Stationary Sets and Infinitary Logic

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October 6, 2003

Abstract

Let K_λ^0 be the class of structures $\langle \lambda, <, A\rangle$, where $A \subseteq \lambda$ is disjoint from a club, and let K_λ^1 be the class of structures $\langle \lambda, <, A\rangle$, where $A \subseteq \lambda$ contains a club. We prove that if $\lambda = \lambda^{<\kappa}$ is regular, then no sentence of $L_{\lambda^+\kappa}$ separates K_λ^0 and K_λ^1. On the other hand, we prove that if $\lambda = \mu^+$, $\mu = \mu^{<\mu}$, and a forcing axiom holds (and $\aleph_1^L = \aleph_1$ if $\mu = \aleph_0$), then there is a sentence of $L_{\lambda\lambda}$ which separates K_λ^0 and K_λ^1.

One of the fundamental properties of $L_{\omega_1\omega}$ is that although every countable ordinal itself is definable in $L_{\omega_1\omega}$, the class of all countable well-ordered structures is not. In particular, the classes

$$\begin{aligned} K^0 &= \{\langle \omega, R\rangle : R \text{ well-orders } \omega\} \\ K^1 &= \{\langle \lambda, R\rangle : \langle \omega, R\rangle \text{ contains a copy of the rationals}\} \end{aligned}$$

cannot be separated by any $L_{\omega_1\omega}$-sentence. In this paper we consider infinite quantifier languages $L_{\kappa\lambda}$, $\lambda > \omega$. Here well-foundedness is readily definable, but we may instead consider the class

$$T_\lambda = \{\langle \lambda, R\rangle : \langle \lambda, R\rangle \text{ is a tree with no branches of length } \lambda\}.$$

*Research partially supported by ? Publication number ?.
†Research partially supported by grant 1011049 of the Academy of Finland

If $\lambda = \lambda^{<\lambda}$, then a result of Hyttinen [1] implies that T_λ cannot be defined in $L_{\lambda^+\lambda}$.

The main topic of this paper is the question whether the classes

$$K^0_\lambda = \{\langle \lambda, <, A\rangle : A \text{ is disjoint from a club of } \lambda\}$$
$$K^1_\lambda = \{\langle \lambda, <, A\rangle : A \text{ contains a club of } \lambda\}$$

can be separated in $L_{\lambda^+\lambda}$ and related languages. Note that a set $A \subseteq \lambda$ contains a club if and only if the tree $T(A)$ of continuously ascending sequences of elements of A has a branch of length λ. We show (Theorem 1) that the classes K^0_λ and K^1_λ cannot be separated by a sentence of $L_{\lambda^+\kappa}$, if $\lambda = \lambda^{<\kappa}$ is regular. The proof of this result uses forcing in a way which seems to be new in the model theory of infinitary languages. It follows from this result that the class

$$S_\lambda = \{\langle \lambda, <, A\rangle : A \text{ is stationary on } \lambda\},$$

that separates K^0_λ and K^1_λ, is undefinable in $L_{\lambda^+\kappa}$, if $\lambda = \lambda^{<\kappa}$ is regular. We complement this result by showing (Theorem 10) that if either $\lambda = \mu^+$ and $\mu = \mu^{<\mu} > \omega$ or $\lambda = \omega_1$ and additionally a forcing axiom holds, then there is a sentence of $L_{\lambda\lambda}$ which defines S_λ and thereby separates K^0_λ and K^1_λ.

Hyttinen [1] actually proves more than undefinability of T_λ in $L_{\lambda^+\lambda}$. He shows that T_λ is undefinable - assuming $\lambda = \lambda^{<\lambda}$ - in $PC(L_{\lambda^+\lambda})$. We show (Theorems 5 and 6) that the related statement that S_{ω_1} is definable in $PC(L_{\omega_2\omega_1})$ is independent of ZFC+CH.

1 The case $\lambda = \lambda^{<\mu}$.

Theorem 1 *If $\lambda = \lambda^{<\kappa}$ is regular, then the classes K^0_λ and K^1_λ cannot be separated by a sentence of $L_{\lambda^+\kappa}$.*

Proof. Assume $\lambda = \lambda^{<\kappa}$ is regular and $\psi \in L_{\lambda^+\kappa}$. Let $\mathcal{P}$ be the forcing notion for adding a Cohen subset to λ. Thus $p \in \mathcal{P}$ if p is a mapping $p : \alpha_p \to 2$ for some $\alpha_p < \lambda$. A condition p extends another condition q, in symbols $p \geq q$, if $\alpha_p \geq \alpha_q$ and $p|a_q = q$. Let G be $\mathcal{P}$-generic and $g = \bigcup G$. Thus

$$V[G] \models g^{-1}(1) \text{ is bi-stationary on } \lambda.$$

Now either ψ or $\neg\psi$ is true in $\langle \lambda, <, g^{-1}(1)\rangle$ in $V[G]$. We may assume, by symmetry, that it is ψ. Let $p \in G$ such that

$$p \Vdash_{\mathcal{P}} \langle \lambda, <, \tilde{g}^{-1}(1)\rangle \models \psi,$$

where $\tilde{g}$ is the canonical name for g. It is easy to use $\lambda = \lambda^{<\kappa}$ and regularity of λ to construct an elementary chain $\langle M_\xi : \xi < \lambda \rangle$ such that

(i) $M_\xi \prec \langle H(beth_7(\lambda)), \in, <^* \rangle$, where $<^*$ is a well-ordering of $H(beth_7(\lambda))$.

(ii) $\lambda + 1 \cup \{p\} \cup \{\mathcal{P}\} \cup TC(\{\psi\}) \subseteq M_0$.

(iii) $\langle M_\eta : \eta < \xi \rangle \in M_{\xi+1}$.

(iv) $M_\nu = \bigcup_{\xi<\nu} M_\xi$ for limit ν.

(v) $(M_\xi)^{<\kappa} \subseteq M_{\xi+1}$.

(vi) $|M_\xi| = \lambda$.

Let $M = \bigcup_{\xi<\lambda} M_\xi$. Note, that $M^{<\kappa} \subseteq M$ because λ is regular. We shall construct two $\mathcal{P}$-generic sets, G^0 and G^1, over M. For this end, list open dense $D \subseteq \mathcal{P}$ with $D \in M$ as $\langle D_\xi : \xi < \lambda \rangle$. Define $G^l = \{p^l_\xi : \xi < \lambda\}$ so that $p^l_0 = p$, $p^l_{\xi+1} \geq p^l_\xi$ with $p^l_{\xi+1} \in D_\xi \cap M$, $p^l_{\xi+1}(\alpha_{p^l_\xi}) = l$, and $p^l_\nu = \bigcup_{\xi<\nu} p^l_\xi$ for limit ν. Clearly, G^l is $\mathcal{P}$-generic over M and

$$M[G^l] \models [\langle \lambda, <, (g^l)^{-1}(1) \rangle \models \psi],$$

where $g^l = \bigcup G^l$. Note also that $M[G^l]^{<\kappa} \subseteq M[G^l]$, because $M^{<\kappa} \subseteq M$ and $\mathcal{P}$ is $< \kappa$-closed.

Lemma 2 *If $\varphi(\vec{x}) \in L_{\lambda^+\kappa}$ such that $TC(\{\varphi(\vec{x})\}) \subseteq M$, $X \in M$, and $\vec{a} \in \lambda^{<\kappa}$, then*

$$\langle \lambda, <, X \rangle \models \varphi(\vec{a}) \iff M[G^l] \models [\langle \lambda, <, X \rangle \models \varphi(\vec{a})].$$

Proof. Easy induction on $\varphi(\vec{x})$. □

By the lemma, $\langle \lambda, <, (g^l)^{-1}(1) \rangle \models \psi$. By construction, $\langle \lambda, <, (g^l)^{-1}(1) \rangle \in K^l_\lambda$. Now we can finish the proof. Suppose $K^0_\lambda \subseteq Mod(\psi)$ and $K^1_\lambda \cap Mod(\psi) = \emptyset$. This contradicts the fact that $\langle \lambda, <, (g^1)^{-1}(1) \rangle \in K^1_\lambda \cap Mod(\psi)$. Suppose $K^1_\lambda \subseteq Mod(\psi)$ and $K^0_\lambda \cap Mod(\psi) = \emptyset$. This contradicts $\langle \lambda, <, (g^0)^{-1}(1) \rangle \in K^0_\lambda \cap Mod(\psi)$. □

Corollary 3 *If $\lambda = \lambda^{<\kappa}$ is regular, then there is no $\varphi \in L_{\lambda^+\kappa}$ such that for all $A \subseteq \lambda$: $\langle \lambda, <, A \rangle \models \varphi \iff A$ is stationary.*

Theorem 1 gives a new proof of the result, referred to above, that if $\lambda = \lambda^{<\lambda}$, then T_λ is not definable in $L_{\lambda\lambda}$. Our proof does not give the stronger result that T_λ is not definable in $PC(L_{\lambda\lambda})$, and there is a good reason: S_{ω_1} may be $PC(L_{\omega_1\omega_1})$-definable, even if $2^{\aleph_0} = \aleph_1$. This is the topic of the next section.

2 An application of Canary trees.

A tree $\mathcal{C}$ is a *Canary tree* if $\mathcal{C}$ has cardinality $\leq 2^\omega$, $\mathcal{C}$ has no uncountable branches, but if a stationary subset of ω_1 is killed by forcing which does not add new reals, then this forcing adds an uncountable branch to $\mathcal{C}$. By [4], this is equivalent to the statement that

($\star$) For every co-stationary $A \subseteq \omega_1$ there is a mapping f with $\mathrm{Rng}(f) \subseteq \mathcal{C}$ such that for all increasing closed sequences s, s' of elements of A, if s is an initial segment of s', then $f(s) <_{\mathcal{C}} f(s')$.

Theorem 4 *(i) Con(ZF)→ Con(ZFC + CH + there is a Canary tree) [3]*

(ii) V=L → there are no Canary trees [6].

Thus the non-existence of Canary trees is consistent with CH, relative to the consistency of ZF. This result was first proved in [3] by the method of forcing.

Theorem 5 *Assuming CH and the existence of a Canary tree, there is a* $\Phi \in PC(L_{\omega_2\omega_1})$ *such that for all* $A \subseteq \omega_1$*:* $\langle \omega_1, <, A\rangle \models \Phi \iff A$ *is stationary.*

Proof. Let $\mathcal{C}$ be a Canary tree. It is easy to construct a $PC(L_{\omega_2\omega_1})$-sentence Ψ such that the following conditions are equivalent for all $A \subseteq \omega_1$:

(i) $\langle \omega_1, <, A\rangle \models \Psi$

(ii) There is a mapping f with $\mathrm{Rng}(f) \subseteq \mathcal{C}$ such that for all increasing closed sequences s, s' of elements of A, if s is an initial segment of s', then $f(s) <_{\mathcal{C}} f(s')$.

We allow predicate symbols with ω-sequences of variables in the $PC(L_{\omega_2\omega_1})$-sentence Ψ. Now the claim follows from the property ($\star$) of Canary trees. □

Theorem 6 *Con(ZF) implies Con(ZFC + CH + there is no* $\Phi \in PC(L_{\omega_2\omega_1})$ *such that for all* $A \subseteq \omega_1$*:* $\langle \omega_1, <, A\rangle \models \Phi \iff A$ *is stationary).*

Proof. We start with a model of GCH and add $\aleph_2$ Cohen subsets to ω_1. In the extension GCH continues to hold. Suppose there is in the extension a $\Phi \in PC(L_{\omega_2\omega_1})$ such that for all $A \subseteq \omega_1$:

$$\langle \omega_1, <, A\rangle \models \Phi \iff A \text{ is stationary.}$$

Since the forcing to add $\aleph_2$ Cohen subsets of ω_1 satisfies the $\aleph_2$-c.c., Φ belongs to the extension of the universe by $\aleph_1$ of the subsets. By first adding all but one of the subsets we can work in $V[A]$ where A is a Cohen subset of ω_1 and Φ is in V. Note that A is a bi-stationary subset of ω_1. Let $\mathcal{P}$ be in V the forcing for adding a Cohen generic subset of ω_1 and let $\tilde{A}$ be the $\mathcal{P}$-name for A. Let p force $\langle \omega_1, <, \tilde{A}\rangle \models \Phi$. By arguing as in the proof of Theorem 1, we can construct in V a model M of cardinality $\aleph_1$ containing $\mathcal{P}$ such that $M^\omega \subseteq M$,

$$M \models [p \Vdash \langle \omega_1, <, \tilde{A}\rangle \models \Phi],$$

and, furthermore, we can extend p to a $\mathcal{P}$-generic set $H \subseteq \omega_1$ over M such that H is non-stationary. Thus $M[H]$ satisfies

$$\langle \omega_1, <, H\rangle \models \Phi. \tag{1}$$

Now (1) is true in V, because $M[H]^\omega \subseteq M[H]$. Since $\mathcal{P}$ is countably closed, we have (1) in $V[A]$, whence H is stationary in $V[A]$, contrary to the fact that H is non-stationary in V. □

3 An application to the topological space ${}^{\omega_1}\omega_1$.

Let $\mathcal{N}_1$ denote the generalized Baire space consisting of all functions $f : \omega_1 \to \omega_1$, with the sets

$$N_s = \{f \in \mathcal{N}_1 : f|\mathrm{Dom}(s) = s\},$$

where $s \in {}^{<\omega_1}\omega_1$, as basic open sets. We call open sets Σ^0_1 and closed sets Π^0_1. A set of the form $\bigcup_{\xi<\omega_1} A_\xi$, where each A_ξ is in $\bigcup_{\beta<\alpha} \Pi^0_\beta$, is called Σ^0_α. Respectively, a set of the form $\bigcap_{\xi<\omega_1} A_\xi$, where each A_ξ is in $\bigcup_{\beta<\alpha} \Sigma^0_\beta$, is called Π^0_α. In $\mathcal{N}_1$ it is natural to define Borel sets as follows: A subset of $\mathcal{N}_1$ is *Borel* if it is Σ^0_α or Π^0_α for some $\alpha < \omega_2$. A set $A \subseteq \mathcal{N}_1$ is Π^1_1 if there is an open set $B \subseteq \mathcal{N}_1 \times \mathcal{N}_1$ such that $\forall f(f \in A \iff \forall g((f,g) \in B)$. A set is Σ^1_1 if its complement is Π^1_1.

Let CUB be the set of characteristic functions of closed unbounded subsets of ω_1, and NON-STAT the set of characteristic functions of non-stationary subsets of ω_1. Clearly, CUB and NON-STAT are disjoint Σ^1_1. It was proved in [4] that, assuming CH, CUB and NON-STAT are Π^1_1 if and only if there is a Canary tree. Another result on [4] says that the sets CUB and NON-STAT cannot be separated by any Π^0_3 or Σ^0_3 set.

Theorem 7 *Assuming CH, the sets CUB and NON-STAT cannot be separated by a Borel set.*

Proof. Let $\{s_\alpha : \alpha < \omega_1\}$ enumerate all $s \in {}^{<\omega_1}\omega_1$. Let $C = \bigcup_{\alpha<\omega_2} C_\alpha$, where

$$\begin{aligned} C_0 &= \{0,1\} \times \mathcal{N}_1 \\ C_\delta &= \{2,3\} \times {}^{\omega_1}(\bigcup_{\alpha<\delta} C_\alpha). \end{aligned}$$

Now we define a Borel set B_c for each $c \in C$ as follows:

$$B_{(0,f)} = \bigcup_{\alpha<\omega_1} N_{s_{f(\alpha)}} \quad , \quad B_{(1,f)} = \bigcap_{\alpha<\omega_1} \mathcal{N}_1 \setminus N_{s_{f(\alpha)}},$$
$$B_{(2,f)} = \bigcup_{\alpha<\omega_1} B_{f(\alpha)} \quad , \quad B_{(3,f)} = \bigcap_{\alpha<\omega_1} B_{f(\alpha)}.$$

Clearly, every Borel subset X of $\mathcal{N}_1$ is of the form B_c for some $c \in C$. Then we call c a *Borel code* of X.

Assume A is a Borel set which separates CUB and NON-STAT. Let c be a Borel code of A. Let $\mathcal{P}$ be the forcing notion for adding a Cohen subset to ω_1. Let G be $\mathcal{P}$-generic and $g = \bigcup G$. Thus

$$V[G] \models g^{-1}(1) \text{ is bi-stationary.}$$

Now either $g^{-1}(1) \in B_c$ or $g^{-1}(1) \in B_c$ in $V[G]$. We may assume, by symmetry, that $g^{-1}(1) \in B_c$. Let $p \in G$ such that

$$p \Vdash_{\mathcal{P}} \tilde{g}^{-1}(1) \in B_c,$$

where $\tilde{g}$ is the canonical name for g. Let $M \prec \langle H(beth_7(\omega_1)), \in, <^* \rangle$, where $<^*$ is a well-ordering of $H(beth_7(\lambda))$, such that $\omega_1 + 1 \cup \{p\} \cup \{\mathcal{P}\} \cup TC(\{c\}) \subseteq M$, $M^{<\omega_1} \subseteq M$ and $|M| = \omega_1$.

We shall construct two $\mathcal{P}$-generic sets, G^0 and G^1, over M. For this end, list open dense $D \subseteq \mathcal{P}$ with $D \in M$ as $\langle D_\xi : \xi < \omega_1 \rangle$. Define $G^l = \{p^l_\xi : \xi < \omega_1\}$ so that $p^l_0 = p$, $p^l_{\xi+1} \geq p^l_\xi$ with $p^l_{\xi+1} \in D_\xi \cap M$, $p^l_{\xi+1}(\alpha_{p^l_\xi}) = l$, and $p^l_\nu = \bigcup_{\xi<\nu} p^l_\xi$ for limit ν. Clearly, G^l is $\mathcal{P}$-generic over M and

$$M[G^l] \models (g^l)^{-1}(1) \in B_c,$$

where $g^l = \bigcup G^l$. Note also that $M[G^l]^{<\omega} \subseteq M[G^l]$, because $M^{<\omega} \subseteq M$ and $\mathcal{P}$ is ω-closed.

Lemma 8 *If $c \in C$ such that $TC(\{c\}) \subseteq M$, and $f \in M$, then*

$$f \in B_c \iff M[G^l] \models [f \in B_c].$$

Proof. Easy induction on c. □

By the lemma, $(g^l)^{-1}(1) \in B_c$. By construction, $(g^0)^{-1}(1) \in$ NON-STAT and $(g^1)^{-1}(1) \in$ CUB. Now we can finish the proof. Suppose CUB $\subseteq A$ and NON-STAT$\cap A = \emptyset$. This contradicts the fact that $(g^0)^{-1}(1) \in$ NON-STAT$\cap A$. Suppose NON-STAT $\subseteq A$ and CUB $\cap A = \emptyset$. This contradicts the fact that $(g^1)^{-1}(1) \in$ CUB $\cap A$. □

4 The case $\lambda^\mu > \lambda$.

Let μ be a cardinal. Sets $A, B \subseteq \mu$ are called *almost disjoint (on μ)* if $\sup(A \cap B) < \mu$. An *almost disjoint λ-sequence* of subsets of μ is a sequence $\mathcal{B} = \langle B_\alpha : \alpha < \lambda \rangle$ such that for all $\alpha \neq \beta$, $|B_\alpha| = \mu$ and the sets B_α and B_β are almost disjoint. The sequence $\mathcal{B}$ is said to be *definable on L_λ* if there is a sequence $\langle \delta_\alpha : \alpha < \lambda \rangle$ such that $\limsup_{\alpha<\lambda} \delta_\alpha = \lambda$ and the predicates $x \in B_y \wedge y < \delta_\alpha$ and $x = \delta_y \wedge x < \alpha \wedge y < \alpha$ are definable on every structure $\langle L_\alpha, \in \rangle$, where $\alpha < \lambda$, that is, there is a first order formula $\varphi_0(x, y)$ of the language of set theory such that for $x, y < \alpha < \lambda$:

$$x \in B_y \wedge y < \delta_\alpha \iff \langle L_\alpha, \in \rangle \models \varphi_0(x, y).$$

Lemma 9 *If $\aleph_1^L = \aleph_1$, then there is an almost disjoint ω_1-sequence of subsets of ω_1, which is definable on L_{ω_1}.*

Proof. There is a set $\{B_i : i < \omega_1^L\}$ of almost disjoint subsets of ω in L. Since $\aleph_1^L = \aleph_1$, this set is really of cardinality $\aleph_1$. Let $\theta(x, y)$ be a Σ_1-formula of set theory such that for all α and $x, y \in L_\alpha$, $x <_L y \iff L_\alpha \models \theta(x, y)$, where $<_L$ is the canonical well-ordering of L. The claim follows easily. □

Theorem 10 *Suppose*

(i) $\lambda = \mu^+$.

(ii) There is an almost disjoint λ*-sequence* $\mathcal{B} = \langle B_\alpha : \alpha < \lambda\rangle$ *of subsets of* μ *which is definable on* L_λ.

(iii) For all club subsets C *of* λ *there is a subset* X *of* μ *such that for all* $\alpha < \lambda$ *we have*

$$\alpha \in C \iff \sup(B_\alpha \setminus C) < \mu.$$

Then there is a sentence $\varphi \in L_{\lambda\lambda}$ *so that for all* $A \subseteq \lambda$*:*

$$\langle\lambda, <, A\rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. Suppose φ_0 defines the almost disjoint sequence, as above. We define a sequence of formulas of $L_{\lambda\lambda}$. The variable vectors $\vec{x}$ in these formulas are always sequences of the form $\langle x_i : i < \mu\rangle$. Let Φ be the conjunction of a large but finite number of axioms of $ZFC + V = L$. If $\psi(\vec{z})$ is a formula of set theory, let $\psi'(\vec{z}, \vec{x}, \vec{u}, \vec{v})$ be the result of replacing every quantifier $\forall y \ldots$ in Φ by $\forall y(\bigvee_{i<\mu} y = x_i \rightarrow \ldots)$, every quantifier $\exists y \ldots$ in Φ by $\exists y(\bigvee_{i<\mu} y = x_i \wedge \ldots)$, and $y \in z$ everywhere in Φ by $\bigvee_{i<\mu}(y = u_i \wedge z = v_i)$. The following formulas pick μ from $\langle\lambda, <\rangle$:

$$\begin{array}{lcl}
\varphi_{\approx\mu}(y) & \iff & \exists\vec{x}((\bigwedge_{i<j<\mu} x_i < x_j) \wedge \forall z(z < y \leftrightarrow \bigvee_{i<\mu} z = x_i)),\\
\varphi_{\in\mu}(y) & \iff & \forall u(\varphi_{\approx\mu}(u) \rightarrow y < u),\\
\psi_{\in\mu}(\vec{y}) & \iff & \bigwedge_{i<\mu} \varphi_{\in\mu}(y_i)\varphi_{B,1}(x, \vec{u}, \vec{v}, z, y)
\end{array}$$

The following formulas are needed to refer to well-founded models of set theory:

$$\begin{array}{lcl}
\varphi_{uni}(\vec{x}, z) & \iff & \bigvee_{i<\mu} z = x_i\\
\varphi_{eps}(\vec{x}, \vec{u}, \vec{v}, z, y) & \iff & \varphi_{uni}(\vec{x}, z) \wedge \varphi_{uni}(\vec{x}, y) \wedge \bigvee_{i<\mu}(z = u_i \wedge y = v_i)\\
\varphi_{wf}(\vec{x}, \vec{u}, \vec{v}) & \iff & \Phi'(\vec{x}, \vec{u}, \vec{v}) \wedge \forall\vec{y}((\bigwedge_{i<\mu} \varphi_{uni}(\vec{x}, y_i)) \rightarrow\\
 & & \bigvee_{i<\mu} \neg\varphi_{eps}(\vec{x}, \vec{u}, \vec{v}, y_{i+1}, y_i))\\
\varphi_{cor}(\vec{x}, \vec{u}, \vec{v}, z) & \iff & \forall s(s < z \leftrightarrow \bigvee_{i<\mu}(s = u_i \wedge z = v_i)
\end{array}$$

Let

$$\begin{aligned}\varphi_B(z,y) \iff & \exists\vec{x}\exists\vec{u}\exists\vec{v}(\varphi_{wf}(\vec{x},\vec{u},\vec{v})\wedge\varphi_{cor}(\vec{x},\vec{u},\vec{v},z)\wedge \\ & \varphi_{cor}(\vec{x},\vec{u},\vec{v},y)\wedge\phi'_0(z,y,\vec{x},\vec{u},\vec{v})).\end{aligned}$$

The point is that if $\alpha \in \mu$ and $\beta \in \lambda$, then $\alpha \in B_\beta$ if and only if $\langle\lambda,<\rangle \models \varphi_B(\alpha,\beta)$. The following formula says that the element y of μ is in the subset of λ coded by $\vec{x}$:

$$\varphi_\varepsilon(y,\vec{x}) \iff \exists u(\varphi_{\in\mu}(u)\wedge\forall z((\varphi_B(z,y)\wedge\textstyle\bigwedge_{i<\mu} z\neq x_i)\to z<u),$$

Finally, if:

$$\begin{aligned}\varphi_{ub}(\vec{x}) &\iff \forall y\exists z(y<z\wedge\varphi_\varepsilon(z,\vec{x})),\\ \varphi_{cl}(\vec{x}) &\iff \forall y(\forall z(z<y\to\exists u(z<u\wedge u<y\wedge\varphi_\varepsilon(u,\vec{x})))\to \\ &\qquad \varphi_\varepsilon(y,\vec{x}))\\ \varphi_{cub}(\vec{x}) &\iff \varphi_{ub}(\vec{x})\wedge\varphi_{cl}(\vec{x})\\ \varphi_{stat} &\iff \forall\vec{x}((\psi_{\in\mu}(\vec{x})\wedge\varphi_{cub}(\vec{x}))\to\exists y(A(y)\wedge\varphi_\varepsilon(y,\vec{x}))),\end{aligned}$$

then $\langle\lambda,<,A\rangle \models \varphi_{stat}$ if and only if A is stationary. □

Corollary 11 *If $2^{\aleph_0} > \aleph_1$, $\aleph_1^L = \aleph_1$ and MA, then there is a $\varphi \in L_{\omega_1\omega_1}$ such that for all $A \subseteq \omega_1$:*

$$\langle\omega_1,<,A\rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. We choose $\lambda = \omega_1$ and $\mu = \omega_0$ in Theorem 10. Condition (ii) holds by Lemma 9. Condition (iii) is a consequence of MA + ¬CH by [2]. □

Note. The proof of Corollary 11 shows that we actually get the following stronger result: If $2^{\aleph_0} > \aleph_1$, $\aleph_1^L = \aleph_1$ and MA, then the full second order extension $L^{II}_{\omega_1\omega_1}$ of $L_{\omega_1\omega_1}$ is reducible to $L_{\omega_1\omega_1}$ in expansions of $\langle\omega_1,<\rangle$. Then, in particular, $T_{\aleph_1}$ is $PC(L_{\omega_1\omega_1})$-definable. This kind of reduction cannot hold on all models. For example, ω_1-like dense linear orders with a first element are all $L_{\infty\omega_1}$-equivalent, but not $L^{II}_{\omega\omega}$-equivalent.

For $\alpha < \lambda = \mu^+$, let $\langle a_i^\alpha : i < \mu\rangle$ be a continuously increasing sequence of subsets of α with $\alpha = \bigcup_{i<\mu} a_i^\alpha$ and $|a_i^\alpha| < \mu$. Define $f_\alpha : \mu \to \mu$ by

$$f_\alpha(i) = otp(a_i^\alpha).$$

Let D_μ be the club-filter on μ. Define for $f, g \in {}^\mu\mu$;

$$f \sim_{D_\mu} g \iff \{i : f(i) = g(i)\} \in D_\mu.$$

Lemma 12 *f_α/D_μ is independent of the choice of the sequence $\langle a_i^\alpha : i < \mu\rangle$.*

Theorem 13 *Suppose*

(i) $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$.

(ii) For every club $C \subseteq \lambda$ there is some $X \subseteq \mu \times \mu$ such that

$$\begin{array}{lcl} \alpha \in C & \to & \{i < \mu : (i, f_\alpha(i)) \in X\} \text{ contains a club} \\ \alpha \notin C & \to & \{i < \mu : (i, f_\alpha(i)) \notin X\} \text{ contains a club.} \end{array}$$

Then there is a sentence $\varphi \in L_{\lambda\lambda}$ such that for all $A \subseteq \lambda$:

$$\langle \lambda, <, A\rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. This is like the proof of Theorem 10. One uses Lemma 12 to refer to the functions f_α. We leave the details to the reader. □

The *Generalized Martin's Axiom for μ* (GMA_μ) from [5] is the following principle:

> Suppose $\mathcal{P}$ is a forcing notion with the properties:
>
> **(GMA1)** Every descending sequence of length $< \mu$ in $\mathcal{P}$ has a greatest lower bound.
>
> **(GMA2)** If $p_\alpha \in \mathcal{P}$ for $\alpha < \mu^+$, then there is a club $C \subseteq \mu^+$ and a regressive function $f : \mu^+ \to \mu^+$ such that if $\alpha \in C$ and $\mathrm{cf}(\alpha) = \mu$, then the set
>
> $$A = \{p_\beta : \mathrm{cf}(\beta) = \mu, f(\alpha) = f(\beta)\}$$
>
> is well-met (i.e. $p, q \in A \to p \vee q \in a$).
>
> Then for any dense open sets $D_\alpha \subseteq \mathcal{P}$, $\alpha < \kappa$, where $\kappa < 2^\mu$, there is a filter in $\mathcal{P}$ which meets every D_α.

Proposition 14 *Suppose $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$, and* GMA_μ. *Then for every club $C \subseteq \lambda$ there is some $X \subseteq \mu \times \mu$ such that*

$$\begin{array}{lcl} \alpha \in C & \to & \{i < \mu : (i, f_\alpha(i)) \in X\} \text{ contains a club} \\ \alpha \notin C & \to & \{i < \mu : (i, f_\alpha(i)) \notin X\} \text{ contains a club.} \end{array}$$

Proof. Let a club $C \subseteq \lambda$ be given. For $\alpha < \beta < \lambda$, let $C_{\alpha\beta} \in D_\mu$ so that $f_\alpha | C_{\alpha\beta} < f_\beta | C_{\alpha\beta}$. Let $\mathcal{P}$ consist of conditions

$$p = (B^p, f^p, \mathbf{c}^p, g^p),$$

where

(i) $B^p \subseteq \lambda$. $|B^p| < \mu$.

(ii) f^p is a partial mapping with $\mathrm{Dom}(f^p) \subseteq \mu \times \mu$, $|\mathrm{Dom}(f^p)| < \mu$, and $\mathrm{Rng}(f^p) \subseteq \{0, 1\}$.

(iii) If $\alpha \in B^p$, then $\{i < \mu : (i, f_\alpha(i)) \in \mathrm{Dom}(f^p)\}$ is an ordinal j^p_α.

(iv) $\mathbf{c}^p = \langle c^p_\alpha : \alpha \in B^p \rangle$, where c^p_α is a closed subset of j^p_α. We denote $\max(c^p_\alpha)$ by δ^p.

(v) If $\alpha \in B^p \cap C$ and $i \in C^p_\alpha$, then $f^p(i, f_\alpha(i)) = 1$. If $\alpha \in B^p \setminus C$ and $i \in C^p_\alpha$, then $f^p(i, f_\alpha(i)) = 0$.

(vi) g^p is a partial mapping with $\mathrm{Dom}(g^p) \subseteq [B^p]^2$ and $\mathrm{Rng}(g^p) \subseteq \mu$.

(vii) If $\alpha < \beta \in \mathrm{Dom}(g^p)$, then $\emptyset \neq c^P_\alpha \setminus g(\alpha, \beta) \subseteq C_{\alpha\beta}$.

The partial ordering "q extends p" is defined as follows:

$$\begin{aligned} p \leq q \quad \Leftrightarrow \quad & B^p \subseteq B^q, f^p \subseteq f^q, g^p \subseteq g^q, \\ & \forall \alpha \in B^p (c^p_\alpha \text{ is an initial segment of } c^q_\alpha), \\ & \text{and if } \delta^p < \delta^q, \text{ then } \mathrm{Dom}(g^q) \supseteq [B^p]^2. \end{aligned}$$

We show now that $\mathcal{P}$ satisfies conditions (GMA1) and (GMA2).

Lemma 15 *$\mathcal{P}$ satisfies (GMA1).*

Proof. Let $p_o \leq \ldots \leq p_i \leq \ldots (i < \gamma)$ in $\mathcal{P}$ with $\gamma < \mu$. We may assume $\delta^{p_0} < \delta^{p_1} < \ldots$. Let $\delta = \sup\{\delta^{p_i} : i < \gamma\}$. Let $B = \bigcup_{i<\gamma} B^{p_i}$. We extend $\bigcup_i f^{p_i}$ to f by defining

$$f(\delta, f_\alpha(\delta)) = \begin{cases} 1 & \text{if } \alpha \in B \cap C \\ 0 & \text{if } \alpha \in B \setminus C. \end{cases}$$

We have to check that this definition is coherent, i.e., if $\alpha \in B \cap C$ and $\beta \in B \setminus C$, then $f_\alpha(\delta) \neq f_\beta(\delta)$. Suppose $\alpha \in B^{p_i}$ and $\beta \in B^{p_{i'}}$ with $\alpha < \beta$

and $i < i'$. Since $\delta^{p_i} < \delta^{p_{i'}}$, $g(\alpha, \beta)$ is defined and $c_\alpha^{p_i} \setminus g(\alpha, \beta) \subseteq C_{\alpha\beta}$. Hence $\delta \in C_{\alpha\beta}$, whence $f_\alpha(\delta) < f_\beta(\delta)$.

Let $\mathbf{c} = \langle c_\alpha : \alpha \in B \rangle$ where $c_\alpha = \bigcup_i c_\alpha^{p_i} \cup \{\delta\}$. Let $j = \bigcup_i j^{p_i} \cup \{\delta\}$. Now the condition $p = (B, f, \mathbf{c}, g)$ is the needed l.u.b. of $(p_i)_{i<\mu}$. □

Lemma 16 *$\mathcal{P}$ satisfies (GMA2).*

Proof. Suppose p_α, $\alpha < \lambda$, are in $\mathcal{P}$. Let h be a one-one mapping from $\mathcal{P}$ to odd ordinals $< \lambda$. By $\mu^{<\mu} = \mu$ there is a club $C \subseteq \lambda$ such that if $\alpha \in C$, $\mathrm{cf}(\alpha) = \mu$, and $B^p \subseteq \alpha$, then $h(p) < \alpha$, and if $\alpha < \beta$, $\alpha, \beta \in C$, then $B^{p_\alpha} \subseteq \beta$. Choose a regressive function g from the complement of C to the even ordinals that is one-one on ordinals of cofinality μ. Suppose $\mathrm{cf}(\alpha) = \mu$. Let $f(\alpha) = g(\alpha)$ if $\alpha \notin C$, and $f(\alpha) = h(p_\alpha|\alpha)$ if $\alpha \in C$. Suppose now $\alpha < \beta$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \mu$, and $f(\alpha) = f(\beta)$. W.l.o.g. $\alpha, \beta \in C$. Thus $h(p_\alpha|\alpha) = h(p_\beta|\beta)$, whence $p_\alpha|\alpha = p_\beta|\beta$. It follows that p_α and p_β have a l.u.b. □

Let

$$D_{\alpha\beta} = \{p \in \mathcal{P} : \alpha \in B^p \text{ and } \delta^p \geq \beta\}$$

where $\alpha < \lambda$, $\beta < \mu$. We show that $D_{\alpha\beta}$ is dense open. Suppose therefore $p \in \mathcal{P}$ is given. We construct $q \in D_{\alpha\beta}$ with $p \leq q$. Let $B^q = B^p \cup \{\alpha\}$. Let

$$E = \bigcap\{C_{\xi\eta} : \xi, \eta \in B^q, \xi < \eta\} (\in D_\mu).$$

Let $\delta^q \in E \setminus \beta$. Define $\mathbf{c}^q = \langle c_\xi^q : \xi \in B^q \rangle$ by

$$c_\xi^q = \begin{cases} c_\xi^p \cup \langle \delta^q \rangle, & \text{if } \xi \neq \alpha \\ \langle \delta^q \rangle, & \text{if } \xi = \alpha. \end{cases}$$

Let

$$f^q = f^p \cup \begin{cases} \{((j, f_\alpha(j)), 1) : j^p \leq j \leq \delta^q\}, & \text{if } \alpha \in C \\ \{((j, f_\alpha(j)), 0) : j^p \leq j \leq \delta^q\}, & \text{if } \alpha \notin C. \end{cases}$$

Let $g^q(\xi, \eta) = \delta^p$ for $(\xi, \eta) \in [M^p]^2 \setminus \mathrm{Dom}(g^p)$. Let $q = (B^q, f^q, g^q, \delta^q)$. Then $q \in D_{\alpha\beta}$, and $p \leq q$.

Let G be a filter that meets every $D_{\alpha\beta}$. Let

$$\begin{aligned} B &= \bigcup\{B^p : p \in G\} \\ f &= \bigcup\{f^p : p \in G\} \\ c_\alpha &= \bigcup\{c_\alpha^p : p \in G\} \end{aligned}$$

Then $B = \lambda$ and each c_α is a club of μ. Let $X = \{(\alpha, \beta) \in \mu \times \mu : f(\alpha, \beta) = 1\}$. Suppose $\alpha \in C$ and $i \in c_\alpha$. Then $f(i, f_\alpha(i)) = 1$ whence $(i, f_\alpha(i)) \in X$. Suppose $\alpha \notin C$ and $i \in c_\alpha$. Then $f(i, f_\alpha(i)) = 0$ whence $(i, f_\alpha(i)) \notin X$. □

Corollary 17 *Suppose $\lambda = \mu^+$, where $\mu = \mu^{<\mu} > \aleph_0$, and* GMA_μ. *Then there is a sentence $\varphi \in L_{\lambda\lambda}$ such that for all $A \subseteq \lambda$:*

$$\langle \lambda, <, A \rangle \models \varphi \iff A \text{ is stationary.}$$

Proof. The claim follows from Theorem 13 and Proposition 14. □

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