\kappa < \lambda$ *for every* $\theta < \lambda$*.*

We give all necessary definitions concerning inverse $\kappa$-systems $\mathcal{A}$ of abelian groups and their special kind of quotient limits $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ in the next section.

## 2 Preliminaries

**Definition 2.1** *Suppose $\gamma$ is a limit ordinal and for every $i < j < \gamma$, $G_i$ is a group and $h_{i,j}$ is a homomorphism from $G_j$ into $G_i$. The family $\mathcal{A} = \langle G_i, h_{i,j} \mid i < j < \gamma \rangle$ is called an inverse $\gamma$-system when the equation $h_{i,j} \circ h_{j,k} = h_{i,k}$ holds for every $i < j < k < \gamma$. As in [She85] we assume that all the groups $G_i$, $i < \gamma$, are additive abelian groups.*

*To simplify our notation we make an agreement that the letters $i$, $j$, $k$, and $l$ always denote ordinals smaller than $\gamma$. Hence "for all $i < j$" means "for all ordinals $i$ and $j$ with $i < j < \gamma$" and so on.*

*The main objects of our study are the following two sets:*

$$\mathrm{Gr}(\mathcal{A}) = \Big\{ \langle \boldsymbol{a}^{i,j} \mid i < j < \gamma \rangle \mid \boldsymbol{a}^{i,j} \in G_i \text{ and for all } k > j,\ \boldsymbol{a}^{i,k} = \boldsymbol{a}^{i,j} + h_{i,j}(\boldsymbol{a}^{j,k}) \Big\};$$
$$\mathrm{Fact}(\mathcal{A}) = \Big\{ \langle \boldsymbol{a}^{i,j} \mid i < j < \gamma \rangle \mid \text{for some } \bar{y} \in \prod_{k<\gamma} G_k,\ \boldsymbol{a}^{i,j} = \bar{y}^i - h_{i,j}(\bar{y}^j) \Big\}.$$

*We consider $\mathrm{Gr}(\mathcal{A})$ and $\mathrm{Fact}(\mathcal{A})$ as additive abelian groups where the group operation $+$ and the unit element $\mathbf{0}$ are pointwise defined. The factor group $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ is well-defined since $\mathrm{Fact}(\mathcal{A}) \subseteq \mathrm{Gr}(\mathcal{A})$ by the requirements $h_{i,j} \circ h_{j,k} = h_{i,k}$ for all $i < j < k$. For any inverse $\gamma$-system $\mathcal{A}$, the group $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$ is called the quotient limit of $\mathcal{A}$.*

**Definition 2.2** *We let $\gamma \star \gamma$ be the set $\{(i,j) \in \gamma \times \gamma \mid i < j\}$. For every subset $I$ of $\gamma \star \gamma$ we define*

$$I^{1st} = \{ i < \gamma \mid (i,j) \in I \text{ for some } j < \gamma \}$$

*and for each $i \in I^{1st}$,*

$$I[i] = \{ j < \gamma \mid (i,j) \in I \}.$$

*We also say that*

*$I$ is cobounded if $\gamma \setminus I^{1st}$ and $\gamma \setminus I[i]$, for all $i \in I^{1st}$, are bounded subsets of $\gamma$;*

*$I$ is coherent if $I^{1st}$ is unbounded in $\gamma$ and for every $i \in I^{1st}$, $I[i] = I^{1st} \setminus (i+1)$;*

*$I$ is eventually coherent if it is unbounded and for every $i \in I^{1st}$, $I^{1st} \setminus I[i]$ is a bounded subset of $\gamma$.*

**Remark.** Suppose $I$ is an eventually coherent subset of $\gamma \star \gamma$ and $S$ is a subset of $I^{1st}$. If $\mathrm{card}(S) < \mathrm{cf}(\gamma)$, then $I^{1st} \setminus (\bigcap_{i \in S} I[i])$ is a bounded subset of $\gamma$. If $S$ is unbounded in $\gamma$, then $I \cap (S \times S)$ is an eventually coherent subset of $I$.

In [She86, Claim 1.12] Shelah proved (note the remark given after the following lemma) that if two sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ from $\mathrm{Gr}(\mathcal{A})$ agree on a coherent set of indices, then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A})$. The following slight improvement of this condition has an essential role in the proof of Theorem 2.

**Lemma 2.3** *Suppose $\mathcal{A}$ is an inverse $\gamma$-system, and $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{Gr}(\mathcal{A})$. Then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A})$ holds if there is an eventually coherent subset $I$ of $\gamma \star \gamma$ such that $\boldsymbol{a}^{i,j} = \boldsymbol{b}^{i,j}$ for all $(i,j) \in I$.*

***Proof.*** We shall need an eventually coherent subset $J$ of $I$ having the property that $\langle J[i] \mid i \in J^{1st} \rangle$ is a decreasing chain of end segments of $J^{1st}$. Let $S$ be an unbounded subset of $I$ having the order type $\mathrm{cf}(\gamma)$. Define a subset $J$ of $I$ by $J^{1st} = S$ and for all $j \in S$,

$$J[j] = S \cap \bigcap_{i \in S \cap (j+1)} \big( I[i] \setminus (i^* + 1) \big),$$

where $i^*$ is the supremum of the bounded subset $I^{1\text{st}} \setminus I[i]$ of $\gamma$. The set $J$ is well-defined since $I$ is eventually coherent and $\operatorname{card}\big(S \cap (j+1)\big) < \operatorname{cf}(\gamma)$ for all $j < \gamma$. Now $J$ is also eventually coherent, and furthermore, for all $i \in J^{1\text{st}}$, $J[i] = S \setminus \min(J[i])$ and for all $j \in J^{1\text{st}} \setminus i$, $\min(J[i]) \le \min(J[j])$.

Define for every $i < \gamma$, $i'$ to be $\min\big(J^{1\text{st}} \setminus (i+1)\big)$ and $i'' = \min(J[i'])$. Then the following are satisfied for all $i < j$:

$i < i' < i''$, $j < j' < j''$, $i' \le j'$, $i'' \le j''$, and also $i' < j''$;

$j'' \in I^{1\text{st}}$ and $(i', i''), (j', j''), (i', j'') \in I$.

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are in $\operatorname{Gr}(\mathcal{A}^T_R)$ we have

$$\begin{aligned} \boldsymbol{a}^{i,j''} &= \boldsymbol{a}^{i,j} + h_{i,j}(\boldsymbol{a}^{j,j''}), \\ \boldsymbol{b}^{i,j''} &= \boldsymbol{b}^{i,j} + h_{i,j}(\boldsymbol{b}^{j,j''}). \end{aligned}$$

Therefore the following equations hold:

(A)
$$\begin{aligned} \boldsymbol{a}^{i,j} - \boldsymbol{b}^{i,j} &= (\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''}) - \big(h_{i,j}(\boldsymbol{a}^{j,j''}) - h_{i,j}(\boldsymbol{b}^{j,j''})\big) \\ &= (\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''}) - h_{i,j}(\boldsymbol{a}^{j,j''} - \boldsymbol{b}^{j,j''}). \end{aligned}$$

Because of $i < i' < j''$ we also have that

$$\begin{aligned} \boldsymbol{a}^{i,j''} &= \boldsymbol{a}^{i,i'} + h_{i,i'}(\boldsymbol{a}^{i',j''}), \\ \boldsymbol{b}^{i,j''} &= \boldsymbol{b}^{i,i'} + h_{i,i'}(\boldsymbol{b}^{i',j''}). \end{aligned}$$

Since $(i', j'') \in I$, $\boldsymbol{a}^{i',j''} = \boldsymbol{b}^{i',j''}$ holds. Hence we get

(B)
$$\boldsymbol{a}^{i,j''} - \boldsymbol{b}^{i,j''} = \boldsymbol{a}^{i,i'} - \boldsymbol{b}^{i,i'}.$$

Moreover, $i < i' < i''$ yields

$$\begin{aligned} \boldsymbol{a}^{i,i''} &= \boldsymbol{a}^{i,i'} + h_{i,i'}(\boldsymbol{a}^{i',i''}), \\ \boldsymbol{b}^{i,i''} &= \boldsymbol{b}^{i,i'} + h_{i,i'}(\boldsymbol{b}^{i',i''}). \end{aligned}$$

Now $(i', i'') \in I$ implies that $\boldsymbol{a}^{i',i''} = \boldsymbol{b}^{i',i''}$, and consequently

$$\boldsymbol{a}^{i,i'} - \boldsymbol{b}^{i,i'} = \boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''}.$$

This equation together with (A) and (B) implies that for all $i < j$

$$\boldsymbol{a}^{i,j} - \boldsymbol{b}^{i,j} = (\boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''}) - h_{i,j}(\boldsymbol{a}^{j,j''} - \boldsymbol{b}^{j,j''}).$$

So the sequence $\bar{y} = \langle \boldsymbol{a}^{i,i''} - \boldsymbol{b}^{i,i''} \mid i < \gamma \rangle \in \prod_{i<\gamma} G_i$ exemplifies that $\boldsymbol{a} - \boldsymbol{b} \in \operatorname{Fact}(\mathcal{A})$, and we have $\boldsymbol{a} \equiv \boldsymbol{b} \mod \operatorname{Fact}(\mathcal{A})$. $\blacksquare_{2.3}$

**Remark.** In [She86, Claim 1.12] the groups of an inverse system $\mathcal{A}$ need not to be abelian groups. Hence instead of the factor group $\operatorname{Gr}(\mathcal{A})/\operatorname{Fact}(\mathcal{A})$ a partition $\operatorname{Gr}(\mathcal{A})/{\approx_{\mathcal{A}}}$ with a special kind of equivalence relation $\approx_{\mathcal{A}}$ were considered there. However, it is straightforward to prove, by means of the preceding proof, also the more general case of Lemma 2.3 where "equivalent modulo $\operatorname{Fact}(\mathcal{A})$" is replaced by $\approx_{\mathcal{A}}$.

In the next section we shall need a notion of a tree, so we shortly describe our notation.

**Definition 2.4** *Suppose $T = \langle T, \lhd \rangle$ is a tree of height $\gamma$. For every $i < \gamma$, $T_i$ is the $i^{th}$ level of the tree. When $i < j < \gamma$ and $\eta \in T_j$, then $\eta{\restriction}i$ denotes the unique element $\nu \in T_i$ for which $\nu \lhd \eta$ holds. For each $i < \gamma$ and $\nu \in T_i$, $T_j[\nu]$ is the set $\{\eta \in T_j \mid \nu \lhd \eta\}$. The set of all $\gamma$-branches of $T$, i.e., the set $\{t \in \prod_{i<\gamma} T_i \mid \text{for all } i < j,\ t(i) \lhd t(j)\}$, is denoted by $\operatorname{Br}_\gamma(T)$.*

## 3 The inverse $\gamma$-system of free $R$-modules

In this section we define special kind of inverse $\gamma$-systems $\mathcal{A}_R^T$ and prove a result concerning cardinalities of their quotient limit $\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)$ (Conclusion 3.12). A direct consequence of the result will be Theorem 2.

**Definition 3.1** *Suppose $\gamma$ is a limit ordinal, $R$ is a ring, and $T$ is a tree of height $\gamma$. We define an inverse $\gamma$-system $\mathcal{A}_R^T = \langle G_i, h_{i,j} \mid i < j < \gamma\rangle$ by the following stipulations:*

a) *for each $i < \gamma$, $G_i$ is the $R$-module freely generated by $\{x_{\nu,l} \mid \nu \in T_i \text{ and } i < l < \gamma\}$;*

b) *for every $i < j < \gamma$, $h_{i,j}$ is the homomorphism from $G_j$ into $G_i$ determined by the values $h_{i,j}(x_{\eta,l}) = x_{\eta\restriction i,l} - x_{\eta\restriction i,j}$, for all $\eta \in T_j$ and $l > j$. (It is easy to check that the equations $h_{i,k} = h_{i,j} \circ h_{j,k}$ are satisfied for all $i < j < k$).*

*We consider $\mathrm{Gr}(\mathcal{A}_R^T)$, $\mathrm{Fact}(\mathcal{A}_R^T)$, and $\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)$ as $R$-modules where the operations $+, \cdot$, and the unit element $\mathbf{0}$ for addition are pointwise defined.*

*For each $t \in \mathrm{Br}_\gamma(T)$, we define $\boldsymbol{t}$ to be the sequence $\langle x_{t(i),j} \mid i < j < \gamma\rangle$. Directly by the definitions of $G_i$ and $h_{i,j}$, $\boldsymbol{t}$ belongs to $\mathrm{Gr}(\mathcal{A}_R^T)$ for every $t \in \mathrm{Br}_\gamma(T)$. We let $\langle \boldsymbol{t}\rangle_R^{t\in\mathrm{Br}_\gamma(T)}$ be the submodule of $\mathrm{Gr}(\mathcal{A}_R^T)$ generated by the elements $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$. When $\mathrm{Br}_\gamma(T)$ is empty $\langle \boldsymbol{t}\rangle_R^{t\in\mathrm{Br}_\gamma(T)}$ is the trivial submodule $\{\mathbf{0}\}$.*

**Remark.** Each $G_i$ is nonempty when $T$ has height $\gamma$. Hence $\prod_{i<\gamma} G_i$ is nonempty, and also

$$\mathrm{Fact}(\mathcal{A}_R^T) = \Big\{\langle \bar{y}^i - h_{i,j}(\bar{y}^j) \mid i < j < \gamma\rangle \mid \bar{y} \in \prod_{i<\gamma} G_i\Big\}$$

is nonempty. So $\mathrm{Gr}(\mathcal{A}_R^T) \supseteq \mathrm{Fact}(\mathcal{A}_R^T)$ is nonempty for every ring $R$ and tree $T$ of height $\gamma$.

Observe also that the inverse $\gamma$-system $\mathcal{A}_R^T$ is the same as used in [She85, Claim 3.8] when $R$ is the trivial ring $\{0,1\}$ and $T$ consists of $\mu$ many disjoint $\gamma$-branches. So the proof given in this section offers an alternative proof for [She85, Claim 3.8], and even more information, namely that $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big)$ must be exactly $\mu$ not only $\geq \mu$.

**Definition 3.2** *Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}_R^T)$ and $i < j < \gamma$. By the definition of $G_i$ and the requirement $\boldsymbol{a}^{i,j} \in G_i$, we define $a_{\nu,l}^{i,j}$ for $\nu \in T_i$ and $l > i$, to be the coefficients from $R$ (with only finitely many of them nonzero) which satisfy the equation*

$$\boldsymbol{a}^{i,j} = \sum\nolimits_{l>i}^{\nu\in T_i} a_{\nu,l}^{i,j} \cdot x_{\nu,l}.$$

*The finite set $\{(\nu,l) \in T_i \times \big(\gamma \setminus (i+1)\big) \mid a_{\nu,l}^{i,j} \neq 0\}$ is called the support of $\boldsymbol{a}^{i,j}$, and it is denoted by $\mathrm{supp}(\boldsymbol{a}^{i,j})$.*

*Suppose $S$ is a subset of $\gamma$, $e \in G_i$, and $e_{\nu,l} \in R$ for every $\nu \in T_i$ and $l > i$ are elements such that*

$$e = \sum\nolimits_{l>i}^{\nu\in T_i} e_{\nu,l} \cdot x_{\nu,l}.$$

*Then we write $e\restriction S$ for the following element of $G_i$:*

$$\sum\nolimits_{l\in S\setminus(i+1)}^{\nu\in T_i} e_{\nu,l} \cdot x_{\nu,l}.$$

The following simple lemma has an important corollary.

**Lemma 3.3**

a) *The restriction* $h_{i,j}(e){\restriction}j$ *equals 0 for every* $i<j$ *and* $e \in G_j$.

b) *For every* $\boldsymbol{a} \in \mathrm{Fact}(\mathcal{A}_R^T) \setminus \{\boldsymbol{0}\}$, *there are* $i<j<\gamma$ *such that* $\boldsymbol{a}^{i,j}{\restriction}j \neq 0$.

***Proof.*** a) Straightforwardly by the definitions of $G_j$ and $h_{i,j}$.

b) By the definition of $\mathrm{Fact}(\mathcal{A}_R^T)$, let $\bar{y} \in \prod_{i<\gamma} G_i$ be such that for all $i<j$, $\boldsymbol{a}^{i,j} = \bar{y}^i - h_{i,j}(\bar{y}^j)$. In addition to that let $y^i_{\nu,l} \in R$, for $i<\gamma$, $\nu \in T_i$ and $l>i$, be such that

$$\bar{y}^i = \sum\nolimits^{\nu\in T_i}_{l>i} y^i_{\nu,l} \cdot x_{\nu,l}.$$

Since $\boldsymbol{a} \neq \boldsymbol{0}$ there must be $i<\gamma$ with $\bar{y}^i \neq 0$. Define $j$ to be $\min\{l>i \mid y^i_{\nu,l} \neq 0 \text{ for some } \nu \in T_i\}+1$. Then $\bar{y}^i{\restriction}j$ is nonzero and because $h_{i,j}(\bar{y}^j){\restriction}j = 0$, we have $\boldsymbol{a}^{i,j}{\restriction}j = \bar{y}^i{\restriction}j - h_{i,j}(\bar{y}^j){\restriction}j = \bar{y}^i{\restriction}j \neq 0$. ∎

**Corollary 3.4** *The elements* $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$, *are independent over* $\mathrm{Fact}(\mathcal{A}_R^T)$, *i.e.,*

$$\langle \boldsymbol{t} \rangle_R^{t\in\mathrm{Br}_\gamma(T)} \cap \mathrm{Fact}(\mathcal{A}_R^T) = \{\boldsymbol{0}\}.$$

*Hence* $\mathcal{A}_R^T$ *satisfies* $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big) \geq \mathrm{card}\big(\langle \boldsymbol{t} \rangle_R^{t\in\mathrm{Br}_\gamma(T)}\big)$.

***Proof.*** Directly by the definition of $\boldsymbol{t}$, $\boldsymbol{t}^{i,j} = x_{t(i),j}$ and hence $\boldsymbol{t}^{i,j}{\restriction}j = 0$, for all $t \in \mathrm{Br}_\gamma(T)$ and $i<j$. So for any nonzero $\boldsymbol{a} = \sum_{1\leq m\leq n} d_m \cdot \boldsymbol{t_m}$, where $n<\omega$, $d_m \in R \setminus \{0\}$, and $t_m \in \mathrm{Br}_\gamma(T)$, the restrictions $\boldsymbol{a}^{i,j}{\restriction}j$ are equal to 0 for all $i<j$. So by the preceding lemma $\boldsymbol{a}$ can not be in $\mathrm{Fact}(\mathcal{A}_R^T)$. ∎

Next we derive equations of weighty significance.

**Lemma 3.5** *Suppose* $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}_R^T)$ *and* $i<j<k<\gamma$. *Then the following equations are satisfied for all* $\nu \in T_i$:

(A) $$b^{i,k}_{\nu,l} = b^{i,j}_{\nu,l} \qquad \text{when } i<l<j;$$

(B) $$b^{i,k}_{\nu,j} = b^{i,j}_{\nu,j} - \sum\nolimits^{\eta\in T_j[\nu]}_{l>j} b^{j,k}_{\eta,l};$$

(C) $$b^{i,k}_{\nu,l} = b^{i,j}_{\nu,l} + \sum\nolimits_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l} \qquad \text{when } l>j.$$

***Proof.*** By dividing the sum into groups we get that

$$\begin{aligned}\boldsymbol{b}^{i,j} &= \sum\nolimits^{\nu\in T_i}_{l>i} b^{i,j}_{\nu,l} \cdot x_{\nu,l} \\ &= \sum_{\nu\in T_i}\Big(\sum_{i<l<j} b^{i,j}_{\nu,l} \cdot x_{\nu,l} + b^{i,j}_{\nu,j} \cdot x_{\nu,j} + \sum_{l>j} b^{i,j}_{\nu,l} \cdot x_{\nu,l}\Big).\end{aligned}$$

Similarly the following equation is satisfied,

$$\boldsymbol{b}^{i,k} = \sum_{\nu\in T_i}\Big(\sum_{i<l<j} b^{i,k}_{\nu,l} \cdot x_{\nu,l} + b^{i,k}_{\nu,j} \cdot x_{\nu,j} + \sum_{l>j} b^{i,k}_{\nu,l} \cdot x_{\nu,l}\Big).$$

From the definition of $h_{i,j}$ we may infer that

$$\begin{array}{rcl} h_{i,j}(\boldsymbol{b}^{j,k}) & = & \sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l} \cdot h_{i,j}(x_{\eta,l}) \\ & = & \sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l} \cdot (x_{\eta\restriction i,l} - x_{\eta\restriction i,j}) \\ & = & \sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l} \cdot x_{\eta\restriction i,l} - \sum_{(\eta\in T_j,\ l>j)} b^{j,k}_{\eta,l} \cdot x_{\eta\restriction i,j} \\ & = & \sum_{\nu\in T_i} \Big( \sum_{l>j} \big( \sum_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l} \big) \cdot x_{\nu,l} - \big( \sum_{(\eta\in T_j[\nu],\ l>j)} b^{j,k}_{\eta,l} \big) \cdot x_{\nu,j} \Big). \end{array}$$

So the equations (A), (B), and (C) for all $i < j < k$ follow by comparing the coefficients of each generator $x_{\nu,l}$ in the equation $\boldsymbol{b}^{i,k} = \boldsymbol{b}^{i,j} + h_{i,j}(\boldsymbol{b}^{j,k})$. ■3.5

**Lemma 3.6** *Suppose* $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}^T_R)$.

a) *For all* $i < j < k$, $\boldsymbol{a}^{i,j}\restriction j = \boldsymbol{a}^{i,k}\restriction j$.

b) $(\mathrm{cf}(\gamma) > \aleph_0)$ *For every* $i < \gamma$, *the union* $\bigcup_{i<j<\gamma} \mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)$ *is of finite cardinality* (*where* $\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) = \mathrm{supp}(\boldsymbol{a}^{i,j}) \cap (T_i \times j)$ *of course*).

c) $(\mathrm{cf}(\gamma) > \aleph_0)$ *There is* $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}^T_R)$ *satisfying the following conditions:*

$\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}^T_R)$,
$I = \{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction j = 0\}$ *is cobounded* (*in fact* $I^{1st} = \gamma$), *and*
*for every* $(i,j) \in I$, $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\}$.

***Proof.*** a) The claim holds directly by Lemma 3.5(A).

b) Suppose the union is infinite. Since $\mathrm{cf}(\gamma) > \aleph_0$ there is some $k < \gamma$ for which already $\bigcup_{j<k} \mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j)$ is infinite. By (a), $\mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) \subseteq \mathrm{supp}(\boldsymbol{a}^{i,k})$ for each $j < k$. Consequently $\bigcup_{j<k} \mathrm{supp}(\boldsymbol{a}^{i,j}\restriction j) \subseteq \mathrm{supp}(\boldsymbol{a}^{i,k})$ contrary to the finiteness of $\mathrm{supp}(\boldsymbol{a}^{i,k})$.

c) By (a) and (b) there must be for every $i < \gamma$ a bound $i^* \in \gamma \setminus (i+1)$ such that for every $j \geq i^*$, $\boldsymbol{a}^{i,i^*}\restriction i^* = \boldsymbol{a}^{i,j}\restriction j$. Define an element $\boldsymbol{c} \in \mathrm{Fact}(\mathcal{A}^T_R)$ by

$$\boldsymbol{c}^{i,j} = \boldsymbol{a}^{i,i^*}\restriction i^* - h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*),$$

for all $i < j$. Let $\boldsymbol{b}$ be $\boldsymbol{a} - \boldsymbol{c}$. Then $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}^T_R)$ and for every $i < \gamma$ and $j \geq i^*$

$$\begin{array}{rcl} \boldsymbol{b}^{i,j} & = & \boldsymbol{a}^{i,j} - \boldsymbol{c}^{i,j} \\ & = & \boldsymbol{a}^{i,j}\restriction(\gamma \setminus j) + \boldsymbol{a}^{i,j}\restriction j - \boldsymbol{a}^{i,i^*}\restriction i^* + h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*) \\ & = & \boldsymbol{a}^{i,j}\restriction(\gamma \setminus j) + h_{i,j}(\boldsymbol{a}^{j,j^*}\restriction j^*). \end{array}$$

It follows from Lemma 3.3(a) that $\boldsymbol{b}^{i,j}\restriction j = 0$ for all $i < \gamma$ and $j \geq i^*$, and thus $I$ is cobounded.

Now suppose, contrary to the last claim in (c), that $b^{i,j}_{\nu,l} \neq 0$ for some $i < \gamma$, $j \geq i^*$, $\nu \in T_i$, and $l > j$. Let $k$ be $\max\{i^*, j^*, l+1\}$. Then both $\boldsymbol{b}^{i,k}\restriction k$ and $\boldsymbol{b}^{j,k}\restriction k$ are 0. By Lemma 3.5(C) the following equation holds:

$$\sum\nolimits_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l} = b^{i,k}_{\nu,l} - b^{i,j}_{\nu,l}.$$

Since $b^{i,j}_{\nu,l} \neq 0$ and $l < k$ implies $b^{i,k}_{\nu,l} = 0$ the sum $\sum_{\eta\in T_j[\nu]} b^{j,k}_{\eta,l}$ must be nonzero. So there is $\eta \in T_j[\nu]$ with $b^{j,k}_{\eta,l} \neq 0$. This contradicts the facts $l < k$ and $\boldsymbol{b}^{j,k}\restriction k$ equals 0. ■

**Lemma 3.7** *Suppose* $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}_R^T)$ *and* $I$ *is a subset of* $\{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j}\}$. *Then for all* $(i,j) \in I$, $\nu \in T_i$, *and* $k \in I[i] \cap I[j]$,

$$b^{i,j}_{\nu,j} = \sum\nolimits_{\eta \in T_j[\nu]} b^{j,k}_{\eta,k} = b^{i,k}_{\nu,k}.$$

***Proof.*** Since $(i,k)$ and $(j,k)$ are in $I$, both $b^{i,k}_{\nu,j}$ and $b^{j,k}_{\eta,l}$ are equal to 0 for all $\eta \in T_j$ when $l \neq k$. Hence Lemma 3.5(B) can be reduced to the form $b^{i,j}_{\nu,j} = \sum_{\eta \in T_j[\nu]} b^{j,k}_{\eta,k}$. Now $(i,j) \in I$ guarantees that $b^{i,j}_{\nu,k} = 0$. Thus the reduced form together with Lemma 3.5(C) (applied for $l = k$) yield $b^{i,j}_{\nu,j} = b^{i,k}_{\nu,k}$. ■

**Lemma 3.8** *Suppose* $\boldsymbol{b}$ *is an element of* $\mathrm{Gr}(\mathcal{A}_R^T)$.

a) *If* $\boldsymbol{b}$ *is not in* $\mathrm{Fact}(\mathcal{A}_R^T)$ *and* $I$ *is an eventually coherent subset of* $\gamma \star \gamma$ *such that* $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\}$ *for all* $(i,j) \in I$, *then there is an eventually coherent subset* $J$ *of* $I$ *with* $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\} \neq 0$ *whenever* $(i,j) \in J$.

b) $(\mathrm{cf}(\gamma) > \aleph_0)$ *If* $J$ *is an eventually coherent subset of* $\gamma \star \gamma$ *such that* $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j}\restriction\{j\} \neq 0$ *for all* $(i,j) \in J$, *then there are a bound* $n^* < \omega$ *and an eventually coherent subset* $K$ *of* $J$ *such that* $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ *for all* $(i,j) \in K$.

***Proof.*** a) Since $\boldsymbol{b} \not\equiv \boldsymbol{0} \mod \mathrm{Fact}(\mathcal{A}_R^T)$ it follows by Lemma 2.3 that there is no subset of $\{(i,j) \in I \mid \boldsymbol{b}^{i,j} = 0\}$ which would be eventually coherent. Hence there is an unbounded subset $S$ of $I^{\mathrm{1st}}$ such that for each $i \in S$ there is $j_i \in I[i]$ with $\boldsymbol{b}^{i,j_i}$ nonzero. Fix any $i \in S$. Since $\boldsymbol{b}^{i,j_i} = \boldsymbol{b}^{i,j_i}\restriction\{j_i\} \neq 0$, let $\nu_i$ be an element of $T_i$ with $b^{i,j_i}_{\nu_i,j_i} \neq 0$. By Lemma 3.7, $b^{i,k}_{\nu_i,k} = b^{i,j_i}_{\nu_i,j_i} \neq 0$ for all $k \in I[i] \cap I[j_i]$. Because $I$ was eventually coherent, we have shown that $J = I \cap (S \times S)$ is an eventually coherent set as wanted in the claim.

b) First of all we claim that for each $i \in J^{\mathrm{1st}}$ the union $\bigcup_{j \in J[i]} \mathrm{supp}(\boldsymbol{b}^{i,j})$ is of finite cardinality. Observe that for every $(i,j) \in J$, $\mathrm{supp}(\boldsymbol{b}^{i,j}) = \mathrm{supp}(\boldsymbol{b}^{i,j}) \cap (T_i \times \{j\})$.

Assume, contrary to this subclaim, that $i \in J^{\mathrm{1st}}$, $\langle j_m \mid m < \omega\rangle$ is an increasing sequence of ordinals in $J[i]$, and $\{\nu_m \mid m < \omega\}$ is a set of distinct elements from $T_i$ such that $b^{i,j_m}_{\nu_m,j_m}$ nonzero for every $m < \omega$. Since $J$ is eventually coherent and $\gamma$ is of uncountable cofinality let $k < \gamma$ be the minimal element in $J[i] \cap \bigcap_{m<\omega} J[j_m]$. Now for each $m < \omega$, the pairs $(i, j_m)$, $(i,k)$, and $(j_m, k)$ are in $J$, and by Lemma 3.7, the equation $b^{i,j_m}_{\nu_m,j_m} = b^{i,k}_{\nu_m,k} \neq 0$ holds. So the infinite set $\{(\nu_m, k) \mid m < \omega\}$ is a subset of $\mathrm{supp}(\boldsymbol{b}^{i,k})$, a contradiction.

It follows from the subclaim that for each $i \in J^{\mathrm{1st}}$, the finite ordinal

$$n_i = \mathrm{card}\Big(\bigcup_{j \in J[i]} \mathrm{supp}(\boldsymbol{b}^{i,j})\Big) + 1$$

satisfies $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n_i$ for all $j \in J[i]$. Since $J^{\mathrm{1st}}$ is uncountable, there are $n^* < \omega$ and an unbounded subset $S$ of $J^{\mathrm{1st}}$ such that $n_i = n^*$ for all $i \in S$. So $n^*$ and the set $K = J \cap (S \times S)$ meet the requirements of the claim. ■

**Lemma 3.9** $(\mathrm{cf}(\gamma) > \mathrm{card}(R))$ *Suppose* $\boldsymbol{b}$ *is in* $\mathrm{Gr}(\mathcal{A}_R^T)$ *and* $I$ *is an eventually coherent subset of* $\{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j} \neq 0\}$. *Then there are* $d \in R$, $t \in \mathrm{Br}_\gamma(T)$, *and an eventually coherent subset* $J$ *of* $I$ *for which* $b^{i,j}_{t(i),j} = d \neq 0$ *whenever* $(i,j) \in J$.

***Proof.*** We define by induction on $\alpha < \mathrm{cf}(\gamma)$ the following objects:

an increasing sequence $\langle i_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle$ of ordinals in $I^{1\mathrm{st}}$ with limit $\gamma$;

an increasing sequence $\langle \nu_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle \in \prod_{\alpha<\mathrm{cf}(\gamma)} T_{i_\alpha}$;

subsets $K_\alpha$ of $I[i_\alpha]$ such that $I^{1\mathrm{st}} \setminus K_\alpha$ are bounded in $\gamma$;

elements $d_\alpha \in R \setminus \{0\}$ such that for every $k \in K_\alpha$, $b^{i_\alpha,k}_{\nu_\alpha,k} = d_\alpha$.

This suffices since $\mathrm{card}(R) < \mathrm{cf}(\gamma)$ implies that there are $d \in R$ and $H \subseteq \mathrm{cf}(\gamma)$ unbounded in $\mathrm{cf}(\gamma)$ such that $d_\alpha = d$ for every $\alpha \in H$. Moreover, the claim is satisfied by $t \in \mathrm{Br}_\gamma(T)$ and $J \subseteq I$ defined as follows. For every $i < \gamma$, $t(i) = \nu_{\beta_i}\restriction i$, where $\beta_i = \min\{\alpha < \mathrm{cf}(\gamma) \mid i_\alpha \geq i\}$, and $J = \bigcup_{\alpha\in H} \big(\{i_\alpha\} \times (S \cap K_\alpha)\big)$, where $S$ is $\{i_\alpha \mid \alpha \in H\}$.

Let $\langle \gamma_\alpha \mid \alpha < \mathrm{cf}(\gamma)\rangle$ be an increasing sequence with limit $\gamma$. Define

$$i_\alpha = \min\big((I^{1\mathrm{st}} \cap \bigcap_{\beta<\alpha} K_\beta) \setminus \gamma_\alpha\big)$$

and

$$j = \min\big(I^{1\mathrm{st}} \cap I[i_\alpha] \cap \bigcap_{\beta<\alpha} I[i_\beta]\big),$$

where both $\bigcap_{\beta<\alpha} K_\beta$ and $\bigcap_{\beta<\alpha} I[i_\beta]$ are equal to $\gamma$ when $\alpha = 0$. This pair $(i_\alpha, j)$ is well-defined since $I$ is eventually coherent, $\alpha < \mathrm{cf}(\gamma)$, and when $\alpha > 0$, $I^{1\mathrm{st}} \setminus K_\beta$ is bounded for each $\beta < \alpha$ by the induction hypothesis.

If $\alpha = 0$, then $(i_0, j) \in I$ guarantees that $\boldsymbol{b}^{i_0,j}\restriction\{j\} = \boldsymbol{b}^{i_0,j} \neq 0$. Hence we can find $\nu_0 \in T_{i_0}$ with $b^{i_0,j}_{\nu_0,j} \neq 0$.

When $\alpha > 0$ we define elements $\eta_\beta \in T_{i_\alpha}[\nu_\beta]$ for each $\beta < \alpha$ as follows. Fix $\beta < \alpha$. Since $i_\alpha \in K_\beta$ we get by the induction hypothesis that $b^{i_\beta,i_\alpha}_{\nu_\beta,i_\alpha} = d_\beta \neq 0$. Furthermore $(i_\beta, i_\alpha) \in I$ (because $K_\beta \subseteq I[i_\beta]$), $(i_\beta, j) \in I$, and $(i_\alpha, j) \in I$ together with Lemma 3.7 yield

$$\sum\nolimits_{\eta\in T_{i_\alpha}[\nu_\beta]} b^{i_\alpha,j}_{\eta,j} = b^{i_\beta,i_\alpha}_{\nu_\beta,i_\alpha} \neq 0.$$

Therefore we can find $\eta_\beta \in T_{i_\alpha}[\nu_\beta]$ for which $b^{i_\alpha,j}_{\eta_\beta,j} \neq 0$.

If $\alpha > 0$ is a successor ordinal define $\nu_\alpha$ to be $\eta_{\alpha-1}$. When $\alpha$ is a limit ordinal, the finiteness of the support $\mathrm{supp}(\boldsymbol{b}^{i_\alpha,j})$ ensures that there are $\nu_\alpha \in T_{i_\alpha}$ and an unbounded subset $H$ of $\alpha$ such that $\eta_{\beta'} = \nu_\alpha$ for all $\beta' \in H$. By the induction hypothesis $\nu_\beta \lhd \nu_{\beta'}$ for all $\beta < \beta' < \alpha$. Hence $\nu_\beta \lhd \nu_{\beta'} \lhd \eta_{\beta'} = \nu_\alpha$ holds for every $\beta < \alpha$ and $\beta' = \min(H \setminus \beta)$.

Let $d_\alpha$ be $b^{i_\alpha,j}_{\nu_\alpha,j}$. By Lemma 3.7, every $k \in I[i_\alpha] \cap I[j]$ satisfies that $b^{i_\alpha,k}_{\nu_\alpha,k} = b^{i_\alpha,j}_{\nu_\alpha,j} = d_\alpha$. Hence $i_\alpha$, $\nu_\alpha$, and $d_\alpha$ together with the set $K_\alpha = I[i_\alpha] \cap I[j]$ meet the requirements given at the beginning of the proof. $\blacksquare_{3.9}$

**Corollary 3.10** $(\mathrm{cf}(\gamma) > \aleph_0)$ *If* $\mathrm{Br}_\gamma(T)$ *is empty, then* $\mathrm{Gr}(\mathcal{A}^T_R) = \mathrm{Fact}(\mathcal{A}^T_R)$.

***Proof.*** Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}^T_R) \setminus \mathrm{Fact}(\mathcal{A}^T_R)$. By Lemma 3.6(c) together with Lemma 3.8(a) there is $\boldsymbol{b} \in \mathrm{Gr}(\mathcal{A}^T_R)$ such that $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}^T_R)$ and the set $\{(i,j) \in \gamma \star \gamma \mid \boldsymbol{b}^{i,j}\restriction\{j\} = \boldsymbol{b}^{i,j} \neq 0\}$ is eventually coherent. By Lemma 3.9 there is a $\gamma$-branch through the tree $T$, i.e., $\mathrm{Br}_\gamma(T) \neq \emptyset$. Observe that the assumption $\mathrm{card}(R) < \mathrm{cf}(\gamma)$ is not needed, as can be seen from the proof of Lemma 3.9. $\blacksquare$

**Lemma 3.11** $(\mathrm{cf}(\gamma) > \max\{\aleph_0, \mathrm{card}(R)\})$ *The elements* $\boldsymbol{t}$, $t \in \mathrm{Br}_\gamma(T)$, *generate* $\mathrm{Gr}(\mathcal{A}_R^T)$ *modulo* $\mathrm{Fact}(\mathcal{A}_R^T)$.

***Proof.*** We show that for every $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}_R^T)$ with $\boldsymbol{a} \notin \mathrm{Fact}(\mathcal{A}_R^T)$ we can find $n < \omega$, $d_1, \dots, d_n \in R \setminus \{0\}$ and $t_1, \dots, t_n \in \mathrm{Br}_\gamma(T)$ satisfying

$$\text{(A)} \qquad \boldsymbol{a} \equiv \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \mod \mathrm{Fact}(\mathcal{A}_R^T).$$

Suppose $\boldsymbol{a} \in \mathrm{Gr}(\mathcal{A}_R^T) \setminus \mathrm{Fact}(\mathcal{A}_R^T)$. By Lemma 3.6(c) and Lemma 3.8(a) let $\boldsymbol{b}$ be an element of $\mathrm{Gr}(\mathcal{A}_R^T)$ and $I_1$ an eventually coherent subset of $\gamma \star \gamma$ such that $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}_R^T)$ and for each $(i,j) \in I_1$, $\boldsymbol{b}^{i,j} = \boldsymbol{b}^{i,j} \restriction \{j\} \neq 0$. Furthermore, we may assume by Lemma 3.8(b) that $n^* < \omega$ is a bound for which $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ hold for all $(i,j) \in I_1$.

By Lemma 3.9 there are $d_1 \in R$, $t_1 \in \mathrm{Br}_\gamma(T)$, and an eventually coherent set $J_1 \subseteq I_1$ having the property that $b^{i,j}_{t_1(i),j} = d_1 \neq 0$ whenever $(i,j) \in J_1$. Since $d_1 \cdot \boldsymbol{t_1} \in \mathrm{Gr}(\mathcal{A}_R^T)$, the sequence $\boldsymbol{c} = \boldsymbol{b} - d_1 \cdot \boldsymbol{t_1}$ is in $\mathrm{Gr}(\mathcal{A}_R^T)$. If $\boldsymbol{c}$ is in $\mathrm{Fact}(\mathcal{A}_R^T)$, then $\boldsymbol{b} \equiv d_1 \cdot \boldsymbol{t_1} \mod \mathrm{Fact}(\mathcal{A}_R^T)$, and because of $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}_R^T)$, also (A) holds for $n = 1$.

Suppose $1 \le n < \omega$ and objects $d_m \in R \setminus \{0\}$, $t_m \in \mathrm{Br}_\gamma(T)$, and $J_m \subseteq J_1$ for $m \le n$ are already defined. Assume also that these objects satisfy the following conditions:

1) $J_{m'} \supseteq J_m$ for all $1 \le m' \le m \le n$;
2) for all $1 \le m' < m \le n$ and $i \in (J_m)^{\mathrm{1st}}$, $t_{m'}(i) \neq t_m(i)$;
3) for every $1 \le m \le n$ and $(i,j) \in J_m$, $b^{i,j}_{t_m(i),j} = d_m \neq 0$;
4) $\boldsymbol{c} = \boldsymbol{b} - \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \notin \mathrm{Fact}(\mathcal{A}_R^T)$.

Clearly $\boldsymbol{c}^{i,j} = \boldsymbol{c}^{i,j} \restriction \{j\}$ and $\mathrm{card}(\mathrm{supp}(\boldsymbol{c}^{i,j})) \le \mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$ for all $(i,j) \in J_n$. Again by Lemma 3.8(a), there is an eventually coherent set $I_{n+1} \subseteq J_n$ such that for each $(i,j) \in I_{n+1}$, $\boldsymbol{c}^{i,j} \neq 0$. Moreover, by Lemma 3.9, there are $d_{n+1} \in R$, $t_{n+1} \in \mathrm{Br}_\gamma(T)$, and an eventually coherent set $J_{n+1} \subseteq \{(i,j) \in I_{n+1} \mid c^{i,j}_{t_{n+1}(i),j} = d_{n+1} \neq 0\}$.

The properties (2), (3) and (4) above imply that $c^{i,j}_{t_m(i),j} = b^{i,j}_{t_m(i),j} - d_m = 0$ for every $m \le n$ and $(i,j) \in J_m$. On the other hand, $c^{i,j}_{t_{n+1}(i),j}$ is nonzero for each $(i,j) \in J_{n+1}$. Thus $t_{n+1}(i)$ can not be in $\{t_m(i) \mid 1 \le m \le n\}$ if $i \in (J_{n+1})^{\mathrm{1st}}$. So for all $(i,j) \in J_{n+1}$, $x_{t_{n+1}(i),j} \notin \{x_{t_m(i),j} \mid 1 \le m \le n\}$, and consequently $b^{i,j}_{t_{n+1}(i),j} = c^{i,j}_{t_{n+1}(i),j}$. Thus also $J_{n+1}$, $t_{n+1}$, and $d_{n+1}$ satisfy the properties (1), (2), and (3) (but not necessarily (4)).

We claim that there must be $n < n^*$ such that

$$\text{(B)} \qquad \boldsymbol{b} - \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \in \mathrm{Fact}(\mathcal{A}_R^T).$$

Assume, contrary to this subclaim, that the process introduced above has been carried out $n^*$ many times and objects $J_m$, $t_m$, $d_m$ for $i \le m \le n^*$ are defined. In addition to that suppose they satisfy the conditions (1), (2), and (3). Define $i = \min\left((J_{n^*})^{\mathrm{1st}}\right)$ and $j = \min(J_{n^*}[i])$. Then for every $m \le n^*$, $(i,j) \in J_m$ yields $b^{i,j}_{t_m(i),j} = d_m \neq 0$. This contradicts the condition $\mathrm{card}(\mathrm{supp}(\boldsymbol{b}^{i,j})) < n^*$, since the set $\{(t_m(i), j) \mid m \le n^*\} \subseteq \mathrm{supp}(\boldsymbol{b}^{i,j})$ is of cardinality $n^*$.

Now suppose $n < \omega$ is a finite ordinal satisfying (B). Then $\boldsymbol{b} \equiv \sum_{1 \le m \le n} d_m \cdot \boldsymbol{t_m} \mod \mathrm{Fact}(\mathcal{A}_R^T)$, and because $\boldsymbol{a} \equiv \boldsymbol{b} \mod \mathrm{Fact}(\mathcal{A}_R^T)$ also (A) is satisfied. ■

**Conclusion 3.12** *For any ordinal $\gamma$ of uncountable cofinality, ring $R$ with* $\mathrm{card}(R) < \mathrm{cf}(\gamma)$, *and tree $T$ of height $\gamma$, the inverse $\gamma$-system $\mathcal{A}_R^T = \langle G_i, h_{i,j} \mid i < j < \gamma \rangle$ has the properties that*

$$\mathrm{card}(G_i) = \max\{\mathrm{card}(\gamma), \mathrm{card}(T_i), \mathrm{card}(R)\}$$

*for all $i < \gamma$, and*

$$\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big) = \mathrm{card}\big(\langle \boldsymbol{t} \rangle_R^{t \in \mathrm{Br}_\gamma(T)}\big).$$

***Proof of Theorem 2.*** Remember that $\lambda$ and $\kappa$ were cardinals with $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda$. We wanted to study possible cardinalities $\mu$ of the quotient limit $\mathrm{Gr}(\mathcal{A})/\mathrm{Fact}(\mathcal{A})$, where $\mathcal{A}$ is an inverse $\kappa$-system consisting of abelian groups having cardinality $< \lambda$. Now Conclusion 3.12 gives a complete solution to this problem because of $\lambda > \mathrm{cf}(\lambda) = \kappa = \mathrm{cf}(\kappa) > \aleph_0$. Namely, in order to meet the requirements $\mathrm{card}(G_i) < \lambda$ for all $i < \kappa$, it is needed only to ensure that $R$ and the $i^{\text{th}}$ level of $T$ are small enough. On the other hand, a suitable choice of $R$ and $T$ yields any desired value for $\mu = \mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big)$. We briefly describe methods to choose suitable $R$ and $T$ for every nonzero $\mu \leq \lambda^\kappa$.

For any $R$, $\mathrm{card}\big(\mathrm{Gr}(\mathcal{A}_R^T)/\mathrm{Fact}(\mathcal{A}_R^T)\big)$ equals 1 when $\mathrm{Br}_\kappa(T)$ is empty. So $\mu = 1$ is possible since obviously there exists a tree of height $\kappa$ without $\kappa$-branches and having levels of cardinality $< \lambda$ when $\lambda$ singular of cofinality $\kappa$. Also all the finite values $\mu > 1$ are possible by taking $T$ with only one $\kappa$-branch and $R$ with $\mathrm{card}(R) = \mu$.

Furthermore the case of infinite $\mu < \lambda$ is satisfied by any $R$ with $\mathrm{card}(R) < \min\{\kappa, \mu\}$ and $T$ with exactly $\mu$ many $\kappa$-branches. The value $\mu = \lambda$ is possible for any $R$ with $\mathrm{card}(R) < \kappa$ because a suitable tree can be constructed, for example, as follows. Let $\langle \lambda_i \mid i < \kappa \rangle$ be an increasing sequence of ordinals $< \lambda$ with limit $\lambda$. Then the tree

$$T = \{t{\restriction}\alpha \mid \alpha < \kappa, t \in \prod_{i<\kappa} \lambda_i, \text{and } t(i) \text{ is nonzero only for finitely many } i < \kappa\},$$

ordered by inclusion, satisfies $\mathrm{card}(\mathrm{Br}_\kappa(T)) = \lambda$ and $\mathrm{card}(T_i) = \lambda_i < \lambda$ for each $i < \kappa$.

Also the cardinalities $\mu$ of the quotient limit, when $\lambda < \mu \leq \lambda^\kappa$, are possible for any ring of cardinality $< \kappa$. Existence of a suitable tree is proved for example in [She89, Fact 10] under the assumption that $2^\kappa < \lambda$ and $\theta^{<\kappa} < \lambda$ for every $\theta < \lambda$ (other sources for a proof are given in [She94, Analytical Guide §10]). ■

## References


[Cha68] C. C. Chang. Some remarks on the model theory of infinitary languages. In J. Barwise, editor, *The syntax and semantics of infinitary languages*, Lecture Notes in Math., 72, pages 36–63. Springer-Verlag, Berlin, 1968.

[Pal77a] E. A. Palyutin. Number of models in $L_{\infty,\omega_1}$ theories, II. *Algebra i Logika*, 16(4):443–456, 1977. English translation in [Pal77b].

[Pal77b] E. A. Palyutin. Number of models in $L_{\infty,\omega_1}$ theories, II. *Algebra and Logic*, 16(4):299–309, 1977.

[Sco65] Dana Scott. Logic with denumerably long formulas and finite strings of quantifiers. In J. W. Addison, Leon Henkin, and Alfred Tarski, editors, *Theory of Models (Proc. 1963 Internat. Sympos. Berkeley)*, pages 329–34. North-Holland, Amsterdam, 1965.

[She81] Saharon Shelah. On the number of nonisomorphic models of cardinality $\lambda$ $L_{\infty,\lambda}$-equivalent to a fixed model. *Notre Dame J. Formal Logic*, 22(1):5–10, 1981.

[She82] Saharon Shelah. On the number of nonisomorphic models in $L_{\infty,\lambda}$ when $\kappa$ is weakly compact. *Notre Dame J. Formal Logic*, 23(1):21–26, 1982.

[She85] Saharon Shelah. On the possible number no$(M)$ = the number of nonisomorphic models $L_{\infty,\lambda}$-equivalent to $M$ of power $\lambda$, for $\lambda$ singular. *Notre Dame J. Formal Logic*, 26(1):36–50, 1985.

[She86] Saharon Shelah. On the no$(M)$ for $M$ of singular power. In *Around classification theory of models*, Lecture Notes in Math., 1182, pages 120–134. Springer-Verlag, Berlin, 1986.

[She89] Saharon Shelah. The number of pairwise non-elementarily-embeddable models. *J. Symbolic Logic*, 54(4):1431–1455, 1989.

[She94] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[SV97] Saharon Shelah and Pauli Väisänen. On the number of $L_{\infty,\omega_1}$-equivalent, non-isomorphic models. Reports of the Department of Mathematics 147, University of Helsinki, 1997.

Saharon Shelah:

Institute of Mathematics
The Hebrew University
Jerusalem. Israel

Rutgers University
Hill Ctr-Busch
New Brunswick. New Jersey 08903
shelah@math.rutgers.edu

Pauli Väisänen:

Department of Mathematics
P.O. Box 4
00014 University of Helsinki
Finland
pauli.vaisanen@helsinki.fi