

Two consistency results on set mappings

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Abstract. It is consistent that there is a set mapping from the four-tuples of ω_n into the finite subsets with no free subsets of size t_n for some natural number t_n . For any $n < \omega$ it is consistent that there is a set mapping from the pairs of ω_n into the finite subsets with no infinite free sets. For any $n < \omega$ it is consistent that there is a set mapping from the pairs of ω_n into ω_n with no uncountable free sets.

In this paper we consider some problems on *set mappings*, that is, for our current purposes, functions of the type $f : [\kappa]^k \rightarrow [\kappa]^{<\mu}$ for some natural number k and cardinals κ, μ , which satisfy $f(x) \cap x = \emptyset$ for $x \in [\kappa]^k$. A subset H of κ is called *free* if $f(x) \cap H = \emptyset$ holds for every $x \in [H]^k$. The most central question of this area of combinatorial set theory is that given k, κ , and μ how large free sets can be guaranteed. The investigation of the case $k = 1$ was started in the thirties by Paul Turán, who asked if there exists an infinite free set if $\mu = \omega$ and κ is the continuum. After G. Grünwald's affirmative answer ([4]) S. Ruziewicz found the right conjecture ([10]); if $\kappa > \mu$ then there is a free set of cardinal κ (remember, $k = 1$ is assumed). Several cases were soon proved, for example S. Piccard solved the case when κ is regular ([9]), but only in 1950 was the full conjecture established by Paul Erdős ([1]) with the assumption of GCH, and ten years later without this assumption, by A. Hajnal ([5]). In the fifties Erdős and Hajnal started the research on the case $k > 1$ following the observation of Kuratowski and Sierpiński (see [4]) that for set mappings on $[\kappa]^k$ there always exists a free set of cardinal $k + 1$ iff $\kappa \geq \mu^{+k}$.

In ZFC alone, Hajnal and Máté extended the Kuratowski-Sierpiński results by showing ([6]) that if $k = 2$ and $\kappa \geq \mu^{+2}$ then there are arbitrarily large finite free sets, and Hajnal proved (see [3]) that a similar result holds for $k = 3, \kappa \geq \mu^{+3}$. One of the problems emphasized in [3] is if the result can be extended to $k = 4, \kappa \geq \mu^{+4}$. In Theorem 1 we show that it is not

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the case; for every natural number n there exists a natural number t_n such that for any given regular μ it is consistent that there is a set mapping $f : [\mu^{+n}]^4 \rightarrow [\mu^{+n}]^{<\mu}$ with no free sets of size t_n . (We assume GCH in the ground model.)

As for the existence of infinite free sets, a special case of a theorem of Erdős and Hajnal states that under CH if $f : [\omega_2]^2 \rightarrow [\omega_2]^{<\omega}$ is a set mapping then there is an uncountable free set for f ([2]). Answering a question of [6] the first author proved that without CH even the existence of an infinite free set cannot be guaranteed [7]. Here we extend that result to arbitrary ω_n . Using this result, we answer another question of Hajnal and Máté, by showing that it is consistent that there exists a set mapping from the pairs of ω_n into $[\omega_n]^1$ with no uncountable free sets.

Theorem 1 was proved by S. Shelah; Theorem 2 and Corollary 3 were subsequently proved by P. Komjáth.

Notation and Definitions. We use the standard axiomatic set theory notation. Cardinals are identified with initial ordinals. If S is a set and κ a cardinal, then $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$, $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$, $[S]^{\leq\kappa} = \{X \subseteq S : |X| \leq \kappa\}$. For a, b, c and r natural numbers, the Ramsey symbol, $a \longrightarrow (b, c)^r$, means that the following statement is true. Whenever the r -element subsets of an a element set are colored with two colors, say 0 and 1, then either there exists a b -element subset with all its r -tuples colored 0 or there exists a c -element subset with all its r -tuples colored 1. The existence of an appropriate a for any given b, c, r is guaranteed by Ramsey's theorem [3].

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To formulate the next result, set $t_0 = 5$, $t_1 = 7$, in general, t_{n+1} is the least number such that $t_{n+1} \longrightarrow (t_n, 7)^5$.

Theorem 1. (GCH) *Assume that $n < \omega$, $\kappa = \tau^{+n}$ for some regular cardinal τ . Then it is consistent that GCH holds below τ , $2^\tau = \kappa$ if $n > 0$, and there is a set mapping $f : [\kappa]^4 \longrightarrow [\kappa]^{<\tau}$ with no free subset of cardinal t_n .*

Proof. By induction on n . Our set mapping will satisfy the additional condition that $f(\{x_0, x_1, x_2, x_3\}) \subseteq (x_1, x_2)$ (the ordinal interval) for all $x_0 < x_1 < x_2 < x_3$.

The case $n = 0$ is obvious, since we can take $f(\{x_0, x_1, x_2, x_3\}) = (x_1, x_2)$.

Assume that V is a model of set theory satisfying the Theorem for n , and for τ^+ in place of τ . That is, for $\mu \leq \tau$, $2^\mu = \mu^+$ holds, and there is a set mapping $F : [\kappa]^4 \rightarrow [\kappa]^{\leq \tau}$ satisfying $F(\{x_0, x_1, x_2, x_3\}) \subseteq (x_1, x_2)$ with no free subset of cardinality t_n . We are going to force with a notion of forcing (P, \leq) in which the conditions will be some pairs of the form (s, g) with $s \in [\kappa]^{< \tau}$, $g : [s]^4 \rightarrow [s]^{< \tau}$ satisfying $g(u) \subseteq F(u)$ for $u \in [s]^4$. Not all pairs as above will be in P but if $(s, g), (s', g')$ are in P then (s', g') will extend (s, g) (in notation $(s', g') \leq (s, g)$) iff $s' \supseteq s$ and $g = g'|[s]^4$.

To describe the condition for $(s, g) \in P$ we introduce two more definitions. If U is a subset of κ then we call U F -closed, if $x_2 \in F(x_0, x_1, x_3, x_4)$ holds whenever $x_0 < x_1 < x_2 < x_3 < x_4$ are in U . If U is a subset of s then we call U g -free, if $x_2 \notin g(\{x_0, x_1, x_3, x_4\})$ holds for all $x_0 < x_1 < x_2 < x_3 < x_4$ in U . Now put (s, g) into P just in case there is no 7-element subset of s which is F -closed and g -free.

Having defined the notion of forcing (P, \leq) , we are going to show some properties of it.

Claim 1. (P, \leq) is $< \tau$ -closed.

Proof of Claim. Immediate from the finite character of the definition. \square

Claim 2. (P, \leq) is τ^+ -c.c.

Proof of Claim. Assume that $p_\xi = (s_\xi, g_\xi) \in P$ for $\xi < \tau^+$. Using the Δ -system lemma we can assume that $s_\xi = a \cup b_\xi$ for some disjoint sets $\{a\} \cup \{b_\xi : \xi < \tau^+\}$. For $\{x_0, x_1, x_2, x_3\} \in [a]^4$, $g_\xi(\{x_0, x_1, x_2, x_3\})$ is a subset of $F(\{x_0, x_1, x_2, x_3\})$ of cardinal $< \tau$. As $|F(\{x_0, x_1, x_2, x_3\})| \leq \tau$, and $|a| < \tau$, we can assume, by $\tau^{< \tau} = \tau$, that $g_\xi|[a]^4$ is the same for $\xi < \tau^+$. We show that any two $p_\xi, p_{\xi'}$ are compatible. Set $q = (a \cup b_\xi \cup b_{\xi'}, g)$ where $g \supseteq g_\xi, g_{\xi'}$ and if

$$\{x_0, x_1, x_2, x_3\} \in [a \cup b_\xi \cup b_{\xi'}]^4 - [a \cup b_\xi]^4 - [a \cup b_{\xi'}]^4$$

then set

$$g(\{x_0, x_1, x_2, x_3\}) = (a \cup b_\xi \cup b_{\xi'}) \cap F(\{x_0, x_1, x_2, x_3\}).$$

We have to show that $q \in P$, that is, there is no 7-element F -closed, g -free subset of s . Assume that B is such a set. As $p_\xi, p_{\xi'}$ are conditions, $B \not\subseteq a \cup b_\xi, B \not\subseteq a \cup b_{\xi'}$. There are, therefore, $\eta_0 \in B \cap b_\xi, \eta_1 \in B \cap b_{\xi'}$.

An easy calculation shows that no matter what position η_0, η_1 occupy in B , there is a five-tuple $y_0 < y_1 < y_2 < y_3 < y_4$ in B such that $\eta_0, \eta_1 \in \{y_0, y_1, y_3, y_4\}$. (This is the point where the choice of 7 plays role.) We get, therefore, that $g(\{y_0, y_1, y_3, y_4\}) = F(\{y_0, y_1, y_3, y_4\}) \cap s \ni y_2$ so B cannot be F -closed and g -free. \square

Let $G \subseteq P$ be a generic subset of P . Set $S = \bigcup\{s : (s, g) \in G\}$ and $f = \bigcup\{g : (s, g) \in G\}$. Clearly, f is a set mapping of the required type on the set S .

Claim 3. *There is a $p \in P$ forcing that $|S| = \kappa$.*

Proof of Claim. Otherwise, 1 forces that S is bounded in κ , and as (P, \leq) is $< \kappa$ -c.c., it forces a bound, say $\xi < \kappa$. But as $(\{\xi\}, \emptyset) \Vdash \xi \in S$, we get a contradiction. \square

Claim 4. *In $V[G]$, f has no free subset of cardinality t_{n+1} .*

Proof of Claim. Assume that $A \subseteq S$ is a free subset of cardinality t_{n+1} . Color the five-tuples of A as follows. If $\{x_0, x_1, x_2, x_3, x_4\} \in [A]^5, x_0 < x_1 < x_2 < x_3 < x_4$ and $x_2 \in F(\{x_0, x_1, x_3, x_4\})$ then color $\{x_0, x_1, x_2, x_3, x_4\}$ by 1, otherwise by 0. As $t_{n+1} \rightarrow (t_n, 7)^5$ either there is a homogeneous subset in color 1 of cardinal 7 or there is a homogeneous subset of color 0 of size t_n . This latter possibility is excluded by the hypothesis on F so we have the former. But that gives a 7-element subset which is F -closed and f -free and this is obviously excluded by the forcing. \square

Now Theorem 1 follows from the claims above by induction on n . \square

Theorem 2. (GCH) *If τ is a regular cardinal, $\kappa < \tau^{+\omega}$, then it is consistent that there is a set mapping $f : [\kappa]^2 \rightarrow [\kappa]^{<\tau}$ with no infinite free sets.*

Proof. For $\kappa \leq \tau$, we can simply take $f(\{x, y\}) = x$.

We are going to show, by induction on positive $n < \omega$ that it is consistent that there exists for $\kappa = \tau^{+n}$ a set mapping f on $[\kappa]^2$ as required. It will also satisfy $f(\{x, y\}) \subseteq x$ for $x < y < \kappa$.

The case $n = 1$ can also be proved in ZFC. If $x < \kappa = \tau^+$, enumerate x as $x = \{\gamma_x(i) : i < \tau\}$. If $x < y$ then let $i(x, y)$ be that index i for which

$x = \gamma_y(i)$ holds. Now set $f(\{x, y\}) = \{\gamma_x(i) : i \leq i(x, y)\}$. If $x_0 < x_1 < \dots$ are the elements of an infinite free set then $i(x_0, x_1) > i(x_1, x_2) > \dots$ which is impossible.

Assume now that $\tau < \kappa$, GCH holds up to and including τ and there is a set mapping $F : [\kappa]^2 \rightarrow [\kappa]^{\leq \tau}$ with no infinite free sets and with $F(\{x, y\}) \subseteq x$ for $x < y < \kappa$. We are going to define a $< \tau$ -closed partial ordering (P, \leq) which adds a set mapping $f : [S]^2 \rightarrow [S]^{< \tau}$ for some $S \in [\kappa]^\kappa$ and with no infinite free sets. It will also satisfy $f(\{x, y\}) \subseteq F(\{x, y\})$ for all $\{x, y\} \subseteq S$.

An element of P will be a triplet of the form $p = (s, g, r)$ where $s \in [\kappa]^{< \tau}$, $g : [s]^2 \rightarrow [s]^{< \tau}$ is a set mapping with $g \subseteq F$. If U is a subset of κ then we call U F -closed, if $x \in F(y, z)$ holds if $x < y < z$ are in U . If U is a subset of s then we call U g -free, if $x \notin g(\{y, z\})$ holds for $x < y < z$ in U . We require that there be no infinite g -free, F -closed subsets of s and r will be a rank function witnessing this. For this, we call a finite subset $u \in [s]^{< \omega}$ secured if $|u| \geq 3$, u is g -free and F -closed. What we assume on r is that it is a function from the secured subsets to τ with $r(u) > r(v)$ if v properly end-extends u . $p' = (s', g', r')$ extends $p = (s, g, r)$ if $s' \subseteq s$, $g' \subseteq g$, $r' \subseteq r$.

It is obvious that (P, \leq) is transitive and $< \tau$ -closed.

Claim 1. (P, \leq) is τ^+ -c.c.

Proof of Claim. Assume, for a contradiction, that we are given τ^+ conditions, $p_\xi = (s_\xi, g_\xi, r_\xi) \in P$ for $\xi < \tau^+$. By the Δ -system lemma we can assume that there are disjoint sets $\{a\} \cup \{b_\xi : \xi < \tau^+\}$ such that $s_\xi = a \cup b_\xi$. As for $x, y \in a$, since $g_\xi(\{x, y\}) \in [F(\{x, y\})]^{< \tau}$, by removing at most τ members from the family we can assume that $F(\{x, y\}) \cap b_\xi = \emptyset$ holds for $x, y \in a$. Then, $g_\xi(\{x, y\}) \subseteq a$, and with one more shrinking, we can assume that $g_\xi(\{x, y\})$ is independent of ξ . We can also assume that the functions r_ξ are identical on the secured subsets of a .

Assume now that $\xi < \xi' < \tau^+$, we want to find a common extension of p_ξ and $p_{\xi'}$. Set $q = (a \cup b_\xi \cup b_{\xi'}, g, r)$ where $g \supseteq g_\xi \cup g_{\xi'}$ is the maximal extension, that is, $g(\{x, y\}) = (a \cup b_\xi \cup b_{\xi'}) \cap F(\{x, y\})$ if $\{x, y\} \cap b_\xi \neq \emptyset$ and $\{x, y\} \cap b_{\xi'} \neq \emptyset$.

We now consider if we can define r . As q is the union of two conditions both omitting infinite g -free, F -closed sets, q won't have such sets, either. So some rank function r can be defined; the question is, if one extending

$r_\xi, r_{\xi'}$ can be given. To show this, it suffices to prove, that if u is a g -free, F -closed set, which is new, that is, has points in b_ξ , as well as in $b_{\xi'}$, then it cannot end extend an “old” secured set (one in p_ξ or in $p_{\xi'}$). Assume that $x_0 < x_1 < x_2 < \dots$ are the elements of u . If $x_i \in b_\xi, x_j \in b_{\xi'}$, and $i, j \neq 0$, then $x_0 \in F(\{x_i, x_j\})$, so $x_0 \in g(\{x_i, x_j\})$ by the definition of g and so our set is not g -free. We get, therefore, that x_0 is the *only* element of $u \cap b_\xi$ (say). The possibility that both x_1 and x_2 are in a is ruled out by our above condition that $b_\xi \cap F(\{x_1, x_2\}) = \emptyset$. This means that $\{x_0, x_1, x_2\}$ is a “new” set, so u is indeed not an end extension of an old secured set as we assumed that secured sets have at least three elements. \square

If $G \subseteq P$ is a generic subset, then define $S = \bigcup\{s : (s, g, r) \in G\}$, $f = \bigcup\{g : (s, g, r) \in G\}$, $R = \bigcup\{r : (s, g, r) \in G\}$, .

Claim 2. $|S| = \kappa$.

Proof of Claim. As in the corresponding proof in Theorem 1. \square

Claim 3. F has no infinite free set in $V[G]$.

Proof of Claim. This is a well-known fact. It follows from the rank characterization of the nonexistence of free sets. \square

Claim 4. f has no infinite free set.

Proof of Claim. Assume that $x_0 < x_1 < \dots$ form an infinite f -free set. By Ramsey’s theorem we can assume that either for every triplet $i < j < k < \omega$, $x_i \in F(x_j, x_k)$ holds or for every triplet $i < j < k < \omega$, $x_i \notin F(x_j, x_k)$ holds. The latter is impossible by Claim 3. Therefore $\{x_0, x_1, \dots\}$ is f -free, F -closed, but then $R(\{x_0, x_1, x_2\}) > R(\{x_0, x_1, x_2, x_3\}) > \dots$, which is impossible. \square

An easy application of Theorem 2 solves another problem of [6].

Corollary 3. For every $n < \omega$ it is consistent that there exists a set mapping $f : [\omega_n]^2 \rightarrow [\omega_n]^1$ with no uncountable free set.

Proof. Applying Theorem 2 assume that $F : [\omega_n]^2 \rightarrow [\omega_n]^{\aleph_0}$ is a set mapping with no infinite free sets so that $F(\{x, y\}) \subseteq x$ for all $x < y < \omega_n$. Define the notion of forcing as follows, $(s, g) \in P$ iff $s \in [\omega_n]^{<\omega}$, $g : [s]^2 \rightarrow [s]^1$, and $g(u) \subseteq F(u)$ for all $u \in [s]^2$. Set $(s', g') \leq (s, g)$ iff $s' \supseteq s, g' \supseteq g$.

Claim 1. If $\alpha < \omega_n$ then the set $D_\alpha = \{(s, g) : \alpha \in s\}$ is dense in (P, \leq) .

Proof of Claim. Straightforward. □

Claim 2. (P, \leq) is c.c.c.

Proof of Claim. Assume that $p_\xi \in P$ for $\xi < \omega_1$. By the usual thinning out procedure we can assume that $p_\xi = (s \cup s_\xi, g_\xi)$ where $s_\xi \cap F(x, y) = \emptyset$ holds for $x, y \in s$, and the functions $g_\xi \upharpoonright [s]^2$ are identical. Now any two p_ξ -s are compatible. □

If $G \subseteq P$ is a generic set, put $f = \bigcup \{g : (s, g) \in G\}$.

Claim 3. f has no uncountable free set.

Proof of Claim. Assume that $p \Vdash X$ is an uncountable free set. There are, for $\xi < \omega_1$, conditions $p_\xi \leq p$ and ordinals α_ξ with $p_\xi \Vdash \alpha_\xi \in X$. Again, we can assume, that $p_\xi = (s \cup s_\xi, g_\xi)$, $\alpha_\xi \in s_\xi$, and the functions $g_\xi \upharpoonright [s]^2$ are identical. As F has no infinite free sets (“no uncountable” suffices) there are ordinals $\xi_0, \xi_1, \xi_2 < \omega_1$ such that $\alpha_{\xi_0} \in F(\alpha_{\xi_1}, \alpha_{\xi_2})$. We can now extend p to a condition $p' = (s', g')$ where

$$s' = s \cup s_{\xi_0} \cup s_{\xi_1} \cup s_{\xi_2},$$

g' extends $g_{\xi_0}, g_{\xi_1}, g_{\xi_2}$ and $g'(\{\alpha_{\xi_1}, \alpha_{\xi_2}\}) = \alpha_{\xi_0}$. □

Now Corollary 3 follows from the claims above. □

References

- [1] P. Erdős: Some remarks on set theory, *Proceedings of the American Mathematical Society*, **1**(1950), 127–141.
- [2] P. Erdős, A. Hajnal: On the structure of set mappings, *Acta Mathematica Acad. Sci. Hung.*, **9**(1958), 111–131.
- [3] P. Erdős, A. Hajnal, A. Máté, R. Rado: *Combinatorial Set Theory: Partition Relations for Cardinals*, North-Holland, Akadémiai Kiadó, 1984.
- [4] G. Grünwald: Egy halmazelméleti tételről, *Mathematikai és Fizikai Lapok*, **44**(1937), 51–53.
- [5] A. Hajnal: Proof of a conjecture of S. Ruziewicz, *Fund. Math.* **50** (1961), 123–128.
- [6] A. Hajnal, A. Máté: Set mappings, partitions, and chromatic numbers, in: *Logic Colloquium '73*, Bristol, North-Holland, 1975, 347–379.

- [7] P. Komjáth: A set mapping with no infinite free subsets, *Journal of Symbolic Logic* **56** (1991), 304–306.
- [8] K. Kuratowski: Sur une caractérisation des alephs, *Fundamenta Mathematicae*, **38**(1951), 14–17.
- [9] S. Piccard: Sur un problème de M. Ruziewicz de la théorie des relations, *Fundamenta Mathematicae*, **29**(1937), 5–9.
- [10] S. Ruziewicz: Une généralisation d’un théorème de M. Sierpiński, *Publ. Math. de l’Université de Belgrade* **5** (1936), 23-27.

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