FORCING FOR hL AND hd

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Abstract. The present paper addresses the problem of attainment of the supremums in various equivalent definitions of hereditary density hd and hereditary Lindelöf degree hL of Boolean algebras. We partially answer two problems of J. Donald Monk, [13, Problems 50, 54], showing consistency of different attainment behaviour and proving that (for the considered variants) this is the best result we can expect.

0. Introduction

We deal with the attainment problem in various definitions of two cardinal functions on Boolean algebras: the hereditary density hd and the hereditary Lindelöf degree hL. These two cardinal functions are closely related, as it is transparent when we pick the right variants of (equivalent) definitions. Also they both are somewhat related to the spread s of Boolean algebras. So, for a Boolean algebra $\mathbb{B}$, we define

- $s(\mathbb{B}) = \sup\{\kappa :$ there is an ideal–independent sequence of length $\kappa\}$,
- $\mathrm{hd}(\mathbb{B}) = \sup\{\kappa :$ there is a left–separated sequence of length $\kappa\}$,
- $\mathrm{hL}(\mathbb{B}) = \sup\{\kappa :$ there is a right–separated sequence of length $\kappa\}$.

Let us recall that a sequence $\langle a_\xi : \xi < \kappa\rangle$ of elements of a Boolean algebra is

- *ideal–independent* if $a_\xi \not\leq \bigvee\limits_{\zeta\in w} a_\zeta$ for each $\xi < \kappa$ and a finite set $w \subseteq \kappa\setminus\{\xi\}$,
- *left–separated* if $a_\xi \not\leq \bigvee\limits_{\zeta\in w} a_\zeta$ for each $\xi < \kappa$ and a finite set $w \subseteq \kappa\setminus(\xi+1)$,
- *right–separated* if $a_\xi \not\leq \bigvee\limits_{\zeta\in w} a_\zeta$ for each $\xi < \kappa$ and a finite set $w \subseteq \xi$.

The above definitions of the three cardinal functions are of special use, see e.g. [15, §1]. However, neither these definitions explain the names of the functions, nor they are good enough justifications for the interest in them. But all three functions originate in the cardinal functions of the topological space $\mathrm{Ult}(\mathbb{B})$ (of ultrafilters on $\mathbb{B}$). And thus, for a Boolean algebra $\mathbb{B}$, we may define (or prove that the following equalities hold true):

- $s(\mathbb{B}) = \sup\{|X| : X \subseteq \mathrm{Ult}(\mathbb{B})$ is discrete in the relative topology $\}$,

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- $\mathrm{hd}(\mathbb{B}) = \sup\{d(X) : X \subseteq \mathrm{Ult}(\mathbb{B})\}$, where
  $d(X) = \min\{|Y| : Y \subseteq X \text{ is dense in } X\}$,
- $\mathrm{hd}(\mathbb{B}) = \sup\{L(X) : X \subseteq \mathrm{Ult}(\mathbb{B})\}$, where
  $L(X) = \min\{\kappa : \text{every open cover of } X \text{ has a subcover of size} \leq \kappa\}$.

The respective pairs of cardinal numbers are defined using sup, so even if we know that they are equal we still may expect different attainment properties: one of the families of cardinals may have the largest member while the other not. Also we may ask if the sup has to be attained. Situation may seem even more complicated if one notices that there are more than just two equivalent definitions of the cardinal functions $s, \mathrm{hd}, \mathrm{hL}$: Monk [13] lists six equivalent definitions for spread (see [13, Theorem 13.1]), nine definitions for hd, and nine for hL (see [13, Theorems 16.1, 15.1]). Fortunately, there is a number of dependencies here.

First, all of the equivalents of spread have the same attainment properties. Moreover, spread is always attained for singular strong limit cardinals and for singular cardinals of countable cofinality (for these and related results see Hajnal and Juhász [3], [4], [5], Juhász [8], [9], Roitman [14], Kunen and Roitman [11], Juhász and Shelah [10]). Then Shelah [20] proved that $2^{\mathrm{cf}(s(\mathbb{B}))} < s(\mathbb{B})$ implies that the spread is attained (see 1.3 here). Finally, it is shown in Shelah [18, §4] that, e.g., if μ is a singular strong limit cardinal such that $\mu < \mathrm{cf}(\lambda) < \lambda \leq 2^\mu$, then there is a Boolean algebra $\mathbb{B}$ such that $|\mathbb{B}| = s(\mathbb{B}) = \lambda$ and the spread is not obtained. Thus, to some extend, the problem of attainment for spread is settled.

Many of the results mentioned above can be carried out for (some) variants of hd and hL. However, the difference between these two cases and the case of the spread is that the various equivalent definitions of the respective cardinal function might have different attainment properties.

Let us introduce some of the equivalents of hL, hd. They will be called $\mathrm{hL}_{(n)}$, $\mathrm{hd}_{(n)}$, with the integer n referring to the respective cardinal κ_n as used in the proofs of [13, 15.1 and 16.1], respectively. Also, we will have $\mathrm{hd}^+_{(n)}$ and $\mathrm{hL}^+_{(n)}$ to have proper language to deal with the attainment questions. Let us start with the hereditary Lindelöf degree hL. First, for a topological space X we define the Lindelöf degree $\mathrm{L}(X)$ of the space X as

$$\mathrm{L}(X) = \min\{\lambda : \text{every open cover of } X \text{ has a subcover of size } \leq \lambda\}.$$

Definition 0.1. Let $\mathbb{B}$ be an infinite Boolean algebra. For an ideal I in a Boolean algebra $\mathbb{B}$ we let

$$\mathrm{cof}(I) = \min\{|A| : A \subseteq I \text{ and } (\forall b \in I)(\exists a \in A)(b \leq a)\}.$$

Now we define

$$\mathrm{hL}^{(+)}_{(0)}(\mathbb{B}) = \sup\{\mathrm{L}(X)^{(+)} : X \text{ is a subspace of } \mathrm{Ult}(\mathbb{B})\},$$
$$\mathrm{hL}^{(+)}_{(1)}(\mathbb{B}) = \sup\{\mathrm{cof}(I)^{(+)} : I \text{ is an ideal of } \mathbb{B}\},$$
$$\mathrm{hL}^{(+)}_{(7)}(\mathbb{B}) = \sup\{\kappa^{(+)} : \text{ there is a right–separated sequence } \langle a_\xi : \xi < \kappa\rangle \text{ in } \mathbb{B}\}.$$

The superscript "(+)" in the above definitions means that each of the formulas has two versions: one with "+" and one without it.

The cardinals mentioned in 0.1 are among those listed in [13, Theorem 15.1], and so $\mathrm{hL}_{(0)}(\mathbb{B}) = \mathrm{hL}_{(1)}(\mathbb{B}) = \mathrm{hL}_{(8)}(\mathbb{B})$. The attainment properties can be described using the versions with "+": $\mathrm{hL}^+_{(i)}(\mathbb{B}) = \mathrm{hL}_{(i)}(\mathbb{B})$ means that the supremum is not obtained; $\mathrm{hL}^+_{(i)}(\mathbb{B}) = \mathrm{hL}^+_{(j)}(\mathbb{B})$ means that the respective two definitions of hL have the same attainment for $\mathbb{B}$. It is not difficult to note that

$$\mathrm{hL}^+_{(7)}(\mathbb{B}) = \mathrm{hL}_{(7)}(\mathbb{B}) \quad \Rightarrow \quad \mathrm{hL}^+_{(1)}(\mathbb{B}) = \mathrm{hL}_{(1)}(\mathbb{B})$$

and

$$\mathrm{hL}_{(0)}(\mathbb{B}) = \mathrm{hL}^+_{(0)}(\mathbb{B}) \text{ is a regular cardinal} \quad \Rightarrow \quad \mathrm{hL}^+_{(7)}(\mathbb{B}) = \mathrm{hL}_{(7)}(\mathbb{B})$$

(and the attainment of hL in senses not listed in 0.1 can be reduced to those three; see [13, p. 190, 191] for details). Also, if $\mathrm{hL}(\mathbb{B})$ is a strong limit cardinal or if it has countable cofinality, then $\mathrm{hL}_{(7)}(\mathbb{B}) < \mathrm{hL}^+_{(7)}(\mathbb{B})$ (see Juhász [9, 4.2, 4.3]).

In 1.4 we will show that if $\mathrm{hL}(\mathbb{B})$ is a singular cardinal such that $2^{\mathrm{cf}(\mathrm{hL}(\mathbb{B}))} < \mathrm{hL}(\mathbb{B})$, then $\mathrm{hL}^+_{(0)}(\mathbb{B}) = \mathrm{hL}^+_{(1)}(\mathbb{B}) = \mathrm{hL}^+_{(7)}(\mathbb{B}) = (\mathrm{hL}(\mathbb{B}))^+$. Thus, e.g., under GCH, the sups in all equivalent definitions of hL are attained at singular cardinals. Next, in section 3, we use forcing to show that, consistently, there is a Boolean algebra $\mathbb{B}$ such that

$$\mathrm{hL}_{(7)}(\mathbb{B}) < \mathrm{hL}^+_{(7)}(\mathbb{B}) \quad \text{and} \quad \mathrm{hL}^+_{(1)}(\mathbb{B}) = \mathrm{hL}_{(1)}(\mathbb{B})$$

(see 3.7). This still leaves some aspects of [13, Problem 50] open: are there any implications between attainment in $\mathrm{hL}_{(0)}$ and $\mathrm{hL}_{(1)}$ sense? Between $\mathrm{hL}_{(0)}$ and $\mathrm{hL}_{(7)}$ sense?

We also carry out the parallel work for the hereditary density. Let us introduce the respective definitions. The density $d(X)$ of a topological space X is defined as the minimal size of a dense subset of X. The *topological density* $d(\mathbb{B})$ of a Boolean algebra $\mathbb{B}$ is the density of the space $\mathrm{Ult}(\mathbb{B})$ of ultrafilters on $\mathbb{B}$. The *algebraic density* (sometimes also called *the π–weight*) of a Boolean algebra $\mathbb{B}$ is

$$\pi(\mathbb{B}) = \min\{|A| : A \subseteq \mathbb{B}\setminus\{\mathbf{0}\} \text{ and } (\forall b \in \mathbb{B}\setminus\{\mathbf{0}\})(\exists a \in A)(a \leq b)\ \}.$$

Definition 0.2. For an infinite Boolean algebra $\mathbb{B}$ we let:

$\mathrm{hd}^{(+)}_{(0)}(\mathbb{B}) = \sup\{d(X)^{(+)} : X \text{ is a subspace of } \mathrm{Ult}(\mathbb{B})\ \}$,
$\mathrm{hd}^{(+)}_{(5)}(\mathbb{B}) = \sup\{\kappa^{(+)} :\ \text{there is a left–separated sequence of length } \kappa\ \}$,
$\mathrm{hd}^{(+)}_{(7)}(\mathbb{B}) = \sup\{\pi(\mathbb{B}^*)^{(+)} : \mathbb{B}^* \text{ is a homomorphic image of } \mathbb{B}\ \}$,
$\mathrm{hd}^{(+)}_{(8)}(\mathbb{B}) = \sup\{d(\mathbb{B}^*)^{(+)} : \mathbb{B}^* \text{ is a homomorphic image of } \mathbb{B}\ \}$.

(Again, the superscripts "(+)" mean that we have two variants for each cardinal: with and without "+".)

Like before, the cardinals mentioned in 0.2 correspond to those listed in [13, Theorem 16.1], and the variants with "+" reflect the attainment properties. The known dependencies here are

$$\begin{aligned} \mathrm{hd}^+_{(5)}(\mathbb{B}) = \mathrm{hd}_{(5)}(\mathbb{B}) \quad &\Rightarrow \quad \mathrm{hd}^+_{(7)}(\mathbb{B}) = \mathrm{hd}_{(7)}(\mathbb{B}) \quad \Rightarrow \\ \mathrm{hd}^+_{(0)}(\mathbb{B}) = \mathrm{hd}_{(0)}(\mathbb{B}) \quad &\Rightarrow \quad \mathrm{hd}^+_{(8)}(\mathbb{B}) = \mathrm{hd}_{(8)}(\mathbb{B}) \end{aligned}$$

and

$$\mathrm{hd}_{(0)}(\mathbb{B}) = \mathrm{hd}^+_{(0)}(\mathbb{B}) \text{ is a regular cardinal} \quad \Rightarrow \quad \mathrm{hd}_{(5)} = \mathrm{hd}^+_{(5)}(\mathbb{B})$$

(and Monk [13, Problem 54] asked for a complete description of dependencies). Like for hL, if $\mathrm{hd}(\mathbb{B})$ is a strong limit cardinal or if it has countable cofinality, then $\mathrm{hd}_{(5)}(\mathbb{B}) < \mathrm{hd}^+_{(5)}(\mathbb{B})$ (see Juhász [9, 4.2, 4.3]).

In 1.5 we note that if $\mathrm{hd}(\mathbb{B})$ is a singular cardinal such that $2^{\mathrm{cf}(\mathrm{hd}(\mathbb{B}))} < \mathrm{hd}(\mathbb{B})$, then $\mathrm{hd}^+_{(8)}(\mathbb{B}) = \mathrm{hd}^+_{(7)}(\mathbb{B}) = \mathrm{hd}^+_{(5)}(\mathbb{B}) = \mathrm{hd}^+_{(0)}(\mathbb{B}) = (\mathrm{hd}(\mathbb{B}))^+$. Consequently, GCH implies that the sups in all equivalent definitions of hd are attained at singular cardinals. Then, in section 4, we show that, consistently, there is a Boolean algebra $\mathbb{B}$ such that

$$\mathrm{hd}_{(5)}(\mathbb{B}) < \mathrm{hd}^+_{(5)}(\mathbb{B}) \quad \text{and} \quad \mathrm{hd}^+_{(7)}(\mathbb{B}) = \mathrm{hd}_{(7)}(\mathbb{B})$$

(see 4.5). This still leaves several aspects of [13, Problem 54] open.

Finally, in the last section of the paper we show that (if we start with the right cardinals μ, λ, $\mathrm{cf}(\lambda) < \lambda$) adding a μ–Cohen real produces a Boolean algebra $\mathbb{B}$ such that $\mathrm{hL}^+_{(7)}(\mathbb{B}) = \mathrm{hd}^+_{(5)}(\mathbb{B}) = s^+(\mathbb{B}) = \lambda$ (put 5.4, 5.6 together). This result is of interest as it shows how easily we may have algebras in which the three cardinal functions do not attain their supremums. (But of course there is the semi-ZFC result of [18, Theorem 4.2].)

Notation: Our notation is standard and compatible with that of classical textbooks on set theory (like Jech [7]) and Boolean algebras (like Monk [12], [13]). However in forcing considerations we keep the older tradition that

the stronger condition is the greater one.

Let us list some of our notation and conventions.

(1) A name for an object in a forcing extension is denoted with a dot above (like $\dot{X}$) with one exception: the canonical name for a generic filter in a forcing notion $\mathbb{P}$ will be called $\Gamma_{\mathbb{P}}$. For a $\mathbb{P}$–name $\dot{X}$ and a $\mathbb{P}$–generic filter G over $\mathbf{V}$, the interpretation of the name $\dot{X}$ by G is denoted by $\dot{X}^G$.
(2) $i, j, \alpha, \beta, \gamma, \delta, \ldots$ will denote ordinals and $\kappa, \mu, \lambda, \theta$ will stand for (always infinite) cardinals.
(3) For a set X and a cardinal λ, $[X]^{<\lambda}$ stands for the family of all subsets of X of size less than λ. If X is a set of ordinals then its order type is denoted by $\mathrm{otp}(X)$.
(4) Sequences of ordinals will be typically called σ, ρ, η, ν; the length of a sequence ρ is $\mathrm{lh}(\rho)$; $\nu \vartriangleleft \eta$ means that the sequence ν in an initial segment of η. The set of all sequences of length μ with values in κ will be denoted by ${}^{\mu}\kappa$. The lexicographic order on sequences of ordinals will be called $<_{\mathrm{lex}}$.
(5) In Boolean algebras we use $\vee$ (and $\bigvee$), $\wedge$ (and $\bigwedge$) and $-$ for the Boolean operations. If $\mathbb{B}$ is a Boolean algebra, $x \in \mathbb{B}$ then $x^0 = x$, $x^1 = -x$. The Stone space of the algebra $\mathbb{B}$ (the space of ultrafilters) is called $\mathrm{Ult}(\mathbb{B})$. When working in the Stone space, we identify the algebra $\mathbb{B}$ with the field of clopen subsets of $\mathrm{Ult}(\mathbb{B})$.
(6) For a subset Y of an algebra $\mathbb{B}$, the subalgebra of $\mathbb{B}$ generated by Y is denoted by $\langle Y \rangle_{\mathbb{B}}$ and the ideal generated by Y is called $\mathrm{id}_{\mathbb{B}}(Y)$.

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1. Golden Oldies: the use of [Sh:233]

In this section we recall how [20] applies to the attainment problems. The proofs of 1.2 and 1.3 were presented in [20], but we recall them here, as we have an impression that those beautiful results went somehow unnoticed. Also, as the results of sections 3 and 4 complement the consequences of [20, Lemma 5.1] presented here, it may be convenient for the reader to have all the proofs presented as well.

Hypothesis 1.1. Let μ, λ be cardinals, and $\bar{\chi} = \langle \chi_i : i < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence of regular cardinals such that

$$\mathrm{cf}(\lambda) < \mu = \left(2^{\mathrm{cf}(\lambda)}\right)^+ < \lambda = \sup_{i<\mathrm{cf}(\lambda)} \chi_i \quad \text{and} \quad \mu < \chi_0.$$

Theorem 1.2 (See [20, Lemma 5.1])**.** *Let X be a topological space with a basis $\mathcal{B}$ consisting of clopen sets. Suppose that Φ is a function assigning cardinal numbers to subsets of X such that $\Phi(X) \geq \lambda$ and*

(i) $\Phi(A) \leq \Phi(A \cup B) \leq \Phi(A) + \Phi(B) + \aleph_0$ *for* $A, B \subseteq X$,

(ii) *for each closed set $Y \subseteq X$ such that $\Phi(Y) \geq \lambda$ and for $i < \mathrm{cf}(\lambda)$, there are $\langle u_\alpha : \alpha < \mu \rangle \subseteq \mathcal{B}$ and $\langle y_\alpha : \alpha < \mu \rangle \subseteq Y$ such that*
  (a) $y_\alpha \in u_\alpha \cap Y$,
  (b) $(\forall v \in \mathcal{B})(y_\alpha \in v \ \Rightarrow\ \Phi(v \cap Y) \geq \chi_i)$,
  (c) $(\forall g : \mu \longrightarrow 2^{\mathrm{cf}(\lambda)})(\exists \alpha, \beta < \mu)(g(\alpha) = g(\beta)\ \&\ y_\alpha \notin u_\beta)$,

(iii) *if $\langle A_\alpha : \alpha < \mu \rangle$ is a sequence of subsets of X such that $\Phi(A_\alpha) \leq \chi_i$ (for $\alpha < \mu$) then $\Phi(\bigcup\limits_{\alpha<\mu} A_\alpha) \leq \chi_i$.*

Then there is a sequence $\langle v_i : i < \mathrm{cf}(\lambda) \rangle \subseteq \mathcal{B}$ such that

$$(\forall i < \mathrm{cf}(\lambda))(\Phi(v_i \setminus \bigcup_{j \neq i} v_j) \geq \chi_i).$$

Proof. First, by induction on $i < \mathrm{cf}(\lambda)$, we choose families K_i of clopen subsets of X, and sets $D_i \subseteq X$ such that $|K_i| = |D_i| = \mu$. So suppose that K_j, D_j have been defined for $j < i$. For each $\mathcal{U} \in [\bigcup\limits_{j<i} K_j]^{<\mathrm{cf}(\lambda)}$ such that $\Phi(X \setminus \bigcup \mathcal{U}) \geq \lambda$ pick $\langle y^{\mathcal{U}}_\alpha : \alpha < \mu \rangle \subseteq X \setminus \mathcal{U}$ and $\langle u^{\mathcal{U}}_\alpha : \alpha < \mu \rangle \subseteq \mathcal{B}$ as guaranteed by (ii) (for i and $Y = X \setminus \bigcup \mathcal{U}$). Let D_i consist of all $y^{\mathcal{U}}_\alpha$ (for $\mathcal{U}$ as above and $\alpha < \mu$); note that $|D_i| = \mu$. Let K_i be a family of clopen sets such that $|K_i| = \mu$ and for each $\mathcal{U}$ as above:

- $u^{\mathcal{U}}_\alpha \in K_i$ for all $\alpha < \mu$,
- if $y^{\mathcal{U}}_\alpha \in u^{\mathcal{U}}_\alpha \setminus u^{\mathcal{U}}_\beta$, $\alpha, \beta < \mu$, then there is $u \in K_i \cap \mathcal{B}$ such that $y^{\mathcal{U}}_\alpha \in u \subseteq u^{\mathcal{U}}_\alpha \setminus u^{\mathcal{U}}_\beta$,
- if $u \in K_i$ then $X \setminus u \in K_i$.

Let $K = \bigcup\limits_{i<\mathrm{cf}(\lambda)} K_i$ (clearly $|K| = \mu$) and let

$$Z_i = \{x \in X :\ \text{if } \{v_\xi : \xi < \mathrm{cf}(\lambda)\} \subseteq K \text{ and } x \in \bigcap_{\xi<\mathrm{cf}(\lambda)} v_\xi \text{ then } \Phi(\bigcap_{\xi<\mathrm{cf}(\lambda)} v_\xi) \geq \chi_i\}.$$

Claim 1.2.1. *If $Y \subseteq X$ is a closed set such that $\Phi(Y) \geq \chi_i$, then $Z_i \cap Y \neq \emptyset$.*

Proof of the claim. Suppose that for each $x \in Y$ we have a sequence $\langle v^x_\xi : \xi < \mathrm{cf}(\lambda) \rangle \subseteq K$ such that $x \in \bigcap\limits_{\xi<\mathrm{cf}(\lambda)} v^x_\xi$ and $\Phi(\bigcap\limits_{\xi<\mathrm{cf}(\lambda)} v^x_\xi) < \chi_i$. There are at most μ possibilities for such sequences, so we get a set $W \in [Y]^{\leq\mu}$ such that

$$Y \subseteq \bigcup_{x \in W} \bigcap_{\xi<\mathrm{cf}(\lambda)} v^x_\xi.$$

Use the assumption (iii) to conclude that $\Phi(\bigcup\limits_{x\in W}\bigcap\limits_{\xi<\mathrm{cf}(\lambda)} v^x_\xi)\leq\chi_i$, and next use (i) to get a contradiction with $\Phi(Y)\geq\lambda$. □

For each $i<\mathrm{cf}(\lambda)$ fix $z_i\in Z_i$.

Now, by induction on $i<\mathrm{cf}(\lambda)$, choose $v_i\in K_i$ and $x_i\in Z_i$ such that

(α) $x_i\in v_i\setminus\bigcup\limits_{j<i} v_j$, $v_i\in\mathcal{B}$,

(β) $x_j\notin v_i$ for $j<i$,

(γ) $z_\varepsilon\notin v_i$ for $i<\varepsilon<\mathrm{cf}(\lambda)$.

Suppose that x_j,v_j have been defined for $j<i$. Let $\mathcal{U}=\{v_j:j<i\}$ and $Y=X\setminus\bigcup\mathcal{U}$ (so it is a closed subset of X). By (γ), for $\varepsilon>i$ we have $z_\varepsilon\in Y$ and thus $\Phi(Y)\geq\chi_\varepsilon$ (just look at the definition of Z_ε; remember $X\setminus v_j\in K$), and hence $\Phi(Y)\geq\lambda$. Consequently, we have sequences $\langle y^{\mathcal{U}}_\alpha:\alpha<\mu\rangle\subseteq D_i$ and $\langle u^{\mathcal{U}}_\alpha:\alpha<\mu\rangle\subseteq K_i$ as chosen before (so they are as in (ii)). Consider a function g defined on μ such that

$$g(\alpha)=u^{\mathcal{U}}_\alpha\cap(\{z_\varepsilon:\varepsilon<\mathrm{cf}(\lambda)\}\cup\{x_j:j<i\}).$$

So by (ii)(c) we find distinct $\alpha,\beta<\mu$ such that $g(\alpha)=g(\beta)$ and $y^{\mathcal{U}}_\alpha\notin u^{\mathcal{U}}_\beta$. Then, by the definition of K_i, we find $v_i\in K_i\cap\mathcal{B}$ such that $y^{\mathcal{U}}_\alpha\in v_i\subseteq u^{\mathcal{U}}_\alpha\setminus u^{\mathcal{U}}_\beta$. It follows from (ii)(b) that $\Phi(v_i\cap Y)=\Phi(v_i\setminus\bigcup\limits_{j<i}v_j)\geq\chi_i$. By claim 1.2.1 we may pick $x_i\in Z_i\cap v_i\cap Y=Z_i\cap v_i\setminus\bigcup\limits_{j<i}v_j$. Since, by our choices, v_i is disjoint from $\{z_\varepsilon:\varepsilon<\mathrm{cf}(\lambda)\}\cup\{x_j:j<i\}$, the inductive step is complete.

After the inductive construction is carried out, look at the sequence $\langle v_i:i<\mathrm{cf}(\lambda)\rangle$. Since $x_i\in Z_i\cap v_i\setminus\bigcup\limits_{j\neq i}v_j$ we easily conclude that $\Phi(v_i\setminus\bigcup\limits_{j\neq i}v_j)\geq\chi_i$. □

Corollary 1.3 (See [20, 3.3., 5.4]). *If $\mathbb{B}$ is a Boolean algebra satisfying $s(\mathbb{B})=\lambda$ then $s^+(\mathbb{B})=\lambda^+$.*

Proof. Suppose $s(\mathbb{B})=\lambda$. Then for each $i<\mathrm{cf}(\lambda)$ we may pick a discrete set $A_i\subseteq\mathrm{Ult}(\mathbb{B})$ of size χ_i. Let $X=\bigcup\limits_{i<\mathrm{cf}(\lambda)}A_i$ (and the topology of X is the one inherited from $\mathrm{Ult}(\mathbb{B})$) and let $\mathcal{B}=\{b\cap X:b\in\mathbb{B}\}$. Finally let $\Phi(A)=|A|$ for $A\subseteq X$. Note that $X,\mathcal{B},\Phi$ clearly satisfy clauses 1.2(i,iii). Suppose that the demand in 1.2(ii) fails for $i<\mathrm{cf}(\lambda)$ and a closed set $Y\subseteq X$ (so $|Y|=\lambda$). Let

$$Y^*_i=\{y\in Y:(\forall v\in\mathcal{B})(y\in v\ \Rightarrow\ |v\cap Y|\geq\chi_i)\}.$$

Case 1: $|Y^*_i|<\mu$.

Then $|Y\setminus Y^*_i|=\lambda$. For each $y\in Y\setminus Y^*_i$ pick $v^y\in\mathcal{B}$ such that $y\in v^y$ and $|v^y\cap Y|<\chi_i$. Consider the function

$$F:Y\setminus Y^*_i\longrightarrow\mathcal{P}(Y\setminus Y^*_i):y\mapsto v^y\cap Y\setminus Y^*_i.$$

By the Hajnal Free Set Theorem (see Hajnal [2]) there is an F–free set $S\subseteq Y\setminus Y^*_i$ of size λ. Then $y\notin F(y')$ for distinct $y,y'\in S$, and thus $v^y\cap S=\{y\}$ for $y\in S$. Consequently S is discrete and $s^+(\mathbb{B})>\lambda$.

Case 2: $|Y^*_i|\geq\mu$.

For some $j<\mathrm{cf}(\lambda)$ we have $|Y^*_i\cap A_j|\geq\mu$, so we may choose distinct $y_\alpha\in Y^*_i\cap A_j$ for $\alpha<\mu$. The set $\{y_\alpha:\alpha<\mu\}$ is discrete (as A_j is so), so we may pick $u_\alpha\in\mathcal{B}$

such that $(\forall\alpha,\beta<\mu)(y_\alpha\in u_\beta \iff \alpha=\beta)$. Then $\langle y_\alpha,u_\alpha:\alpha<\mu\rangle$ is as required in 1.2(ii), contradicting our assumption that this clause fails.

So we may assume that the assumptions of 1.2 are satisfied, and therefore we may find $\langle v_i : i<\mathrm{cf}(\lambda)\rangle\subseteq\mathcal{B}$ such that $|v_i\setminus\bigcup_{j\neq i} v_j|\geq\chi_i$ for each $i<\mathrm{cf}(\lambda)$. Then, for every $i<\mathrm{cf}(\lambda)$, there is $\xi(i)<\mathrm{cf}(\lambda)$ such that

$$|A_{\xi(i)}\cap v_i\setminus\bigcup_{j\neq i} v_j|\geq\chi_i.$$

Let

$$A=\bigcup_{i<\mathrm{cf}(\lambda)}(A_{\xi(i)}\cap v_i\setminus\bigcup_{j\neq i}v_j).$$

Clearly $|A|=\lambda$ and easily A is discrete. □

Theorem 1.4. *If* $\mathbb{B}$ *is a Boolean algebra satisfying* $\mathrm{hL}(\mathbb{B})=\lambda$ *then*

$$\mathrm{hL}^+_{(0)}(\mathbb{B})=\mathrm{hL}^+_{(1)}(\mathbb{B})=\mathrm{hL}^+_{(7)}(\mathbb{B})=\lambda^+.$$

Proof. So assume $\mathrm{hL}(\mathbb{B})=\lambda$.

If $s^+(\mathbb{B})>\lambda$, that is if $\mathbb{B}$ has an ideal independent sequence of length λ, then easily all sups in the equivalent definitions of hL are obtained. So we may assume

(⊛) $s^+(\mathbb{B})\leq\lambda$ and thus, by 1.3, $s^+(\mathbb{B})<\lambda$. We may also assume that $s^+(\mathbb{B})<\chi_0$.

Let $X=\mathrm{Ult}(\mathbb{B})$, $\mathcal{B}=\mathbb{B}$, and for $Y\subseteq X$ let

$$\Phi(Y)=\sup\{\kappa:\text{ there is a right separated sequence in } Y \text{ of length } \kappa\ \}.$$

(Recall that in a topological space Y, a sequence $\langle y_\xi:\xi<\kappa\rangle$ is right separated whenever all initial segments of the sequence are open in the relative topology.) We are going to apply 1.2 to $X,\mathcal{B},\Phi$, and for that we need to check the assumptions there. Clauses (i) and (iii) are obvious, and let us verify 1.2(ii).

Let $i<\mathrm{cf}(\lambda)$ and let $Y\subseteq\mathrm{Ult}(\mathbb{B})$ be a closed set such that $\Phi(Y)=\lambda$. Let $\langle x_\xi:\xi<\chi_i^+\rangle\subseteq Y$ be a right separated sequence, and let $b_\xi\in\mathbb{B}$ be such that $x_\xi\in b_\xi$ and $x_\zeta\notin b_\xi$ for $\xi<\zeta<\chi_i^+$. Let

$$Z=\{\xi<\chi_i^+:\mathrm{cf}(\xi)=\chi_i\ \&\ (\exists a\in\mathbb{B})(x_\xi\in a\ \&\ \Phi(a\cap Y)<\chi_i)\}.$$

Claim 1.4.1. *Z is not stationary in χ_i^+.*

Proof of the claim. Assume Z is stationary. For $\xi\in Z$ pick $a_\xi\in\mathbb{B}$ such that $x_\xi\in a_\xi$ and $\Phi(a_\xi\cap Y)<\chi_i$. Note that then for some $\zeta(\xi)<\xi$ we have

$$(\forall\varepsilon<\xi)(x_\varepsilon\in a_\xi \Rightarrow \varepsilon<\zeta(\xi)).$$

By the Fodor lemma, for some ζ^* the set $Z^*=\{\xi\in Z:\zeta(\xi)=\zeta^*\}$ is stationary. Now look at the set $Y^*=\{x_\xi:\xi\in Z^*\ \&\ \xi>\zeta^*\}$: we have

$$(\forall\xi\in Z^*\setminus(\zeta^*+1))((a_\xi\cap b_\xi)\cap Y^*=\{x_\xi\}).$$

Consequently Y^* is a discrete set of size χ_i^+, contradicting (⊛). □

Thus we may pick an increasing sequence $\langle\xi(\alpha):\alpha<\mu\rangle$ of ordinals below χ_i^+ such that $\mathrm{cf}(\xi(\alpha))=\chi_i$ and $\xi(\alpha)\notin Z$ (for $\alpha<\mu$). Let $y_\alpha=x_{\xi(\alpha)}$ and $u_\alpha=b_{\xi(\alpha)}$. Then $\langle y_\alpha,u_\alpha:\alpha<\mu\rangle$ is as required in 1.2(ii) (for Y,i).

Consequently we may apply 1.2 to choose a sequence $\langle v_i : i < \mathrm{cf}(\lambda)\rangle \subseteq \mathbb{B}$ such that

$$(\forall i < \mathrm{cf}(\lambda))(\Phi(v_i \setminus \bigcup_{j\neq i} v_j) \geq \chi_i).$$

For $i < \mathrm{cf}(\lambda)$ choose a right separated sequence $\langle y^i_\xi : \xi < \chi_i\rangle \subseteq v_{i+1} \setminus \bigcup\limits_{j\neq i+1} v_j$. Let I consist of those $b \in \mathbb{B}$ that for some finite set $W \subseteq \mathrm{cf}(\lambda)$ and a sequence $\langle\zeta(i) : i \in W\rangle \in \prod\limits_{i\in W} \chi_i$ we have

$$(\forall i < \mathrm{cf}(\lambda))(\forall \xi < \chi_i)(y^i_\xi \in b \ \Rightarrow\ i \in W \ \&\ \xi < \zeta(i)).$$

Claim 1.4.2. *I is an ideal in $\mathbb{B}$ and $\mathrm{cof}(I) = \lambda$. Consequently $\mathrm{hL}^+_{(1)}(\mathbb{B}) = \lambda^+$ and hence $\mathrm{hL}^+_{(7)}(\mathbb{B}) = \lambda^+$.*

Proof of the claim. Plainly, I is an ideal in $\mathbb{B}$. Suppose that $A \subseteq I$ is of size less than λ, and for $b \in A$ let $W_b \in [\mathrm{cf}(\lambda)]^{<\omega}$, $\langle\zeta_b(i) : i \in W_b\rangle \in \prod\limits_{i\in W_b} \chi_i$ witness $b \in I$. Let $i < \mathrm{cf}(\lambda)$ be such that $\chi_i > |A|$ and let $\sup\{\zeta_b(i) : (\exists b \in A)(i \in W_b)\} < \xi < \chi_i$. Take $b \in \mathbb{B}$ such that $y^i_\xi \in b$ and $(\forall\zeta < \chi_i)(\xi < \zeta \ \Rightarrow\ y^i_\zeta \notin b)$. Then

$$y^j_\varepsilon \in b \cap v_{i+1} \quad \Rightarrow \quad j = i \ \&\ \varepsilon \leq \xi,$$

so $v_{i+1} \wedge b \in I$, but it is not included in any member of Z. □

Let $Y = \{y^i_\zeta : i < \mathrm{cf}(\lambda) \ \&\ \zeta < \chi_i\}$.

Claim 1.4.3. *$L(Y) = \lambda$, and consequently $\mathrm{hL}^+_{(0)}(\mathbb{B}) = \lambda^+$.*

Proof of the claim. For $i < \mathrm{cf}(\lambda)$ and $\xi < \chi_i$, let $U_{i,\xi}$ be an open subset of v_{i+1} such that

$$(\forall\zeta < \chi_i)(y^i_\zeta \in U_{i,\xi} \ \Leftrightarrow\ \zeta \leq \xi).$$

Put $\mathcal{U}_i = \{U_{i,\xi} : \xi < \chi_i\}$, $\mathcal{U} = \bigcup\limits_{i<\mathrm{cf}(\lambda)} \mathcal{U}_i$. It should be clear that if $\mathcal{U}' \subseteq \mathcal{U}_i$ is of size less than χ_i then $Y \cap \bigcup\mathcal{U}' \neq Y \cap \bigcup\mathcal{U}_i$. Also $y^i_\xi \notin \bigcup\mathcal{U}_j \subseteq v_j$ for $i \neq j$, so we may conclude that no subfamily of $\mathcal{U}$ of size less than λ covers Y, showing the claim. □

□

Theorem 1.5. *If $\mathrm{hd}(\mathbb{B}) = \lambda$ then $\mathrm{hd}^+_{(8)}(\mathbb{B}) = \lambda^+$ (and thus also $\mathrm{hd}^+_{(0)}(\mathbb{B}) = \mathrm{hd}^+_{(7)}(\mathbb{B}) = \mathrm{hd}^+_{(5)}(\mathbb{B}) = \lambda^+$).*

Proof. We may follow like in 1.4 and use 1.2 to get our conclusion. However, an alternative way is to use a result of Šapirovskiĭ that for every compact space X, $\mathrm{hd}(X) \leq s(X)^+$ (see Šapirovskiĭ [17] or Hodel [6, 7.17]). Consequently, in our situation, $\mathrm{hd}(\mathbb{B}) = s(\mathbb{B})$ and by 1.3 we conclude that $s^+(\mathbb{B}) = \lambda^+$. But this implies that there is a homomorphic image $\mathbb{B}^*$ of $\mathbb{B}$ with the cellularity $c(\mathbb{B}^*) = \lambda$ (see [13, Theorem 3.25 and p. 175]). Clearly $d(\mathbb{B}^*) \geq c(\mathbb{B}^*)$, so we get our conclusion. □

2. Some combinatorics

Arguments based on the Δ–lemma are very important in forcing considerations. The result quoted below is a variant of the Δ–lemma and in various forms was presented, proved and developed in [21, §6], [19, §6] and [23, §7].

Lemma 2.1 (see [19, 6.1]). *Assume that:*

(i) *σ, θ are regular cardinals and κ is a cardinal,*
(ii) $(\forall \alpha < \sigma)(|\alpha|^\kappa < \sigma)$,
(iii) *$\mathcal{D}$ is a σ–complete filter on θ containing all co-bounded subsets of θ,*
(iv) *$\langle \beta^\alpha_\varepsilon : \varepsilon < \kappa\rangle$ is a sequence of ordinals (for $\alpha < \theta$),*
(v) *$X \subseteq \theta$ is such that $X \neq \emptyset \mod \mathcal{D}$.*

Then there are a sequence $\langle \beta^*_\varepsilon : \varepsilon < \kappa\rangle$ and a set $w \subseteq \kappa$ such that:

(a) $(\forall \varepsilon \in \kappa \setminus w)(\sigma \leq \mathrm{cf}(\beta^*_\varepsilon) \leq \theta)$,
(b) *the set*

$$B \stackrel{\text{def}}{=} \{\alpha \in X : \begin{array}{l} \textit{if } \varepsilon \in w \textit{ then } \beta^\alpha_\varepsilon = \beta^*_\varepsilon, \\ \textit{if } \varepsilon \in \kappa \setminus w \textit{ then } \sup\{\beta^*_\zeta : \zeta < \kappa,\ \beta^*_\zeta < \beta^*_\varepsilon\} < \beta^\alpha_\varepsilon < \beta^*_\varepsilon\} \end{array}$$

is not $\emptyset$ modulo the filter $\mathcal{D}$,
(c) *if $\beta'_\varepsilon < \beta^*_\varepsilon$ (for $\varepsilon \in \kappa \setminus w$) then*

$$\{\alpha \in B : (\forall \varepsilon \in \kappa \setminus w)(\beta'_\varepsilon < \beta^\alpha_\varepsilon)\} \neq \emptyset \mod \mathcal{D}.$$

The above version of the Δ–lemma will have multiple use in our proofs in the next two sections. Among others, it will be applied to filters given by 2.2, 2.3 below.

Lemma 2.2. *Suppose that $\mathbb{B}$ is a Boolean algebra generated by $\langle x_\xi : \xi < \chi\rangle$. Let $I \subseteq \mathbb{B}$ be an ideal with $\mathrm{cof}(I) = \lambda$ and let $\aleph_0 < \mu < \lambda$. Then there are a regular cardinal $\theta \in [\mu, \lambda]$, a $(<\theta)$–complete filter $\mathcal{D}$ on θ and a sequence $\langle a_\alpha : \alpha < \theta\rangle \subseteq I$ such that*

($*_1$) *all co-bounded subsets of θ are in the filter $\mathcal{D}$, and for every $b \in I$:*

$$\{\alpha < \theta : a_\alpha \leq b\} = \emptyset \mod \mathcal{D},$$

($*_2$) *for each $\alpha < \theta$, $a_\alpha \notin \mathrm{id}_{\mathbb{B}}(\{a_\beta : \beta < \alpha\})$,*
($*_3$) *every a_α (for $\alpha < \theta$) is of the form*

$$a_\alpha = \bigwedge_{\ell < n} x^{t(\alpha,\ell)}_{\xi(\alpha,\ell)} \qquad \textit{(where } n < \omega,\ \xi(\alpha,\ell) < \chi,\ t(\alpha,\ell) < 2\textit{)}.$$

Proof. It is basically like [22, 2.2, 2.3], but for reader's convenience we present the proof fully.

Claim 2.2.1. *Assume $\mu_0 < \lambda$. Then there are a regular cardinal $\theta \in [\mu_0, \lambda]$ and a set $Y \in [I]^\theta$, such that*

$$(\forall Z \in [I]^{<\theta})(\exists b \in Y)(\forall a \in Z)(b \not\leq a).$$

Proof of the claim. Assume not. By induction on $|Y|$ we show that then

(⊛) if $Y \in [I]^{\leq\lambda}$ then there is $Y^* \subseteq I$ such that $|Y^*| = \mu_0$ and

$$(\forall b \in Y)(\exists a \in Y^*)(b \leq a).$$

If $|Y| \leq \mu_0$, then there is nothing to do.
Suppose now that $Y \subseteq I$ and $|Y| > \mu_0$ is a regular cardinal. Then, using the assumption that the claim fails, we may find a set $Z \subseteq I$ such that $|Z| < |Y|$ and $(\forall b \in Y)(\exists a \in Z)(b \leq a)$. Now apply the induction hypothesis to Z and get a set $Z^* \subseteq I$ of size μ_0 such that $(\forall a \in Z)(\exists c \in Z^*)(a \leq c)$ – clearly the set Z^* works for Y too.
So suppose now that $Y \subseteq I$ and $|Y|$ is a singular cardinal $> \mu_0$. Let $Y = \bigcup\limits_{\xi < \mathrm{cf}(|Y|)} Y_\xi$, where $|Y_\xi| < |Y|$ (for $\xi < \mathrm{cf}(|Y|)$). For each ξ apply the inductive hypothesis to get $Y^*_\xi \subseteq I$ such that $|Y^*_\xi| = \mu_0$ and $(\forall b \in Y_\xi)(\exists a \in Y^*_\xi)(b \leq a)$. Put $Y^+ = \bigcup\limits_{\xi < \mathrm{cf}(|Y|)} Y^*_\xi$ and note that $|Y^+| \leq \mathrm{cf}(|Y|) \cdot \mu_0 < |Y|$. Again, apply the inductive hypothesis ($\circledast$), this time to Y^+, to get the respective Y^* and note that it works for Y too.

To finish the proof of the claim note that the statement in ($\circledast$) contradicts the assumption that $\mu_0 < \lambda = \mathrm{cof}(I)$. $\square$

If a set $Y \subseteq I$ is given by 2.2.1 for I, μ_0, θ then we say that it is *temporarily (I, μ_0, θ)–good.*

Claim 2.2.2. *Suppose that $Y \subseteq I$ is temporarily (I, μ, θ)–good, $\kappa < |Y|$. Assume $Y = \bigcup\limits_{\xi<\kappa} Y_\xi$. Then for some $\xi < \kappa$ the set Y_ξ is temporarily (I, μ, θ)–good.*

Proof of the claim. Suppose that $Y = \bigcup\limits_{\xi<\kappa} Y_\xi$, $\kappa < |Y|$ and no Y_ξ is temporarily (I, μ, θ)–good. For $\xi < \kappa$ choose $Z_\xi \subseteq I$ such that $|Z_\xi| < |Y| = \theta$ and

$$(\forall b \in Y_\xi)(\exists a \in Z_\xi)(b \leq a),$$

and put $Z = \bigcup\limits_{\xi<\kappa} Z_\xi$. Then Z contradicts "Y is temporarily (I, μ, θ)–good". The claim is shown. $\square$

Now, let $Y \subseteq I$ be a temporarily (I, μ, θ)–good set, $\theta = |Y|$, and let $Y = \{b_\alpha : \alpha < \theta\}$ be an enumeration. Each b_α can be represented as

$$b_\alpha = \bigvee_{j<j_\alpha} \bigwedge_{\ell<n_\alpha} x^{t(\alpha,j,\ell)}_{\xi(\alpha,j,\ell)}.$$

By 2.2.2 we find n^*, j^* and $A \in [\theta]^\theta$ such that $(\forall \alpha \in A)(j_\alpha = j^* \ \&\ n_\alpha = n^*)$ and the set $Y^* = \{b_\alpha : \alpha \in A\}$ is temporarily (I, μ, θ)–good. For $j < j^*$ and $\alpha \in A$ let $b^j_\alpha = \bigwedge\limits_{\ell<n^*} x^{t(\alpha,j,\ell)}_{\xi(\alpha,j,\ell)}$ and let $Y^j = \{b^j_\alpha : \alpha \in A\}$. We claim that for some $j < j^*$ the set Y^j is temporarily (I, μ, θ)–good. If not, then we find $Z_j \subseteq I$ (for $j < j^*$) such that $|Z_j| < \theta$ and $(\forall \alpha \in A)(\exists a \in Z_j)(b^j_\alpha \leq a)$. Put

$$Z = \{a_0 \vee \ldots \vee a_{j^*-1} : a_0 \in Z_0, \ldots, a_{j^*-1} \in Z_{j^*-1}\}$$

and note that this set contradicts "Y^* is temporarily (I, μ, θ)–good".

So let $j_0 < j^*$ be such that the set $Y^{**} \stackrel{\rm def}{=} \{b^{j_0}_\alpha : \alpha \in A\}$ is temporarily (I, μ, θ)–good and let $Y^{**} = \{a_\alpha : \alpha < \theta\}$ be an enumeration.

For $b \in I$ let $F_b = \{\alpha < \theta : a_\alpha \not\leq b\}$ and let $\mathcal{D}_0$ be the $(< \theta)$–complete filter of subsets of θ generated by $\{F_b : b \in I\}$.

First note that if $\kappa < \theta$ and $\langle b_\xi : \xi < \kappa\rangle \subseteq I$ then (by the choice of Y^{**}) we may find $\alpha < \theta$ such that $(\forall \xi < \kappa)(a_\alpha \not\leq b_\xi)$. Consequently $\bigcap\limits_{\xi<\kappa} F_{b_\xi} \neq \emptyset$ and we may conclude that $\mathcal{D}_0$ is a proper filter on θ. Since $\alpha \notin F_{a_\alpha}$, we get that $\mathcal{D}_0$ extends the filter of co-bounded subsets of θ.

Claim 2.2.3. *The set* $A^+ \stackrel{\text{def}}{=} \{\alpha < \theta : a_\alpha \in \mathrm{id}_{\mathbb{B}}(\{a_\beta : \beta < \alpha\})\}$ *does not belong to the filter* $\mathcal{D}_0$.

Proof of the claim. Assume toward contradiction that $A^+ \in \mathcal{D}_0$. Thus we have a sequence $\langle b_\xi : \xi < \kappa\rangle \subseteq I$, $\kappa < \theta$, such that $\bigcap\limits_{\xi<\kappa} F_{b_\xi} \subseteq A^+$. It follows from the choice of Y^{**} that $Y^{**} \not\subseteq \mathrm{id}_{\mathbb{B}}(\{b_\xi : \xi < \kappa\})$. So let $\alpha < \theta$ be the first such that $a_\alpha \notin \mathrm{id}_{\mathbb{B}}(\{b_\xi : \xi < \kappa\})$. This implies that $a_\alpha \in \bigcap\limits_{\xi<\kappa} F_{b_\xi} \subseteq A^+$, and thus $a_\alpha \in \mathrm{id}_{\mathbb{B}}(\{a_\beta : \beta < \alpha\})$. By the minimality of α we have $\mathrm{id}_{\mathbb{B}}(\{a_\beta : \beta < \alpha\}) \subseteq \mathrm{id}_{\mathbb{B}}(\{b_\xi : \xi < \kappa\})$, and we get a contradiction. □

Take the set A^+ from 2.2.3 and let $\mathcal{D} = \{X \setminus A^+ : X \in \mathcal{D}_0\}$. Then the filter $\mathcal{D}$ and $\langle a_\alpha : \alpha \in \theta \setminus A^+\rangle$ satisfy the demands $(*_1)$–$(*_3)$ (after taking the increasing enumeration of $\theta \setminus A^+$). □

Lemma 2.3 (see [22, 2.2, 2.3])**.** *Suppose* $\mathrm{cf}(\lambda) < \lambda$, $\mu < \lambda$. *Assume that* $\mathbb{B}$ *is a Boolean algebra generated by* $\langle x_\xi : \xi < \chi\rangle$ *and* $I \subseteq \mathbb{B}$ *is an ideal such that* $\pi(\mathbb{B}/I) = \lambda$. *Then there are a regular cardinal* $\theta \in [\mu, \lambda]$, *a* $(<\theta)$*–complete filter* $\mathcal{D}$ *on* θ *and a sequence* $\langle a_\alpha : \alpha < \theta\rangle \subseteq \mathbb{B} \setminus I$ *such that*

$(\otimes_1)$ *the filter* $\mathcal{D}$ *contains all co-bounded subsets of* θ *and for every* $b \in \mathbb{B} \setminus I$*:*
$$\{\alpha < \theta :\ b \leq a_\alpha \mod I\} = \emptyset \mod \mathcal{D},$$
$(\otimes_2)$ *if* $\beta < \alpha < \theta$ *then* $a_\beta \wedge (-a_\alpha) \notin I$,

$(\otimes_3)$ *every* a_α *(for* $\alpha < \theta$*) is of the form*
$$a_\alpha = \bigwedge_{\ell<n} x^{t(\alpha,\ell)}_{\xi(\alpha,\ell)} \qquad \textit{(where } n < \omega,\ \xi(\alpha,\ell) < \chi,\ t(\alpha,\ell) < 2\textit{).}$$

Proof. It is an easy modification of [22, 2.2, 2.3] (and the proof is fully parallel to that of Lemma 2.2 here). □

One of the ways of describing Boolean algebras is giving a dense set of ultrafilters (equivalently: homomorphisms from the algebra into 2). This is useful when we want to force a Boolean algebra by smaller approximations (see the forcing notions used in [22], [16]).

Definition 2.4. For a set w and a family $F \subseteq 2^w$ we define
$\mathrm{cl}(F) = \{g \in 2^w : (\forall u \in [w]^{<\omega})(\exists f \in F)(f \restriction u = g \restriction u)\}$,
$\mathbb{B}_{(w,F)}$ is the Boolean algebra generated freely by $\{x_\alpha : \alpha \in w\}$ except that
if $u_0, u_1 \in [w]^{<\omega}$ and there is no $f \in F$ such that $f \restriction u_0 \equiv 0$, $f \restriction u_1 \equiv 1$
then $\bigwedge\limits_{\alpha\in u_1} x_\alpha \wedge \bigwedge\limits_{\alpha\in u_0} (-x_\alpha) = 0$.

Proposition 2.5 (see [22, 2.6])**.** *Let* $F \subseteq 2^w$. *Then:*

(1) *each* $f \in F$ *extends (uniquely) to a homomorphism from* $\mathbb{B}_{(w,F)}$ *to* $\{0,1\}$ *(i.e. it preserves the equalities from the definition of* $\mathbb{B}_{(w,F)}$*),*

(2) *if* $\tau(y_0,\dots,y_\ell)$ *is a Boolean term and* $\alpha_0,\dots,\alpha_\ell \in w$ *are distinct then*
$$\mathbb{B}_{(w,F)} \models \tau(x_{\alpha_0},\dots,x_{\alpha_\ell}) \neq 0 \qquad \textit{if and only if}$$
$$(\exists f \in F)(\{0,1\} \models \tau(f(\alpha_0),\dots,f(\alpha_k)) = 1),$$

(3) *if* $w \subseteq w^*$, $F^* \subseteq 2^{w^*}$ *and*

$$(\forall f \in F)(\exists g \in F^*)(f \subseteq g) \quad \textit{and} \quad (\forall g \in F^*)(g \restriction w \in \mathrm{cl}(F))$$

then $\mathbb{B}_{(w,F)}$ *is a subalgebra of* $\mathbb{B}_{(w^*,F^*)}$.

Remark 2.6. Let $F \subseteq 2^w$. We will use the same notation for a member f of F and the homomorphism from $\mathbb{B}_{(w,F)}$ determined by it. Hence, for a Boolean term τ, a finite set $v \subseteq w$ and $f \in F$, we may write $f(\tau(x_\alpha : \alpha \in v))$ etc.

Proposition 2.7. *Let* $\mathbb{B}$ *be a Boolean algebra.*

(1) *A sequence* $\bar{a} = \langle a_\alpha : \alpha < \kappa \rangle$ *of elements of* $\mathbb{B}$ *is:*
- ideal independent *if and only if for each* $\alpha < \kappa$ *there is a homomorphism* $f_\alpha : \mathbb{B} \longrightarrow \{0,1\}$ *such that*

$$f_\alpha(a_\alpha) = 1 \quad \textit{and} \quad (\forall \beta < \kappa)(\alpha \neq \beta \ \Rightarrow\ f_\alpha(a_\beta) = 0);$$

- left–separated *if and only if for each* $\alpha < \kappa$ *there is a homomorphism* $f_\alpha : \mathbb{B} \longrightarrow \{0,1\}$ *such that*

$$f_\alpha(a_\alpha) = 1 \quad \textit{and} \quad (\forall \beta < \kappa)(\alpha < \beta \ \Rightarrow\ f_\alpha(a_\beta) = 0);$$

- right–separated *if and only if for each* $\alpha < \kappa$ *there is a homomorphism* $f_\alpha : \mathbb{B} \longrightarrow \{0,1\}$ *such that*

$$f_\alpha(a_\alpha) = 1 \quad \textit{and} \quad (\forall \beta < \alpha)(f_\alpha(a_\beta) = 0).$$

(2) *If the algebra* $\mathbb{B}$ *is generated by a sequence* $\langle x_\xi : \xi < \chi \rangle$, *and there is an ideal independent (left–separated, right–separated, respectively) sequence of elements of* $\mathbb{B}$ *of length* κ, *then there is such a sequence with terms of the form*

$$a_\alpha = \bigwedge_{k < k_\alpha} x_{\xi(\alpha,k)}^{t(\alpha,k)}$$

and where $\xi(\alpha,k) < \chi$, $t(\alpha,k) \in \{0,1\}$ *and* $\xi(\alpha,k) \neq \xi(\alpha,k')$ *whenever* $k < k' < k_\alpha$.

3. Forcing for hL

In this section we show that consistently there is a Boolean algebra $\mathbb{B}$ of size λ in which there is a strictly increasing λ–sequence of ideals but every ideal in $\mathbb{B}$ is generated by less than λ elements. This answers [12, Problem 43] (and thus a part of [13, Problem 50]). The problem if the respective example can be constructed just from cardinal arithmetic assumptions remains open.

Definition 3.1. (1) *A good parameter* is a tuple $S = (\mu, \lambda, \bar{\chi})$ such that μ, λ are cardinals satisfying

$$\mu = \mu^{<\mu} < \mathrm{cf}(\lambda) < \lambda \quad \text{and} \quad (\forall \alpha < \mathrm{cf}(\lambda))(\forall \xi < \mu)(\alpha^\xi < \mathrm{cf}(\lambda)),$$

and $\bar{\chi} = \langle \chi_i : i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals such that $\mathrm{cf}(\lambda) < \chi_i < \lambda$, $(\forall i < \mathrm{cf}(\lambda))(\chi_i^{<\mu} = \chi_i)$ and $\lambda = \sup\limits_{i < \mathrm{cf}(\lambda)} \chi_i$.

(2) A good parameter $S = (\mu, \lambda, \bar{\chi})$ is *a convenient parameter* if additionally $\mathrm{cf}(\lambda) = \mu^+$.

Definition 3.2. Let $S=(\mu,\lambda,\bar{\chi})$ be a convenient parameter and let the set

$$\mathcal{X}_S \stackrel{\rm def}{=} \{(i,\xi): i<{\rm cf}(\lambda)\ \&\ \xi<\chi_i\}$$

be equipped with the lexicographic order $\prec_S$ (i.e., $(i,\xi)\prec_S(i',\xi')$ if and only if either $i<i'$, or $i=i'$ and $\xi<\xi'$).

(1) We define a forcing notion $\mathbb{Q}_S$ as follows.
A condition is a tuple $p=\langle w^p,u^p,\langle f^p_{i,\xi}:(i,\xi)\in u^p\rangle\rangle$ such that
(a) $w^p\in[{\rm cf}(\lambda)]^{<\mu}$, $\quad u^p\in[\mathcal{X}_S]^{<\mu}$,
(b) $(\forall i\in w^p)((i,0)\in u^p)$ and if $(i,\xi)\in u^p$ then $i\in w^p$,
(c) for $(i,\xi)\in u^p$, $\quad f^p_{i,\xi}:u^p\longrightarrow 2$ is a function such that

$$(j,\zeta)\in u^p\ \&\ (j,\zeta)\prec_S(i,\xi)\quad\Rightarrow\quad f^p_{i,\xi}(j,\zeta)=0,$$

and $f^p_{i,\xi}(i,\xi)=1$,
the order is given by: $p\le q$ if and only if
(α) $w^p\subseteq w^q$, $\quad u^p\subseteq u^q$, and
(β) $(\forall(i,\xi)\in u^p)(f^p_{i,\xi}\subseteq f^q_{i,\xi})$, and
(γ) for each $(i,\xi)\in u^q$ one of the following occurs:
either $f^q_{i,\xi}\restriction u^p=\mathbf{0}_{u^p}$,
or $i\in w^p$ and for some $\zeta,\varepsilon\le\chi_i$ we have $(i,\zeta)\in u^p$ and $f^q_{i,\xi}\restriction u^p=(f^p_{i,\zeta})_\varepsilon$, where $(f^p_{i,\zeta})_\varepsilon:u^p\longrightarrow 2$ is defined by

$$(f^p_{i,\zeta})_\varepsilon(j,\gamma)=\begin{cases}0 & \text{if } j=i,\ \gamma<\varepsilon,\\ f^p_{i,\zeta}(j,\gamma) & \text{otherwise,}\end{cases}$$

or $i\notin w^p$ and $f^q_{i,\xi}\restriction u^p=(f^p_{j,\zeta})_\varepsilon$ for some $(j,\zeta)\in u^p$ and $\zeta,\varepsilon\le\chi_j$, where $(f^p_{j,\zeta})_\varepsilon$ is defined as above.

(2) We say that conditions $p,q\in\mathbb{Q}_S$ are *isomorphic* if the linear orders

$$(u^p,\prec_S\restriction u^p)\ \text{ and }\ (u^q,\prec_S\restriction u^q)$$

are isomorphic, and if $H:u^p\longrightarrow u^q$ is the $\prec_S$–isomorphism then:
(α) $H(i,\xi)=(j,0)$ if and only if $\xi=0$,
(β) $f^p_{i,\xi}=f^q_{H(i,\xi)}\circ H$ (for $(i,\xi)\in u^p$).
In this situation we may call H *an isomorphism from p to q*.

Remark 3.3. (1) Of course, $\prec_S$ is a well ordering of $\mathcal{X}_S$ in the order type λ.
(2) The forcing notion $\mathbb{Q}_S$ is a relative of the one used in [16, §7].
(3) There are μ isomorphism types of conditions in $\mathbb{Q}_S$ (remember $\mu^{<\mu}=\mu$). A condition $p\in\mathbb{Q}_S$ is determined by its isomorphism type and the set u^p.

Proposition 3.4. *Let $S=(\mu,\lambda,\bar{\chi})$ be a convenient parameter. Then $\mathbb{Q}_S$ is a $(<\mu)$–complete μ^+–cc forcing notion.*

Proof. First we should check that $\mathbb{Q}_S$ is really a partial order and for this we have to verify the transitivity of $\le$. So suppose that $p\le q$ and $q\le r$ and let us justify that $p\le r$. The only perhaps unclear demand is clause 3.2(1γ). Assume that $(i,\xi)\in u^r$ and $f^r_{i,\xi}\restriction u^p\ne\mathbf{0}_{u^p}$ and consider two cases.

Case 1: $\quad i\in w^p$.
Then $i\in w^q$ and, by the definition of $\le$ (clause (γ)), we may pick $\zeta\le\varepsilon\le\chi_i$ such that $(i,\zeta)\in u^q$ and $f^r_{i,\xi}\restriction u^q=(f^q_{i,\zeta})_\varepsilon$. Again by clause (γ), for some ζ',ε' we have

$(i,\zeta')\in u^p$ and $f^q_{i,\zeta}\restriction u^p=(f^p_{i,\zeta'})_{\varepsilon'}$. Now look at the definition of the operation $(\cdot)_\varepsilon$ – it should be clear that $f^r_{i,\xi}\restriction u^p=(f^p_{i,\zeta'})_{\varepsilon''}$ for some ε''.

Case 2: $i\notin w^p$.
If $i\in w^q$ then for some ζ,ε we have $f^r_{i,\xi}\restriction u^q=(f^q_{i,\zeta})_\varepsilon$ and $f^q_{i,\zeta}\restriction u^p=(f^p_{j,\zeta'})_{\varepsilon'}$ for some j,ζ',ε'. Now, since $i\notin w^p$ we may write $f^r_{i,\xi}\restriction u^p=(f^q_{i,\zeta})_\varepsilon\restriction u^p=(f^p_{j,\zeta'})_{\varepsilon'}$ and we are done. Suppose now that $i\notin w^q$. Then $f^r_{i,\xi}\restriction u^q=(f^q_{j,\zeta})_\varepsilon$ (for some j,ζ,ε) and we ask if $j\in w^p$. If so, then for some ζ',ε' we have $f^q_{j,\zeta}\restriction u^p=(f^p_{j,\zeta'})_{\varepsilon'}$ and hence $f^r_{i,\xi}\restriction u^p=(f^p_{j,\zeta'})_{\varepsilon''}$ (for some ε''). If not (i.e., if $j\notin w^p$) then like before we easily conclude that $f^r_{i,\xi}\restriction u^p=(f^q_{j,\zeta})_\varepsilon\restriction u^p=f^q_{j,\zeta}\restriction u^p=(f^p_{j',\zeta'})_{\varepsilon'}$ (for some j',ζ',ε').

Thus $\mathbb{Q}_S$ is a forcing notion. To check that it is $(<\mu)$–complete suppose that $\gamma<\mu$ and $\langle p_\alpha:\alpha<\gamma\rangle\subseteq\mathbb{Q}_S$ is an increasing sequence of conditions. Put $w^p=\bigcup\limits_{\alpha<\gamma}w^{p_\alpha}$, $u^p=\bigcup\limits_{\alpha<\gamma}u^{p_\alpha}$ and for $(i,\xi)\in u^p$ let

$$f^p_{i,\xi}=\bigcup\{f^{p_\alpha}_{i,\xi}:(i,\xi)\in u^{p_\alpha},\ \alpha<\gamma\}.$$

Plainly, $\langle w^p,u^p,\langle f^p_{i,\xi}:(i,\xi)\in u^p\rangle\rangle\in\mathbb{Q}_S$ is an upper bound to $\langle p_\alpha:\alpha<\gamma\rangle$.

Now assume that $\mathcal{A}\subseteq\mathbb{Q}_S$ is of size μ^+. Since $\mu^{<\mu}=\mu$ and $\mathrm{cf}(\lambda)=\mu^+$ we may use Δ–lemma and "standard cleaning" and find conditions $p,q\in\mathcal{A}$ such that

(i) p,q are isomorphic (and let $H:u^p\longrightarrow u^q$ be the isomorphism),
(ii) $H\restriction(u^p\cap u^q)$ is the identity on $u^p\cap u^q$,
(iii) $\sup(w^p\cap w^q)<\min(w^p\setminus w^q)\le\sup(w^p\setminus w^q)<\min(w^q\setminus w^p)$.

Now we are going to define an upper bound r to p,q. To this end we put $w^r=w^p\cup w^q$, $u^r=u^p\cup u^q$ and for $(i,\xi)\in u^r$ we define $f^r_{i,\xi}:u^r\longrightarrow 2$ as follows.

- If $(i,\xi)\in u^p$, $i\in w^p\cap w^q$ then $f^r_{i,\xi}=f^p_{i,\xi}\cup(f^q_{H(i,\xi)})_\xi$,
- if $(i,\xi)\in u^q$, $i\in w^p\cap w^q$ then $f^r_{i,\xi}=(f^p_{H^{-1}(i,\xi)})_\xi\cup f^q_{i,\xi}$,
- if $(i,\xi)\in u^p$, $i\in w^p\setminus w^q$ then $f^r_{i,\xi}=f^p_{i,\xi}\cup f^q_{H(i,\xi)}$,
- if $(i,\xi)\in u^q$, $i\in w^q\setminus w^p$ then $f^r_{i,\xi}=\mathbf{0}_{u^p}\cup f^q_{i,\xi}$.

It should be clear that in all cases the functions $f^r_{i,\xi}$ are well defined and that they satisfy the demand 3.2(1c). Hence $r=\langle w^r,u^r,\langle f^r_{i,\xi}:(i,\xi)\in u^r\rangle\rangle\in\mathbb{Q}_S$ and one easily checks that it is a condition stronger than both p and q. So we may conclude that $\mathbb{Q}_S$ satisfies the μ^+–chain condition. □

For a condition $p\in\mathbb{Q}_S$ let $F^p=\{\mathbf{0}_{u^p}\}\cup\{(f^p_{i,\xi})_\zeta:\xi,\zeta\le\chi_i,\ (i,\xi)\in u^p\}$, where $(f^p_{i,\xi})_\zeta:u^p\longrightarrow 2$ is defined like in 3.2(1γ):

$$(f^p_{i,\xi})_\zeta(j,\gamma)=\begin{cases}0 & \text{if } j=i,\ \gamma<\zeta,\\ f^p_{i,\xi}(j,\gamma) & \text{otherwise.}\end{cases}$$

Further, let $\mathbb{B}_p$ be the Boolean algebra $\mathbb{B}_{(u^p,F^p)}$ (as defined in 2.4). Note that $p\le q$ implies that $\mathbb{B}_p$ is a subalgebra of $\mathbb{B}_q$ (remember 2.5). Let $\dot{\mathbb{B}}^0_S$ be a $\mathbb{Q}_S$–name such that $\Vdash_{\mathbb{Q}_S}$ " $\dot{\mathbb{B}}^0_S=\bigcup\{\mathbb{B}_p:p\in\Gamma_{\mathbb{Q}_S}\}$ " and for $(i,\xi)\in\mathcal{X}_S$ let $\dot{f}_{i,\xi}$ be a $\mathbb{Q}_S$–name such that

$$\Vdash_{\mathbb{Q}_S}\text{ `` }\dot{f}_{i,\xi}=\bigcup\{f^p_{i,\xi}:(i,\xi)\in u^p,\ p\in\Gamma_{\mathbb{Q}_S}\}\text{ ''}.$$

Proposition 3.5. *Assume that $S=(\mu,\lambda,\bar{\chi})$ is a convenient parameter. Then in $\mathbf{V}^{\mathbb{Q}_S}$:*

(1) *$\dot{f}_{i,\xi} : \mathcal{X}_S \longrightarrow 2$ (for $(i,\xi) \in \mathcal{X}_S$) is such that $\dot{f}_{i,\xi}(i,\xi) = 1$ and*

$$(\forall (j,\zeta) \in \mathcal{X}_S)((j,\zeta) \prec_S (i,\xi) \ \Rightarrow\ \dot{f}_{i,\xi}(j,\zeta) = 0).$$

(2) *$\dot{\mathbb{B}}^0_S$ is the Boolean algebra $\mathbb{B}_{(\mathcal{X}_S,\dot{F})}$ (see 2.4), where*

$$\dot{F} = \{(\dot{f}_{i,\xi})_\zeta : (i,\xi) \in \mathcal{X}_S,\ \xi \le \zeta \le \chi_i\}$$

and $(\dot{f}_{i,\xi})_\zeta : \mathcal{X}_S \longrightarrow 2$ is such that

$$(\dot{f}_{i,\xi})_\zeta(j,\gamma) = \begin{cases} 0 & \text{if } j = i,\ \gamma < \zeta, \\ \dot{f}_{i,\xi}(j,\gamma) & \text{otherwise,} \end{cases} \qquad \text{(for } (j,\gamma) \in \mathcal{X}_S).$$

(3) *The sequence $\langle x_{i,\xi} : (i,\xi) \in \mathcal{X}_S \rangle$ is right–separated in $\dot{\mathbb{B}}^0_S$ (when we consider $\mathcal{X}_S$ with the well ordering $\prec_S$).*

Proof. Should be clear (for the third clause remember that each $\dot{f}_{i,\xi}$ extends to a homomorphism from $\dot{\mathbb{B}}^0_S$ to $\{0,1\}$, see 2.5; remember 2.7). □

Theorem 3.6. *Assume $S = (\mu, \lambda, \bar{\chi})$ is a convenient parameter. Then*

$$\Vdash_{\mathbb{Q}_S} \text{“ there is no ideal } I \subseteq \dot{\mathbb{B}}^0_S \text{ such that } \mathrm{cof}(I) = \lambda \text{ ”}.$$

Proof. Let $\dot{I}$ be a $\mathbb{Q}_S$–name for an ideal in $\dot{\mathbb{B}}^0_S$, $p \in \mathbb{Q}_S$, and suppose that $p \Vdash_{\mathbb{Q}_S} \mathrm{cof}(\dot{I}) = \lambda$.

Fix $i < \mathrm{cf}(\lambda)$ for a moment.

It follows 2.2 that we may choose p_i, θ_i, n_i, $\dot{\mathcal{D}}_i$, $\dot{e}_i$ and $\dot{t}_i$ such that

(α) $p_i \in \mathbb{Q}_S$ is a condition stronger than p, θ_i is a regular cardinal, $\chi_i^+ < \theta_i < \lambda$ and $n_i \in \omega$,

(β) $\dot{\mathcal{D}}_i$ is a $\mathbb{Q}_S$–name for a $(<\theta_i)$–complete filter on θ_i extending the filter of co-bounded subsets of θ_i,

(γ) $\Vdash_{\mathbb{Q}_S}$ “ $\dot{e}_i : \theta_i \times n_i \longrightarrow \mathcal{X}_S$ and $\dot{t}_i : \theta_i \times n_i \longrightarrow 2$ ”;
for $\alpha < \theta_i$ let $\dot{a}^i_\alpha$ be a $\mathbb{Q}_S$–name for an element of $\dot{\mathbb{B}}^0_S$ such that

$$\Vdash_{\mathbb{Q}_S} \text{“ } \dot{a}^i_\alpha = \bigwedge_{\ell < n_i} x^{\dot{t}_i(\alpha,\ell)}_{\dot{e}_i(\alpha,\ell)} \text{ ”},$$

(δ) $p_i \Vdash_{\mathbb{Q}_S}$ “ $\dot{a}^i_\alpha \in \dot{I}$ ” for each $\alpha < \theta_i$,

(ε) $p_i \Vdash_{\mathbb{Q}_S}$ “ if $b \in \dot{I}$ then $\{\alpha < \theta_i : \dot{a}^i_\alpha \le b\} = \emptyset \mod \dot{\mathcal{D}}_i$ and $\dot{a}^i_\alpha \notin \mathrm{id}_{\dot{\mathbb{B}}^0_S}(\{\dot{a}^i_\beta : \beta < \alpha\})$ for each $\alpha < \theta_i$ ”.

For each $\alpha < \theta_i$ choose an antichain $\{p^i_{\alpha,\zeta} : \zeta < \mu\}$ of conditions stronger than p_i, maximal above p_i, and such that each $p^i_{\alpha,\zeta}$ decides the values of $\dot{e}_i(\alpha,\cdot)$, $\dot{t}_i(\alpha,\cdot)$. Let

$$p^i_{\alpha,\zeta} \Vdash_{\mathbb{Q}_S} \text{“ } \dot{e}_i(\alpha,\ell) = e^\zeta_i(\alpha,\ell) \quad \& \quad \dot{t}_i(\alpha,\ell) = t^\zeta_i(\alpha,\ell) \text{ ”} \qquad (\text{for } \ell < n_i).$$

Plainly, we may demand that $i \in w^{p^i_{\alpha,\zeta}}$ and $e^\zeta_i(\alpha,\ell) \in u^{p^i_{\alpha,\zeta}}$ (for $\alpha < \theta_i$, $\zeta < \mu$, $\ell < n_i$).

Suppose now that $G \subseteq \mathbb{Q}_S$ is a generic filter (over $\mathbf{V}$) such that $p_i \in G$ and work in $\mathbf{V}[G]$ for a while. Since the filter $\dot{\mathcal{D}}^G_i$ is $(<\theta_i)$–complete we find ordinals $\dot{\gamma}^G_i < \theta_i$

and $\dot\zeta_i^G < \mu$ such that the set

$$\dot X_i^G \stackrel{\rm def}{=} \Big\{\beta<\theta_i:\ \ \dot\gamma_i^G\le\beta \text{ and } p^i_{\dot\gamma_i^G,\dot\zeta_i^G}, p^i_{\beta,\dot\zeta_i^G}\in G \text{ and } w^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}} = w^{p^i_{\beta,\dot\zeta_i^G}}, \text{ the conditions } p^i_{\dot\gamma_i^G,\dot\zeta_i^G}, p^i_{\beta,\dot\zeta_i^G} \text{ are isomorphic, and if } H: u^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}} \longrightarrow u^{p^i_{\beta,\dot\zeta_i^G}} \text{ is the isomorphism then } (\forall\ell<n_i)(H(e_i^{\dot\zeta_i^G}(\dot\gamma_i^G,\ell)) = e_i^{\dot\zeta_i^G}(\beta,\ell)\ \&\ t_i^{\dot\zeta_i^G}(\dot\gamma_i^G,\ell) = t_i^{\dot\zeta_i^G}(\beta,\ell)) \text{ and if } j\le i,\ (j,\xi)\in\mathcal{X}_S \text{ then } (j,\xi)\in u^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}} \Leftrightarrow (j,\xi)\in u^{p^i_{\beta,\dot\zeta_i^G}}\Big\}$$

is not $\emptyset$ modulo $\dot{\mathcal{D}}_i^G$ (remember that in $\mathbf{V}[G]$ we still have $\mathrm{cf}(\lambda)^{<\mu}=\mathrm{cf}(\lambda)$ and $\chi_i^{<\mu}=\chi_i$). Let $\dot\delta_i^G = \mathrm{otp}(u^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}}, \prec_S)$ and for $\alpha\in\dot X_i^G$ let $\langle s_\varepsilon^{\alpha,i}:\varepsilon<\dot\delta_i^G\rangle$ be the $\prec_S$–increasing enumeration of $u^{p^i_{\alpha,\dot\zeta_i^G}}$. Apply Lemma 2.1 to μ^+, θ_i, $\dot\delta_i^G$, $\dot{\mathcal{D}}_i^G$ and $\langle s_\varepsilon^{\alpha,i}:\varepsilon<\dot\delta_i^G\rangle$ here standing for $\sigma,\theta,\kappa,\mathcal{D}$ and $\langle\beta_\varepsilon^\alpha:\varepsilon<\kappa\rangle$ (respectively) there. (Remember $\prec_S$ is a well ordering of $\mathcal{X}_S$ in the order type λ.) So we find a sequence $\langle s_\varepsilon^{*,i}:\varepsilon<\dot\delta_i^G\rangle\subseteq\mathcal{X}_S$ and a set $\dot v_i^G\subseteq\dot\delta_i^G$ such that

(i) $(\forall\varepsilon\in\dot\delta_i^G\setminus\dot v_i^G)(\mu^+\le\mathrm{cf}(\{s\in\mathcal{X}_S: s\prec_S s_\varepsilon^{*,i}\},\prec_S)\le\theta_i)$,

(ii) the set

$$\dot B_i^G \stackrel{\rm def}{=} \Big\{\beta\in\dot X_i^G:\ \text{if } \varepsilon\in\dot v_i^G \text{ then } s_\varepsilon^{\beta,i}=s_\varepsilon^{*,i}, \text{ and if } \varepsilon\in\dot\delta_i^G\setminus\dot v_i^G \text{ then } \sup_{\prec_S}\{s_\zeta^{*,i}:\zeta<\dot\delta_i^G,\ s_\zeta^{*,i}\prec_S s_\varepsilon^{*,i}\}\prec_S s_\varepsilon^{\beta,i}\prec_S s_\varepsilon^{*,i}\Big\}$$

is not $\emptyset$ modulo the filter $\dot{\mathcal{D}}_i^G$,

(iii) if $s'_\varepsilon\prec_S s_\varepsilon^{*,i}$ for $\varepsilon\in\dot\delta_i^G\setminus\dot v_i^G$ then

$$\big\{\beta\in\dot B_i^G: (\forall\varepsilon\in\dot\delta_i^G\setminus\dot v_i^G)(s'_\varepsilon\prec_S s_\varepsilon^{\beta,i})\big\}\ne\emptyset \mod \dot{\mathcal{D}}_i^G.$$

As there was no special role assigned to $\dot\gamma_i^G$ (other than determining the order type of a condition) we may assume that $\dot\gamma_i^G\in\dot B_i^G$.

Now we go back to $\mathbf{V}$ and we choose a condition $q_i\in\mathbb{Q}_S$, ordinals $\gamma_i,\zeta_i,\delta_i$, a set v_i and a sequence $\langle s_\varepsilon^{*,i}:\varepsilon<\delta_i\rangle\subseteq\mathcal{X}_S$ such that $q_i\ge p^i_{\gamma_i,\zeta_i}$ and q_i forces that these objects have the properties listed in (i)–(iii) above. Note that if some condition stronger than q_i forces that $\beta\in\dot B_i$, then any condition stronger than both q_i and p^i_{β,ζ_i} does so. Then the conditions p^i_{β,ζ_i} and $p^i_{\gamma_i,\zeta_i}$ are isomorphic and the isomorphism is the identity on $u^{p^i_{\beta,\zeta_i}}\cap u^{p^i_{\gamma_i,\zeta_i}}$, and it preserves $e_i^{\zeta_i}, t_i^{\zeta_i}$. Also then $w^{p^i_{\beta,\zeta_i}} = w^{p^i_{\gamma_i,\zeta_i}}$ and $u^{p^i_{\beta,\zeta_i}}\cap(\{j\}\times\chi_j) = u^{p^i_{\gamma_i,\zeta_i}}\cap(\{j\}\times\chi_j)$ for $j\le i$. In this situation we will use $\langle s_\varepsilon^{\beta,i}:\varepsilon<\delta_i\rangle$ to denote the $\prec_S$–increasing enumeration of $u^{p^i_{\beta,\zeta_i}}$ (and so $s_\varepsilon^{\beta,i}=s_\varepsilon^{*,i}$ for $\varepsilon\in v_i$, and $\sup_{\prec_S}\{s_\zeta^{*,i}:\zeta<\delta_i,\ s_\zeta^{*,i}\prec_S s_\varepsilon^{*,i}\}\prec_S s_\varepsilon^{\beta,i}\prec_S s_\varepsilon^{*,i}$ for $\varepsilon\in\delta\setminus v_i$).

Claim 3.6.1. *If $j\le i<\mathrm{cf}(\lambda)$, $\ell<n_i$ and $e_i^{\zeta_i}(\gamma_i,\ell)=(j,\varepsilon)$ (for some ε) then $t_i^{\zeta_i}(\gamma_i,\ell)=1$.*

Proof of the claim. Suppose that the claim fails for some $j_0\le i$, $\varepsilon_0<\chi_{j_0}$ and $\ell_0<n_i$ (i.e., $t_i^{\zeta_i}(\gamma_i,\ell_0)=0$ and $e_i^{\zeta_i}(\gamma_i,\ell_0)=(j_0,\varepsilon_0)$). Choose α such that $\gamma_i<\alpha<\theta_i$ and letting $r_1=p^i_{\gamma_i,\zeta_i}$, $r_2=p^i_{\alpha,\zeta_i}$ we have:

- the conditions r_1, r_2 are isomorphic and if H is the isomorphism from r_1 to r_2 then $H(e_i^{\zeta_i}(\gamma_i,\ell)) = e_i^{\zeta_i}(\alpha,\ell)$ and $t_i^{\zeta_i}(\gamma_i,\ell) = t_i^{\zeta_i}(\alpha,\ell)$ (for $\ell < n_i$),
- $w^{r_1} = w^{r_2}$ and the isomorphism H is the identity on $u^{r_1} \cap u^{r_2}$,
- $(j,\xi) \prec_S H(j,\xi)$ for $(j,\xi) \in u^{r_1} \setminus u^{r_2}$, and
- if $j \le i$, $(j,\xi) \in \mathcal{X}_S$ then $(j,\xi) \in u^{r_1} \Leftrightarrow (j,\xi) \in u^{r_2}$.

Why is the choice possible? Let $G \subseteq \mathbb{Q}_S$ be generic over $\mathbf{V}$ such that $q_i \in G$. It follows from clauses (ii), (iii) that we may find $\alpha \in \dot{B}_i^G \setminus (\gamma_i + 1)$ such that $(\forall \varepsilon \in \delta_i \setminus v_i)(s_\varepsilon^{\gamma_i,i} \prec_S s_\varepsilon^{\alpha,i})$. Then the two ordinals γ_i, α have the required properties in $\mathbf{V}[G]$, and hence clearly in $\mathbf{V}$ too.

Next we let $w^r = w^{r_1} = w^{r_2}$, $u^r = u^{r_1} \cup u^{r_2}$ and for $(j,\xi) \in u^r$ we define $f_{j,\xi}^r : u^r \longrightarrow 2$ as follows.

- If $(j,\xi) \in u^{r_1} \cap u^{r_2}$ then $f_{j,\xi}^r = f_{j,\xi}^{r_1} \cup f_{j,\xi}^{r_2}$,
- if $(j,\xi) \in u^{r_1} \setminus u^{r_2}$ then $f_{j,\xi}^r = f_{j,\xi}^{r_1} \cup f_{H(j,\xi)}^{r_2}$,
- if $(j,\xi) \in u^{r_2} \setminus u^{r_1}$ then $f_{j,\xi}^r = (f_{H^{-1}(j,\xi)}^{r_1})_\xi \cup f_{j,\xi}^{r_2}$.

Check that the functions $f_{j,\xi}^r$ are well defined and that

$$r = \langle w^r, u^r, \langle f_{j,\xi}^r : (j,\xi) \in u^r \rangle \rangle \in \mathbb{Q}_S$$

is a condition stronger than r_1, r_2. Let $\tau_1 = \bigwedge\limits_{\ell < n_i} x_{e_i^{\zeta_i}(\gamma_i,\ell)}^{t_i^{\zeta_i}(\gamma_i,\ell)}$ and $\tau_2 = \bigwedge\limits_{\ell < n_i} x_{e_i^{\zeta_i}(\alpha,\ell)}^{t_i^{\zeta_i}(\alpha,\ell)}$. Suppose that $(j,\xi) \in u^r$ and $\xi \le \zeta < \chi_j$. If $j \le i$ then $(\{j\} \times \chi_j) \cap u^{r_1} = (\{j\} \times \chi_j) \cap u^{r_2}$ and therefore $(f_{j,\xi}^r)_\zeta(\tau_1) = (f_{j,\xi}^r)_\zeta(\tau_2)$. If $j > i$ then necessarily $(f_{j,\xi}^r)_\zeta(j_0,\xi_0) = 0$, so $(f_{j,\xi}^r)_\zeta(\tau_1) = (f_{j,\xi}^r)_\zeta(\tau_2) = 0$. Consequently $\mathbb{B}_r \models \tau_1 = \tau_2$ and hence $r \Vdash \dot{a}_{\gamma_i}^i = \dot{a}_\alpha^i$, contradicting clause (ε) (and so finishing the proof of the claim). □

Take $n < \omega$, $\delta < \mu$, $v \subseteq \delta$ and an unbounded set $Y \subseteq \mathrm{cf}(\lambda)$ such that for $i, j \in Y$:

- $n_i = n$, $\delta_i = \delta$, $v_i = v$, and
- the conditions $p_{\gamma_i,\zeta_i}^i, p_{\gamma_j,\zeta_j}^j$ are isomorphic, and the isomorphism maps $e_i^{\zeta_i}(\gamma_i,\cdot)$ and $t_i^{\zeta_i}(\gamma_i,\cdot)$ onto $e_j^{\zeta_j}(\gamma_j,\cdot), t_j^{\zeta_j}(\gamma_j,\cdot)$, respectively.

Now apply Lemma 2.1 to find a sequence $\langle s_{*,\varepsilon} : \varepsilon < \delta \rangle \subseteq \mathcal{X}_S \cup \{(\mathrm{cf}(\lambda), 0)\}$ and a set $v^* \subseteq \delta$ such that

(a) $(\forall \varepsilon \in \delta \setminus v^*)(\mathrm{cf}(\{s \in \mathcal{X}_S : s \prec_S s_{*,\varepsilon}\}, \prec_S) = \mu^+)$,

(b) the set

$$C \stackrel{\text{def}}{=} \Big\{ i \in Y : \text{ if } \varepsilon \in v^* \text{ then } s_\varepsilon^{*,i} = s_{*,\varepsilon}, \quad \text{and} \\ \text{if } \varepsilon \in \delta \setminus v^* \text{ then} \\ \sup_{\prec_S}\{s_{*,\zeta} : \zeta < \delta,\ s_{*,\zeta} \prec_S s_{*,\varepsilon}\} \prec_S s_\varepsilon^{*,i} \prec_S s_{*,\varepsilon} \Big\}$$

is unbounded in $\mathrm{cf}(\lambda)$,

(c) if $s'_\varepsilon \prec_S s_{*,\varepsilon}$ for $\varepsilon \in \delta \setminus v^*$, then the set

$$\{i \in C : (\forall \varepsilon \in \delta \setminus v^*)(s'_\varepsilon \prec_S s_\varepsilon^{*,i})\}$$

is unbounded in $\mathrm{cf}(\lambda)$.

[So $\sigma, \theta, \kappa, \mathcal{D}$ and $\langle \langle \beta_\varepsilon^\alpha : \varepsilon < \kappa \rangle : \alpha < \theta \rangle$ in 2.1 correspond to $\mathrm{cf}(\lambda) = \mu^+$, δ^* and the filter of co-bounded subsets of $\mathrm{cf}(\lambda)$ and $\langle \langle s_\varepsilon^{*,i} : \varepsilon < \delta \rangle : i < \mathrm{cf}(\lambda) \rangle$ here.]

Next we use clauses (c), (a) and (iii), (i) to choose inductively a set $C^+ \subseteq C$ of size $\mathrm{cf}(\lambda)$ and ordinals $\alpha_i < \theta_i$ (for $i \in C^+$) such that for every $i \in C^+$:

(d) if $\varepsilon\in\delta\setminus v^*$ then for all $j\in C^+\cap i$ and $\zeta<\delta$ we have
$$s^{*,j}_\zeta\prec_S s_{*,\varepsilon}\ \Rightarrow\ s^{*,j}_\zeta\prec_S s^{*,i}_\varepsilon\quad\text{ and }\quad s^{\alpha_j,j}_\zeta\prec_S s_{*,\varepsilon}\ \Rightarrow\ s^{\alpha_j,j}_\zeta\prec_S s^{*,i}_\varepsilon,$$

(e) some condition stronger than q_i forces that $\alpha_i\in\dot B_i$ (see clause (ii) earlier),

(f) if $\varepsilon\in\delta\setminus v$ then for all $j\in C^+\cap i$ and $\zeta<\delta$ we have
$$s^{*,j}_\zeta\prec_S s^{*,i}_\varepsilon\ \Rightarrow\ s^{*,j}_\zeta\prec_S s^{\alpha_i,i}_\varepsilon\quad\text{ and }\quad s^{\alpha_j,j}_\zeta\prec_S s^{*,i}_\varepsilon\ \Rightarrow\ s^{\alpha_j,j}_\zeta\prec_S s^{\alpha_i,i}_\varepsilon,$$

(g) if $\varepsilon\in v^*$, $s_{*,\varepsilon}=(j,\zeta)$ then $j<\min(C^+)$.

Note that then
$$i,j\in C^+\ \&\ \zeta,\varepsilon<\delta\ \&\ s^{\alpha_j,j}_\zeta=s^{\alpha_i,i}_\varepsilon\quad\Rightarrow\quad\varepsilon=\zeta\in v\cap v^*.$$

So $\langle\langle s^{\alpha_i,i}_\varepsilon:\varepsilon<\delta\rangle:i\in C^+\rangle$ is a Δ–system of sequences with the heart $\langle s_{*,\varepsilon}:\varepsilon\in v\cap v^*\rangle$. Let $u^*=\{s_{*,\varepsilon}:\varepsilon\in v\cap v^*\}$ and $w^*=\{j<\mathrm{cf}(\lambda):(j,0)\in u^*\}$.

Pick $i^*\in C^+$ such that $|C^+\cap i^*|=\mu$.

Claim 3.6.2.
$$q_{i^*}\Vdash_{\mathbb{Q}_S}\text{ “ }(\forall\alpha\in\dot B_{i^*})(\exists j_1,j_2\in C^+)(\dot a^{i^*}_\alpha\le\dot a^{j_1}_{\alpha_{j_1}}\vee\dot a^{j_2}_{\alpha_{j_2}}\ \&\ p_{j_1},p_{j_2}\in\Gamma_{\mathbb{Q}_S})\text{ ”.}$$

Proof of the claim. We are going to show that for every condition $q\ge q_{i^*}$ and an ordinal $\alpha<\theta_{i^*}$ such that $q\Vdash\alpha\in\dot B_{i^*}$, there are a condition $r\ge q$ and ordinals $j_1,j_2\in C^+$ such that
$$r\Vdash\text{ “ }\dot a^{i^*}_\alpha\le\dot a^{j_1}_{\alpha_{j_1}}\vee\dot a^{j_2}_{\alpha_{j_2}}\ \&\ p_{j_1},p_{j_2}\in\Gamma_{\mathbb{Q}_S}\text{ ”.}$$

So suppose $q\ge q_{i^*}$ and $q\Vdash\alpha\in\dot B_{i^*}$. We may assume that $p^{i^*}_{\alpha,\zeta_{i^*}}\le q$ (see the definition of $\dot X_{i^*},\dot B_{i^*}$). Choose $j_1\in C^+\cap i^*$ and $j_2\in C^+\setminus(i^*+1)$ such that
$$u^q\cap u^{p^{j_1}_{\alpha_{j_1},\zeta_{j_1}}}=u^q\cap u^{p^{j_2}_{\alpha_{j_2},\zeta_{j_2}}}=u^*\quad\text{ and }\quad\sup(w^q)<\min(w^{p^{j_2}_{\alpha_{j_2},\zeta_{j_2}}}\setminus w^*).$$

(Remember that $\{u^{p^j_{\alpha_j,\zeta_j}}:j\in C^+\}$ forms a Δ–system with heart u^* and hence $\{w^{p^j_{\alpha_j,\zeta_j}}:j\in C^+\}$ forms a Δ–system with heart w^*.)

To make the notation somewhat simpler let $q^0=p^{i^*}_{\alpha,\zeta_{i^*}}$, $q^1=p^{j_1}_{\alpha_{j_1},\zeta_{j_1}}$ and $q^2=p^{j_2}_{\alpha_{j_2},\zeta_{j_2}}$. Note that the conditions q^0,q^1,q^2 are pairwise isomorphic, and the isomorphisms are the identity on the u^* (which is the common part of any two u^{q^k}'s). Put
$$\tau_0=\bigwedge_{\ell<n}x^{t^{\zeta_{i^*}}_{i^*}(\alpha,\ell)}_{e^{\zeta_{i^*}}_{i^*}(\alpha,\ell)}\quad\text{ and }\quad\tau_k=\bigwedge_{\ell<n}x^{t^{\zeta_{j_k}}_{j_k}(\alpha_{j_k},\ell)}_{e^{\zeta_{j_k}}_{j_k}(\alpha_{j_k},\ell)}\qquad(\text{for }k=1,2).$$

Thus τ_k is an element of the algebra $\mathbb{B}_{q^k}$. Clearly, for $k,k'<3$, the isomorphism $H^{k,k'}$ from q^k to $q^{k'}$ carries τ_k to $\tau_{k'}$.

Now we are going to define a condition $r\in\mathbb{Q}_S$ stronger than q,q^1 and q^2. For this we put $w^r=w^q\cup w^{q^1}\cup w^{q^2}$, $u^r=u^q\cup u^{q^1}\cup u^{q^2}$ and we define functions $f^r_{i,\xi}:u^r\longrightarrow 2$ considering several cases.

(1) If $(i,\xi)\in u^{q^1}$ and $i\in w^*$ then we put $f^r_{i,\xi}=f^q_{H^{1,0}(i,\xi)}\cup f^{q^1}_{i,\xi}\cup f^{q^2}_{H^{1,2}(i,\xi)}$ (note that this includes the case $(i,\xi)\in u^*$).

(2) If $(i,\xi)\in u^{q^1}$, $i\notin w^*$ then we put $f^r_{i,\xi}=\mathbf{0}_{u^q}\cup f^{q^1}_{i,\xi}\cup\mathbf{0}_{u^{q^2}}$.

(3) If $(i,\xi) \in u^q \setminus u^*$ then we look at $f^q_{i,\xi} \restriction u^{q^0}$. If it is $\mathbf{0}_{u^{q^0}}$ then we let $f^r_{i,\xi} = f^q_{i,\xi} \cup \mathbf{0}_{u^{q^1}} \cup \mathbf{0}_{u^{q^2}}$. Otherwise we find $(j,\zeta) \in u^{q^0}$ and $\zeta \le \varepsilon \le \chi_j$ such that $f^q_{i,\xi} \restriction u^{q^0} = (f^{q^0}_{j,\zeta})_\varepsilon$ and if $i \in w^{q^0}$ then $i = j$, and we define:

(α) if $j \in w^*$, $j < i \le \sup(w^*)$ then $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(j,\zeta)})_{\chi_j} \cup (f^{q^2}_{H^{0,2}(j,\zeta)})_{\chi_j}$,

(β) if $i = j \in w^*$ then $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(j,\zeta)})_{\varepsilon^*} \cup (f^{q^2}_{H^{0,2}(j,\zeta)})_{\varepsilon^*}$, where $\varepsilon^* = \max\{\varepsilon, \xi\}$,

(γ) if $j \in w^*$, $i < j$ then $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(j,\zeta)})_\varepsilon \cup (f^{q^2}_{H^{0,2}(j,\zeta)})_\varepsilon$,

(δ) if either $i > \sup(w^*)$ or $j \notin w^*$ then we first choose $j' \in w^{q^2}$ and $\zeta' \le \varepsilon' \le \chi_{j'}$ such that $(j',\zeta') \in u^{q^2}$ and $(f^{q^2}_{j',\xi'})_{\varepsilon'}(j'',\xi'') = 0$ whenever $(j'',\xi'') \in u^{q^2}$, $j'' \in w^*$, and $(f^{q^2}_{j',\xi'})_{\varepsilon'}(\tau_2) = 1$ if possible (under our conditions); next we let $f^r_{i,\xi} = f^q_{i,\xi} \cup \mathbf{0}_{u^{q^1}} \cup (f^{q^2}_{j',\zeta'})_{\varepsilon'}$.

(4) If $(i,\xi) \in u^{q^2} \setminus u^*$, $i \in w^*$ then we let $f^r_{i,\xi} = (f^q_{H^{2,0}(i,\xi)})_\xi \cup (f^{q^1}_{H^{2,1}(i,\xi)})_\xi \cup f^{q^2}_{i,\xi}$.

(5) If $(i,\xi) \in u^{q^2}$, $i \notin w^*$ then we put $f^r_{i,\xi} = \mathbf{0}_{u^q} \cup \mathbf{0}_{u^{q^1}} \cup f^{q^2}_{i,\xi}$.

It should be a routine to check that in all cases the function $f^r_{i,\xi}$ is well defined and that $r = \langle w^r, u^r, \langle f^r_{i,\xi} : (i,\xi) \in u^r\rangle\rangle \in \mathbb{Q}_S$ is a condition stronger than q, q^1, q^2 (and thus stronger than p_{j_1}, p_{j_2}). [Remember that $w^* \subseteq \min(C^+)$, so for $j \in w^*$ we have $(j,\xi) \in u^{q^0} \Leftrightarrow (j,\xi) \in u^{p^{i^*}_{\alpha_{i^*},\zeta_{i^*}}}$ and hence, when checking clause 3.2(1c) in Case 1, we may use clauses (d), (f) of the choice of the set C^+. They imply that if $(i,\xi) \in u^{q^1}$, $i \in w^*$ then $(i,\xi) \preceq_S H^{1,0}(i,\xi) \preceq_S H^{1,2}(i,\xi)$. Considering Case 3($\delta$) with $j \notin w^*$, use the fact that $\min(w^{q^0} \setminus w^*) \ge \sup(w^*)$ (it follows from our choices). Similarly in Case 2 remember $\min(w^{q^1} \setminus w^*) \ge \sup(w^*)$.]

We claim that $\mathbb{B}_r \models \tau_0 \le \tau_1 \vee \tau_2$ and for this we have to show that there is no function $f \in F^r$ with $f(\tau_0) = 1$ and $f(\tau_1) = f(\tau_2) = 0$ (see 2.5). So suppose toward contradiction that $f \in F^r$ is such a function. Note that f cannot be $\mathbf{0}_{u^r}$ as then the values given to all the terms would be the same (remember they are isomorphic). So for some $(i,\xi) \in u^r$ and $\xi \le \varepsilon \le \chi_i$ we have $f = (f^r_{i,\xi})_\varepsilon$. Let us look at all the cases appearing in the definition of the functions $f^r_{j,\zeta}$'s (we keep labeling as there so we do not repeat the descriptions of the cases).

Case 1: Clearly $f^r_{i,\xi}(\tau_0) = f^r_{i,\xi}(\tau_1)$. It follows from the demands (d), (f) of the choice of C^+ that if $i \in w^*$, $(i,\zeta) \in u^{q^0}$, $(i',\zeta') = H^{0,1}(i,\zeta)$, then $i' = i$ and $\zeta' \le \zeta$. Consequently, we may use 3.6.1 to conclude that $(f^r_{i,\xi})_\varepsilon(\tau_0) \le (f^r_{i,\xi})_\varepsilon(\tau_1)$, what contradicts the choice of f.

Case 2: Plainly $(f^r_{i,\xi})_\varepsilon(\tau_0) = (f^r_{i,\xi})_\varepsilon(\tau_2)$.

Case 3α: Note that $f^r_{i,\xi}(\tau_0) = f^r_{i,\xi}(\tau_1)$ and, as $j < i \le \sup(w^*)$, necessarily $i \notin w^{q^0} \cup w^{q^1}$. Hence easily $(f^r_{i,\xi})_\varepsilon(\tau_0) = (f^r_{i,\xi})_\varepsilon(\tau_1)$.

Cases 3β,γ, 4: Like in cases 1, 3α we conclude $(f^r_{i,\xi})_\varepsilon(\tau_0) \le (f^r_{i,\xi})_\varepsilon(\tau_1)$.

Case 3δ: It follows from the choice of ζ', ε', j' there that $f^r_{i,\xi}(\tau_0) \le f^r_{i,\xi}(\tau_2)$. If $i \notin w^{q^0}$ then (as also $i \notin w^{q_2}$) we have $f(\tau_0) = f^r_{i,\xi}(\tau_0)$ and $f(\tau_2) = f^r_{i,\xi}(\tau_2)$, so we are done. If $i \in w^{q^0}$ then $i = j$ and we easily finish by the choice of ζ', ε', j'.

Case 5: Clearly $(f^r_{i,\xi})_\varepsilon(\tau_0) = (f^r_{i,\xi})_\varepsilon(\tau_1)$, a contradiction.

Thus we may conclude that $r \Vdash$"$\dot{a}^{i^*}_\alpha \leq \dot{a}^{j_1}_{\alpha_{j_1}} \vee \dot{a}^{j_2}_{\alpha_{j_2}}$", finishing the proof of the claim. $\square$

Now we may easily finish the theorem: take a generic filter $G \subseteq \mathbb{Q}_S$ over $\mathbf{V}$ such that $q_{i^*} \in G$ and work in $\mathbf{V}[G]$. Since the filter $\dot{\mathcal{D}}^G_{i^*}$ is $(<\theta_{i^*})$–complete and $\mathrm{cf}(\lambda) < \theta_{i^*}$, we find $j_1, j_2 \in C^+$ such that $p_{j_1}, p_{j_2} \in G$ and

$$\{\alpha \in \dot{B}^G_{i^*} : (\dot{a}^{i^*}_\alpha)^G \leq (\dot{a}^{j_1}_{\alpha_{j_1}})^G \vee (\dot{a}^{j_2}_{\alpha_{j_2}})^G\} \neq \emptyset \mod \dot{\mathcal{D}}^G_{i^*}$$

(remember $\dot{B}^G_{i^*} \neq \emptyset \mod \dot{\mathcal{D}}^G_{i^*}$ by (ii)). But then also $(\dot{a}^{j_1}_{\alpha_{j_1}})^G \vee (\dot{a}^{j_2}_{\alpha_{j_2}})^G \in \dot{I}^G$, so we get a contradiction to clause (ε). $\square$

Conclusion 3.7. It is consistent that there is a Boolean algebra $\mathbb{B}$ of size λ such that there is a right–separated sequence of length λ in $\mathbb{B}$, (so $\mathrm{hL}^+_{(7)}(\mathbb{B}) = \lambda^+$), but there is no ideal $I \subseteq \mathbb{B}$ with the generating number λ (and thus $\mathrm{hL}^+_{(1)}(\mathbb{B}) = \mathrm{hL}_{(1)}(\mathbb{B}) = \lambda$).

Problem 3.1. Does there exist a Boolean algebra $\mathbb{B}$ as in 3.7 in semi-ZFC? I.e., can one construct such an algebra for λ from cardinal arithmetic assumptions?

4. Forcing for hd

Here we deal with a problem parallel to the one from the previous section and related to the attainment question for hd. We introduce a forcing notion $\mathbb{P}_S$ complementary to $\mathbb{Q}_S$ and we use it to show that, consistently, there is a Boolean algebra $\mathbb{B}$ of size λ in which there is a strictly decreasing λ–sequence of ideals but every homomorphic image of $\mathbb{B}$ has algebraic density less than λ. This gives a partial answer to [13, Problem 54]. Again, we do not know if an example like that can be constructed from cardinal arithmetic assumptions.

Definition 4.1. Let $S = (\mu, \lambda, \bar{\chi})$ be a good parameter (see 3.1) and let $\mathcal{X}_S, \prec_S$ be as defined in 3.2.

(1) We define a forcing notion $\mathbb{P}_S$ as follows.
A condition is a tuple $p = \langle w^p, u^p, \langle f^p_{i,\xi} : (i,\xi) \in u^p\rangle\rangle$ such that
 (a) $w^p \in [\mathrm{cf}(\lambda)]^{<\mu}$, $u^p \in [\mathcal{X}_S]^{<\mu}$,
 (b) $(\forall i \in w^p)((i,0) \in u^p)$ and if $(i,\xi) \in u^p$ then $i \in w^p$,
 (c) for $(i,\xi) \in u^p$, $f^p_{i,\xi} : u^p \longrightarrow 2$ is a function such that

$$(j,\zeta) \in u^p \ \& \ (i,\xi) \prec_S (j,\zeta) \quad \Rightarrow \quad f^p_{i,\xi}(j,\zeta) = 0,$$

 and $f^p_{i,\xi}(i,\xi) = 1$,
the order is given by: $p \leq q$ if and only if
 (α) $w^p \subseteq w^q$, $u^p \subseteq u^q$, and
 (β) $(\forall (i,\xi) \in u^p)(f^p_{i,\xi} \subseteq f^q_{i,\xi})$, and
 (γ) for each $(i,\xi) \in u^q$ one of the following occurs:
 either $f^q_{i,\xi} \restriction u^p = \mathbf{0}_{u^p}$,
 or $i \in w^p$ and for some $\zeta, \varepsilon < \chi_i$ we have $(i,\zeta) \in u^p$ and $f^q_{i,\xi} \restriction u^p = (f^p_{i,\zeta})^\varepsilon$, where $(f^p_{i,\zeta})^\varepsilon : u^p \longrightarrow 2$ is defined by

$$(f^p_{i,\zeta})^\varepsilon(j,\gamma) = \begin{cases} 0 & \text{if } j = i,\ \varepsilon \leq \gamma < \chi_i, \\ f^p_{i,\zeta}(j,\gamma) & \text{otherwise,} \end{cases}$$

or $i \notin w^p$ and *either* $f^q_{i,\xi} \restriction u^p = (f^p_{j,\zeta})^\varepsilon$ (defined above) for some $(j,\zeta) \in u^p$, $\varepsilon < \chi_j$ *or* $f^q_{i,\xi} \restriction u^p = (f^p_{j,\zeta})_{j'}$ for some $(j,\zeta) \in u^p$ and $j' \leq j$, where $(f^p_{j,\zeta})_{j'} : u^p \longrightarrow 2$ is defined by

$$(f^p_{j,\zeta})_{j'}(j^*,\gamma^*) = \begin{cases} 0 & \text{if } j' \leq j^*, \\ f^p_{j,\zeta}(j^*,\gamma^*) & \text{otherwise.} \end{cases}$$

(2) Conditions $p, q \in \mathbb{P}_S$ are said to be *isomorphic* if the well orderings

$$(u^p, \prec_S\restriction u^p) \quad \text{and} \quad (u^q, \prec_S\restriction u^q)$$

are isomorphic, and if $H : u^p \longrightarrow u^q$ is the $\prec_S$–isomorphism then:
(α) $H(i,\xi) = (j,0)$ if and only if $\xi = 0$,
(β) $f^p_{i,\xi} = f^q_{H(i,\xi)} \circ H$ (for $(i,\xi) \in u^p$).

Proposition 4.2. *Let $S = (\mu,\lambda,\bar\chi)$ be a good parameter. Then $\mathbb{P}_S$ is a $(<\mu)$–complete μ^+–cc forcing notion.*

Proof. Plainly $\mathbb{P}_S$ is a $(<\mu)$–complete forcing notion (compare the proof of 3.4). To verify the chain condition suppose that $\mathcal{A} \subseteq \mathbb{P}_S$, $|\mathcal{A}| = \mu^+$. Apply the Δ–lemma and "standard cleaning" to choose isomorphic conditions $p, q \in \mathcal{A}$ such that if $H : u^p \longrightarrow u^q$ is the isomorphism from p to q then $H \restriction (u^p \cap u^q)$ is the identity on $u^p \cap u^q$. Put $w^r = w^p \cup w^q$, $u^r = u^p \cup u^q$ and for $(i,\xi) \in u^r$ define a function $f^r_{i,\xi} : u^r \longrightarrow 2$ as follows.

- If $(i,\xi) \in u^p$, $i \in w^p \cap w^q$ then $f^r_{i,\xi} = f^p_{i,\xi} \cup (f^q_{H(i,\xi)})^{\xi+1}$,
- if $(i,\xi) \in u^q$, $i \in w^p \cap w^q$ then $f^r_{i,\xi} = (f^p_{H^{-1}(i,\xi)})^{\xi+1} \cup f^q_{i,\xi}$,
- if $(i,\xi) \in u^p$, $i \in w^p \setminus w^q$ then $f^r_{i,\xi} = f^p_{i,\xi} \cup (f^q_{H(i,\xi)})_i$,
- if $(i,\xi) \in u^q$, $i \in w^q \setminus w^p$ then $f^r_{i,\xi} = (f^p_{H^{-1}(i,\xi)})_i \cup f^q_{i,\xi}$.

It is a routine to check that the functions $f^r_{i,\xi}$ are well defined and that they satisfy the demand 4.1(1c). Hence $r = \langle w^r, u^r, \langle f^r_{i,\xi} : (i,\xi) \in u^r\rangle\rangle \in \mathbb{P}_S$ and one easily checks that it is an upper bound to both p and q. ☐

For a condition $p \in \mathbb{P}_S$ let

$$F^p = \{(f^p_{i,\xi})^\varepsilon, (f^p_{i,\xi})_j : (i,\xi) \in u^p,\ \varepsilon < \chi_i,\ j \leq i\},$$

where $(f^p_{i,\xi})^\varepsilon, (f^p_{i,\xi})_j : u^p \longrightarrow 2$ are defined like in 4.1(1γ):

$$(f^p_{i,\xi})^\varepsilon(i',\zeta') = \begin{cases} 0 & \text{if } i = i',\ \varepsilon \leq \zeta', \\ f^p_{i,\xi}(i',\zeta') & \text{otherwise,} \end{cases}$$

$$(f^p_{i,\xi})_j(i',\zeta') = \begin{cases} 0 & \text{if } j \leq i', \\ f^p_{i,\xi}(i',\zeta') & \text{otherwise.} \end{cases}$$

Like in the previous section, $\mathbb{B}_p$ is the Boolean algebra $\mathbb{B}_{(u^p,F^p)}$ (see 2.4) (note that $p \leq q$ implies that $\mathbb{B}_p$ is a subalgebra of $\mathbb{B}_q$). Let $\dot{\mathbb{B}}^1_S$ be a $\mathbb{P}_S$–name such that

$$\Vdash_{\mathbb{P}_S} \text{“ } \dot{\mathbb{B}}^1_S = \bigcup\{\mathbb{B}_p : p \in \Gamma_{\mathbb{P}_S}\} \text{ ”},$$

and for $s \in \mathcal{X}_S$ let $\dot f_s$ be a $\mathbb{P}_S$–name such that

$$\Vdash_{\mathbb{P}_S} \text{“ } \dot f_s = \bigcup\{f^p_s : s \in u^p,\ p \in \Gamma_{\mathbb{P}_S}\} \text{ ”}.$$

Proposition 4.3. *Assume that $S = (\mu,\lambda,\bar\chi)$ is a good parameter. Then in $\mathbf{V}^{\mathbb{P}_S}$:*

(1) *For* $s\in\mathcal{X}_S$, $\dot{f}_s:\mathcal{X}_S\longrightarrow 2$ *is such that* $\dot{f}_s(s)=1$ *and*

$$(\forall s'\in\mathcal{X}_S)(s\prec_S s'\ \Rightarrow\ \dot{f}_s(s')=0).$$

(2) $\dot{\mathbb{B}}^1_S$ *is the Boolean algebra* $\mathbb{B}_{(\mathcal{X}_S,\dot{F})}$ *(see 2.4), where*

$$\dot{F}=\{(\dot{f}_{i,\xi})^\varepsilon,(\dot{f}_{i,\xi})_j:(i,\xi)\in\mathcal{X}_S,\ \varepsilon<\chi_i,\ j\leq i\},$$

and $(\dot{f}_{i,\xi})^\varepsilon,(\dot{f}_{i,\xi})_j:\mathcal{X}_S\longrightarrow 2$ *are such that*

$$(\dot{f}_{i,\xi})^\varepsilon(i',\zeta')=\begin{cases}0 & \text{if } i=i',\ \varepsilon\leq\zeta',\\ \dot{f}_{i,\xi}(i',\zeta') & \text{otherwise,}\end{cases}$$

$$(\dot{f}_{i,\xi})_j(i',\zeta')=\begin{cases}0 & \text{if } j\leq i',\\ \dot{f}_{i,\xi}(i',\zeta') & \text{otherwise.}\end{cases}$$

(3) *The sequence* $\langle x_s:s\in\mathcal{X}_S\rangle$ *is left–separated in* $\dot{\mathbb{B}}^1_S$ *(when we consider* $\mathcal{X}_S$ *with the well ordering* $\prec_S$*).*

Theorem 4.4. *Assume* $S=(\mu,\lambda,\bar{\chi})$ *is a good parameter. Then*

$$\Vdash_{\mathbb{P}_S}\text{“ there is no ideal } I\subseteq\dot{\mathbb{B}}^1_S \text{ such that } \pi(\dot{\mathbb{B}}^1_S/I)=\lambda\text{ ”.}$$

Proof. Not surprisingly, the proof is similar to the one of 3.6. Let $\dot{I}$ be a $\mathbb{P}_S$–name for an ideal in $\dot{\mathbb{B}}^1_S$, $p\in\mathbb{P}_S$, and suppose that $p\Vdash_{\mathbb{P}_S}\pi(\dot{\mathbb{B}}^1_S/\dot{I})=\lambda$.

Fix $i<\mathrm{cf}(\lambda)$. Use 2.3 to choose p_i, θ_i, n_i, $\dot{\mathcal{D}}_i$, $\dot{e}_i$ and $\dot{t}_i$ such that

(α) $p_i\in\mathbb{P}_S$ is a condition stronger than p, θ_i is a regular cardinal, $\chi_i^+<\theta_i<\lambda$ and $n_i\in\omega$,
(β) $\dot{\mathcal{D}}_i$ is a $\mathbb{P}_S$–name for a $(<\theta_i)$–complete filter on θ_i extending the filter of co-bounded subsets of θ_i,
(γ) $\Vdash_{\mathbb{P}_S}$“ $\dot{e}_i:\theta_i\times n_i\longrightarrow\mathcal{X}_S$ and $\dot{t}_i:\theta_i\times n_i\longrightarrow 2$ ”;
for $\alpha<\theta_i$ let $\dot{a}^i_\alpha$ be a $\mathbb{P}_S$–name for an element of $\dot{\mathbb{B}}^1_S$ such that

$$\Vdash_{\mathbb{P}_S}\text{“ }\dot{a}^i_\alpha=\bigwedge_{\ell<n_i}x^{\dot{t}_i(\alpha,\ell)}_{\dot{e}_i(\alpha,\ell)}\text{ ”},$$

(δ) $p_i\Vdash_{\mathbb{P}_S}$“ $\dot{a}^i_\alpha\in\dot{\mathbb{B}}^1_S\setminus\dot{I}$ ” for each $\alpha<\theta_i$,
(ε) $p_i\Vdash_{\mathbb{P}_S}$“ if $b\in\dot{\mathbb{B}}^1_S\setminus\dot{I}$ then $\{\alpha<\theta_i:b\leq\dot{a}^i_\alpha\mod\dot{I}\}=\emptyset\mod\dot{\mathcal{D}}_i$ and $(\forall\alpha<\theta_i)(\forall\beta<\alpha)(\dot{a}^i_\beta\wedge(-\dot{a}^i_\alpha)\notin\dot{I})$ ”.

For each $\alpha<\theta_i$ choose a maximal above p_i antichain $\{p^i_{\alpha,\zeta}:\zeta<\mu\}$ such that each $p^i_{\alpha,\zeta}\geq p_i$ decides the values of $\dot{e}_i(\alpha,\cdot)$, $\dot{t}_i(\alpha,\cdot)$. Let

$$p^i_{\alpha,\zeta}\Vdash_{\mathbb{P}_S}\text{“ }\dot{e}_i(\alpha,\ell)=e^\zeta_i(\alpha,\ell)\quad\&\quad\dot{t}_i(\alpha,\ell)=t^\zeta_i(\alpha,\ell)\text{ ”}\qquad(\text{for }\ell<n_i),$$

and we may assume that $(i,0),e^\zeta_i(\alpha,\ell)\in u^{p^i_{\alpha,\zeta}}$ for $\alpha<\theta_i$, $\ell<n_i$ and $\zeta<\mu$. Take a generic filter $G\subseteq\mathbb{P}_S$ such that $p_i\in G$ and work in $\mathbf{V}[G]$. Choose ordinals $\dot{\gamma}^G_i<\theta_i$

and $\dot\zeta_i^G<\mu$ such that the set

$$\begin{array}{rl}\dot X_i^G \stackrel{\rm def}{=} \Big\{\beta<\theta_i: & \dot\gamma_i^G\le\beta \ \text{ and } \ p^i_{\dot\gamma_i^G,\dot\zeta_i^G}, p^i_{\beta,\dot\zeta_i^G}\in G \ \text{ and } \ w^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}} = w^{p^i_{\beta,\dot\zeta_i^G}},\\ & \text{the conditions } p^i_{\dot\gamma_i^G,\dot\zeta_i^G}, p^i_{\beta,\dot\zeta_i^G} \text{ are isomorphic, and}\\ & \text{if } H: u^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}} \longrightarrow u^{p^i_{\beta,\dot\zeta_i^G}} \text{ is the isomorphism then}\\ & (\forall\ell<n_i)(H(e_i^{\dot\zeta_i^G}(\dot\gamma_i^G,\ell))=e_i^{\dot\zeta_i^G}(\beta,\ell)\ \&\ t_i^{\dot\zeta_i^G}(\dot\gamma_i^G,\ell)=t_i^{\dot\zeta_i^G}(\beta,\ell))\\ & \text{and if } j\le i,\ (j,\xi)\in\mathcal{X}_S \ \text{ then}\\ & (j,\xi)\in u^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}} \ \Leftrightarrow\ (j,\xi)\in u^{p^i_{\beta,\dot\zeta_i^G}}\Big\}\end{array}$$

is not $\emptyset$ modulo $\dot{\mathcal D}_i^G$. Let $\dot\delta_i^G=\operatorname{otp}(u^{p^i_{\dot\gamma_i^G,\dot\zeta_i^G}},\prec_S)$ and for $\alpha\in\dot X_i^G$ let $\langle s^{\alpha,i}_\varepsilon:\varepsilon<\dot\delta_i^G\rangle$ be the $\prec_S$–increasing enumeration of $u^{p^i_{\alpha,\dot\zeta_i^G}}$. Apply Lemma 2.1 to find a sequence $\langle s^{*,i}_\varepsilon:\varepsilon<\dot\delta_i^G\rangle\subseteq\mathcal{X}_S$ and a set $\dot v_i^G\subseteq\dot\delta_i^G$ such that

(i) $(\forall\varepsilon\in\dot\delta_i^G\setminus\dot v_i^G)(\chi_i^+\le\operatorname{cf}(\{s\in\mathcal{X}_S: s\prec_S s^{*,i}_\varepsilon\},\prec_S)\le\theta_i)$,
(ii) the set

$$\begin{array}{rl}\dot B_i^G \stackrel{\rm def}{=} \Big\{\beta\in\dot X_i^G: & \text{if } \varepsilon\in\dot v_i^G \text{ then } s^{\beta,i}_\varepsilon=s^{*,i}_\varepsilon, \quad\text{and}\\ & \text{if } \varepsilon\in\dot\delta_i^G\setminus\dot v_i^G \text{ then}\\ & \sup_{\prec_S}\{s^{*,i}_\zeta:\zeta<\dot\delta_i^G,\ s^{*,i}_\zeta\prec_S s^{*,i}_\varepsilon\}\prec_S s^{\beta,i}_\varepsilon\prec_S s^{*,i}_\varepsilon\Big\}\end{array}$$

is not $\emptyset$ modulo the filter $\dot{\mathcal D}_i^G$,
(iii) if $s'_\varepsilon\prec_S s^{*,i}_\varepsilon$ for $\varepsilon\in\dot\delta_i^G\setminus\dot v_i^G$ then

$$\{\beta\in\dot B_i^G:(\forall\varepsilon\in\dot\delta_i^G\setminus\dot v_i^G)(s'_\varepsilon\prec_S s^{\beta,i}_\varepsilon)\}\neq\emptyset \mod \dot{\mathcal D}_i^G.$$

We may assume that $\dot\gamma_i^G\in\dot B_i^G$.

Now, in $\mathbf{V}$, we choose a condition $q_i\in\mathbb{P}_S$, ordinals $\gamma_i,\zeta_i,\delta_i$, a set v_i and a sequence $\langle s^{*,i}_\varepsilon:\varepsilon<\delta_i\rangle\subseteq\mathcal{X}_S$ such that $q_i\ge p^i_{\gamma_i,\zeta_i}$, and q_i forces that these objects are as described above. If some condition stronger than q_i forces that $\alpha\in\dot B_i$, then we will use $\langle s^{\alpha,i}_\varepsilon:\varepsilon<\delta_i\rangle$ to denote the $\prec_S$–increasing enumeration of $u^{p^i_{\alpha,\zeta_i}}$.

Next, like in the proof of 3.6, we pick an unbounded set $Y\subseteq\operatorname{cf}(\lambda)$ and $n<\omega$, $\delta<\mu$, $v\subseteq\delta$ such that for $i,j\in Y$:

- $n_i=n$, $\delta_i=\delta$, $v_i=v$, and
- the conditions $p^i_{\gamma_i,\zeta_i},p^j_{\gamma_j,\zeta_j}$ are isomorphic, and the isomorphism maps $e_i^{\zeta_i}(\gamma_i,\cdot)$ and $t_i^{\zeta_i}(\gamma_i,\cdot)$ onto $e_j^{\zeta_j}(\gamma_j,\cdot),t_j^{\zeta_j}(\gamma_j,\cdot)$, respectively.

Now use Lemma 2.1 to find a sequence $\langle s_{*,\varepsilon}:\varepsilon<\delta\rangle\subseteq\mathcal{X}_S\cup\{(\operatorname{cf}(\lambda),0)\}$ and a set $v^*\subseteq\delta$ such that

(a) $(\forall\varepsilon\in\delta\setminus v^*)(\operatorname{cf}(\{s\in\mathcal{X}_S:s\prec_S s_{*,\varepsilon}\},\prec_S)=\operatorname{cf}(\lambda))$,
(b) the set

$$\begin{array}{rl}C \stackrel{\rm def}{=} \Big\{i\in Y: & \text{if } \varepsilon\in v^* \text{ then } s^{*,i}_\varepsilon=s_{*,\varepsilon}, \quad\text{and}\\ & \text{if } \varepsilon\in\delta\setminus v^* \text{ then}\\ & \sup_{\prec_S}\{s_{*,\zeta}:\zeta<\delta,\ s_{*,\zeta}\prec_S s_{*,\varepsilon}\}\prec_S s^{*,i}_\varepsilon\prec_S s_{*,\varepsilon}\Big\}\end{array}$$

is unbounded in $\operatorname{cf}(\lambda)$,

(c) if $s'_\varepsilon \prec_S s_{*,\varepsilon}$ for $\varepsilon\in\delta\setminus v^*$, then the set

$$\{i\in C: (\forall\varepsilon\in\delta\setminus v^*)(s'_\varepsilon\prec_S s^{*,i}_\varepsilon)\}$$

is unbounded in $\mathrm{cf}(\lambda)$.

Next choose a set $C^+\in[C]^{\mathrm{cf}(\lambda)}$ and ordinals $\alpha_i<\beta_i<\theta_i$ (for $i\in C^+$) such that for every $i\in C^+$:

(d) if $\varepsilon\in\delta\setminus v^*$ then for all $j\in C^+\cap i$ and $\zeta<\delta$ we have

$$\begin{array}{lcll} s^{*,j}_\zeta\prec_S s_{*,\varepsilon} & \Rightarrow & s^{*,j}_\zeta\prec_S s^{*,i}_\varepsilon, & \text{and}\\ s^{\alpha_j,j}_\zeta\prec_S s_{*,\varepsilon} & \Rightarrow & s^{\alpha_j,j}_\zeta\prec_S s^{*,i}_\varepsilon, & \text{and}\\ s^{\beta_j,j}_\zeta\prec_S s_{*,\varepsilon} & \Rightarrow & s^{\beta_j,j}_\zeta\prec_S s^{*,i}_\varepsilon, & \end{array}$$

(e) some condition stronger than q_i forces that $\alpha_i,\beta_i\in\dot{B}_i$,

(f) if $\varepsilon\in\delta\setminus v$ and $x\in\{\alpha_i,\beta_i\}$, then for all $j\in C^+\cap i$ and $\zeta<\delta$ we have

$$\begin{array}{lcl} s^{*,j}_\zeta\prec_S s^{*,i}_\varepsilon \Rightarrow s^{*,j}_\zeta\prec_S s^{x,i}_\varepsilon & \text{and} & s^{\alpha_j,j}_\zeta\prec_S s^{*,i}_\varepsilon \Rightarrow s^{\alpha_j,j}_\zeta\prec_S s^{x,i}_\varepsilon, \quad \text{and}\\ s^{\beta_j,j}_\zeta\prec_S s^{*,i}_\varepsilon \Rightarrow s^{\beta_j,j}_\zeta\prec_S s^{x,i}_\varepsilon & \text{and} & s^{\alpha_i,i}_\zeta\prec_S s^{*,i}_\varepsilon \Rightarrow s^{\alpha_i,i}_\zeta\prec_S s^{\beta_i,i}_\varepsilon, \end{array}$$

(g) if $\varepsilon\in v^*$, $s_{*,\varepsilon}=(j,\zeta)$ then $j<\min(C^+)$.

Then $\langle\langle s^{\alpha_i,i}_\varepsilon, s^{\beta_i,i}_\varepsilon:\varepsilon<\delta\rangle: i\in C^+\rangle$ forms a Δ–system of sequences with heart $\langle s_{*,\varepsilon}:\varepsilon\in v\cap v^*\rangle$; but note that $s^{\alpha_i,i}_\varepsilon=s^{\beta_i,i}_\varepsilon$ for $\varepsilon\in v$. Let $u^*=\{s_{*,\varepsilon}:\varepsilon\in v\cap v^*\}$ and $w^*=\{j<\mathrm{cf}(\lambda):(j,0)\in u^*\}$.

Claim 4.4.1. *For each $i_0\in C^+$:*

$$q_{i_0}\Vdash_{\mathbb{P}_S} \text{“}(\forall\alpha\in\dot{B}_{i_0})(\exists i^*\in C^+)(\dot{a}^{i^*}_{\alpha_{i^*}}\wedge(-\dot{a}^{i^*}_{\beta_{i^*}})\le\dot{a}^{i_0}_\alpha\ \&\ p_{i^*}\in\Gamma_{\mathbb{P}_S})\text{”}$$

(where $\dot{B}_{i_0}$ was defined in (ii)*).*

Proof of the claim. Let $i_0\in C^+$. We will show that for every condition $q\ge q_{i_0}$ and an ordinal $\alpha<\theta_{i_0}$ such that $q\Vdash\alpha\in\dot{B}_{i_0}$, there are $i^*\in C^+$ and a condition r stronger than both q and p_{i^*}, and such that $r\Vdash$“ $\dot{a}^{i^*}_{\alpha_{i^*}}\wedge(-\dot{a}^{i^*}_{\beta_{i^*}})\le\dot{a}^{i_0}_\alpha$ ”.

So suppose $q\ge q_i$, $q\Vdash\alpha\in\dot{B}_{i_0}$. We may assume that $p^{i_0}_{\alpha,\zeta_{i_0}}\le q$. Choose $i^*\in C^+\setminus(i_0+1)$ such that

$$u^q\cap u^{p^{i^*}_{\alpha_{i^*},\zeta_{i^*}}}=u^q\cap u^{p^{i^*}_{\beta_{i^*},\zeta_{i^*}}}=u^*\qquad\text{and}\qquad w^q\subseteq i^*,$$

Let $p^{i_0}_{\alpha,\zeta_{i_0}}=q^0$, $p^{i^*}_{\alpha_{i^*},\zeta_{i^*}}=q^1$, $p^{i^*}_{\beta_{i^*},\zeta_{i^*}}=q^2$, and

$$\tau_0=\bigwedge_{\ell<n}x^{t^{\zeta_{i_0}}_{i_0}(\alpha,\ell)}_{e^{\zeta_{i_0}}_{i_0}(\alpha,\ell)},\qquad \tau_1=\bigwedge_{\ell<n}x^{t^{\zeta_{i^*}}_{i^*}(\alpha_{i^*},\ell)}_{e^{\zeta_{i^*}}_{i^*}(\alpha_{i^*},\ell)}\quad\text{and}\quad \tau_2=\bigwedge_{\ell<n}x^{t^{\zeta_{i^*}}_{i^*}(\beta_{i^*},\ell)}_{e^{\zeta_{i^*}}_{i^*}(\beta_{i^*},\ell)}$$

(so $q^0\le q$ and $\tau_0\in\mathbb{B}_{q^0}\subseteq\mathbb{B}_q$, $\tau_1\in\mathbb{B}_{q^1}$, $\tau_2\in\mathbb{B}_{q^2}$). Note that the conditions q^0,q^1,q^2 are pairwise isomorphic and the isomorphism $H^{k,k'}$ from q^k to $q^{k'}$ carries τ_k to $\tau_{k'}$. Moreover, $H^{k,k'}$ is the identity on $u^{q^k}\cap u^{q^{k'}}$. Also note that $w^{q^1}=w^{p^{i^*}_{\gamma_{i^*},\zeta_{i^*}}}=w^{q^2}$ and, as $w^q\subseteq i^*$, our choices imply $H^{k,0}(i,\xi)\preceq_S(i,\xi)$ for $k=1,2$, $(i,\xi)\in u^{q^k}$.

Now we define a condition r stronger than q,q^1,q^2. We put $w^r=w^q\cup w^{q^1}$, $u^r=u^q\cup u^{q^1}\cup u^{q^2}$ and we define functions $f^r_{i,\xi}:u^r\longrightarrow 2$ as follows.

(1) If $(i,\xi)\in u^{q^1}\cap u^{q^2}$, $i\in w^q$ then we let $f^r_{i,\xi}=f^q_{H^{1,0}(i,\xi)}\cup f^{q^1}_{i,\xi}\cup f^{q^2}_{i,\xi}$.
[Note that by (d)+(ii) we have $(i,0)\preceq_S H^{1,0}(i,\xi)\preceq_S(i,\xi)$.]

(2) If $(i,\xi) \in u^{q^1} \cap u^{q^2}$, $i \notin w^q$ then we first choose ε^* such that, if possible, $(f^{q^0}_{H^{1,0}(i,\xi)})^{\varepsilon^*}(\tau_0) = 1$, and then we let $f^r_{i,\xi} = (f^q_{H^{1,0}(i,\xi)})^{\varepsilon^*} \cup f^{q^1}_{i,\xi} \cup f^{q^2}_{i,\xi}$.
[Note that $H^{1,0}(i,\xi) \prec_S (i,\xi)$, and thus if $H^{1,0}(i,\xi) = (j,\zeta)$ then $j < i$, $j \notin w^{q^1}$.]

(3) If $(i,\xi) \in u^{q^2} \setminus u^{q^1}$ (so $i > i^* \geq \sup(w^q)$) then we first choose ε^* such that, if possible, $(f^{q^0}_{H^{2,0}(i,\xi)})^{\varepsilon^*}(\tau_0) = 1$, and then we let $f^r_{i,\xi} = (f^q_{H^{2,0}(i,\xi)})^{\varepsilon^*} \cup f^{q^1}_{H^{2,1}(i,\xi)} \cup f^{q^2}_{i,\xi}$.
[Note that $H^{2,0}(i,\xi) \prec_S (i,0) \prec_S H^{2,1}(i,\xi) \prec_S (i,\xi)$; remember $w^{q^1} = w^{q^2}$. Also, if $H^{2,0}(i,\xi) = (j,\zeta)$, then $j \notin w^{q^1}$.]

(4) If $(i,\xi) \in u^{q^1} \setminus u^{q^2}$ then, like above, we choose ε^* such that if possible then $(f^{q^0}_{H^{1,0}(i,\xi)})^{\varepsilon^*}(\tau_0) = 1$, and next we put $f^r_{i,\xi} = (f^q_{H^{1,0}(i,\xi)})^{\varepsilon^*} \cup f^{q^1}_{i,\xi} \cup (f^{q^2}_{H^{1,2}(i,\xi)})^{\xi+1}$.

(5) If $(i,\xi) \in u^q \setminus u^{q^1}$ then we look at $f^q_{i,\xi} \restriction u^{q^0}$. If it is $\mathbf{0}_{u^{q^0}}$ then we let $f^r_{i,\xi} = f^q_{i,\xi} \cup \mathbf{0}_{u^{q^1}} \cup \mathbf{0}_{u^{q^2}}$. Otherwise, we consider the following three cases.

(α) Suppose $i \in w^{q^0}$. Then for some $\varepsilon \leq \zeta < \chi_i$, $\varepsilon \leq \xi + 1$ we have $f^q_{i,\xi} \restriction u^{q^0} = (f^{q^0}_{i,\zeta})^\varepsilon$ and we let:
– if $i \in w^{q^1}$ then $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(i,\zeta)})^\varepsilon \cup (f^{q^2}_{H^{0,2}(i,\zeta)})^\varepsilon$,
– if $i \notin w^{q^1}$ then $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(i,\zeta)})_i \cup (f^{q^2}_{H^{0,2}(i,\zeta)})_i$.

[Note that if $i \in w^{q^1}$ then $(i,\zeta) \preceq_S H^{0,1}(i,\zeta) = H^{0,2}(i,\zeta) \prec_S (i+1,0)$, and if $i \notin w^{q^1}$ then $(j,0) \preceq_S H^{0,1}(i,\zeta) \preceq_S H^{0,2}(i,\zeta) \prec_S (j+1,0)$ for some $j > i$.]

(β) Suppose $i \notin w^{q^0}$ (so $i \notin w^{q^1}$) and $f^q_{i,\xi} \restriction u^{q^0} = (f^{q^0}_{i',\zeta'})^{\varepsilon'}$, $(i',\zeta') \in u^{q^0}$, $\varepsilon' \leq \zeta' < \chi_{i'}$.
– If $i' \in w^{q^1}$ and $i' < i$, then put $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(i',\zeta')})^{\varepsilon'} \cup (f^{q^2}_{H^{0,2}(i',\zeta')})^{\varepsilon'}$.
– If $i' \in w^{q^1}$ and $i < i'$, then we put $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(i',\zeta')})_i \cup (f^{q^2}_{H^{0,2}(i',\zeta')})_i$.
– If $i' \notin w^{q^1}$, then let $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(i',\zeta')})_i \cup (f^{q^2}_{H^{0,2}(i',\zeta')})_i$.

(γ) Suppose $i \notin w^{q^0}$ and $f^q_{i,\xi} \restriction u^{q^0} = (f^{q^0}_{i',\zeta'})_{j'}$, $j' \leq \min\{i,i'\}$, $(i',\zeta') \in u^{q^0}$. Let $f^r_{i,\xi} = f^q_{i,\xi} \cup (f^{q^1}_{H^{0,1}(i',\zeta')})_{j'} \cup (f^{q^2}_{H^{0,2}(i',\zeta')})_{j'}$.

Verifying that the functions $f^r_{i,\xi}$ are well defined and that $r = \langle w^r, u^r, \langle f^r_{i,\xi} : (i,\xi) \in u^r \rangle\rangle \in \mathbb{P}_S$ is a condition stronger than q, q^1, q^2 is left to the reader. Let us argue that $\mathbb{B}_r \models \tau_1 \wedge (-\tau_2) \leq \tau_0$. If not then we have a function $f \in F^r$ such that $f(\tau_0) = f(\tau_2) = 0$ and $f(\tau_1) = 1$. Clearly f cannot be $\mathbf{0}_{u^r}$, so it is either $(f^r_{i,\xi})^\varepsilon$ or $(f^r_{i,\xi})_j$. Let us look at the definition of the functions $f^r_{i,\xi}$ and consider each case there separately.

Cases 1, $5\alpha,\beta,\gamma$: Plainly $f^r_{i,\xi}(\tau_1) = f^r_{i,\xi}(\tau_2)$ and also $(f^r_{i,\xi})_j(\tau_1) = (f^r_{i,\xi})_j(\tau_2)$ (remember $w^{q^1} = w^{q^2}$). As far as the operation $(\cdot)^\varepsilon$ is concerned, note that $(\{i\} \times$

$\chi_i) \cap u^{q^1} = (\{i\} \times \chi_i) \cap u^{q^2}$, so (in these cases) we easily get $(f^r_{i,\xi})^\varepsilon(\tau_1) = (f^r_{i,\xi})^\varepsilon(\tau_2)$, a contradiction.

CASE 2: Again, $f^r_{i,\xi}(\tau_1) = f^r_{i,\xi}(\tau_2)$ and $(f^r_{i,\xi})_j(\tau_1) = (f^r_{i,\xi})_j(\tau_2)$ (for each j). So suppose that $f = (f^r_{i,\xi})^\varepsilon$ for some ε, and look at the choice of ε^* in the current case. Since $1 = (f^r_{i,\xi})^\varepsilon(\tau_1) = (f^{q^1}_{i,\xi})^\varepsilon(\tau_1)$, we conclude that $1 = (f^{q^0}_{H^{1,0}(i,\xi)})^{\varepsilon^*}(\tau_0) = f^r_{i,\xi}(\tau_0) = (f^r_{i,\xi})^\varepsilon(\tau_0)$, a contradiction.

CASE 3: Note that $f^r_{i,\xi}(\tau_1) = f^r_{i,\xi}(\tau_2)$ (and also $(f^r_{i,\xi})_j(\tau_1) = (f^r_{i,\xi})_j(\tau_2)$). Now, if for some ε we have $(f^r_{i,\xi})^\varepsilon(\tau_1) = 1$, then look at the choice of ε^* – necessarily $(f^r_{i,\xi})^\varepsilon(\tau_0) = f^r_{i,\xi}(\tau_0) = 1$ (remember $(i,0) \prec_S H^{2,1}(i,\xi) \prec_S (i,\xi)$).

CASE 4: Like above: if for some ε we have $(f^r_{i,\xi})^\varepsilon(\tau_1) = 1$, then necessarily $f^r_{i,\xi}(\tau_0) = (f^r_{i,\xi})^\varepsilon(\tau_0) = 1$. Moreover, $(f^r_{i,\xi})_j(\tau_1) = (f^r_{i,\xi})_j(\tau_2)$ for all $j \leq i$.

In all cases we get a contradiction showing that $\mathbb{B}_r \models \tau_1 \wedge (-\tau_2) \leq \tau_0$, and hence $r \Vdash$ " $\dot{a}^{i^*}_{\alpha_{i^*}} \wedge (-\dot{a}^{i^*}_{\beta_{i^*}}) \leq \dot{a}^{i_0}_\alpha$ ", finishing the proof of the claim. □

Finally we note that 4.4.1 and clauses (β), (ε) give an immediate contradiction, showing the theorem. □

Conclusion 4.5. It is consistent that there is a Boolean algebra $\mathbb{B}$ of size λ such that there is a left–separated sequence of length λ in $\mathbb{B}$ (and thus $\mathrm{hd}^+_{(5)}(\mathbb{B}) = \lambda^+$), but there is no ideal $I \subseteq \mathbb{B}$ with $\pi(\mathbb{B}/I) = \lambda$ (so $\mathrm{hd}^+_{(7)}(\mathbb{B}) = \mathrm{hd}_{(7)}(\mathbb{B}) = \lambda$).

Problem 4.1. Can one construct a Boolean algebra $\mathbb{B}$ as in 4.5 for λ from any cardinal arithmetic assumptions?

5. More on the attainment problem

In this section we will assume the following:

Hypothesis 5.1. $S = (\mu, \lambda, \bar{\chi})$ is such that μ, λ are cardinals satisfying

$$\mu = \mu^{<\mu} < \mathrm{cf}(\lambda) < \lambda \leq 2^\mu,$$

and $\bar{\chi} = \langle \chi_i : i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing continuous sequence of cardinals such that

$$\chi_0 = 0, \quad \mathrm{cf}(\lambda) < \chi_1, \quad \mathrm{cf}(\chi_{i+1}) = \chi_{i+1}, \qquad \text{and} \qquad \sup_{i<\mathrm{cf}(\lambda)} \chi_i = \lambda.$$

For $\alpha < \lambda$ let $j(\alpha) < \mathrm{cf}(\lambda)$ be such that $\chi_{j(\alpha)} \leq \alpha < \chi_{j(\alpha)+1}$.

Definition 5.2. (1) A pair $(\bar{\eta}, A)$ is *a base for* $S = (\mu, \lambda, \bar{\chi})$ if
  (a) $A \subseteq \mu^{<\mu}$, $\bar{\eta} = \langle \eta_\alpha : \alpha < \lambda \rangle \subseteq \mu^\mu$,
  (b) if $\alpha < \beta < \lambda$, $j(\alpha) = j(\beta)$ then $\eta_\alpha \cap \eta_\beta \notin A$, and
  (c) if $Y \in [\lambda]^\lambda$ then there are distinct $\alpha, \beta \in Y$ such that $\eta_\alpha \cap \eta_\beta \in A$.

(2) $(\bar{\eta}, A)$ is called *a base*$^+$ *for* S if it satisfies demands (a), (b) (stated above) and
  (c$^+$) if $Y \in [\lambda]^\lambda$ and $\mathbf{t} \in \{0,1\}$, then there are $\alpha, \beta \in Y$ such that

$$\alpha < \beta, \qquad \eta_\alpha \cap \eta_\beta \in A, \qquad \text{and} \qquad \eta_\alpha <_{\mathrm{lex}} \eta_\beta \ \text{ iff } \ \mathbf{t} = 0.$$

For a topological space X, a (κ_0,κ_1)–Lusin set in X is a set $L\subseteq X$ such that $|L|=\kappa_0$ and for every meager subset Z of X the intersection $Z\cap L$ is of size less than κ_1. (See, e.g., Cichoń [1] for a discussion of sets of this type.) Below, the space μ^μ is equipped with the topology generated by sets of the form

$$[\rho]=\{\eta\in\mu^\mu:\rho\vartriangleleft\eta\}$$

for $\rho\in\mu^{<\mu}$.

Proposition 5.3. *Assume that for some $i^*<\mathrm{cf}(\lambda)$ there is a (λ,χ_{i^*})–Lusin set L in μ^μ. Then there is a base$^+$ for S.*

Proof. Choose sequences $\langle\nu_i:i<\mathrm{cf}(\lambda)\rangle\subseteq\mu^\mu$ and $\langle\rho_\alpha:\alpha<\lambda\rangle\subseteq L$, both with no repetitions. For $\alpha<\lambda$ let $\eta_\alpha\in\mu^\mu$ be defined by

$$\eta_\alpha(2\cdot\xi)=\nu_{j(\alpha)}(\xi)\quad\text{ and }\quad\eta_\alpha(2\cdot\xi+1)=\rho_\alpha(\xi)$$

(for $\xi<\mu$), and let $A=\bigcup\limits_{\xi<\mu}\mu^{2\cdot\xi}$. We claim that $(\langle\eta_\alpha:\alpha<\lambda\rangle,A)$ is a base$^+$ for S. The conditions 5.2(1)(a,b) should be clear. Let us verify 5.2(2)(c$^+$). So suppose that $Y\in[\lambda]^\lambda$ and $\mathbf{t}\in\{0,1\}$. Choose sequences $\langle Y_i:i<\mathrm{cf}(\lambda)\rangle$ and $\langle j_i:i<\mathrm{cf}(\lambda)\rangle$ such that

- $Y_i\subseteq Y$, $(\forall\alpha\in Y_i)(j(\alpha)=j_i)$, and $|Y_i|=\chi_{i^*}$ (so $\{\rho_\alpha:\alpha\in Y_i\}$ is not meager),
- the sequence $\langle j_i:i<\mathrm{cf}(\lambda)\rangle$ is strictly increasing.

For each $i<\mathrm{cf}(\lambda)$ pick $\sigma_i\in\mu^{<\mu}$ such that

$$(\forall\sigma\in\mu^{<\mu})(\sigma_i\vartriangleleft\sigma\quad\Rightarrow\quad[\sigma]\cap\{\rho_\alpha:\alpha\in Y_i\}\neq\emptyset).$$

We may pick $i_0<i_1<\mathrm{cf}(\lambda)$ such that

$$\sigma_{i_0}=\sigma_{i_1}=\sigma^*\qquad\text{and}\qquad\nu_{j_{i_0}}<_{\rm lex}\nu_{j_{i_1}}\ \text{ iff }\ \mathbf{t}=0.$$

(Remember that, under the assumptions of 5.1, $(\mu^\mu,<_{\rm lex})$ contains no monotonic sequences of length $\mathrm{cf}(\lambda)$.) Let $\xi=\mathrm{lh}(\nu_{j_{i_0}}\cap\nu_{j_{i_1}})$ and take $\sigma'\in\mu^{<\mu}$ such that $\sigma^*\trianglelefteq\sigma'$ and $\xi<\mathrm{lh}(\sigma')$. Now pick $\alpha_0\in Y_{i_0}$ and $\alpha_1\in Y_{i_1}$ such that $\sigma'\vartriangleleft\rho_{\alpha_0}\cap\rho_{\alpha_1}$ (there are such α_0,α_1 by the choice of $\sigma_{i_0}=\sigma_{i_1}=\sigma^*$). Note that then necessarily $\alpha_0<\alpha_1$, $\mathrm{lh}(\eta_{\alpha_0}\cap\eta_{\alpha_1})=2\cdot\xi$ (so $\eta_{\alpha_0}\cap\eta_{\alpha_1}\in A$) and $\eta_{\alpha_0}<_{\rm lex}\eta_{\alpha_1}$ iff $\mathbf{t}=0$. $\square$

Proposition 5.4. *Let $\mathbb{P}=(2^{<\mu},\vartriangleleft)$ be the μ–Cohen forcing notion. Then*

$$\Vdash_{\mathbb{P}}\text{ “ there is a base}^+\text{ for } S \text{ (and } S \text{ is still as in 5.1) ”.}$$

Proof. Pick sequences $\langle\nu_i:i<\mathrm{cf}(\lambda)\rangle$ and $\langle\rho_\alpha:\alpha<\lambda\rangle$ of pairwise distinct elements of μ^μ. Let $\dot{A}^*$ be a $\mathbb{P}$–name for the generic subset of μ (added by $\mathbb{P}$) and let $\dot{A}$ be a $\mathbb{P}$–name such that

$$\Vdash_{\mathbb{P}}\text{ “ }\dot{A}=\{\nu\in\mu^{<\mu}:\mathrm{lh}(\nu)\in\dot{A}^*\}\text{ ”.}$$

For $\alpha<\lambda$, let $\dot{\eta}_\alpha$ be a $\mathbb{P}$–name for a function in μ^μ such that

$$\Vdash_{\mathbb{P}}(\forall\xi\in\dot{A}^*)(\dot{\eta}_\alpha(\xi)=\nu_{j(\alpha)}(\mathrm{otp}(\dot{A}^*\cap\xi))\ \&\ (\forall\xi\in\mu\setminus\dot{A}^*)(\dot{\eta}_\alpha(\xi)=\rho_\alpha(\mathrm{otp}(\xi\setminus\dot{A}^*))).$$

We claim that

$$\Vdash_{\mathbb{P}}\text{ “ }(\langle\dot{\eta}_\alpha:\alpha<\lambda\rangle,A)\text{ is a base}^+\text{ for } S\text{ ”.}$$

Clauses 5.2(1)(a,b) should be clear, so let us prove 5.2(2)(c$^+$) only. Let $\langle\dot{\alpha}_\gamma:\gamma<\lambda\rangle$ be a $\mathbb{P}$–name for an increasing λ–sequence of elements of λ, and let $\mathbf{t}\in\{0,1\}$, $p\in\mathbb{P}$. For each $\gamma<\lambda$ pick a condition $p_\gamma\geq p$ and an ordinal α_γ such that $p_\gamma\Vdash\dot{\alpha}_\gamma=\alpha_\gamma$.

Necessarily, there are $X \in [\lambda]^{\lambda}$ and $p^* \in \mathbb{P}$ such that $p^* = p_\gamma$ for $\gamma \in X$. Then also $\alpha_{\gamma_0} < \alpha_{\gamma_1}$ for $\gamma_0 < \gamma_1$ from X. Shrinking X a little we may also demand that for some sequences $\sigma_j \in \mu^{\mathrm{lh}(p^*)+2}$ (for $j < \mathrm{cf}(\lambda)$) we have

$$\gamma \in X \ \&\ j(\alpha_\gamma) = j \quad \Rightarrow \quad \sigma_j \vartriangleleft \rho_{\alpha_\gamma}.$$

Now pick $\gamma_0 < \gamma_1$ from X such that letting $j_0 = j(\alpha_{\gamma_0})$ and $j_1 = j(\alpha_{\gamma_1})$ we have

$$j_0 < j_1 \quad \text{and} \quad \sigma_{j_0} = \sigma_{j_1} \quad \text{and} \quad \nu_{j_0} <_{\mathrm{lex}} \nu_{j_1} \ \text{ iff } \ \mathbf{t} = 0.$$

Let a condition $q \geq p^*$ be such that $\mathrm{lh}(q) = \mathrm{lh}(p^*) + \mathrm{lh}(\nu_{j_0} \cap \nu_{j_1}) + 2$ and $q(\xi) = 1$ for all $\xi \in \mathrm{lh}(q) \setminus \mathrm{lh}(p^*)$. It should be clear that $\alpha_{\gamma_0} < \alpha_{\gamma_1}$ and

$$q \Vdash \text{“ } \dot\eta_{\alpha_{\gamma_0}} \cap \dot\eta_{\alpha_{\gamma_1}} \in \dot A \quad \text{and} \quad \dot\eta_{\alpha_{\gamma_0}} <_{\mathrm{lex}} \dot\eta_{\alpha_{\gamma_1}} \ \text{ iff } \ \mathbf{t} = 0 \text{ ”}.$$

□

Definition 5.5. Let $\mathbf{b} = (\bar\eta, A)$ be a base for S, $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$. We define the Boolean algebra $\mathbb{B}^{\mathbf{b}}$ determined by $\mathbf{b}$. First, functions $f^{\mathbf{b}}_\alpha : \lambda \longrightarrow 2$ (for $\alpha < \lambda$) are such that

$$f^{\mathbf{b}}_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \ \text{ or } \ \alpha \neq \beta \ \&\ \eta_\alpha \cap \eta_\beta \in A \ \&\ \eta_\alpha <_{\mathrm{lex}} \eta_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we let $F^{\mathbf{b}} = \{f^{\mathbf{b}}_\alpha : \alpha < \lambda\}$ and $\mathbb{B}^{\mathbf{b}} = \mathbb{B}_{(\lambda, F^{\mathbf{b}})}$ (see 2.4).

Theorem 5.6. *If* $\mathbf{b}$ *is a base for* $S = (\mu, \lambda, \bar\chi)$*, then*

$$\mathrm{hL}(\mathbb{B}^{\mathbf{b}}) = \mathrm{hd}(\mathbb{B}^{\mathbf{b}}) = s^+(\mathbb{B}^{\mathbf{b}}) = \lambda.$$

If additionally $\mathbf{b}$ *is a base*$^+$ *for* S *then also*

$$\mathrm{hL}^+_{(7)}(\mathbb{B}^{\mathbf{b}}) = \mathrm{hd}^+_{(5)}(\mathbb{B}^{\mathbf{b}}) = \lambda.$$

Proof. Let $\mathbf{b} = (\bar\eta, A)$, $\bar\eta = \langle \eta_\alpha : \alpha < \lambda \rangle$. Clearly $|\mathbb{B}^{\mathbf{b}}| = \lambda$.

Claim 5.6.1. $\mathrm{hL}(\mathbb{B}^{\mathbf{b}}) = \mathrm{hd}(\mathbb{B}^{\mathbf{b}}) = s(\mathbb{B}^{\mathbf{b}}) = \lambda$.

Proof of the claim. By 5.2(1)(b), $f^{\mathbf{b}}_\alpha(\beta) = 0$ whenever $\alpha \neq \beta$ and $j(\alpha) = j(\beta)$. Therefore, by 2.7(1), the sequence $\langle x_\alpha : \chi_i \leq \alpha < \chi_{i+1} \rangle$ is ideal independent (for each $i < \mathrm{cf}(\lambda)$). □

The main part is to show that $s^+(\mathbb{B}^{\mathbf{b}}) = \lambda$ (and/or under the additional assumption, that $\mathrm{hL}^+_{(7)}(\mathbb{B}^{\mathbf{b}}) = \mathrm{hd}^+_{(5)}(\mathbb{B}^{\mathbf{b}}) = \lambda$), and for this we will need the following technical claim.

Claim 5.6.2. *Suppose that* $k^*, \ell^* < \omega$, $\alpha_k, \alpha_{\ell,k} < \lambda$ *(for* $k < k^*$, $\ell < \ell^*$*) and* $\sigma_0, \ldots, \sigma_{k^*-1} \in \mu^{<\mu}$ *are such that*

- (α) $\sigma_0, \ldots, \sigma_{k^*-1}$ *are pairwise incomparable,*
- (β) $\sigma_k \vartriangleleft \eta_{\alpha_k}$, $\sigma_k \vartriangleleft \eta_{\alpha_{\ell,k}}$ *(for* $\ell < \ell^*$, $k < k^*$*),*
- (γ) *for each* $k < k^*$ *one of the following occurs:*
  - (i) $\alpha_k = \alpha_{\ell,k}$ *for some* $\ell < \ell^*$*, or*
  - (ii) *there are* $\ell_1, \ell_2, \ell_3 < \ell^*$ *such that*
    - $\eta_{\alpha_k} \cap \eta_{\alpha_{\ell_1,k}} \vartriangleleft \eta_{\alpha_k} \cap \eta_{\alpha_{\ell_2,k}} \vartriangleleft \eta_{\alpha_k} \cap \eta_{\alpha_{\ell_3,k}}$*, and*
    - $\eta_{\alpha_k} \cap \eta_{\alpha_{\ell_1,k}}, \eta_{\alpha_k} \cap \eta_{\alpha_{\ell_2,k}} \in A$*, and*
    - $\eta_{\alpha_{\ell_1,k}} <_{\mathrm{lex}} \eta_{\alpha_k} <_{\mathrm{lex}} \eta_{\alpha_{\ell_2,k}}$.

Let $t(k)\in\{0,1\}$ *for* $k<k^*$. *Then*

$$\mathbb{B}^{\mathbf{b}}\models\bigwedge_{k<k^*}x^{t(k)}_{\alpha_k}\leq\bigvee_{\ell<\ell^*}\bigwedge_{k<k^*}x^{t(k)}_{\alpha_{\ell,k}}.$$

Proof of the claim. We are going to show that, under our assumptions, for each $f\in F^{\mathbf{b}}$ there is $\ell<\ell^*$ such that $(\forall k<k^*)(f(\alpha_k)=f(\alpha_{\ell,k}))$. So let us fix $\beta<\lambda$, and we consider $f^{\mathbf{b}}_\beta$. First note that

($\boxtimes_k$) if σ_k is not an initial segment of η_β, then $f^{\mathbf{b}}_\beta(\alpha_k)=f^{\mathbf{b}}_\beta(\alpha_{\ell,k})$ for all $\ell<\ell^*$.

[Why? Suppose $\sigma_k\ntriangleleft\eta_\beta$. Then clearly $\alpha_k\neq\beta\neq\alpha_{\ell,k}$ (for $\ell<\ell^*$) and

$$\eta_{\alpha_k}\cap\eta_\beta=\eta_{\alpha_{\ell,k}}\cap\eta_\beta\quad\text{ and }\quad\eta_{\alpha_k}<_{\rm lex}\eta_\beta\ \Leftrightarrow\ \eta_{\alpha_{\ell,k}}<_{\rm lex}\eta_\beta.$$

Now look at the definition of $f^{\mathbf{b}}_\beta$.]

If no σ_k is an initial segment of η_β, then (by ($\boxtimes_k$)) we conclude $f^{\mathbf{b}}_\beta(\alpha_k)=f^{\mathbf{b}}_\beta(\alpha_{\ell,k})$ for all $\ell<\ell^*$, $k<k^*$. So suppose that $\sigma_m\vartriangleleft\eta_\beta$, $m<k^*$. Then for all $k<k^*$, $k\neq m$, we have $\sigma_k\ntriangleleft\eta_\beta$ and thus $f^{\mathbf{b}}_\beta(\alpha_k)=f^{\mathbf{b}}_\beta(\alpha_{\ell,k})$ (for all $\ell<\ell^*$). Thus it is enough to find $\ell<\ell^*$ such that $f^{\mathbf{b}}_\beta(\alpha_m)=f^{\mathbf{b}}_\beta(\alpha_{\ell,m})$. If $\alpha_m=\alpha_{\ell,m}$ for some $\ell<\ell^*$, then this ℓ works. So suppose $\alpha_m\neq\alpha_{\ell,m}$ for all $\ell<\ell^*$. Then clause (γ)(ii) holds true for m, and let ℓ_1,ℓ_2,ℓ_3 be as there. If $\eta_{\alpha_m}\cap\eta_\beta\vartriangleleft\eta_{\alpha_m}\cap\eta_{\alpha_{\ell_3,m}}$, then easily $f^{\mathbf{b}}_\beta(\alpha_m)=f^{\mathbf{b}}_\beta(\alpha_{\ell_3,m})$. Otherwise $\eta_{\alpha_m}\cap\eta_{\alpha_{\ell_3,m}}\trianglelefteq\eta_{\alpha_m}\cap\eta_\beta$, and $f^{\mathbf{b}}_\beta(\alpha_{\ell_1,m})\neq f^{\mathbf{b}}_\beta(\alpha_{\ell_2,m})$, so either ℓ_1 or ℓ_2 works. □

Claim 5.6.3. $s^+(\mathbb{B}^{\mathbf{b}})=\lambda$.

Proof of the claim. Suppose that $\langle a_\xi:\xi<\lambda\rangle$ is an ideal independent sequence in $\mathbb{B}^{\mathbf{b}}$. We may assume that $a_\xi=\bigwedge\limits_{k<k_\xi}x^{t(\xi,k)}_{\alpha(\xi,k)}$ and $\alpha(\xi,k)\neq\alpha(\xi,k')$ whenever $k<k'<k_\xi$ (remember 2.7(2)). Also we may assume that $k_\xi=k^*$ for all $\xi<\lambda$ (as $\mathrm{cf}(\lambda)>\omega$).

Fix $i<\mathrm{cf}(\lambda)$ for a moment.

After possibly re-enumerating the sequences $\langle\alpha(\xi,k):k<k^*\rangle$, we may find a set $S_i\subseteq[\chi_i,\chi_{i+1})$, an ordinal $\varepsilon^*_i<\mu$, a sequence $\langle\nu^i_k:k<k^*\rangle$ of pairwise distinct elements of $\mu^{\varepsilon^*_i}$, and $t^i_k\in\{0,1\}$ and $j^i_k<\mathrm{cf}(\lambda)$ (for $k<k^*$) such that

(i) S_i is unbounded in χ_{i+1},
(ii) $t(\xi,k)=t^i_k$ and $j(\alpha(\xi,k))=j^i_k$ for all $\xi\in S_i$ and $k<k^*$,
(iii) $\nu^i_k\vartriangleleft\eta_{\alpha(\xi,k)}$ for $k<k^*$ and $\xi\in S_i$,
(iv) $\langle\langle\alpha(\xi,k):k<k^*\rangle:\xi\in S_i\rangle$ is a Δ–system of sequences with heart $\langle\alpha^i_k:k<k(i)\rangle$,
(v) the sequence $\langle\alpha(\xi,k):\xi\in S_i\rangle$ is strictly increasing for $k(i)\leq k<k^*$,
(vi) $j^i_k\geq i$ for $k(i)\leq k<k^*$ (it follows from (ii)+(iv)).

Next pick a set $S\subseteq[\mathrm{cf}(\lambda)]^{\mathrm{cf}(\lambda)}$ such that (possibly after some re-enumerations)

(vii) $k(i)=k^+$, $t^i_k=t_k$, $\varepsilon^*_i=\varepsilon^*$ and $\nu^i_k=\nu^*_k$ for $k<k^*$, $i\in S$,
(viii) $\langle\langle\alpha^i_k:k<k^+\rangle:i\in S\rangle$ is a Δ–system of sequences with heart $\langle\alpha_k:k<k^{**}\rangle$,
(ix) $\langle\langle j^i_k:k<k^*\rangle:i\in S\rangle$ is a Δ–system of sequences with heart $\langle j_k:k\in w\rangle$, $w\subseteq k^*$.

Note that then $k^{**}\subseteq w\subseteq k^+$. Also, possibly further shrinking S and the S_i's (for $i\in S$), we may demand that

(x) if $i_1 < i_2$, $i_1, i_2 \in S$, then $j^{i_1}_k < i_2$ (for $k < k^*$),

(xi) if $i_1, i_2 \in S$ are distinct, $\xi_1 \in S_{i_1}$ and $\xi_2 \in S_{i_2}$, then

$$\{\alpha(\xi_1,k) : k < k^*\} \cap \{\alpha(\xi_2,k) : k < k^*\} = \{\alpha_k : k < k^{**}\}.$$

Let $S^* = \bigcup\limits_{i\in S} S_i$. For $\varepsilon < \mu$ and $k^+ \leq k < k^*$ let

$$S^L_{\varepsilon,k} = \big\{\xi \in S^* : (\forall \zeta \in S^*)\big(\varepsilon > \mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \ \text{ or } \ \eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \notin A \ \text{ or } \ \eta_{\alpha(\xi,k)} \leq_{\mathrm{lex}} \eta_{\alpha(\zeta,k)}\big)\big\},$$
$$S^R_{\varepsilon,k} = \big\{\xi \in S^* : (\forall \zeta \in S^*)\big(\varepsilon > \mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \ \text{ or } \ \eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \notin A \ \text{ or } \ \eta_{\alpha(\zeta,k)} \leq_{\mathrm{lex}} \eta_{\alpha(\xi,k)}\big)\big\}.$$

We claim that both $|S^L_{\varepsilon,k}| < \lambda$ and $|S^R_{\varepsilon,k}| < \lambda$. Why? Assume, e.g., $|S^L_{\varepsilon,k}| = \lambda$. Note that, by (v)+(vi)+(x), $\alpha(\xi,k) < \alpha(\zeta,k)$ for $\xi < \zeta$ from S^*. Pick $\nu \in \mu^\varepsilon$ and a set $X \in [S^L_{\varepsilon,k}]^\lambda$ such that $(\forall \xi \in X)(\nu \lhd \eta_{\alpha(\xi,k)})$. By 5.2(1)(c), there are distinct $\xi, \zeta \in X$ such that $\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \in A$. Clearly $\mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \geq \varepsilon$ and we easily get a contradiction with $\xi, \zeta \in S^L_{\varepsilon,k}$. Similarly for $S^R_{\varepsilon,k}$.

For $k^+ \leq k < k^*$ let

$$S^\otimes_k = \Big\{\xi \in S^* : \begin{array}{l} \text{for all } \varepsilon < \mu \text{ there exists } \zeta \in S^* \text{ such that } \eta_{\alpha(\xi,k)} <_{\mathrm{lex}} \eta_{\alpha(\zeta,k)}, \\ \text{and } \varepsilon \leq \mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \ \text{ and } \ \eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \in A, \\ \text{and} \\ \text{for all } \varepsilon < \mu \text{ there exists } \zeta \in S^* \text{ such that } \eta_{\alpha(\zeta,k)} <_{\mathrm{lex}} \eta_{\alpha(\xi,k)}, \\ \text{and } \varepsilon \leq \mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \ \text{ and } \ \eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \in A \end{array}\Big\}.$$

Note that $S^* \setminus S^\otimes_k = \bigcup\limits_{\varepsilon<\mu} (S^L_{\varepsilon,k} \cup S^R_{\varepsilon,k})$, and hence $|S^* \setminus S^\otimes_k| < \lambda$ for each $k \in [k^+, k^*)$.

Fix distinct $\xi^*, \xi_* \in \bigcap\limits_{k=k^+}^{m-1} S^\otimes_k$ such that $j(\xi^*) = j(\xi_*)$. For each $k \in [k^+, k^*)$ pick $\xi^k_1, \xi^k_2, \xi^k_3 \in S^* \setminus \{\xi^*, \xi_*\}$ such that

$$\nu^*_k \lhd \eta_{\alpha(\xi^*,k)} \cap \eta_{\alpha(\xi^k_1,k)} \lhd \eta_{\alpha(\xi^*,k)} \cap \eta_{\alpha(\xi^k_2,k)} \lhd \eta_{\alpha(\xi^*,k)} \cap \eta_{\alpha(\xi^k_3,k)},$$
$$\eta_{\alpha(\xi^*,k)} \cap \eta_{\alpha(\xi^k_1,k)}, \eta_{\alpha(\xi^*,k)} \cap \eta_{\alpha(\xi^k_2,k)} \in A,$$
$$\eta_{\alpha(\xi^k_1,k)} <_{\mathrm{lex}} \eta_{\alpha(\xi^*,k)} <_{\mathrm{lex}} \eta_{\alpha(\xi^k_2,k)}.$$

Now look: letting $\alpha_k = \alpha(\xi^*,k)$, $\{\alpha_{\ell,k} : \ell < \ell^*\}$ be the suitable enumeration of $\big\{\alpha(\xi^{k'}_n, k) : k^+ \leq k' < k^* \ \&\ n \in \{1,2,3\}\big\} \cup \{\alpha(\xi_*, k)\}$, and $\sigma_k = \nu^*_k$, we get that the clauses (α)–(γ) of 5.6.2 are satisfied. Hence

$$a_{\xi^*} = \bigwedge_{k<k^*} x^{t_k}_{\alpha(\xi^*,k)} \leq \bigwedge_{k<k^*} x^{t_k}_{\alpha(\xi_*,k)} \vee \bigvee_{n=1}^{3} \bigvee_{k'=k^+}^{k^*-1} \bigwedge_{k<k^*} x^{t_k}_{\alpha(\xi^{k'}_n,k)} = a_{\xi_*} \vee \bigvee_{n=1}^{3} \bigvee_{k'=k^+}^{k^*-1} a_{\xi^{k'}_n}.$$

Since clearly $\xi^* \notin \{\xi_*\} \cup \{\xi^{k'}_n : k^+ \leq k' < k^*, n = 1,2,3\}$, we get a contradiction. □

Claim 5.6.4. *If* $\mathbf{b}$ *is a base*$^+$ *then also* $\mathrm{hL}^+_{(7)}(\mathbb{B}^{\mathbf{b}}) = \mathrm{hd}^+_{(5)}(\mathbb{B}^{\mathbf{b}}) = \lambda$.

Proof of the claim. It is similar to 5.6.3. Suppose that $\langle a_\xi : \xi < \lambda\rangle$ is a right separated sequence in $\mathbb{B}^{\mathbf{b}}$. Like before we may assume that $a_\xi = \bigwedge\limits_{k<k^*} x^{t(\xi,k)}_{\alpha(\xi,k)}$ and $\alpha(\xi,k) \neq \alpha(\xi,k')$ whenever $k < k' < k^*$. Next we apply the same "cleaning procedure" as in 5.6.3 getting $S, S_i, \varepsilon^*, \nu^*_k, t_k, j^i_k$ etc such that clauses (i)—(xi) are satisfied. We let $S^* = \bigcup\limits_{i\in S} S_i$ and for $\varepsilon < \mu$ and $k^+ \le k < k^*$ we define

$$S^+_{\varepsilon,k} = \big\{\xi \in S^* : (\forall \zeta \in S^* \cap \xi)\big(\varepsilon > \mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \text{ or } \eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \notin A \text{ or } \eta_{\alpha(\xi,k)} \le_{\mathrm{lex}} \eta_{\alpha(\zeta,k)}\big)\big\},$$
$$S^-_{\varepsilon,k} = \big\{\xi \in S^* : (\forall \zeta \in S^* \cap \xi)\big(\varepsilon > \mathrm{lh}(\eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)}) \text{ or } \eta_{\alpha(\xi,k)} \cap \eta_{\alpha(\zeta,k)} \notin A \text{ or } \eta_{\alpha(\zeta,k)} \le_{\mathrm{lex}} \eta_{\alpha(\xi,k)}\big)\big\}.$$

Then both $|S^+_{\varepsilon,k}| < \lambda$ and $|S^-_{\varepsilon,k}| < \lambda$. [It is like before: assume, e.g., $|S^+_{\varepsilon,k}| = \lambda$. Pick $\nu \in \mu^\varepsilon$ and a set $X \in [S^+_{\varepsilon,k}]^\lambda$ such that $(\forall \xi \in X)(\nu \vartriangleleft \eta_{\alpha(\xi,k)})$. Note that $\alpha(\zeta,k) < \alpha(\xi,k)$ for $\zeta < \xi$ from S^*. Use 5.2(2)(c$^+$) to find $\zeta < \xi$, both from X, such that $\eta_{\alpha(\zeta,k)} \cap \eta_{\alpha(\xi,k)} \in A$ and $\eta_{\alpha(\zeta,k)} <_{\mathrm{lex}} \eta_{\alpha(\xi,k)}$. A clear contradiction.]

Next for $k^+ \le k < k^*$ we let $S^\otimes_k = S^* \setminus \bigcup\limits_{\varepsilon<\mu} (S^+_{\varepsilon,k} \cup S^-_{\varepsilon,k})$. Choose $\xi_* < \xi^*$ from $\bigcap\limits_{k=k^+}^{m-1} S^\otimes_k$ such that $j(\xi^*) = j(\xi_*)$. And next for each $k \in [k^+, k^*)$ pick $\xi^k_1, \xi^k_2, \xi^k_3 \in S^* \cap \xi^*$ like those in the proof of 5.6.3. Finish in the same way. □

□

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