Dimension blocking categoricity in higher logics

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Abstract

We generalize the well-known Hart-Shelah example [HS90] to higher infinitary logics. We build, for each $k < \omega$ and for each infinite cardinal λ, a sentence ψ_k^λ of $L_{(2^\lambda)^+,\omega}$ that is categorical in $\lambda, \lambda^+, \ldots, \lambda^{+k}$ but not in $\beth_{k+1}(\lambda)^+$; we study the dimensional encoding of combinatorics involved in the construction of this sentence and study various properties of the resulting abstract elementary class $\mathcal{K}^*(\lambda, k) = (\mathrm{Mod}(\psi_k^\lambda), \prec_{(2^\lambda)^+,\omega})$. We obtain in particular λ-good frames that are not λ^+-good, generalizing results of Boney and Vasey [BV16].

The study of categoricity transfer has been central to model theory since Morley's theorem; the question of finding extensions of this theorem to infinitary contexts and to abstract elementary classes has been a major source of results. Many central concepts of stability theory, both in first order and in its generalizations, have been derived from the various attempts at generalizing Morley's theorem.

One of the most important landmarks along this path was Shelah's Categoricity Transfer result for $L_{\omega_1,\omega}$: if a sentence ψ is categorical in $\aleph_n$ for all $n < \omega$ and the weak GCH holds for the $\aleph_n$'s ($2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$) then ψ is categorical in all cardinals (see [She83a] and [She83b]).

The necessity of assuming the categoricity at all the $\aleph_n$'s becomes natural in light of the previous results; a few years later Hart and Shelah [HS90] provided (for each $k \in$

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$[3, \omega)$) an example of a sentence ψ_k in $L_{\omega_1,\omega}$ categorical in $\aleph_0, \aleph_1, \cdots, \aleph_{k-2}$ but not categorical in any cardinal above or at $\aleph_{k-1}$. This important example has later been referred to as **the Hart-Shelah example.** The present paper presents a strong generalization of this example to other logics.

Two natural questions at this point are:

- How much does this example depend on $L_{\omega_1,\omega}$? In particular, can it be lifted up to other infinitary logics such as $L_{\mu,\omega}$ for some μ?
- Can the "interval" of failure of categoricity be extended beyond finite length?

Our construction provides an answer to the first question, for higher infinitary logics: for each infinite cardinal λ and each $k \in (2, \omega)$ we provide a sentence ψ_k of $L_{(2^\lambda)^+,\omega}$ that is categorical in $\lambda, \lambda^+, \ldots, \lambda^{+k}$ but is not categorical in any cardinality $\mu \geq \beth_{k+1}(\lambda)^+$.

The papers [She83a] and [She83b] opened a path toward stability theory beyond first order, originally for $L_{\omega_1,\omega}$ but later quite widely generalized to abstract elementary classes (AECs, see [She09a] and [She09b]) over the past two decades by several authors.

The shift of focus from infinitary logic to abstract elementary classes entails in many cases using Galois (orbital) types instead of syntactic types; although this shift is natural, compactness and locality properties in general do not transfer to Galois types. In particular, *tameness* and *type-shortness* do not hold in general for Galois types. Tameness was isolated by Grossberg and VanDieren [GV06]; later, Baldwin and Shelah [BS08] produced the first systematic study of the failure of tameness.

In[BK09], Baldwin and Kolesnikov study again the Hart-Shelah example: they prove that for the sentence ψ_k of $L_{\omega_1,\omega}$ of the example, the corresponding AEC

$$\mathcal{K}^{HS}(\omega_1, k) = (\mathrm{Mod}(\psi_k), \prec_{\omega_1,\omega})$$

- has disjoint amalgamation,
- is Galois stable exactly in $\aleph_0, \aleph_1, \ldots, \aleph_{k-3}$,
- is $(< \aleph_0, \leq \aleph_{k-3})$-tame,
- but is not $(\aleph_{k-3}, \aleph_{k-2})$-tame.

Baldwin and Kolesnikov really study a slight variant of the Hart-Shelah example, presented in the language of group actions and revealing the filiation to the early Baldwin-Lachlan example of an $\aleph_1$-categorical theory which is not almost strongly minimal.

More recently, Boney [Bon14] has continued this study of the behavior of the Hart-Shelah example; he has proved that the class $\mathcal{K}^{HS}(\omega_1, k)$ has a "good $\aleph_m$-frame" for all $m \leq k - 3$ but cannot have a good frame above by the failure of stability. Then, Boney and Vasey [BV16] continue this study and show first that the frame at $\aleph_{k-3}$ cannot be "successful". They study good frames in connection with the Hart-Shelah example: for frames around the $\aleph_n$'s ($n < \omega$) the Hart-Shelah example is a natural place to look for "boundary properties": being "successful up to some point" but failing to be successful above.

Our generalization of the Hart-Shelah example addresses the question of beginning to lift these properties above the $\aleph_n$'s. Our sentence ψ_k^λ is in $L_{(2^\lambda)^+,\omega}$, is categorical in $\lambda, \lambda^+, \ldots, \lambda^{+k}$ but is not categorical in $\beth_{k+1}(\lambda)^+$. The sentence resembles the original Hart-Shelah one (or the Baldwin-Kolesnikov variant) but there are substantial differences:

- The sentences are constructed in all cases by first building a "standard model" and then extracting the sequence from it. In the Hart-Shelah example, one predicate Q "ties together" various copies of groups in a way that ends up linking the "dimension" of the predicate to the length of induction in the proof of categoricity. In our example, we need a large family of predicates Q_s, $s \in S = [\lambda]^{<\aleph_0}$.
- The "failure of categoricity" argument at cardinals greater than or equal to $\beth_{k+1}(\lambda)^+$ here is done by using a regular filter $\mathfrak{D}$.

With the results of this paper, the analysis of tameness, disjoint amalgamation, Galois-stability and interaction with good frames of the abstract elementary class

$$\mathcal{K}^*(\lambda, k) = (\mathrm{Mod}(\psi_k^\lambda), \prec_{(2^\lambda)^+,\omega})$$

will follow a pattern along the lines of [BV16]: we can get

- $(< \lambda, \lambda^{+k-2})$-tame and $(< \lambda, \lambda^{+k-2})$-typeshort over models of size λ^{+k-3},
- for each $m \leq k-3$ there is a frame $\mathfrak{s}^*(\lambda, k)_m$ such that type-full and good, on $\mathrm{Mod}(\psi_k^\lambda)$,
- the last frame $\mathfrak{s}^*(\lambda, k)_m$ is not weakly successful.

In the last section we briefly elaborate on this.

A first draft of this paper was circulated around 2005. This version includes a streamlined presentation of the example, together with connections with later presentations of the Hart-Shelah example due to Baldwin and Kolesnikov [BK09] and to the study of good frames in the light of this family of examples, started by Boney and Vasey [BV16].

Our notation is standard.

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1 Building the sentence ψ_k in $L_{(2^\lambda)^+,\omega}$

Context 1.1. *From now on, fix an infinite cardinal* λ *and a natural number* $k \geq 2$.

We will build a sentence ψ_k in $L_{(2^\lambda)^+,\omega}$. This will take the rest of this section.

To build the sentence ψ_k, we first give a description of a **canonical model** M_I, built along a set I, the "spine" of the model. The set I can be arbitrary (but infinite) - the model M_I will be built "around" this I, as a sort of hull of I. The first part of the construction is similar to the first part of the Hart-Shelah example, with one important difference in the construction of the group.

1.1 The spine I, the hull M_I

We will define the 'canonical models' by using groups as 'supports' - these groups are defined next, and are based on the set S of finite subsets of λ.

Definition 1.2. Notation/Building bricks. *We fix the following basic objects to use in the construction later.*

- $S = S_\lambda := [\lambda]^{<\aleph_0} = \{u \subset \lambda | u$ *is finite*$\}$,
- $\mathfrak{D} = \mathfrak{D}_\lambda := \{A \subset S | \exists u_A \in S \forall v \in S (u_A \subset v \to v \in A)\}$, *the regular filter on* S *generated by sets of the form* $\langle u \rangle = \{v \in S | u \subset v\}$.
- $G^+ = G^+_\lambda := {}^S(\mathbb{Z}_2)$, *as a group with the natural operation* $(f+g)(v) = f(v) +_{\mathbb{Z}_2} g(v)$,
- $G = G_\lambda := \{f \in {}^S(\mathbb{Z}_2) | \ker(f) = \{u \in S | f(u) = 0\} \in \mathfrak{D}\}$, *as a subgroup of* G^+*:* $G \leq G^+$, *as if* $f, g \in G$ *then* $\ker(f), \ker(g) \in \mathfrak{D}$, *so* $\ker(f+g) \supset \ker(f) \cap \ker(g) \in \mathfrak{D}$, *so* $\ker(f+g) \in \mathfrak{D}$ *and* $f + g \in G$.

Note that $|G| = 2^\lambda$.

It is worth keeping in mind that the vocabulary for the construction of M_I and the idea of the definition of ψ^λ_k depends on these basic notions.

Definition 1.3. The construction of the model M_I**.**

For a fixed set I we define first a group $H = H_I$ and then the model M_I.

(a) $H = H_I = [[I]^k]^{<\aleph_0} = \{S = \{u_0, u_1, \ldots\} | |S| < \aleph_0, u_i \in [I]^k\}$, with the group operation given by $S + T := S \Delta T$ (symmetric difference).

[Note: We can, equivalently, construe H_I as $(\mathbb{Z}_2)^{[[I]^k]^{<\aleph_0}}$, by setting, for $h \in [[I]^k]^{<\aleph_0}$, $g_h : [I]^k \to \mathbb{Z}_2$ with

$$g_h(u) = \begin{cases} 0 & \text{if } u \notin [I]^k \cap h \\ 1 & \text{if } u \in [I]^k \cap h \end{cases}$$

with the group operation

$$(f+g)(u) := f(u) +_{\mathbb{Z}_2} g(u).]$$

(b) the model $M = M_I$: we will next describe the universe, basic relations, projections, other partial functions, and most crucially, **the family of predicates** Q_s (for $s \in S$).

Universe: the universe of M_I consists of the following union of different sorts:

$$|M_I| = I \cup [I]^k \cup [I]^{k+1} \cup ([I]^k \times S \times H) \cup ([I]^k \times S \times \mathbb{Z}_2) \cup H \cup ([I]^{k+1} \times G).$$

Before continuing to the definition of the relations and functions of M_I, the following remarks on the universe of M_I are important:

- One way of thinking about the model is as

$$|M_I|$$

consisting of

$$\overbrace{I \cup [I]^k \cup [I]^{k+1}}^{\text{'control part'}} \cup \overbrace{([I]^k \times S \times H) \cup ([I]^k \times S \times \mathbb{Z}_2) \cup H \cup ([I]^{k+1} \times G)}^{\text{H and 'zeroless' copies of } H_I,\ \mathbb{Z}_2 \text{ and } G}.$$

- Notice that the intersection between all those pieces of the model is empty.
- The universe of M_I depends directly on I, as is clear from the various pieces. In particular, when the cardinality of I is $\geq \lambda$, the cardinality of M_I will be equal to $|I|$.
- The universe "depends" in an indirect way on k as well. Of course in our standard model this dependence is immediate, as seen from the superindices. However, we will need projections among the functions in the model to actually axiomatize the connections between pieces such as I, $[I]^k$, etc. *This dependence on* k *will be crucial in the "dimension" analysis later.*
- The universe *also* depends on λ, through the appearance of S and G among the pieces.

Before putting structure on $|M_I|$, we try to provide the reader with a general description of what will our standard model be.

In a **first stage,** we provide a 'combinatorial' description of our model. The relations and functions we will define below will do the following: in addition to the indices I (and in addition to the k and $k+1$ tuples of indices) from the 'control region' and a copy of the group H, we will have access to affine versions of the group H_I, of $\mathbb{Z}_2$ as well as a copy of G for each $k+1$-tuple, one copy of the affine version of $\mathbb{Z}_2$ for each k-tuple from I and finite set $s \in S$.

In a **second stage,** we add a family of predicates Q_s for $s \in S$. Each predicate Q_s will connect three things:

- one affine copy of H,
- k-many affine copies of $\mathbb{Z}_2$, and
- one copy of G

This part of the construction can alternatively be described in at least two additional ways:

- group action structures (generalizing Baldwin-Kolesnikov's reconstruction of the Hart-Shelah example [BK09]),
- Kuratowski's Perfect Set Theorem combinatorics (this observation is due to Velickovic).

However, we do not elaborate here those other connections.

Basic Relations: these consist of a family of λ-many predicates

$$P_0^M, P_{1,1}^M, P_{1,2}^M, P_2^M, (P_{2,s}^M)_{s\in S}, P_3^M, (P_{3,s}^M)_{s\in S}, P_4^M, P_5^M$$

defined by

- $P_0^M = I$,
- $P_{1,1}^M = [I]^k$,
- $P_{1,2}^M = [I]^{k+1}$,
- $P_2^M = [I]^k \times S \times H$,
- for $s \in S$, $P_{2,s}^M = \{(u, s, h) \in P_2^M | u \in [I]^k, h \in H\} = [I]^k \times \{s\} \times H$,
- $P_3^M = [I]^k \times S \times \mathbb{Z}_2$ (a copy of $\mathbb{Z}_2$ for each $b \in [I]^k$, $s \in S$),
- for $s \in S$, $P_{3,s}^M = \{(u, s, i) \in P_3^M | u \in [I]^k, i \in \mathbb{Z}_2\} = [I]^k \times \{s\} \times \mathbb{Z}_2$,
- $P_4^M = H$,
- $P_5^M = [I]^{k+1} \times G$

Remark 1.4. *The meaning of* $P_0^M, P_{1,1}^M, P_{1,2}^M, P_2^M$, P_3^M, P_4^M *is clear. In the case of* $P_{2,s}^M$, *the idea is that we stack "copies" of* H *for each* $b \in [I]^k$ *and each* $s \in S$, *and similarly for* P_3^M, $P_{3,s}^M$. *Another way of seeing this is thinking of the predicates as codifying families, as follows:*

- P_2^M *corresponds to* $(H_{v,s})_{v\in[I]^k, s\in S}$,
- P_3^M *corresponds to* $((\mathbb{Z}_2)_{v,s})_{v\in[I]^k, s\in S}$,
- P_5^M *corresponds to* $(G_u)_{u\in[I]^{k+1}}$.

Projections: We also include, for $\ell < k$, all the projections $\pi^0_\ell : P^M_{1,1} \to P^M_0$:

$$\pi^0_\ell(\bar{a}) = a_\ell,$$

and for $\ell < k+1$, the projections $\pi^1_\ell : P^M_{1,2} \to P^M_0$:

$$\pi^1_\ell(\bar{a}) = a_\ell.$$

The role of these projections is to tie the predicates $P^M_{1,1}$ and $P^M_{1,2}$ to P^M_0 making them behave as the corresponding sets of ktuples or $k+1$-tuples.

Other Partial Functions: We also include λ-many functions in M_I,

$$F^M_2, F^M_3, F^M_4, F^M_5, (F^M_{3,g^*})_{g^* \in G} :$$

- A unary function F^M_2 with domain P^M_2, given by

$$F^M_2(u, s, h) = u,$$

- A unary function F^M_3 with domain P^M_3, given by

$$F^M_3(u, s, i) = u,$$

- for $g^* \in G$, a unary function F^M_{3,g^*} with domain P^M_5, given by

$$F^M_{3,g^*}(u, g) = (u, g^* + g),$$

- A binary function F^M_4 with domain $P^M_2 \times P^M_4$, given by

$$F^M_4\Big((v, s, h), h_1\Big) = (v, s, h +_H h_1)$$

- A unary function F^M_5 with domain P^M_5, given by

$$F^M_5(u, g) = u,$$

Remark 1.5.
- *The role of* F^M_2 *is to project* P^M_2 *(essentially* $(H_{v,s})_{v \in [I]^k, s \in S}$*) onto its first coordinate; to trace the* k*-element subset of* I *it corresponds to. Similarly for* F^M_3 *and* F^M_5*.*
- *The functions* F^M_{3,g^*} *and* F^M_4 *are the closest we get to see the parts* P^M_5 *and* P^M_2 *as groups: it is important to note that* we do not add the group operations *but just the actions corresponding to each element of the group over the appropriate fibers!*

- *Notice that $+^H$ is definable - so in this case there is no need to add an analog of F_4 for copies of $\mathbb{Z}_2$:*

$$F_4^M(F_4^M((u,s,h),h_1),h_2) = F_4^M((u,s,h),h_3) \Leftrightarrow H \models h_1 + h_2 = h_3.$$

- *Notice also that if we replace $\mathbb{Z}_2$ by a larger group, we may need an analog to F_4 for copies of $\mathbb{Z}_2$!*

A $(3k+4)$**-ary predicate** Q_s, for each $s \in S$. This is the crux of the construction of the model M_I. The predicate will encode interactions between the different parts of the model, in a way that will involve *dimensional* interactions between them. This predicate on the one hand *enables* later to move up in the proof of categoricity by induction $k-1$ times from λ to λ^k and on the other *blocks* the proof from moving up to λ^{k+1}. It is interpreted in M_I as the set of tuples

$$\langle a_0, \dots, a_k, u_0, \dots, u_k, x_0, \dots, x_{k-1}, y_k, z\rangle$$

satisfying (for fixed $s \in S$!!) $h_k \in H$, $i_\ell \in \mathbb{Z}_2 (\ell < k)$, $g \in G$:

(α) $a_\ell \in I$ with no repetitions ($\ell \leq k$),

(β) $u_\ell = \langle a_m | m \neq \ell\rangle \in P_{1,1}^M$ ($\ell \leq k$),

(γ) $y_k = (u_k, s, h_k) \in P_2^M$,

(δ) x_ℓ has the form $(u_\ell, s, i_\ell) \in P_3^M$ ($\ell < k$) so $i_\ell \in \mathbb{Z}_2$,

(ϵ) z is of the form $(u, g) \in P_5^M$, where $u = (a_0, \dots, a_k) \in [I]^{k+1}$ and

(ζ) **(main point)**

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s).$$

Remark 1.6. (ζ) *is the crucial part of the definition. It provides the connection between* k *copies of* $\mathbb{Z}_2$*, one copy of* H*, one copy of* G *and the* $k+1$ k*-element subsets of a set of size* $k+1$ *in* I.

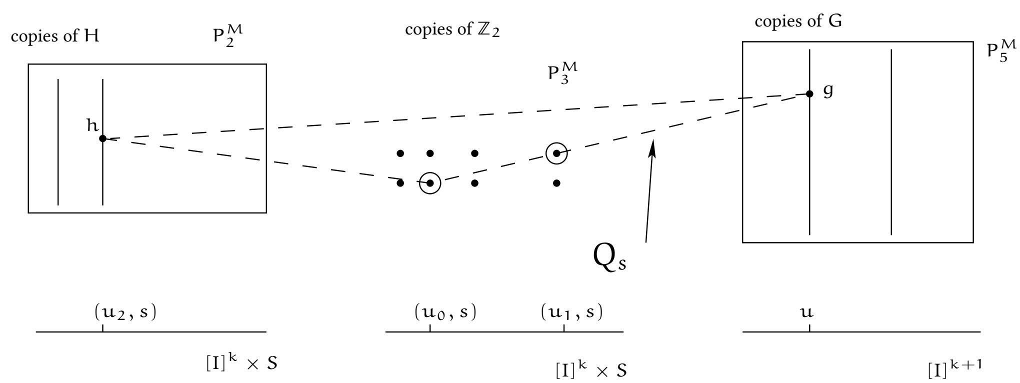
1.1.1 The case $k = 2$, in detail

As an example to visualize the situation, we momentarily fix $k = 2$. We also fix $s \in S$ and choose some $u \in [I]^{k+1} = [I]^{2+1}$. So, $u = \langle a_0, a_1, a_2\rangle$. This determines automatically (using the projections) in the models we have described so far a_0, a_1, a_2 and $u_0 = \langle a_1, a_2\rangle$, $u_1 = \langle a_0, a_2\rangle$, $u_2 = \langle a_0, a_1\rangle$. Our 'affine groups' hinge on $\mathcal{P}^-(3)$ in the following sense: we have copies of the affine version of $\mathbb{Z}_2$ over u_0 and u_1, a copy of H over u_2 and a copy of G over u.

Now, the interactions between these elements also depend strongly on their placement with respect to one another in $\mathcal{P}^-(3)$. The definition of Q_s in (ζ) above looks as follows when $k = 2$:

$$\mathbb{Z}_2 \models i_{\langle a_1, a_2 \rangle} + i_{\langle a_0, a_2 \rangle} = h_{\langle a_0, a_1 \rangle}(\langle a_1, a_2 \rangle) + g(s).$$

This may be read as follows: for fixed $s \in S$, the copy of H placed over $\langle a_0, a_1 \rangle$ is determined by a combination of the two *other sides* ($i_{\langle a_1, a_2 \rangle}$ and $i_{\langle a_0, a_2 \rangle}$) *and* the 'vertex' $\langle a_0, a_1, a_2 \rangle$ copy of G. The predicate Q_s provides then simultaneously for all $\mathcal{P}^-(3)$-diagrams from our index set I, actions of the full copy of H on the affine copies of H - and the copies of $\mathbb{Z}_2$ act as "switches" for membership in Q_s.



The previous picture illustrates the case $k = 2$ ("triangular interactions") - for arbitrary $k \geq 2$ the interactions coded by the predicate Q_s take the shape of tetrahedra, etc.

The fact that in (ζ) we choose 0 as subindex for u is not important; it could have been any $\ell \leq k$; note that u_k appears only in y_k.

1.2 The language, the sentence ψ_k^λ, the AEC $\mathcal{K}^*(\lambda, k)$

We now build our sentence ψ_k^λ.

Definition 1.7. *First, the vocabularies.*

- *Let* τ^- *be the vocabulary of all the construction above, except the predicates* $\{Q_s | s \in S\}$ *and*
- *let* τ *be the full vocabulary used in the construction of* M_I.

Specifically,

$$\textit{let } \tau^- = \langle P_0, P_{1,1}, P_{1,2}, P_2, (P_{2,s})_{s \in S}, P_3, (P_{3,s})_{s \in S}, P_4, P_5,$$
$$\pi_0^0, \ldots, \pi_{k-1}^0, \pi_0^1, \ldots, \pi_k^1, F_2, F_3, F_4, F_5, (F_{3,g^*})_{g^* \in G} \rangle$$

and let $\tau = \tau^- \cup \{Q_s | s \in S\}$.

Notice that $|\tau| = |G_\lambda| + |S| + \aleph_0 = 2^\lambda$, since $|G_\lambda| = 2^\lambda$.

Definition 1.8. The models $M_{I,f}$ and $M^-_{I,f}$

We will also use the following variants on the standard model: for a set I and a function

$$f : [I]^{k+1} \times S \to \mathbb{Z}_2$$

$M_{I,f}$ is the following model:

Vocabulary: τ, as for M_I. The model will be constructed just like M_I, **except we change the interpretation of** Q_s. So, we define

$$M_{I,f} \restriction \tau^- := M_I \restriction \tau^-,$$

and in $M_{I,f}$ the interpretation of $Q_s^{M_{I,f}}$ is the set of tuples

$$\langle a_0, \ldots, a_k, u_0, \ldots, u_k, x_0, \ldots, x_{k-1}, y_k, z \rangle$$

as in 1.3, but condition (ζ) becomes here

$$(\zeta)^*_f \qquad \mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s) + f(u, s).$$

(The role of the new function $f(u, s)$ will be clarified later. It clearly gives us tools to describe many possible models of the sentence psi^λ_k.)

$M^-_{I,f}$ is defined like M_I but omitting the predicates Q_s. Its vocabulary is τ^- from 1.7.

An arbitrary τ-structure M is **strongly standard** if $M \restriction \tau^- = M_I \restriction \tau^-$ for $I = P_0^M$.

Definition 1.9. (The abstract classes and the sentence ψ^λ_k)

1. Let $K_1 := \{M | M \approx M_{I,f}$ for some infinite set I, for some f as in 1.8$\}$. Then K_1 is a class of τ-models.

2. We now construct the sentence $\psi^\lambda_k \in L_{(2^\lambda)^+,\omega}(\tau)$:

    (a) Let first T_0 be the set of **all** the first order sentences which every M_I satisfies (as I is infinite, all the M_I are elementarily equivalent in first order),

    (b) $\psi_G \equiv \forall z_1 z_2([P_5(z_1) \wedge P_5(z_2) \wedge F_5(z_1) = F_5(z_2)] \to \bigvee_{g^* \in G} F_{3,g^*}(z_1) = z_2)$,

    (c) $\psi_{\mathbb{Z}_2} \equiv \forall y(P_2(y) \leftrightarrow \bigvee_{s \in S} P_{2,s}(y))$,

    (d) $\psi_H \equiv \forall y(P_3(y) \leftrightarrow \bigvee_{s \in S} P_{3,s}(y))$, Then we define the sentence $\psi_k^\lambda \in L_{(2^\lambda)^+,\omega}(\tau)$ as the conjunction

$$\psi_k^\lambda \equiv \bigwedge T_0 \wedge \psi_G \wedge \psi_{\mathbb{Z}_2} \wedge \psi_H$$

ψ_G above 'says' that G acts transitively (through the functions F_{3,g^*}) on copies of G (fibers of P_5).

$\psi_{\mathbb{Z}_2}$ says that the copies of $\mathbb{Z}_2$ (in the predicate P_2) do behave as copies of $\mathbb{Z}_2$

ψ_H says that there are no "non-standard fibers" in P_3: every element of P_3 is in some $P_{3,s}$

Note that, although there are 2^{2^λ} sentences in the logic, we are only using 2^λ of them, as witnessed by $|G| = 2^\lambda$.

3. Let $K^*(\lambda, k) :=$ Mod (ψ_k^λ) with the strong substructure relation

$$\prec_{K^*(\lambda,k)} := \prec_{L_{(2^\lambda)^+,\omega}} .$$

4. M from $K^*(\lambda, k)$ is **standard** if $P_{1,1}^M = [P_0^M]^k$ and $P_{1,2}^M = [P_0^M]^{k+1}$ and the π_ℓ^t's correspond to the actual projections sending $u \in [I]^k$ to its ℓ'th coordinate in I.

Claim 1.10. *For any $M_I \models \psi_k^\lambda$, $M_I \approx M_{I,\mathbf{0}}$, for the function $\mathbf{0} : [I]^{k+1} \times S \to \mathbb{Z}_2$ of constant value 0.*

The proof is immediate from the definition.

Claim 1.11. *Every $M \models \psi_k^\lambda$ is isomorphic to a strongly standard M.*

Next, a straightforward observation.

Claim 1.12. *$M_{I,f}$ is strongly standard.*

Proposition 1.13. *$(\mathcal{K}^*(\lambda, k), \prec_{K^*(\lambda,k)})$ is an abstract elementary class with Löwenheim-Skolem number λ.*

We will later study the degree to which our class $\mathcal{K}^*(\lambda, k)$ satisfies disjoint amalgamation, Galois-stability, etc. We will also connect briefly our results with the study of good frames, generalizing Boney and Vasey [BV16].

2 Canonical models, categoricity, amalgamation

2.1 Solutions, choices and correction functions

We now start the second stage of our proof: we build tools that will enable us to study the categoricity spectrum of ψ^λ_k. For this, it is enough to show that every model in the suitable cardinals is isomorphic to a standard one. We will describe choices and correction functions, that will be used in counting the models of cardinality $\lambda, \lambda^+, \dots, \lambda^{+m}$, for $m \leq k$.

This may be described as trying to recover the 'lost' zero of the copies of the groups. To this end, we define 'choices' (depending on the model M and on various subsets of $P^M_{1,1}$) of relevant elements for the crucial equation, and 'correction functions' for these equations.

Expanding choices from partial to global ones is the crux of the proof.

Definition 2.1.

1. For $M \models \psi^\lambda_k$, we say $(\bar{x}, \bar{y}, \bar{z})$ is a **partial** M-(J_0, J_1, J_2)-**choice** if

   (a) $J_0, J_1 \subset P^M_{1,1}$, $J_2 \subset P^M_{1,2}$,
   (so, in the case of standard models, $J_0, J_1 \subset [I]^k$, $J_2 \subset [I]^{k+1}$)

   (b) $\bar{x} = \langle x_{u,s} | s \in S, u \in J_0 \rangle$, where

   $$x_{u,s} \in (P^M_{3,s})^{-1}(u) \subset P^M_{3,s}.$$

   (c) $\bar{y} = \langle y_{u,s} | s \in S, u \in J_1 \rangle$,

   $$y_{u,s} \in H^M_{u,s} := (P^M_{2,s})^{-1}(u) \subset P^M_{2,s}.$$

   (d) $\bar{z} = \langle z_u | u \in J_2 \rangle$,

   $$z_u \in G^M_u := (F^M_5)^{-1}(u) \subset P^M_5.$$

   (So, informally, $\bar{x}$ chooses an element i in each copy of $\mathbb{Z}_2$, $\bar{y}$ chooses a h in each copy of H, $\bar{z}$ chooses a g in each copy of G, for each relevant (u, s), so $x_{u,s}$ is *some* element in the 'fiber' of u via F^M_3, and analogously for $\bar{y}$ and $\bar{z}$)

2. Call $(\bar{x}, \bar{y}, \bar{z})$ a **partial** M-J-**choice** if it is an M-(J, J, J^M_*)-choice, where

$$J^M_* := \Big\{ a \in P^M_{1,2} \Big| \bigwedge_{m \leq k} \exists b \in J[\bigwedge_{\ell < m} (\pi^1_\ell(a) = \pi^0_\ell(b) \wedge \bigwedge_{\ell \in [m,k[} \pi^0_\ell(b) = \pi^1_{\ell+1}(a)] \Big\}.$$

   The previous is a way of describing, in our language of projections, that (in the standard case) J^M_* consists of the $k+1$-element sets such that *all* their ($k+1$-many) k-element subsets are in J).

So, when M is standard, we have that

$$J_*^M = \left\{ \langle a_\ell | \ell \le k \rangle \middle| \bigwedge_{m \le k} \langle a_\ell | \ell \ne m \rangle \in J \right\}.$$

Finally, we say that $(\bar{x}, \bar{y}, \bar{z})$ is a **global** M-**choice** if it is a partial M-$P^M_{1,1}$-choice. We will sometimes just say "M-choice" (if clear from context).

3. Fix a standard M and a M-(J_0, J_1, J_2)-choice $(\bar{x}, \bar{y}, \bar{z})$. Then we let the **correction function** f for M and $(\bar{x}, \bar{y}, \bar{z})$ be the function such that
   (a) Dom (f) is the set of pairs (u, s) such that
      (α) $u = \langle a_\ell | \ell \le k \rangle \in J_2 \subset P^M_{1,2}$,
      (β) if $u_m := \langle a_\ell | \ell \le k, \ell \ne m \rangle$, $u_\ell \in J_0$ for $\ell < k$, $u_k \in J_1 \subset P^M_{1,1}$,
   (b) rng $(f) \subset \mathbb{Z}_2$, and
   (c) (recall $x_{u_\ell, s}, y_{u_k, s}, z_{u_k}$ are from the choice)
   $$f(u, s) = 0 \Leftrightarrow \langle a_0, \ldots, a_k, u_0, \ldots, u_k, x_{u_0, s}, \ldots, x_{u_{k-1}, s}, y_{u_k, s}, z_{u_k} \rangle \in Q^M_s.$$

   The definition makes sense as the λ's are from the choice.

4. If f is an correction function for some M_I and a M_I-(J_0, J_1, J_2)-choice $(\bar{x}, \bar{y}, \bar{z})$, then we call f a (I, J_0, J_1, J_2)-correction function.

5. $C(I, J_0, J_1, J_2)$ denotes the set of all (I, J_0, J_1, J_2)-correction functions.

The next three claims are general observations on correction functions and choices.

Claim 2.2. *1. If $(\bar{x}, \bar{y}, \bar{z})$ is a global M-choice, $M \models \psi^\lambda_k$, and f is the M-correction function for $(\bar{x}, \bar{y}, \bar{z})$, and f is identically zero, then $M \approx M_I$ for some I.*

2. If f above is zero on $P^M_{1,1}$, $P^M_{1,2}$ and $f = f' \restriction J_2 \times S$, then $M \approx M_{P_1, f'}$.

Proof Part (1) is a consequence of 2.5 below. Part (2) is clear. □

Corollary 2.3. *The correction function for $M_{I,f}$ and the standard M-choice $(\bar{x}, \bar{y}, \bar{z})$ is f.*

Proof Similar to the above: add zeroes to f as in 2.2. □

Claim 2.4. *For every $M \in$ Mod (ψ^λ_k), there is an M-choice $(\bar{x}, \bar{y}, \bar{z})$.*

Proof Immediate: just construct the tuples. There the demands are on each choice separately. There are no demands connecting different choices. □

The next lemma is a crucial step. It shows how to build possible isomorphisms from arbitrary N in the class K_2 to canonical models $M_{I,f}$.

Lemma 2.5. *For every* $N \in \operatorname{Mod}(\psi_k^\lambda)$ *and global* N*-choice* $(\bar{x}, \bar{y}, \bar{z})$ *with correction function* f*, there are* I *and* $\mathbf{h}$ *such that* $\mathbf{h}$ *is an isomorphism from* N *onto* $M_{I,f}$ *and*

$$\mathbf{h}(x_{u,s}) = (\mathbf{h}(u), s, 0_{\mathbb{Z}_2}), \quad \mathbf{h}(y_{u,s}) = (\mathbf{h}(u), s, 0_{H_I}), \quad \mathbf{h}(z_u) = (\mathbf{h}(u), 0_G).$$

Proof Let $N \models \psi_k^\lambda$, and fix a global N-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function f. We build I and $\mathbf{h}$ as in the statement.

First, extract the predicates for the model $M = M_{I,f}$: let $I := P_0^N$. Clearly, $P_0^M = P_0^N$. Then, by the definition of $M_{I,f}$, we get that

$$\left.\begin{array}{l} x \in P_{1,1}^N \cup P_{1,2}^N \\ \ell < k, \pi_\ell(x) = x_\ell \end{array}\right\} \Rightarrow \mathbf{h}(x) = (\mathbf{h}(x_0), \dots, \mathbf{h}(x_{k-1}))$$ i.e., $\mathbf{h}$'s value at the predicates corresponding to tuples is defined as the tuple of values of $\mathbf{h}$ on elements of P_0^N.

Furthermore, the construction of $\mathbf{h}$ on P_2^N, P_3^N, P_4^N and P_5^N should respect the predicates $P_{2,s}^N$ and $P_{3,s}^N$. So we have

$$\left.\begin{array}{l} x \in P_2^N \\ F_2^N(x) = u \in P_{1,1}^N \\ F_4^N(x, h) = x' \end{array}\right\} \Rightarrow F_2^N(\mathbf{h}(x)) = \mathbf{h}(F_2^N(x)) = \mathbf{h}(u)$$

and if $F_4^N(x, h) = x'$ then $F_4^N(\mathbf{h}(x), \mathbf{h}(h)) = \mathbf{h}(x')$.

So until now we know that if $x \in P_2^N$, then $\mathbf{h}(x)$ must have the form

$$\mathbf{h}(x) = (\mathbf{h}(u), s, -).$$

Since the definition of $\mathbf{h} \restriction P_2^N$ is still unrestricted in the choice of the 'third coordinate' element of H, this part will only be tied by the correction function.

In a last stage, we just need to check that this definition works together fine with the predicates Q_s. Fix $s \in S$. Checking the equivalence

$$Q_s^N(a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k z)$$

$$\Updownarrow$$

$$Q_s^{M_{I,f}}(\mathbf{h}(a_0) \dots \mathbf{h}(a_k) \mathbf{h}(u_0) \dots \mathbf{h}(u_k) \mathbf{h}(x_0) \dots \mathbf{h}(x_{k-1}) \mathbf{h}(y_k) \mathbf{h}(z)).$$

amounts to answering the question

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s) + f(u, s)$$

$$\Updownarrow ?$$

$$\mathbb{Z}_2 \models \sum_{\ell<k} \mathbf{h}(i_\ell) = h_k(\mathbf{h}(u_0)) + g(\mathbf{h}(s)) + f(\mathbf{h}(u), \mathbf{h}(s))$$

But choosing

$$\begin{cases} \mathbf{h}(x_{u,s}) = (\mathbf{h}(u), s, 0_{\mathbb{Z}_2}), \\ \mathbf{h}(y_{u,s}) = (\mathbf{h}(u), s, 0_H), \\ \mathbf{h}(z_u) = (\mathbf{h}(u), 0_G) \end{cases}$$

works for these equations: we are 'choosing' 0 on the third coordinates – at the $x_{u,s}$, $y_{u,s}$, $z(u)$ that had already been selected by the choice function.
So, this turns the equation at the choices into

$$\mathbb{Z}_2 \models 0 = \sum_{\ell<k} 0 = 0(\star) + 0(\star) + f(\star).$$

But, since f was a correction function *for our choice*,

$$f(u,s) = 0 \Leftrightarrow \langle a_0, \ldots, a_k, u_0, \ldots, u_k, x_{u_0,s}, \ldots, x_{u_{k-1},s}, y_{u_k,s}, z_{u_k} \rangle \in Q_s^N,$$

the definition of $\mathbf{h}$ works. □

2.2 Canonical choices

Definition 2.6. *Fix $M = M_{I,f}$, and let $(\bar{x}, \bar{y}, \bar{z})$ be the M-choice given by*

$$x_{u,s} = (u, s, 0_{\mathbb{Z}_2}),$$

$$y_{u,s} = (u, s, 0_{H_I}),$$

$$z_u = (u, 0_G).$$

This is by definition the **canonical** *M-choice*[1]*.*

Here is the crucial lemma.

Lemma 2.7. *If M_1 and M_2 are strongly standard, and $(\bar{x}, \bar{y}, \bar{z})_\ell$ is an M_ℓ-choice for M_ℓ ($\ell = 1, 2$), $P_0^{M_1} = P_0^{M_2}$ with correction function f_ℓ for $\ell = 1, 2$ then the following are equivalent:*

(a) *there is an isomorphism from M_1 onto M_2 over the identity on $P_0^{M_1} \cup P_1^{M_1}$*

(b)$_1$ *there is an M_2-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is f_1,*

(b)$_2$ *there is an M_1-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is f_2,*

(c) *there are functions g_0, g_1, g_2 (this is to correct the choice of zeros), with*

[1] So the choices act like 'variations' on the group structure of G, H and $\mathbb{Z}_2$.

1. $g_1 : [I]^k \times S \to \mathbb{Z}_2$ *(like the* $x_{u,s}$*'s above),*
2. $g_2 : [I]^k \times S \to I_H$ *(like the* $y_{u,s}$*'s above),*
3. $g_3 : [I]^{k+1} \to G$ *(like the* z_u*'s above),*
4. *if* $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle$ *are like in Definition 1.3 for* M_1*, or* M_2 then

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$$

Proof

(a) $\to$ **(b)**$_1$ Recall that $M_1 \restriction \tau^- = M_2 \restriction \tau^-$, so M_1 and M_2 have the same universes. Fix $F : M_1 \xrightarrow{\approx}_{P_0^{M_1} \cup P_1^{M_1}} M_2$. We have, since f_1 is a correction function for M for the choice $(\bar{x}, \bar{y}, \bar{z})_1$, that

$$f_1(u, s) = 0 \Leftrightarrow \langle a_0 \dots a_k u_0 \dots u_k x^1_{u_0 s} \dots x^1_{u_{k-1} s} y^1_{u_k s} z^1_{u_k}\rangle \in Q_s^{M_1}.$$

But the right hand side holds iff

$$\langle a_0 \dots a_k u_0 \dots u_k F(x^1_{u_0 s}) \dots F(x^1_{u_{k-1} s}) F(y^1_{u_k s}) F(z^1_{u_k})\rangle \in Q_s^{M_2},$$

since F is an isomorphism fixing $P_0^{M_1} \cup P_1^{M_1}$, and $a_0, \dots, a_k \in P_0^{M_1}$. This gives us the M_2-choice for which f_1 is a correction function: given $u_\ell \subset u$, $u_\ell \in [I]^k$, $u \in [I]^{k+1}$, let $x'_{u_\ell,s} = F(x^1_{u_\ell,s})$, $y'_{u,s} = F(y^1_{u_k,s})$, $z'_{u_k} = F(z^1_{u_k})$.

(a) $\to$ **(b)**$_2$ Same.

(b)$_\ell$ $\to$ **(c)** $(\ell = 1, 2)$ The point of (c) is that we may find concrete representations g_1, g_2, g_3, that act *independently from* M *or* N as 'corrected choice functions' for the zeros for f_1 and f_2. So, suppose we have a M_2-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function f_1. Then for any $u \in P_0^{M_2}$ and any $s \in S$, if $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle$ are like in Definition 1.3

$$\langle a_0 \dots a_k u_0 \dots u_k x_{u_0 s} \dots x_{u_{k-1} s} y_{u_k s} z_{u_k}\rangle \in Q_s^{M_2}$$

$$\Updownarrow$$

$$f_1(u, s) = 0.$$

But since f_1 is also a correction function for the M_1-choice $(\bar{x}, \bar{y}, \bar{z})_1$,

$$f_1(u, s) = 0$$

$$\Updownarrow$$

$$\langle a_0 \dots a_k u_0 \dots u_k x^1_{u_0 s} \dots x^1_{u_{k-1} s} y^1_{u_k s} z^1_{u_k} \rangle \in Q_s^{M_1}.$$

So, we have both $\mathbb{Z} \models \sum_{\ell<k} i_\ell = h_k(u_0) + g(s)$ and $\mathbb{Z} \models \sum_{\ell<k} i^1_\ell = h^1_k(u_0) + g^1(s)$, so setting

$$g_1(u_\ell, s) = i^1_\ell, \quad g_2(u_k, s) = h^1_k, \quad g_3(u) = g^1$$

yields

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s).$$

Since f_1 does this for all possible $k+1$-tuples, we have all the compability we need.

(c) $\to$ **(a)** If the predicates are the same modulo g_1, g_2 and g_3 then obtaining **(a)** becomes a matter of building $F : M_1 \xrightarrow{\approx}_{P_0^{M_1} \cup P_1^{M_1}} M_2$. Clearly we can start by $F \restriction P_0^{M_1} = id$, and then extend its definition to all the other portions of the model. The only strong restriction to the extension of this to the whole model is given by the relations $Q_s^{M_1}$ and $Q_s^{M_2}$.

$\square$

Remark 2.8. *1. We shall use 'simple' versions of $\langle g_1, g_2, g_3 \rangle$, usually to prove isomorphism (two of them zero).*

2. *Counting the number of isomorphism types count has some similarity to $\mathrm{Ext}(G, \mathbb{Z})$, in particular to the work of Shelah and Väisänen in [SV]. Here $I(\lambda, \psi)$ is counted by the group of correction functions, derived from some g_1, g_2, g_3:*

$$I(\lambda, \psi^\lambda_k) = \left\{ f \in C(I, J_0, J_1, J_2) \middle| f(u, s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_0, s) - g_3(u) \right\}.$$

3. *In the isomorphism proof, we will use the regularity of the filter: we will put together λ demands.*

The next lemma is the first step in the categoricity proof. It provides conditions for extending partial M-choices to global M-choices for combination (λ, m), $m < k$.

Lemma 2.9. (Extension property for W of size $m < k$, $|P_0^M| \leq \lambda$.)
Assume $m < k$, $M \models \psi$, M is strongly standard, $|P_0^M| \leq \lambda$, $W \subset P_0^M$, $W = \{b_\ell | \ell < m\}$ with no repetition, $J = \{u \in P_{1,1}^M | W \not\subset u\}$ (note that $u \in [P_0^M]^k$, as M is standard), $(\bar{x}, \bar{y}, \bar{z})$ is an M-J-choice with correction function f_0, identically zero. Then, *we can extend $(\bar{x}, \bar{y}, \bar{z})$ to an M-choice with correction function identically zero.*

PROOF

Part A: Without loss of generality, by 1.11, since M is strongly standard, $I = P_0^M$. Let $\langle \bar{a}^\alpha | \alpha < \beta^* \rangle$ list $P_{1,1}^M$ with $\langle \bar{a}^\alpha | \alpha < \alpha^* \rangle$ listing J (we have also used u for naming these $\bar{a}^\alpha$'s). Let $\langle \bar{b}^\gamma | \gamma < \gamma^* \rangle$ list $\{\bar{a} \in {}^{k+1}I | \bar{a}$ with no repetition and $W \subset \text{rng}(\bar{a})\}$ and $\gamma^* < \lambda^+$. Let, for $\alpha < \alpha^*$,

$$x_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, i_{\alpha,s}) \in (\mathbb{Z}_2)_{\bar{a}^\alpha,s},\ i_{\alpha,s} \in \mathbb{Z}_2,$$
$$y_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, h_{\alpha,s}) \in (H)_{\bar{a}^\alpha,s},\ h_{\alpha,s} \in H_I,$$
$$z_{\bar{b}^\gamma} = (\bar{b}^\gamma, g),\ g \in G.$$

Our hypothesis is then that we have choice functions for all $u \in P_{1,1}^M$ such that $u \not\supset W$. We use a zero correction function - as we don't know yet how to take care of $u \supset W$.

We will now choose $x_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, i_{\alpha,s})$, $y_{\bar{a}^\alpha,s} = (\bar{a}^\alpha, s, h_{\alpha,s})$, $z_{\bar{b}^\gamma} = (\bar{b}^\gamma, g)$ for $\alpha^* \le \alpha < \beta^*$ and appropriate γ.

Without loss of generality, $\beta^* \le \alpha^* + \lambda$, $\gamma^* \le \lambda$. (Remember $S = [\lambda]^{<\aleph_0}$.)

Part B: **First,** we choose $i_{\alpha,s} = 0_{\mathbb{Z}_2}$ for $\alpha^* \le \alpha < \beta^*$, $s \in S$.

Second, we are now at the level of consistently 'choosing h'. We try to choose $h_{\alpha,s}$ for $\alpha^* \le \alpha < \beta^*$ and $s \in S$ such that

(*) if $\gamma \in s \subset \lambda$, $\bar{b}^\gamma = \langle b_\ell^\gamma | \ell \le k \rangle$, $u_n^\gamma = \langle b_\ell^\gamma | \ell \le k, \ell \ne n \rangle$, let $\varepsilon(\gamma, n) < \beta^*$ be such that $u_n^\gamma = \bar{a}^{\epsilon(\gamma,n)}$ then

$$h_{\epsilon(\gamma,k),s}(\bar{a}^{\epsilon(\gamma,0)}) = 0$$
$$\Updownarrow$$
$$\langle b_0^\gamma, \ldots, b_k^\gamma, u_0^\gamma, \ldots, u_k^\gamma, x_{\epsilon(\gamma,0)}, \ldots, x_{\epsilon(\gamma,k-1)}, (\bar{a}^{\epsilon(\gamma,k)}, s, 0_H), (u^\gamma, 0_G) \rangle \in Q_s^M.$$

Note that all the elements in the bottom part are defined.

Let $t(\gamma, s)$ be 0 if the bottom statement is true, 1 otherwise (so we are using $\mathbb{Z}_2$ to code). This gives $|s|$ demands, one for each $\gamma \in s$. The sequence $\langle \bar{a}^{\varepsilon(\gamma,0)} | \gamma \in s \rangle$ is without repetition (see part **B**). So we have to show that the set of equations in the variable h varying on $H = [{}^kI]^{<\aleph_0}$, considered as a set of characteristic functions is

$$\{h(\bar{a}^{\varepsilon(\gamma,0)}) = t(\gamma, s) | \gamma \in s\}.$$

By the definition, it is solvable by the characteristic function of the subset $\{\bar{a}^{\varepsilon(\gamma,0)} | \gamma \in s\}$ of $[I]^k$.

The decisions are done for each s separately, also fixing s we can deal with one $\alpha \in [\alpha^*, \beta^*] \setminus \{\beta^*\}$ e.g. choosing $h_{\alpha,s}$ we have to consider only $\gamma < \gamma^*$ such that $\{\epsilon(\gamma,\ell) | \ell < k\} \subset s$; there are here only finitely many γ's, and if $\gamma_1 \neq \gamma_2 \in s$ (and $\epsilon(\gamma_1, k) = \alpha = \epsilon(\gamma_2, k)$ necessarily $\epsilon(\gamma_1, 0) \neq \epsilon(\gamma_2, 0)$ (as $\bar{a}^\gamma$ is reconstructible from α and $\epsilon(\gamma_1, 0)$), i.e. if equality holds then $\bar{b}^{\gamma_1} = \bar{b}^{\gamma_2}$) and by the choice of H we can find $h_{\epsilon(\gamma,k),s}$.

Part C: We now 'glue' the choices, for fixed γ. For each $\bar{b} \in {}^{k+1}I$, $\bar{b} = \bar{b}^\gamma$ for some $\gamma < \gamma^*$, so

$$S^*_\gamma = \Big\{ s \in S \Big| M \models Q_s(b^\gamma_0, \dots, b^\gamma_k, \bar{a}^{\epsilon(\gamma,0)}, \dots, \bar{a}^{\epsilon(\gamma,k)}, x_{u^\gamma_0,s}, \dots, x_{u^\gamma_{k-1},s}, y_{u^\gamma_k,s}, (u^\gamma, 0_S)) \Big\}$$

belongs to $\mathfrak{D}$ (by the regularity of $\mathfrak{D}$).

Next choose $z_{\bar{b}_1} = (\bar{b}, g)$ by

$$g(s) = \begin{cases} 0 & \text{if } s \in S^*_\gamma \\ 1 & \text{if } s \notin S^*_\gamma \end{cases}$$

Now then, with these x, y and z, the equation holds.

$\square_{2.9}$

We now prove the general extension property.

Lemma 2.10. (Full extension)
Let $M \models \psi$ be strongly canonical, $J_1 \subset J_2 \subset P^M_0$, with $|J_2| < \lambda^{+k-1}$ and $(\bar{x}, \bar{y}, \bar{z})$ an M-J_1-choice with correction function identically zero. Then $(\bar{x}, \bar{y}, \bar{z})$ can be extended to an M-J_2-choice with correction function identically zero.

Proof Without loss of generality, $J_2 = J_1 \cup \{b\}$. If J_1 has size $\leq \lambda$, this is lemma 2.9. Now suppose $|J_1| = \lambda^{+m_1}$ (for $m_1 < k$), so enumerate J_1 as $\langle a_\beta | \beta < \lambda^{+m_1} \rangle$. Let $J^\alpha_1 = \{a_\beta | \beta < \alpha\}$, and let $(\bar{x}, \bar{y}, \bar{z})_\alpha$ be the restriction of $(\bar{x}, \bar{y}, \bar{z})$ to an M-J^α_1-choice. We define by induction M-J^α_1 choices with correction function identically zero $(\bar{x}, \bar{y}, \bar{z})'_\alpha \supset (\bar{x}, \bar{y}, \bar{z})_\alpha$. Use lemma 2.14 for $m_2 = 2$ to extend $(\bar{x}, \bar{y}, \bar{z})'_\alpha \cup (\bar{x}, \bar{y}, \bar{z})_{\alpha+1}$ to an M-$J^{\alpha+1}_1 \cup \{b\}$-choice with correction function identically zero. At limits take unions; finally,

$$\Big(\bigcup_{\alpha < \lambda^{+m_1}} \bar{x}'_\alpha, \bigcup_{\alpha < \lambda^{+m_1}} \bar{y}'_\alpha, \bigcup_{\alpha < \lambda^{+m_1}} \bar{z}'_\alpha \Big)$$

turns out to be an M-J_2-solution extending $(\bar{x}, \bar{y}, \bar{z})$. $\square$

Claim 2.11. *In 2.9, we can allow* $|P_0^M| \leq \lambda^{+m_1}$ *if* $W = \{b_\ell | \ell < m_2\}$ *and* $m_1 + m_2 < k$.

Proof We prove this by induction on m_1. The proof is quite parallel to some of the proofs in [She83a] and to [HS90]. This part has few changes. We include versions of those proofs adapted to our context.

For $m_1 = 0$, this was done in 2.9. Suppose it holds for m_1 ($< k$), and m_2 is such that $m_1 + m_2 < k$. Consider $W \subset P_0^M$,

$$W = \{b_\ell | \ell < m_2\},$$

$(m_1 + 1) + m_2 < k$, $J = \{u \in P_1^M | W \not\subset u\}$, and let $(\bar{x}, \bar{y}, \bar{z})$ be an M-J-choice with correction function identically 0.

Suppose, for $M \models \psi$ strongly standard, $A_\emptyset \subset P_0^M$, $a_0, \dots, a_{m_2-1}$ different elements of $P_0^M \setminus A_\emptyset$,

Definition 2.12. $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2)\rangle$ *is a compatible* λ^{+m_1}*-*$\mathcal{P}^-(m_2)$*-system of choices iff*

1. $\bigcup_{s \in \mathcal{P}^-(m_2)} A_s = A_\emptyset \cup \{a_0, \dots, a_{m_2-1}\}$, $|A_\emptyset| \leq \lambda^{+m_1}$, $A_s = A_\emptyset \cup \{a_t | t \in s\}$.
2. $(\bar{x}, \bar{y}, \bar{z})_s$ *is a* M-A_s*-choice,* $\forall s \in \mathcal{P}^-(m_2)$.
3. *For every* $s, t \in \mathcal{P}^-(m_2)$, $s \subset t \Rightarrow (\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})_t$[2].

Lemma 2.13. *If* $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2)\rangle$ *is a compatible* λ*-*$\mathcal{P}^-(m_2)$*-system with* $m_2 < k$ *then there is an* M*-*$\bigcup_{s \in \mathcal{P}^-(m_2)} A_s$*-choice* $(\bar{x}, \bar{y}, \bar{z})$ *extending all the* $(\bar{x}, \bar{y}, \bar{z})_s$*, for* $s \in \mathcal{P}^-(m_2)$.

Proof If $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2)\rangle$ is a compatible λ-$\mathcal{P}^-(m_2)$-system with $m_2 < k$, if

$$u \in [\bigcup_{s \in \mathcal{P}^-(m_2)} A_s]^k \setminus \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k,$$

then $\{a_0 \dots a_{m_2-1}\} \subset u$. As $m_2 < k$, there must be some $b \in u \setminus \{a_0 \dots a_{m_2-1}\} \subset u$. Now, if $c \in \bigcup_{s \in \mathcal{P}^-(m_2)} A_s \setminus u$ then

$$(u \setminus \{b\}) \cup \{c\} \notin \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k$$

hence if $u \subset v$ where $v \in [\bigcup_{s \in \mathcal{P}^-(m_2)} A_s]^{k+1}$ then there must exist $u' \subset v$, $|u'| = k$, $u \neq u'$ such that $u' \notin \bigcup_{s \in \mathcal{P}^-(m_2)} [A_s]^k$. $\square_{2.13}$

[2] here, of course, we are abusing notation - by $(\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})_t$ we mean $\bar{x}_s \subset \bar{x}_t$, $\bar{y}_s \subset \bar{y}_t$ and $\bar{z}_s \subset \bar{z}_t$.

Lemma 2.14. *If* $\langle A_s, (\bar{x}, \bar{y}, \bar{z})_s | s \in \mathcal{P}^-(m_2)\rangle$ *is a compatible* λ^{+m_1}*-*$\mathcal{P}^-(m_2)$*-system of choices with* $m_1 + m_2 < k$ *then there is a* $\bigcup_{s\in\mathcal{P}^-(m_2)} A_s$*-choice* $(\bar{x}, \bar{y}, \bar{z})$ *such that* $(\bar{x}, \bar{y}, \bar{z})_s \subset (\bar{x}, \bar{y}, \bar{z})$ *for every* $s \in \mathcal{P}^-(m_2)$.

Proof Induct on m_1. For $m_1 = 0$, this is lemma 2.13. For $m_1 > 0$, suppose $A_s = A_\emptyset \cup \{b_t | t \in s\}$. Enumerate $A_\emptyset$, $\langle a_\beta | \beta < \lambda^{+m_1}\rangle$ and let $A^\alpha_\emptyset = \{a_\beta | \beta < \alpha\}$. Now let $A^\alpha_s = A^\alpha_\emptyset \cup \{b_t | t \in s\}$ for every $s \in \mathcal{P}^-(m_2)$ and $(\bar{x}, \bar{y}, \bar{z})^\alpha_s$ the restriction of $(\bar{x}, \bar{y}, \bar{z})_s$ to an M-A^α_s-choice. The point is to get $(\bar{x}, \bar{y}, \bar{z})_\alpha$ (increasing with α for $\alpha < \lambda^{+m_1}$) such that $(\bar{x}, \bar{y}, \bar{z})_\alpha$ is an M-$\bigcup_{s\in\mathcal{P}^-(m_2)} A^\alpha_s$-choice, with $(\bar{x}, \bar{y}, \bar{z})_\alpha \supset (\bar{x}, \bar{y}, \bar{z})^\alpha_s$ (the obvious meaning, again) for every $s \in \mathcal{P}^-(m_2)$. With this, taking $(\bar{x}, \bar{y}, \bar{z}) = \bigcup_{\alpha<\lambda^{+m_1}} (\bar{x}, \bar{y}, \bar{z})_\alpha$ we are done. [Why is the construction possible? At limits, just take unions. At successors, use the induction hypothesis.] $\square_{2.14}$

Theorem 2.15. *If* $M \models \psi^\lambda_k$ *is strongly canonical and* $|M| < \lambda^{+k}$ *then there is an* M*-choice with correction function identically zero.*

Proof Use the previous lemmas and induction .

Conclusion 2.16. (Categoricity and amalgamation up to $\lambda^{+(k-1)}$**)**

1. *For* $m < k$*,* Mod (ψ^λ_k) *has a unique model* M*,* $|P^M_0| = \lambda^{+m}$.
2. *For* $m < k - 2$*, if* $2^\lambda \leq \lambda^{+m}$*, then* $\mathcal{K}^*(\lambda, k)$ *has amalgamation in* λ^{+m}.
3. *If* $m < k$*,* $\lambda^{+m} > 2^\lambda$*, then* $\mathcal{K}^*(\lambda, k)$ *is categorical in* λ^{+m}.

Proof This is just a summary of the previous arguments. Notice, however, that we require $2^\lambda < \lambda^{+m}$ or $2^\lambda \leq \lambda^{+m}$ for our conclusions. This is due to the fact that our models are large: they contain copies of G, so they have size at least 2^λ. $\square$

3 ψ^λ_k is not categorical above $\beth_{k+1}(\lambda)^+$

We have proved in 2.16 that ψ is categorical in λ^{+m} if $m < k$ and $2^\lambda < \lambda^{+m}$. We now prove that our sentence is not categorical in any cardinality $\kappa \geq \mu = \beth_{k+1}(\lambda)^+$: ψ^λ_k has the maximal number of models possible in μ.

As before, we use our terminology of "solutions and corrections functions" to count the number of models

The following fact is a consequence of Fact 2.7.

Fact 3.1. *If* M_{1,f_1} *and* M_{2,f_2} *are models of* ψ*, and* $h : I_1 \to I_2$ *is one-to-one and onto,* then *there is an isomorphism* $h^+ : M_{1,f_1} \to M_{2,f_2}$ *extending* h *iff there are functions* g_0, g_1, g_2 *(this is to correct the choice of zeros), with*

1. $g_1 : [I]^k \times S \to \mathbb{Z}_2$ *(like the* $x_{u,s}$*'s above),*
2. $g_2 : [I]^k \times S \to I_H$ *(like the* $y_{u,s}$*'s above),*
3. $g_3 : [I]^{k+1} \to G$ *(like the* z_u*'s above),*
4. *if* $\langle a_0 \dots a_k u_0 \dots u_k x_0 \dots x_{k-1} y_k, z\rangle$ *are like in Definition 1.3 for* M_1*, or* M_2 then

$$\mathbb{Z}_2 \models \sum_{\ell<k} i_\ell - h_k(u_0) - g(s) = \sum_{\ell<k} g_1(u_\ell, s) - g_2(u_k, s)(u_0) - g_3(u)(s)$$

Proof   We start with models of the form M_{1,f_1} and M_{2,f_2}. These are strongly standard (see Definition 1.8). But the proof of Fact 2.7 applies then to these models. □

We now use our filter $\mathfrak{D}$ (recall that $\mathfrak{D}$ is the regular filter on S generated by sets of the form $\langle u\rangle = \{v \in S | u \subset v\}$, where $S = [\lambda]^{\aleph_0}$, $\mathfrak{D} = \mathfrak{D}_\lambda := \{A \subset S | \exists u_A \in S \forall v \in S(u_A \subset v \to v \in A)\}$ (more details in definition 1.2).

Claim 3.2. *Let* $f : [I]^{k+1} \times S \to \mathbb{Z}_2$ *be such that*

$$\otimes \qquad \{s \in S | f_u \neq 0\} \in \mathfrak{D}, \text{ for all } u \in [I]^{k+1},$$

for $f_u : S \to \mathbb{Z}_2$ *with* $f_u(s) = f(u, s)$.
Then, *the following is a sufficient condition for*

$$M_{I,f} \not\approx M_I :$$

(*) *if* $F_1 : [I]^k \to [I]^{\leq\lambda}$, $F_2 : [I]^k \to {}^S(\mathbb{Z}_2)$ π *a permutation of* I, then *we can find* $t_0, \dots, t_k \in I$ *with no repetitions such that*

- **(α)** $t_\ell \notin F_1[\{t_0 \dots t_k\} \setminus \{t_\ell\}]$ *if* $\ell = k$,
- **(β)** $f_{\pi\{t_0,\dots,t_k\}} - \sum_{\ell<k} F_2(\{t_0, \dots, t_k\} \setminus \{t_\ell\}) \notin G$

So, by the definition of G,

$$\{s \in S | f_{\pi(t_0,\dots,t_k)}(s) = \sum_{\ell<k} F_2(\{t_0, \dots, t_k\} \setminus \{t_\ell\})(s)\} \notin \mathfrak{D}.$$

Before proving 3.2, we note some facts.

Definition 3.3. $f : [I]^{k+1} \times S \to \mathbb{Z}_2$ *is an* I*-function iff it satisfies* $\bigotimes$ *above.*

Fact 3.4. *If* f_1, f_2 *are* I*-functions, and* $f = f_1 - f_2$ *(coordinatewise) satisfies (*), then* $M_{I,f_1} \not\approx M_{I,f_2}$.

Proof By 2.7, we can apply the criterion (*) to the model. □

Fact 3.5. *In 3.2, the assumption on* f *does not entail a loss of generality, as for every* f *there is a* f' *as above such that* $M_{I,f} \approx M_{I,f'}$.

Notice the role of the permutation π of I in the combinatorics that follows.

Proof of 3.2. Assume that $(\bar{x}, \bar{y}, \bar{z})$ witnesses $M_{I,f} \approx M_I$, with correction function identically zero. We show that $(*)$ of 3.2 cannot hold, for the following choice of F_1 and F_2.

Define $F_1 : [I]^k \to [I]^{\leq\lambda}$ by

$$F_1(u) = \bigcup\{v \in [I]^k | \text{ for some } s_1 \in S, y_{u,s_1}(v) \neq 0\}.$$

This is well defined, as $F_1(u)$ is a union of $|S|$ finite sets. Also, set

$$F_2(u) = \langle x_{u,s} | s \in S\rangle.$$

Let now $t_0, \ldots, t_k \in I$ (with no repetitions) satisfy $(\alpha)+(\beta)$. Let as usual $u = \{t_0, \ldots, t_k\}$, $u_\ell = u \setminus \{t_\ell\}$. By (α),

$$y_{u_k,s}(u_0) = 0.$$

> [Just notice that (α) asks that $t_k \notin F_1(u_k) = \bigcup\{v \in [I]^k| \text{ for some } s_1 \in S, y_{u,s_1}(v) \neq 0\}$, so for all $v \in [I]^k$, if $t_k \in v$, then for all $s_1 \in S$ we have $y_{u_k,s_1}(v) = 0$. In particular, as $t_k \in u_0$, $y_{u_k,s_1}(u_0) = 0$.]

So, by the choice $(\bar{x}, \bar{y}, \bar{z})$ and since we chose our correction function to be identically zero, for each z, we have that

$$\mathbb{Z}_2 \models x_{u_0,s} + x_{u_1,s} + \ldots x_{u_{k-1},s} - y_{u_k,s} - z_u = f_{\pi(u_0),s} + \cdots + f_{\pi(u_{k-1}),s} - f_{u_k,s}(u_0) - f_u(s).$$

But we also have that $f_u(s)$ is not zero (initial assumption) and $z_u(s) = 0$ for the $\mathfrak{D}$-majority of $s \in S$ (by the definition of G). Also, $y_{u_k,s}(u_0) = 0$, by the choice of the t's (clause (α)). So,

(*) For the $\mathfrak{D}$-majority of $s \in S$

$$\sum_{\ell<k} x_{u_\ell,s} = f_{\pi(u)}(s).$$

But this contradicts (β). □

Remark 3.6. *We can then regard* F_2 *as*

$$F_2 : [I]^k \to {}^S(\mathbb{Z}_2)/G.$$

Conclusion 3.7. *For* $\mu = \beth_{k+1}(\lambda)^+$, ψ^λ_k *is not categorical.*

This is not optimal (μ is large) but is enough for our main aim. In a possible continuation, we will address this issue.

Proof We take advantage of the combinatorial reduction from 3.2.

Stage A: First, let k be even. There is f an I-function (as in 3.2). Now, assume that F_1, F_2 are as in 3.2, and derive a contradiction. First find $E \subset \mu$ club such that

$$\alpha_0 < \cdots < \alpha_k \in E \Longrightarrow \begin{cases} F_1(\alpha_0, \dots, \alpha_{k-1}) \subset \alpha_k, \\ \pi(\alpha_0), \dots, \pi(\alpha_{k-1}) < \alpha_k. \end{cases}$$

Apply then Erdös-Rado to F_2 in order to get $\alpha_0 < \cdots < \alpha_k$ in E with $u = \{\alpha_0, \dots, \alpha_k\}$, $u_\ell = u \setminus \{\alpha_\ell\}$, $\langle F_2(u_\ell) | \ell < k \rangle$ constant.

But then

$$\sum_{\ell<k} F_2(u_\ell) = 0,$$

as our group is of order 2 (and k was chosen to be an even number).

Stage B: More generally, choose μ such that

$\bigotimes_1$ $\mu \to (\omega)^k_{2^\lambda}$,

$\bigotimes_2$ $\mu \not\to (\omega)^{k+1}_{2^\lambda}$,

$\bigotimes_3$ μ regular.

Then, use f as in 3.2 exemplifying $\bigotimes_2$. Looking at f as $(u \mapsto f_u/G \in ({}^S(\mathbb{Z}_2)/G)$, toward contradiction, if F_1, F_2 are as in 3.2, we let $E \subset \mu$ club as above, $\alpha_0, \dots, \alpha_n, \dots$ exemplifying $\mu \to (\omega)^k_{2^\lambda}$ for the coloring F_2. So $F_2 \restriction [\{\alpha_0, \dots, \alpha_n, \dots\}]^k$ is constant, say for increasing k-tuples from E, hence also the coloring by c by the argument as above, contradicting its choice.

Stage C: For larger cardinals this obviously works, as the criterion is monotonic. □

Remark 3.8. *Here are some of the main differences between the structure of this proof and that of [HS90]:*

1. *The use of the filter* $\mathfrak{D}$ *- it is not needed there.*
2. *The way the group itself is used is slightly different at the end of the proof.*

Remark 3.9. *The class has the maximal number of models at all* $\mu > \beth_{k+1}(\lambda)^+$.

3.1 On good λ^{+m}-frames

We now briefly discuss the connections between our work and the recent results in [BV16]. We provide no details, as the argument there relies on general structural features of the classes constructed, and these lift to our situation (the argument in [BV16] depends on two different ideas: first, an *external* analysis of the implication of the failure of categoricity of the Hart-Shelah sentence and second, an internal analysis of *solutions* and *choices* – the terminology for "recovering zeros" in the groups). For definitions of frames, good frames, etc. we refer the reader to [BV16].

Proposition 3.10. *For infinite* λ *and finite* k*, we have for each* $m \in [2, k)$*:*

- *The abstract elementary class*

$$\mathcal{K}^*(\lambda, k) = (\mathrm{Mod}(\psi_k^\lambda), \prec_{(2^\lambda)^+,\omega})$$

  is $(< \lambda, \lambda^{+k-2})$*-tame and* $(< \lambda, \lambda^{+k-2})$*-typeshort over models of size* λ^{+k-3}*,*

- *for each* $m \leq k - 3$ *there is a frame* $\mathfrak{s}^*(\lambda, k)_m$ *such that type-full and good, on* $\mathrm{Mod}(\psi_k^\lambda)$*,*

- *The frame* $\mathfrak{s}^*(\lambda, k)_m$ *is not weakly successful.*

We do not provide the details of the proof; but these follow along the lines of Boney and Vasey's (elaborate) arguments. In particular, we have the construction of the frames (and checking that the independence notion ("non-forking") corresponds to nonalgebraicity. This part does not really differ (although our construction uses the more complex family of predicates Q_s, for $s \in S$).

The frame $\mathfrak{s}^*(\lambda, k)_m = (\mathcal{K}^*(\lambda, k)_{\lambda^{+m}}, \downarrow, gS^{bs})$ is constructed as in [BV16]:

- The class of models is $\mathcal{K}^*(\lambda, k)_{\lambda^{+m}}$, the class of model of size λ^{+m} insider our main AEC,

- The independence notion $\downarrow$ is nonalgebraicity: that is, $p \in gS^{bs}(M)$ if and only if $p = \mathrm{gatp}(a/M; N)$ for $a \in I(N) \setminus I(M)$,

- $a \downarrow_{M_0}^{M_2} M_1$ if and only if $a \in I(M_2) \setminus I(M_1)$.

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