STRONG DICHOTOMY
OF CARDINALITY
SH664

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Abstract. We investigate strong dichotomical behaviour of the number of equivalence classes and related cardinal.



Saharon: compare with Journal proofs!



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## Annotated Content

§0 Introduction

§1 Countable Groups

[We present a result on a sequence of analytic equivalence relations on $\mathscr{P}(\omega)$ and apply it to $\aleph_0$-system of groups getting a strong dichotomy: being infinite implies cardinality continuum sharpening [GrSh 302a].]

§2 On $\lambda$-analytic equivalence relations

[We generalize theorems on the number of equivalence classes for analytic equivalent relations replacing $\aleph_0$ by $\lambda$ regular, unfortunately this is only consistent. Noting that if we just add many Cohen subsets to $\lambda$ we get something, but first the dichotomy is $\leq \lambda^+, = 2^\lambda$ rather than $\leq \lambda, = 2^\lambda$, second we assume much less.]

§3 On $\lambda$-systems of groups

[This relates to §2 as the application relates to the lemma in §1.]

§4 Back to the $p$-rank of Ext

[We show that we can put the problem in the title to the previous context, and show that in Easton model, §2 and §3 apply to every regular $\lambda$.]

§5 Strong limit of countable cofinality

[We generalize the theorem on $\aleph_0$ systems of groups from §1, replace $\aleph_0$ by a strong limit uncountable cardinal of countable cofinality; this continues [GrSh 302a].]

## §0

A usual dichotomy is that in many cases, reasonably definable sets, satisfies the continuum hypothesis, i.e. if they are uncountable they have cardinality continuum. A strong dichotomy is when: if the cardinality is infinite it is continuum, as in [Sh 273]. We are interested in such phenomena when $\lambda = \aleph_0$ is replaced by $\lambda$ regular uncountable and also by $\lambda = \beth_\omega$ or more generally by strong limit of cofinality $\aleph_0$.

Question: Does the parallel of 1.2 holds for e.g. $\beth_\omega$? portion?

This continues Grossberg Shelah [GrSh 302], [GrSh 302a] and see history there. We also generalize results on the number of analytic equivalence relations, continuing Harrington Shelah [HrSh 152] and [Sh 202] and see history there.
On the connection to the rank of the $p$-torsion subgroup see [MRSh 314] and history there. See more [ShVs 719].

On $\mathrm{Ext}(G,\mathbb{Z})$, $\mathrm{rk}_p(\mathrm{Ext}(G,\mathbb{Z})$ see [EM].

## §1 Countable groups

Here we give a complete proof of a strengthening of the theorem of [GrSh 302a], for the case $\lambda = \aleph_0$ using a variant of [Sh 273].

**1.1 Theorem.** *1) Suppose*

- $(A)$ *$\lambda$ is $\aleph_0$. Let $\langle G_m, \pi_{m,n} : m \leq n < \omega \rangle$ be an inverse system whose inverse limit is $G_\omega$ with $\pi_{n,\omega}$ such that $|G_n| < \lambda$. (So $\pi_{m,n}$ is a homomorphism from $G_n$ to $G_m, \alpha \leq \beta \leq \gamma \leq \omega \Rightarrow \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$ and $\pi_{\alpha,\alpha}$ is the identity).*
- $(B)$ *Let $\mathbf{I}$ be an index set. For every $t \in \mathbf{I}$, let $\langle H^t_m, \pi^t_{m,n} : m \leq n < \omega \rangle$ be an inverse system of groups and $H^t_\omega$ with $\pi^t_{n,\omega}$ be the corresponding inverse limit and $H^t_m$ of cardinality $\leq \lambda$.*
- $(C)$ *Let for every $t \in \mathbf{I}, \sigma^t_n : H^t_n \to G_n$ be a homomorphism such that all diagrams commute (i.e. $\pi_{m,n} \circ \sigma^t_n = \sigma^t_m \circ \pi^t_{m,n}$ for $m \leq n < \omega$), and let $\sigma^t_\omega$ be the induced homomorphism from $H^t_\omega$ into $G_\omega$.*
- $(D)$ *$\mathbf{I}$ is countable*[1]
- $(E)$ *For every $\mu < \lambda$ and $t \in \mathbf{I}$ there is a sequence $\langle f_i \in G_\omega : i < \mu \rangle$ such that $i < j \Rightarrow f_i f_j^{-1} \notin \ Rang(\sigma^t_\omega)$.*

*Then there is $\langle f_i \in G_\omega : i < 2^\lambda \rangle$ such that*
*$i \neq j \ \& \ t \in \mathbf{I} \Rightarrow f_i f_j^{-1} \notin \ Rang(\sigma^t_\omega)$.*
*2) We can weaken in clause (A) to $(A)^-$ replacing $|G_n| < \lambda$ by $|G_n| \leq \lambda$, if we change clause (E) to*

- $(E)^*$ *for every $t \in \mathbf{I}, m < \omega$ there are $n, f$ such that $f$ is a member of $G_\omega, n < k < \omega \Rightarrow \pi_{k,\omega}(f) \notin \ Rang(\sigma^t_\omega)$ and $e_{G_n} = \pi_{n,\omega}(f)$.*

We shall show below that 1.1 follows from 1.2.

**1.2 Lemma.** *Assume for every $n < \omega, \mathscr{E}_n$ is an analytic two place transitive relation on $\mathscr{P}(\omega) = \{A : A \subseteq \omega^+\}$ which satisfies, for each $m < \omega$ for some infinite $Z_m \subseteq \omega$ we have*

- $(*)_{m,Z_m}$ *if $A, B \subset \mathbf{Z}^+, n \in Z_m, n \notin B, A = B \cup \{n\}$, then $\neg(A \mathscr{E}_m B) \vee \neg(B \mathscr{E}_m A)$*
- $(**)$ *if $m < \omega, A' \mathscr{E}_m B$ and $A'' \mathscr{E}_m B$ then $A' \mathscr{E}_m A''$.*

*Then there is a perfect subset $\mathbf{P}$ of $\mathscr{P}(\omega)$ of pairwise $\mathscr{E}_m$-nonrelated $A \subseteq \omega$, simultaneously for all $n$, that is $A \neq B \ \& \ A \in \mathbf{P} \ \& \ B \in \mathbf{P} \ \& \ m < \omega \Rightarrow \neg(A \mathscr{E}_m B)$.*

[1]this is stronger, earlier $\mathbf{I}$ was finite

*1.3 Remark.* 1) The proof uses some knowledge of set theory and is close to [Sh 273, Lemma 1.3].
2) We say $A, B$ are $\mathscr{E}$-related if $A \mathscr{E} B$, and we say $A, B$ are non-$\mathscr{E}$-related if $\neg(A \mathscr{E} B)$.

*Proof.* Let $r_m \in {}^\omega 2$ be the real parameter involved in a definition $\varphi_m(x, y, r_m)$ of $\mathscr{E}_m$. Let $\bar{\varphi} = \langle \varphi_m : m < \omega \rangle, \bar{r} = \langle r_m : m < \omega \rangle, \bar{\mathscr{E}} = \langle \bar{\mathscr{E}}_m : m < \omega \rangle$. Let $N$ be a countable elementary submodel of $(\mathscr{H}((2^{\aleph_0})^+), \in)$ to which $\bar{\varphi}, \bar{r}, \bar{\mathscr{E}}$ belong. Now we shall show

$(***)$ if $\langle A_1, A_2 \rangle$ be a pair of subsets of $\omega$ which is Cohen generic over $N$ [this means that it belongs to no first category subset of $\mathscr{P}(\omega) \times \mathscr{P}(\omega)$ which belongs to $N$] then

- $(\alpha)$ $A_1, A_2$ are $\mathscr{E}_m$-related in $N[A_1, A_2]$ if they are $\mathscr{E}_m$-related
- $(\beta)$ $A_1, A_2$ are non-$\mathscr{E}_m$-related in $N[A_1, A_2]$.

*Proof of* $(***)$.

- $(\alpha)$ by the absoluteness criterions (Levy Sheönfied)
- $(\beta)$ if not, then some finite information forces this, hence for some $n$
  - $\circledast$ if $\langle A'_1, A'_2 \rangle$ is Cohen generic over $N$ and $A'_1 \cap \{0, 1, \ldots, n\} = A_1 \cap \{0, 1, \ldots, n\}$ and $A'_2 \cap \{0, 1, \ldots, n\} = A_2 \cap \{1, \ldots, n\}$ then $A'_1, A'_2$ are $\mathscr{E}_m$-related in $N[A'_1, A'_2]$.

Choose $k \in Z_m \backslash \{0, 1, \ldots, n+1\}$. Let $A''_1$ be $A_1 \cup \{k\}$ if $k \notin A_1$ and $A_1 \backslash \{k\}$ if $k \in A_1$.

Trivially also $\langle A''_1, A_2 \rangle$ is Cohen generic over $N$, hence by $\circledast$ above $A''_1, A_2$ are $\mathscr{E}_m$-related in $N[A''_1, A_2]$. By $(***)(\alpha)$ we know that really $A''_1, A_2$ are $\mathscr{E}_m$-related. By $(**)$ clearly $A_1, A''_1$ are $\mathscr{E}_m$-related and also $A''_1, A_1$ are $\mathscr{E}_m$-related. But this contradicts the hypothesis $(*)_{m, Z_m}$. So $(***)$ holds.

We can easily find a perfect (nonempty) subset $\mathbf{P}$ of $\{A : A \subseteq \omega\}$ such that for any distinct $A, B \in \mathbf{P}, (A, B)$ is Cohen generic over $N$. So for each $m$ for $A \neq B \in \mathbf{P}$ we have $N[A, B] \models$ "$A, B$ are not $\mathscr{E}_m$-equivalent" and by $(***)(\alpha)$ clearly $A, B$ are not $\mathscr{E}_m$-equivalent. This finishes the proof. $\square_{1.2}$

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*1.4 Proof of 1.1.* 1) Follows from part (2) as $(E) \Rightarrow (E)^+$ when the $G_n$'s are finite (use $(E)$ for $\mu^* = |G_n| + 1$).
2) Let $k_n = n^2$ and we choose $\langle f_n : n < \omega \rangle$ such that:

- $(a)$ $f_n \in G_\omega$
- $(b)$ $k_n \le i < k_{n+1} \Rightarrow e_{G_n} = \pi_{n,\omega}(f_i)$
- $(c)$ for every $t \in \mathbf{I}$, for arbitrarily large $k$ we have $\pi_{k+1,\omega}(f_k) \notin \text{Rang}(\sigma^t_{k+1})$.

Clearly $(a), (b)$ are straight for $(c)$ use assumption $(E)^+$ and bookkeeping.

By induction on $n$ for every $\eta \in {}^n 2$ we choose $f_\eta \in G_\omega$ as follows: for $n = 0, f_\eta = e_{G_\omega}$, for $\eta = \nu\char`^\langle 0 \rangle, \nu \in {}^{n+}2$ let $f_\eta = f_\nu$ and for $\eta = \nu\char`^\langle 1 \rangle$ let $f_\eta = f_\nu f^{-1}_{n-1}$. Clearly $m \le n < \omega \ \& \ \eta \in {}^n 2 \Rightarrow \pi_{m,\omega}(f_{\eta \restriction m}) = \pi_{m,\omega}(f_\eta)$.

Lastly, for $A \subseteq \omega$, let $\eta_A \in {}^\omega 2$ be its characteristic function and $g_A \in G_\omega$ be the unique $f \in G_\omega$ satisfying $m \le n < \omega \Rightarrow \pi_{m,\omega}(f_{\eta \restriction n}) = \pi_{m,\omega}(f_A)$. Let $\mathbf{I} = \{t_m : m < \omega\}$ (well we can add trivial $H$'s) and let $\mathscr{E}_m$ be $A \mathscr{E}_m B \Leftrightarrow A \subseteq \omega \ \& \ B \subseteq \omega \ \& \ g^{-1}_A g_B \in \text{Rang}(\sigma^{t_m}_\omega)$. Clearly $\mathscr{E}_m$ is an equivalence relation hence it satisfies condition $(**)$ of 1.2. Lastly, let $Z_m =: \{k : \pi_{k+1,\omega}(f_k) \notin \text{Rang}(\sigma^{t_m}_\omega)\}$. If $A, B, m, k$ are as in $(*)$ of 1.2 then $\pi_{k+1,\omega}(g^{-1}_A g_B) = \pi_{k+1,\omega}(f_k) \notin \text{Rang}(\sigma^t_{k+1})$. We have the assumptions of 1.2, hence get its conclusion. $\square_{1.1}$

## §2 On $\lambda$-analytic equivalence relations

*2.1 Hypothesis.* $\lambda = \operatorname{cf}(\lambda)$ is fixed.

**2.2 Definition.** 1) A sequence of relations $\bar R = \langle R_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ on ${}^\lambda 2$ (equivalently $\mathscr{P}(\lambda)$) i.e. a sequence of definitions of such relations in $(\mathscr{H}(\lambda^+), \in)$ and with parameters in $\mathscr{H}(\lambda^+)$ is called $\lambda$-w.c.a. sequence (weakly Cohen absolute) if: for any $A \subseteq \lambda$ we have

- $(*)_A$ there are $N, r$ such that:
  - $(\alpha)$ $N$ is a transitive model
  - $(\beta)$ $N^{<\lambda} \subseteq N, \lambda + 1 \subseteq N$, the sequence of the definitions of $\bar R$ (including the parameters) belongs to $N$
  - $(\gamma)$ $A \in N$
  - $(\delta)$ $r \in {}^\lambda 2$ is Cohen over $N$; that is generic for $({}^{\lambda>}2, \triangleleft)$ over $N$
  - $(\varepsilon)$ $R_\varepsilon$ and $\neg R_\varepsilon$ are absolute from $N[r]$ to $V$ for each $\varepsilon < \varepsilon(*)$.

2) We say $\bar R$ is $(\lambda,\mu)$-w.c.a. if for $A \subseteq \lambda$ we can find $N, r_\alpha$ (for $\alpha < \mu$) satisfying clauses $(\alpha), (\beta), (\gamma)$ from above and

- $(\delta)'$ for $\alpha \neq \beta < \mu, (r_\alpha, r_\beta)$ is a pair of Cohens over $N$
- $(\varepsilon)'$ $R_\varepsilon$ and $\neg R_\varepsilon$ are absolute from $N[r_\alpha, r_\beta]$ to $V$ for each $\alpha \neq \beta < \mu$ and $\varepsilon < \varepsilon(*)$.

3) We say $\lambda$ is $(\lambda,\mu)$-w.c.a. <u>if</u> every $\lambda$-analytic relation $R$ on ${}^\lambda 2$ is $(\lambda,\mu)$-w.c.a.
Analytic means that it has the form $R(X_1,\dots,X_n) = (\exists Y_1,\dots,Y_m \subseteq \lambda\times\lambda)\varphi(Y_1,\dots,Y_m; X_1,\dots,X_n)$

**2.3 Claim.** *Assume*

- $(A)$ *$\varepsilon(*) \le \lambda$ and $\langle \mathscr{E}_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is a $(\lambda,\mu)$-w.c.a. sequence, each $\mathscr{E}_\varepsilon$ an equivalence relation on $\mathscr{P}(\lambda)$, more exactly a definition of one and*
- $(B)$ *if $\varepsilon < \varepsilon(*)$ and $A, B \subseteq \lambda$ and $\alpha \in A \backslash B \backslash \varepsilon, A = B \cup \{\alpha\}$, <u>then</u> $A, B$ are not $\mathscr{E}$-equivalent.*

*<u>Then</u> there is a set $\mathscr{P} \subseteq \mathscr{P}(\lambda)$ of $\mu$-pairwise non-$\mathscr{E}_\varepsilon$-equivalent members of $\mathscr{P}(\lambda)$ for all $\varepsilon < \varepsilon(*)$ simultaneously.*

*2.4 Remark.* If in 2.2 we ask that $\{r_\eta : \eta \in {}^\lambda 2\}$ perfect (see 2.5 below), then we can demand that so is $\mathscr{P}$.

**2.5 Definition.** 1) $\mathscr{P} \subseteq \mathscr{P}(\lambda)$ is perfect if there is a $\lambda$-perfect tree $T \subseteq {}^{\lambda>}2$ (see below) such that $\mathscr{P} = \{\{\alpha < \lambda : \eta(\alpha) = 1\} : \eta \in \lim_\lambda(T)\}$.
2) $T$ is a $\lambda$-perfect tree if:

- $(a)$ $T \subseteq {}^{\lambda>}2$ is non-empty
- $(b)$ $\eta \in T \ \& \ \alpha < \ell g(\eta) \Rightarrow \eta \restriction \alpha \in T$
- $(c)$ if $\delta < \lambda$ is a limit ordinal, $\eta \in {}^\delta 2$ and $(\forall \alpha < \delta)(\eta \restriction \alpha \in T)$, <u>then</u> $\eta \in T$
- $(d)$ if $\eta \in T, \ell g(\eta) < \alpha < \lambda$ then there is $\nu, \eta \triangleleft \nu \in T \cap {}^\alpha 2$
- $(e)$ if $\eta \in T$ then there are $\triangleleft$-incomparable $\nu_1, \nu_2 \in T$ such that $\eta \triangleleft \nu_1 \ \& \ \eta \triangleleft \nu_1$.

3) $\mathrm{Lim}_\delta(T) = \{\eta : \ell g(\eta) = \delta$ and $(\forall \alpha < \delta)(\eta \restriction \alpha \in T)\}$.

*Proof of 2.3.*
Let $T^* = {}^{\lambda>}2$.
Let $N$ and $r_\alpha \in {}^\lambda 2$ for $\alpha < \mu$ be as in Definition 2.2. We identify $r_\alpha$ with $\{\gamma < \lambda : r_\alpha(\gamma) = 1\}$.
By clause $(\varepsilon)'$ of Definition 2.2(2) clearly

$(*)_0$ if $\varepsilon < \varepsilon(*)$, and $\alpha \neq \beta < \mu$, <u>then</u> $\mathscr{E}_\varepsilon$ define an equivalence relation in $N[r_\alpha, r_\beta]$ on $\mathscr{P}(\lambda)^{N[r_\alpha, r_\beta]}$.

It is enough to prove assuming $\alpha \neq \beta < \mu$ and $\varepsilon < \varepsilon(*)$ that,

$(*)_1$ $\neg r_\alpha \mathscr{E}_\varepsilon r_\beta$.

By clause $(\varepsilon)'$ of Definition 2.2(2) it is enough to prove

$(*)_2$ $N[r_\alpha, r_\beta] \models \neg r_\alpha \mathscr{E}_\varepsilon \nu_\beta$.

Assume this fails, so $N[r_\alpha, r_\beta] \models r_\alpha \mathscr{E}_\varepsilon r_\beta$ then for some $i < \lambda$

$$(r_\alpha \restriction i, r_\beta \restriction i) \Vdash_{({}^{\lambda>}2) \times ({}^{\lambda>}2)} \text{“}\underset{\sim}{r}_1 \mathscr{E}_\varepsilon \underset{\sim}{r}_2\text{”}$$

and without loss of generality $i > \varepsilon$. Define $r \in {}^\lambda 2$ by

$$r(j) = \begin{cases} r_\beta(j) & \underline{\text{if}}\ j \neq i \\ 1 - r_\beta(j) & \underline{\text{if}}\ j = i \end{cases}$$

So also $(r_\alpha, r)$ is a generic pair for ${}^{\lambda>}2 \times {}^{\lambda>}2$ over $N$ and $(r_\alpha \restriction i, r \restriction i) = (r_\alpha \restriction i, r_\beta \restriction i)$ hence by the forcing theorem

$(*)_3$ $N[r_\alpha, r] \models \underset{\sim}{r}_\alpha \mathscr{E}_\varepsilon r$.

But $r_\alpha, r_\beta, r \in N[r_\alpha, r_\beta] = N[r_\alpha, r]$. As we are assuming that $(*)_2$ fail (toward contradiction) we have $N[r_\alpha, r_\beta] \models r_\alpha \mathscr{E}_\varepsilon r_\beta$ and by $(*)_3$ and the previous sentence we have $N[r_\alpha, r_\beta] \models r \mathscr{E}_\varepsilon r$ so together by $(*)_0$ we have $N[r_\alpha, r_\beta] \models r_\beta \mathscr{E}_\varepsilon r$ hence $V \models r_\beta \mathscr{E}_\varepsilon r$, a contradiction to assumption (b). $\square_{2.3}$

**2.6 Definition.** We call $Q$ a pseudo $\lambda$-Cohen forcing if:

- $(a)$ $Q$ is a nonempty subset of $\{p : p$ a partial function from $\lambda$ to $\{0,1\}\}$
- $(b)$ $p \leq_Q q \Rightarrow p \subseteq q$
- $(c)$ $\mathscr{I}_i = \{p \in Q : i \in \mathrm{Dom}(p)\}$ is a dense subset for $i < \lambda$
- $(d)$ define $F_i : \mathscr{I}_i \to \mathscr{I}_i$ by: $\mathrm{Dom}(F_i(p)) = \mathrm{Dom}(F_i(p))$ and

$$(F_i(p))(j) = \begin{cases} p(j) & \underline{\text{if}}\ j = i \\ 1 - p(j) & \underline{\text{if}}\ j \neq i \end{cases}$$

then $F_i$ is an automorphism of $(\mathscr{I}_i, <^Q \restriction \mathscr{I}_i)$.

**2.7 Claim.** *In 2.2, 2.5 we can replace* $({}^{\lambda>}2, \triangleleft)$ *by* $Q$.

<u>2.8 Observation</u>: So if $V \models G.C.H., P$ is Easton forcing, <u>then</u> in $V^P$ for every regular $\lambda$, for $Q = (({}^{\lambda>}2)^V, \triangleleft)$ we have: $Q$ is pseudo $\lambda$-Cohen and in $V^P$ we have $\lambda$ is $(\lambda, 2^\lambda)$-w.c.a.

<u>2.9 Discussion</u>: But in fact $\lambda$ being $(\lambda, 2^\lambda)$- w.c.a. is a weak condition.

We can generalize further using the following definition

**2.10 Definition.** 1) For $r_0, r_1 \in {}^\lambda 2$ we say $(r_0, r_1)$ or $r_0, r_1$ is an $\bar{R}$-pseudo Cohen pair over $N$ if ($\bar{R}$ is a definition (in $(\mathscr{H}(\lambda^+), \in)$) of a relation on $\mathscr{P}(\lambda)$ (or ${}^\lambda 2$), the definition belongs to $N$ and) for some forcing notion $Q \in N$ and $Q$-names $\underset{\sim}{r}_0, \underset{\sim}{r}_1$ and $G \subseteq Q$ ($G \in V$) generic over $N$ we have:

- $(a)$ $\underset{\sim}{r}_0[G] = r_0$ and $\underset{\sim}{r}_1[G] = r_1$
- $(b)$ for every $p \in G$, for every $i < \lambda$ large enough and $\ell(*) < 2$ there is $G' \subseteq Q$ generic over $N$ such that: $p \in G$ and $(\underset{\sim}{r}_\ell[G'])(j) = (\underset{\sim}{r}_\ell[G])(j) \Leftrightarrow (j, \ell) \neq (i, \ell(*))$
- $(c)$ for $\varepsilon < \varepsilon(*)$, $R_\varepsilon$ is absolute from $N[G]$ and from $N[G']$ to $V$.

2) We say $\lambda$ is $\mu$-p.c.a for $\bar R$ <u>if</u> for every $x \in \mathscr{H}(\lambda^+)$ there are $N, \langle r_i : i < \mu \rangle$ such that:

(a) $N$ is a transitive model of $ZFC^-$

(b) for $i \neq j < \mu, (r_i, r_j)$ is an $\bar R$-pseudo Cohen pair over $N$.

3) We omit $\bar R$ if this holds for any $\lambda$-sequence of $\sum^1_1$ formula in $\mathscr{H}(\lambda^+)$.

Clearly

**2.11 Claim.** *1) If $\lambda$ is $\mu$-p.c.a for $\mathscr{E}, \mathscr{E}$ an equivalence relation on $\mathscr{P}(\lambda)$ and $A \subseteq B \subseteq \lambda$ & $|B \backslash A| = 1 \Rightarrow \neg A \mathscr{E} B$, <u>then</u> $\mathscr{E}$ has $\geq \mu$ equivalence classes.*
*2) Similarly if $\mathscr{E} = \bigvee\limits_{\varepsilon < \varepsilon(*)} \mathscr{E}_\varepsilon, \varepsilon(*) \leq \lambda$ and $\lambda$ is $\mu$-p.c.a. for $\langle \mathscr{E}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ and $A \subseteq B \subseteq \lambda$ & $|B \backslash A| = |B \backslash A \backslash \varepsilon| = 1 \Rightarrow \neg A \mathscr{E}_\varepsilon B$, <u>then</u> there are $A_\alpha \subseteq \lambda$ for $\alpha < \mu$ such that $\varepsilon < \varepsilon(*)$ & $\alpha < \beta < \mu \Rightarrow \neg(A_\alpha \mathscr{E}_\varepsilon A_\beta)$.*

## §3 On $\lambda$-systems of groups

*3.1 Hypothesis.* $\lambda = \mathrm{cf}(\lambda)$.

We may wonder does 2.3 have any cases it covers?

**3.2 Definition.** 1) We say $\mathscr{Y} = (\bar{A}, \bar{K}, \bar{G}, \bar{D}, \bar{g}^*)$ is a $\lambda$-system if

$(A)$ $\bar{A} = \langle A_i : i \le \lambda \rangle$ is an increasing sequence of sets, $A = A_\lambda = \{A_i : i < \lambda\}$

$(B)$ $\bar{K} = \langle K_t : t \in A \rangle$ is a sequence of finite groups

$(C)$ $\bar{G} = \langle G_i : i \le \lambda \rangle$ is a sequence of groups, $G_i \subseteq \prod\limits_{t \in A_i} K_t$, each $G_i$ is closed and $i < j \le \lambda \Rightarrow G_i = \{g \restriction A_i : g \in G_j\}$ and $G_\lambda = \{g \in \prod\limits_{t \in A_\lambda} K_t : (\forall i < \lambda)(g \restriction A_i \in G_i)\}$

$(D)$ $\bar{D} = \langle D_\delta : \delta \le \lambda$ (a limit ordinal) $\rangle, D_\delta$ an ultrafilter on $\delta$ such that $\alpha < \delta \Rightarrow [\alpha, \delta) \in D_\delta$

$(E)$ $\bar{g}^* = \langle g_i^* : i < \lambda \rangle, g_i^* \in G_\lambda$ and $g_i^* \restriction A_i = e_{G_i} = \langle e_{K_t} : t \in A_i \rangle$.

Of course, formally we should write $A_i^{\mathscr{Y}}, K_t^{\mathscr{Y}}, G_i^{\mathscr{Y}}, D_\delta^{\mathscr{Y}}, g_i^{\mathscr{Y}}$, etc., if clear from the context we shall not write this.

2) Let $\mathscr{Y}^-$ be the same omitting $D_\lambda$ and we call it a lean $\lambda$-system.

**3.3 Definition.** For a $\lambda$-system $\mathscr{Y}$ and $j \le \lambda + 1$ we say $\bar{f} \in \mathrm{cont}(j, \mathscr{Y})$ if:

$(a)$ $\bar{f} = \langle f_i : i < j \rangle$

$(b)$ $f_i \in G_\lambda$

$(c)$ if $\delta < j$ is a limit ordinal then $f_\delta = \mathrm{Lim}_{D_\delta}(\bar{f} \restriction \delta)$ which means:

$$\text{for every } t \in A, f_\delta(t) = \mathrm{Lim}_{D_\delta} \langle f_i(t) : i < \delta \rangle$$

which means

$$\{i < \delta : f_\delta(t) = f_i(t)\} \in D_\delta.$$

<u>3.4 Fact</u>: 1) If $\bar{f} \in \mathrm{cont}(j, \mathscr{Y}), i < j$ <u>then</u> $\bar{f} \restriction i \in \mathrm{cont}(i, \mathscr{Y})$.

2) If $\bar{f} \in \mathrm{cont}(j, \mathscr{Y})$ and $j < \lambda$ is non-limit, and $f_j \in G_\lambda$ <u>then</u>

$$\bar{f}\hat{\ }\langle f_j \rangle \in \mathrm{cont}(j+1, \mathscr{Y}).$$

3) If $\bar f \in$ cont$(j, \mathscr{Y})$ and $j$ is a limit ordinal $\le \lambda$, <u>then</u> for some unique $f_j \in G_\lambda$ we have $\bar f\char`^\langle f_j \rangle \in$ cont$(j+1, \mathscr{Y})$.
4) If $j \le \lambda + 1, f \in G$ <u>then</u> $\bar f = \langle f : i < j \rangle \in$ cont$(j, \mathscr{Y})$.
5) If $\bar f, \bar g \in$ cont$(j, \mathscr{Y})$, <u>then</u> $\langle f_i g_i : i < j \rangle$ and $\langle f_i^{-1} : i < j \rangle$ belongs to cont$(j, \mathscr{Y})$.

*Proof.* Straight (for part (3) we use each $K_t$ is finite).

**3.5 Definition.** Let $\mathscr{Y}$ be a $\lambda$-system.
1) For $\bar g \in {}^j(G_\lambda)$ and $j \le \lambda$ we define $f_{\bar g} \in G_\lambda$ by induction on $j$ for all such $\bar g$ as follows:

<u>$j = 0$</u>: $f_{\bar g} = e_G = \langle e_{K_t} : t \in A \rangle$

<u>$j = i + 1$</u>: $f_{\bar g} = f_{\bar g \restriction i} g_i$

<u>$j$ limit</u>: $f_{\bar g} =$ Lim$_{D_\delta} \langle f_{\bar g \restriction i} : i < j \rangle$

2) We say $\bar g$ is trivial on $X$ if $i \in X \cap \ell g(\bar g) \Rightarrow g_i = e_{G_\lambda}$.
3) For $\eta \in {}^{\lambda \ge}2$ let $\bar g^\eta = \langle g_i^\eta : i < \ell g(\eta) \rangle$, where

$$g_i^\eta = \begin{cases} g_i^* & \text{if } \eta(i) = 1 \\ e_{G_\lambda} & \text{if } \eta(i) = 0 \end{cases}$$

recall $g_i^*$ is part of $\mathscr{Y}$ (see Definition 3.2).

**3.6 Claim.** *1) If $i \le j$ and $\bar g, \bar g', \bar g'' \in {}^j(G_\lambda), \bar g' \restriction i = \bar g \restriction i, \bar g'$ is trivial on $[i, j)$, $\bar g'' \restriction [i, j) = \bar g \restriction [i, j)$ and $\bar g''$ is trivial on $i$, <u>then</u>:*

$$f_{\bar g} = f_{\bar g'} f_{\bar g''} \text{ and } f_{\bar g'} = f_{\bar g \restriction i}.$$

*2) For $\eta \in {}^\lambda 2, f_{(\bar g^\eta)} =$ Lim$\langle f_{(\bar g^{\eta \restriction i})} : i < \lambda \rangle$ (i.e. any ultrafilter $D'_\lambda$ on $\lambda$ containing the co-bounded sets will do), so $\mathscr{Y}^-$, a lean $\lambda$-system, is enough.*

*Proof.* Straight.

**3.7 Claim.** *Let $\mathscr{Y}$ be a $\lambda$-system (or just a lean one), $H_\varepsilon$ a subgroup of $G_\lambda$ for $\varepsilon < \varepsilon(*) \leq \lambda$ and $\mathscr{E}_\varepsilon$ the equivalence relation $[f'(f'')^{-1} \in H_\varepsilon]$ and assume: $\lambda > i \geq \varepsilon \Rightarrow g_i^* \notin H_\varepsilon$.*

(1) *The assumption (B) of 2.3 holds with $f_A = f_{(\bar{g}^\eta)}$ when $A \subseteq \lambda, \eta \in {}^\lambda 2, A = \{i : \eta(i) = 1\}$*

(2) *if in addition $\bar{A}, \bar{K}, \bar{G} \restriction K, \bar{D}, \bar{g}^* \in \mathscr{H}(\lambda^+)$ and $\langle H_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ is $(\lambda, \mu)$-w.c.a., then also assumption (A) of 3.3 holds (hence its conclusion).*

*Proof.* Straight.

**3.8 Claim.** *Assume*

(A) *$\mathscr{Y}$ a $\lambda$-system (or just a lean one), $A_i \subseteq \lambda^+, |A_i| \leq \lambda, G_i \in \mathscr{H}(\lambda^+)$*

- (i) $\varepsilon(*) \leq \lambda$,
- (ii) $\bar{H} = \langle H_i^\varepsilon : i \leq \lambda, \varepsilon < \varepsilon(*)\rangle$,
- (iii) $\pi_{i,j}^\varepsilon : H_j^\varepsilon \to H_i^\varepsilon$ *a homomorphism,*
- (iv) *for $i_0 \leq i_1 \leq i_2$ we have $\pi_{i_0,i_1}^\varepsilon \circ \pi_{i_1,i_2}^\varepsilon = \pi_{i_0,i_2}^\varepsilon$,*
- (v) $\sigma_i^\varepsilon : H_t^\varepsilon \to G_i$,
- (vi) $\sigma_i^\varepsilon \pi_{i,j}^\varepsilon(f) = (\sigma_j^\varepsilon(f)) \restriction A_i$,
- (vii) *$H_\lambda^\varepsilon, \sigma_\lambda^\varepsilon$ is the inverse limit (with $\pi_{i,\lambda}^\varepsilon$) of $\langle H_i^\varepsilon, \pi_{i,j}^\varepsilon, \sigma_i^\varepsilon : i \leq j < \lambda\rangle$ and*
- (viii) $i < \lambda \Rightarrow H_i^\varepsilon \in \mathscr{H}(\lambda^+)$

(B) $H_\varepsilon = \mathit{Rang}(\sigma_\lambda^\varepsilon)$.

Then

($\alpha$) *the assumptions of 3.7 holds*

($\beta$) *if $\lambda$ is $(\lambda, \mu)$-w.c.a. then also the conclusion of 3.7, 2.3 holds so there are $h_\alpha \in G_\lambda$ for $\alpha M \mu$ such that $\alpha \neq \beta < \mu \ \& \ \varepsilon < \varepsilon(*) \Rightarrow f_\alpha f_\beta^{-1} \notin H_\varepsilon$.*

*Proof.* Straight.

* * *

We can go one more step in concretization.

**3.9 Claim.** *1) Assume*

- $(a)$ *$L$ is an abelian group of cardinality $\lambda$*
- $(b)$ *$p$ a prime number*
- $(c)$ *if $L' \subseteq L, |L'| < \lambda$, then $Ext_p(L', \mathbb{Z}) \neq 0$*
- $(d)$ *$\lambda$ is $\mu$-w.c.a. (in $V$).*

*Then $\mu \leq r_p(Ext(L, \mathbb{Z}))$, see definition below.*
*2) If $(a), (b), (d)$ above, $\mu > \lambda, \lambda$ strongly inaccessible then $r_p(Ext(L, \mathbb{Z})) \notin [\lambda, \mu)$.*

*3.10 Remark.* 1) For an abelian group $M$ let prime $p$ and $r_p(M)$ be the dimension of the subgroup of $\{x \in M : px = 0\}$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$.
2) For an abelian group $M$ let $r_0(M)$ be $\max\{|X| : X \subseteq M \backslash \mathrm{Tor}(M)$ and is independent in $M/\mathrm{Tor}(M)$.

*Proof.* Without loss of generality $L$ is $\aleph_1$-free (so torsion free).
Without loss of generality the set of elements of $G$ is $\lambda$. Let $A = A_\lambda = \lambda, L_\lambda = L$, for $j < \lambda, A_j$ a proper initial segment of $\lambda$ such that $L_j = L \restriction A_j$ is a pure subgroup of $L$, increasing continuously with $j$.
Let $K_t = \mathbb{Z}/p\mathbb{Z}, G_i = \mathrm{HOM}(L_i, \mathbb{Z}/p\mathbb{Z})$.
Let $\varepsilon(*) = 1$, so$\varepsilon = 0$; let $H_i = \mathrm{HOM}(L_i, \mathbb{Z})$ and $(\sigma^\varepsilon_i(f))(x) = f(x) + p\mathbb{Z}, M_\varepsilon = \mathrm{Rang}(\sigma^\varepsilon_\lambda)$ for $i \leq j$ let $\pi_{i,j} : G_j \to G_i$ is $\pi_{i,j}(f) = f \restriction G_i$. We know that $r_p(\mathrm{Ext}(G, \mathbb{Z}))$ is $(G_\lambda : M_0)$. By assumption $(d)$ for each $i < \lambda$ we can choose $g^*_i \in G_\lambda \backslash M_\varepsilon$ such that $g^*_i \restriction L_i$ is zero. The rest is left to the reader (using 3.8 using any lean $\lambda$-system $\mathscr{Y}$ with $G_i, K_t, \varepsilon(*), \pi_{i,j}, \sigma^\varepsilon_\lambda$ as above (and $D_\delta$ for limit ordinal $< \lambda$, any ultrafilter as in 3.2). $\square_{3.9}$

## §4 Back to the $p$-rank of Ext

For consistency of "no examples" see [MRSh 314].

**4.1 Definition.** 1) Let

$$\Xi_{\mathbb{Z}} = \big\{\bar\lambda : \bar\lambda = \langle \lambda_p : p < \omega \text{ prime or zero}\rangle \text{ and for some abelian } (\aleph_1\text{-free}) \text{ group } L, \lambda_p = r_p(\mathrm{Ext}(G,\mathbb{Z}))\big\}.$$

2) For an abelian group $G$ let $\mathrm{rk}(G) = \mathrm{Min}\{\mathrm{rk}(G') : G/G' \text{ is free}\}$.
Clearly $\Xi_{\mathbb{Z}}$ is closed under products. Let $\mathbf{P}$ be the set of primes.

Remember that (see [Sh:f, AP], 2.7, 2.7A, 2.13(1),(2)).
4.2 Fact: In the Easton model if $G$ is $\aleph_1$-free not free, $G' \subseteq G, |G'| < |G| \Rightarrow G/G'$ not free then $r_0(\mathrm{Ext}(G,\mathbb{Z})) = 2^{|G|}$.

4.3 Fact: 1) Assume $\mu$ is a strong limit cardinal $> \aleph_0$, $\mathrm{cf}(\mu) = \aleph_0, \lambda = \mu^+, 2^\mu = \mu^+$ and some $Y \subseteq [{}^\omega\mu]^{\lambda^+}$ is $\mu$-free, (equivalently $\mu^+$-free, see in proof).
Let $\mathbf{P}_0, \mathbf{P}_1$ be a partition of the set of primes.
Then for some $\aleph_1$-free abelian group $L, |L| = \mu^+, 2^\lambda = r_0(\mathrm{Ext}(G,\mathbb{Z}))$ and $p \in \mathbf{P}_1 \Rightarrow r_p(\mathrm{Ext}(G,\mathbb{Z})) = 2^\lambda$ and $p \in \mathbf{P}_0 \Rightarrow r_p(\mathrm{Ext}(G,\mathbb{Z})) = 0$.

*Remark.* On other cardinals see [MRSh 314], close to [MkSh 418, Th.12].

*Proof.* For notational simplicity assume $\mathbf{P}_0 \neq \emptyset$. Let $Y = \{\eta_i : i < \lambda\}$. Let pr:$\mu^2 \to \mu$ be a pairing function, so $pr(pr_1(\alpha), pr_2(\alpha)) = \alpha$. Without loss of generality $\eta_i(n) = \eta_j(m) \Rightarrow n = m \;\&\; \eta_i \restriction m = \eta_j \restriction m$. Let $L$ be $\bigoplus\limits_{\alpha<\lambda} \mathbb{Z}x_\alpha$. Let $\langle (p_i, f_i) : i < \lambda^+\rangle$ list the pairs $(p,f)$ where $p \in \mathbf{P}_0$ and $f \in \mathrm{HOM}(L, \mathbb{Z}/p\mathbb{Z})$. We shall choose $(g_i, \nu_i, \rho_i)$ by induction on $i < \lambda$ such that:

$\boxtimes(\alpha)$ $g_i \in \mathrm{HOM}(L,\mathbb{Z})$

$(\beta)$ $(\forall x \in L)[g_i(x)/p\mathbb{Z} = f_i(x)]$

$(\gamma)$ $\rho_i, \nu_i \in {}^\omega\mu$ and $\eta_i(n) = pr_1(\nu_i(n)) = pr_1(\rho_i(n))$

$(\delta)$ $(\forall j \leq i)(\exists n < \omega)(\forall m)[n \leq m < \omega \to g_j(x_{\nu_i(m)}) = g_j(x_{\rho_i(m)})$

$(\varepsilon)$ $(\forall j < i)(\exists n < \omega)$ [for some sequence $\langle b_m : m \in [n,\omega)\rangle$ of natural numbers we have $n \leq m < \omega \Rightarrow (\prod\limits_{p \in \mathbf{P}_0 \cap n} p) b_{m+1} = b_m + g_i(x_{\nu_j(m)}) - g_i(x_{\rho_j(m)})$]

$(\zeta)$ $\nu_i(m) \neq \rho_i(m)$ for $m < \omega$.

Arriving to $i$ first choose a function let $h_i : i \to \omega$ be such that $j < i \Rightarrow h_i(j) > p_j$ and $\langle \{\eta_j \restriction \ell : \ell \in [h_i(j), \omega)\} : j < i \rangle$ is a sequence of pairwise disjoint sets (possible as $Y$ is $\mu^+$-free). Second choose $g_i$ such that clauses $(\varepsilon) + (\beta)$ holds with $n = h_i(j)$, this is possible as the choice of $h$ splits the problem, that is, the various cases of $(\varepsilon)$ (one for each $j$) does not conflict. More specifically, first choose $g \restriction \{x_\alpha : (\forall j < i)(\forall \ell)(h_i(j) \le \ell < \omega \to \alpha \ne \eta_j(\ell))$ as required in $(\beta)$, possible as $L$ is free. Second by induction on $m \ge h_i(j)$ we choose $b_{m+1}$ such that $0/p\mathbb{Z} = b_{m+1}/p_i\mathbb{Z} + f_i(x_{\nu_j(m)}) - f_i(x_{\rho_j(m)})$ and then choose $g_i(x_{\nu_j(m)}), g(x_{\rho_j(m)})$ such that the $m$-th equation in clause $(\varepsilon)$ for $j$ holds. Let $i = \bigcup\limits_{n<\omega} A^i_n$ be such that $A^i_n \subseteq A^i_{n+1}$ and $|A^i_n| < \mu$. Now choose by induction on $n, \rho_i(n), \nu_i(n)$ as distinct ordinals $\in \{\alpha \in \mu : \alpha \notin \{\nu_i(m), \rho_i(m) : m < m\}$ and $pr_1(\alpha) = \eta_i(n)\}$ such that $\langle g_j(x_{\nu_i(\alpha)}) : j \in A^i_n\rangle = \langle g_j(x_{\rho_i(m)}) : j \in A^i_n\rangle$. So we have carried the induction.
Let $G$ be generated by $L \cup \{y_{i,m} : i < \lambda, m < \omega\}$ freely except that (the equations of $L$ and) $(\prod\limits_{p \in \mathbf{P}_0 \cap n} p) y_{i,n+1} = y_{i,n} + x_{\nu_i(n)} - x_{\rho_i(n)}$.

Why is the abelian group as required?

$\boxtimes_1$ $G$ is $\mu^+$-free
[Why? As $\langle \eta_\alpha : \alpha < \mu^+\rangle$ is and clause $(\gamma)$.]

$\boxtimes_2$ if $p \in \mathbf{P}_0$, <u>then</u> $r_p(Ext(G,\mathbb{Z})) = 0$.
[Why? So let $f \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ and we should find $g \in \mathrm{Hom}(G,\mathbb{Z})$ such that $f = g/p\mathbb{Z}$. Clearly for some $i < \mu^+$ we have $(p_i, f_i) = (p, f)$, now $g_i$ was chosen such that we can extend $g_i$ to a homomorphism $g_{i,i}$ from $G_i =: \langle L \cup \{y_{j,n} : j < i, n < \omega\}\rangle_G$ to $\mathbb{Z}$ such that $g_{i,i}(x)/p\mathbb{Z} = f(x)$ and if $j < i$ we choose $n^{i,j}$ and $\langle b^{i,j}_m : m \in [n^{i,j}, \omega)\rangle$ are as required in closed $(\varepsilon)$, we let $g_{i,i}(y_{j,m}) = b_m$ for $m \in [n^{ij}, \omega)$. Lastly, we define by induction on $j \in [i, \mu^+]$ a homomorphism $g_{i,j}$ from $G_j$ into $\mathbb{Z}$ such that $g_{i,j}(x)/p\mathbb{Z} = f(x)$ for $x \in G_j, g_{i,j}$ is increasing with $j$. For $j = i$ this was done, for limit take union and for $j = \varepsilon + 1$, by clause $(\delta)$ of $\boxtimes$ we know that for some $n = n^{i,j}$ we have $m[n,\omega) \Rightarrow g_i(x_{\nu_i(m)}) = g_i(x_{\rho_i(n)})$, so for $m \in [n, \omega)$ we let $g_{i,\varepsilon+1}(y_{\varepsilon,n}) = 0$ and solve the equations to determine $g_{i,\varepsilon+1}(y_{\varepsilon,n})$ by downward induction.]

$\boxtimes_3$ if $p \in \mathbf{P}_1$, <u>then</u> $r_p(Ext(G,\mathbb{Z})) = 2^\mu$.
[Why? Because every $h \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z}))$ has $> 1$ extensions to $h' \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$ hence $\mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ has cardinality $2^{\mu^+} > 2^\mu$, whereas every $h \in \mathrm{Hom}(L, \mathbb{Z})$ has at most one extension to $h^+ \in \mathrm{Hom}(G,\mathbb{Z})$, so the result follows.]

$\boxtimes_4$ $r_0(\mathrm{Ext}(G,\mathbb{Z})) = 2^{\mu^+}$
[Why? Similar to $\boxtimes_3$, i.e. using cardinality considerations.]

$\square_{4.3}$

<u>4.4 Question</u>: Do we have compactness for singular for $\mathrm{Ext}_p(G, \mathbb{Z}) = 0$?

**4.5 Claim.** *[Omitted, see [Sh 724] and x.x.]*

<u>4.6 Question</u>: If $\bar{\lambda} \in \Xi_{\mathbb{Z}}$ can we derive $\bar{\lambda}' \in \Xi_{\mathbb{Z}}$ by increasing some $\lambda_p$'s?

<u>4.7 Fact</u>: If $\bar{\lambda}^i = \langle \lambda_p^i : p \in \mathbf{P} \cup \{0\} \rangle \in \Xi_{\mathbb{Z}}$ for $i < \alpha$ and $\lambda_p = \prod\limits_{i<\alpha} \lambda_p^i$, then $\langle \lambda_p : p \in \mathbf{P} \cup \{0\} \rangle \in \Xi_{\mathbb{Z}}$.

*Proof.* As if $G = \bigoplus\limits_{i<\alpha} G_i$ then $\mathrm{Ext}(G, \mathbb{Z}) = \prod\limits_{i<\lambda} \mathrm{Ext}(G, \mathbb{Z})$ hence $r_p(\mathrm{Ext}(G, \mathbb{Z})) = \prod\limits_{i<\alpha} r_p(\mathrm{Ext}(G_i, \mathbb{Z}))$.

<u>4.8 Concluding Remark</u>: In [EkSh 505] the statement "there is a $W$-abelian group" is characterized.

We can similarly characterize "there is a separable group". We have the same characterization for "there is a non-free abelian group" such that for some $p$, $r_p(\mathrm{Ext}(G, \mathbb{Z})) = 0$.

<u>Question</u>: What can $\mathbf{P}^* = \{p : p \text{ prime and } \bar{\lambda} \in \Xi_{\mathbb{Z}} \ \& \ \lambda_0 > 0 \Rightarrow \lambda_p > 0\}$ be (if $V = L$ it is $\emptyset$, in 4.5 it is $\mathbf{P}$, are there other possibilities?)

**4.9 Claim.** *If $\lambda$ is strong inaccessible or $\lambda = \mu^+, \mu$ strong limit singular of cofinality $\aleph_0, S \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$ is stationary not reflecting and $\diamondsuit^*_S$ and $\mathbf{P}_0$ a set of primes, <u>then</u> there is a $\lambda$-free abelian group $G$ such that $r_0(Ext(G, \mathbb{Z})) = 2^\lambda = 0$ and: $p \in \mathbf{P}_0 \Rightarrow r_p(Ext(G, \mathbb{Z})) = 2^\lambda$ and $p$ prime and $p \notin \mathbf{P}_0 \Rightarrow r_p(Ext(G, \mathbb{Z}) = 0$.*

## §5 Strong limit of countable cofinality

We continue [GrSh 302] and [GrSh 302a].

**5.1 Definition.** 1) We say $\mathscr{A}$ is a $(\lambda, \mathbf{I})$-system if $\mathscr{A} = (\lambda, \mathbf{I}, \bar{G}, \bar{H}^*, \bar{\pi}, \bar{\sigma})$ where $\bar{G} = \langle G_\alpha : \alpha \le \omega \rangle, \bar{H} = \langle \bar{H}^t : t \in \mathbf{I} \rangle, \bar{H}^t = \langle H^t_\alpha : \alpha \le \omega \rangle, \bar{\pi} = \langle \pi_{\alpha,\beta}, \pi^t_{\alpha,\beta} : \alpha \le \beta \le \omega, t \in \mathbf{I} \rangle, \bar{\sigma} = \langle \sigma^t_\alpha : t \in \mathbf{I}, \alpha \le \omega \rangle)$ satisfies (we may write $\lambda^{\mathscr{A}}, \pi^{t,\mathscr{A}}_{\alpha,\beta}$, etc.)

- $(A)$ $\lambda$ is $\aleph_0$ or generally a cardinal of cofinality $\aleph_0$
- $(B)$ $\langle G_m, \pi_{m,n} : m \le n < \omega \rangle$ is an inverse system of groups whose inverse limit is $G_\omega$ with $\pi_{n,\omega}$ such that $|G_n| \le \lambda$. (So $\pi_{m,n}$ is a homomorphism from $G_n$ to $G_m, \alpha \le \beta \le \gamma \le \omega \Rightarrow \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\beta}$ and $\pi_{\alpha,\alpha}$ is the identity).
- $(C)$ $\mathbf{I}$ is an index set of cardinality $\le \lambda$. For every $t \in \mathbf{I}$ we have $\langle H^t_m, \pi^t_{m,n} : m \le n < \omega \rangle$ is an inverse system of groups and $H^t_\omega$ with $\pi^t_{n,\omega}$ being the corresponding inverse limit $H^t_\omega$ with $\pi^t_{m,\omega}$ and $H^t_m$ has cardinality $\le \lambda$.
- $(D)$ for every $t \in \mathbf{I}, \sigma^t_n : H^t_n \to G_n$ is a homomorphism such that all diagrams commute (i.e. $\pi_{m,n} \circ \sigma^t_n = \sigma^t_m \circ \pi^t_{m,n}$ for $m \le n < \omega$), and let $\sigma^t_\omega$ be the induced homomorphism from $H^t_\omega$ into $G_\omega$
- $(E)$ $G_0 = \{e_{G_0}\}, H^t_0 = \{e_{H^t_0}\}$ (just for simplicity).

2) We say $\mathscr{A}$ is strict if $|G_n| < \lambda, |H^t_n| < \lambda, |\mathbf{I}| < \lambda$. Let $\mathscr{E}_t$ be the following equivalence relation on $G_\omega : f \mathscr{E}_t g$ iff $fg^{-1} \in$ Rang$(\sigma^t_\omega)$.
3) Let nu$(\mathscr{A}) = \sup\{\mu :$ for each $n < \omega,$ there is a sequence $\langle f_i : i < \mu \rangle$ such that $f_i \in G_\omega$ and $\mu \le \lambda \Rightarrow \pi_{n,\omega}(f_i) = \pi_{n,\omega}(f_0)$ for $i < \mu$ and $i < j < \mu \ \& \ t \in I \Rightarrow \neg f_i \mathscr{E}_t f_j\}$.
We write nu$(\mathscr{A}) =^+ \mu$ to mean that moreover the supremum is obtained. Let nu$^+(\mathscr{A})$ be the first $\mu$ such that for $n = 0$, there is no $\langle f_i : i < \mu \rangle$ as above (so nu$(\mathscr{A}) \le$ nu$^+(\mathscr{A})$ and if nu$(\mathscr{A}) > \mu$ then nu$(\mathscr{A}) \le$ nu$^+(\mathscr{A}) \le$ nu$(\mathscr{A})^+$ and nu$(\mathscr{A}) <$ nu$^+(\mathscr{A})$ implies nu$(\mathscr{A})$ is a limit cardinal and the supremum not obtained).
4) We say $\mathscr{A}$ is an explicit $(\bar{\lambda}, \bar{\mathbf{J}})$-system if: $\mathscr{A} = (\bar{\lambda}, \bar{\mathbf{J}}, \bar{G}, \bar{H}, \bar{\pi}, \bar{\sigma})$ and

- $(\alpha)$ $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle, \bar{\mathbf{J}} = \langle \mathbf{J}_n : n < \omega \rangle$
- $(\beta)$ $\lambda_n < \lambda_{n+1}, \mathbf{J}_n \subseteq \mathbf{J}_{n+1}$,
- $(\gamma)$ letting $\lambda^{\mathscr{A}} = \sum_{n<\omega} \lambda_n, \mathbf{I}^{\mathscr{A}} = \bigcup_{n<\omega} \mathbf{J}_n$ we have sys$(\mathscr{A}) =: (\lambda, \mathbf{I}, \bar{G}, \bar{H}, \bar{\pi}, \bar{\sigma})$ is a $(\lambda, \mathbf{I})$-system
- $(\delta)$ $|\mathbf{J}_n| \le \lambda_n, |G_n| \le \lambda_m, |H^t_n| < \lambda$ and $|H^n_t| \le |H^{n+1}_t|$.

5) We add in (4), full if

($\varepsilon$) $|H^t_n| \le \lambda_n$.

6) For an explicit $(\lambda, \bar{\mathbf{J}})$-system $\mathscr{A}$ let $\mathrm{nu}^+_*(\mathscr{A}) = \sup\{\mu^+$:for every $n < \omega$ there is a sequence $\langle f_i : i < \mu\rangle$ such that $f_i \in G$, and $\mu \le \lambda \Rightarrow \pi_{n,\omega}(p_i) = \pi_{n,\omega}(f_0)$ for $i < \mu$ and $i < j < \mu \;\&\; t \in \mathbf{J}_n \Rightarrow \neg f_i \mathscr{E}_t f_j\}$.
7) For a $\lambda$-system $\mathscr{A}$, we define $\mathrm{nu}^+_*(\mathscr{A})$ similarly, except we say: for some $\bar{\mathbf{J}} = \langle \mathbf{J}_n : n < \omega\rangle$ such that $\mathbf{I} = \bigcup\limits_{n<\omega} \mathbf{J}_n, \mathbf{J}_n \subseteq \mathbf{J}_{n+1}$.

**5.2 Claim.** *1) For any strict $(\lambda, \mathbf{I})$-system $\mathscr{A}$ there are $\bar\lambda, \bar{\mathbf{J}}$ and an explicit $(\bar\lambda, \bar{\mathbf{J}})$-system $\mathscr{B}$ such that sys$(\mathscr{B}) = \mathscr{A}$ so*

$$\lambda = \sum_{n<\omega} \lambda_n, \mathbf{I} = \bigcup_{n<\omega} \mathbf{J}_n,\ nu(\mathscr{B}) = nu(\mathscr{A})$$

*(and if in one side the supremum is obtained, so in the other).*
*2) For any $(\lambda, \mathbf{I})$-system $\mathscr{A}$ such that $\lambda > 2^{\aleph_0}$ and $nu^+(\mathscr{A}) \ge \mu \ge \lambda$ and $cf(\mu) \notin [\aleph_1, 2^{\aleph_0}]$ there is an explicit $(\bar\lambda, \bar{\mathbf{J}})$-system $\mathscr{B}$ such that $\lambda^{\mathscr{A}} = \sum\limits_{n<\omega} \lambda^{\mathscr{B}}_n, \mathbf{I}^{\mathscr{A}} = \bigcup\limits_{n<\omega} \mathbf{J}^{\mathscr{B}}_n$ and $nu^+(\mathscr{A}) \ge\ nu^+(\mathscr{B}) \ge \mu$.*
*3) In part (2) if $f : Card \cap \lambda \to Card$ is increasing we can demand $\lambda_n \in\ Rang(f)$, $f(\lambda_n) < \lambda_{n+1}$. So if $\lambda$ is strong limit $> \aleph_0$, <u>then</u> we can demand $2^{\lambda^{\mathscr{B}}_n} < \lambda^{\mathscr{B}}_{n+1} = cf(\lambda^{\mathscr{B}}_{n+1})$.*
*4) As in (2), (3) above with $nu^+_*$ instead of $nu^+$.*

*Proof.* 1) Straight.
2) Let $\bar\lambda = \langle \lambda_n : n < \omega\rangle$ be such that $\lambda = \sum\limits_{n<\omega} \lambda_n, 2^{\aleph_0} < \lambda_n < \lambda_{n+1}$, $\mathrm{cf}(\lambda_n) = \lambda_n$. Let $\langle G_{n,\ell} : \ell < \omega\rangle$ be increasing, $G_{n,\ell}$ a subgroup of $G_n$ of cardinality $\le \lambda_\ell$ and $G_n = \bigcup\limits_{\ell<\omega} G_{n,\ell}$. Let $\langle H^t_{n,\ell} : \ell < \omega\rangle$ be an increasing sequence of subgroups of $H^t_n$ with union $H^t_n, |H^t_{n,\ell}| \le \lambda_\ell$. Let $\langle \mathbf{J}_n : n < \omega\rangle$ be an increasing sequence of subsets of $\mathbf{I}$ with union $\mathbf{I}$ such that $|\mathbf{J}_n| \le \lambda_n$.
Without loss of generality $\pi_{m,n}$ maps $G_{n,\ell}$ into $G_{m,\ell}$ and $\pi^t_{m,n}$ maps $H^t_{n,\ell}$ into $H^t_{m,\ell}$ and $\sigma^t_n$ maps $H^t_{n,\ell}$ into $G^t_{n,\ell}$ (why? just close the witness).
Now for every increasing $\eta \in {}^\omega\omega$ we let

$$G^\eta_\omega = \{g \in G_\omega :\ \text{for every } n < \omega \text{ we have } \pi_{n,\omega}(g) \in G_{n,\eta(n)}\}.$$

Clearly

$(*)_1(\alpha)$ $G^\eta_\omega$ is a subgroup of $G_\omega$

$(\beta)$ $\{G^\eta_\omega : \eta \in {}^\omega\omega$ increasing$\}$ is directed, i.e. if $(\forall n < \omega)\eta(n) \le \nu(n))$ where $\eta, \nu \in {}^\omega\omega$ then $G^\eta_\omega \subseteq G^\nu_\omega$

$(\gamma)$ $G_\omega = \cup\{G^\eta_\omega : \eta \in {}^\omega\omega$ (increasing)$\}$.

First assume $\mathrm{cf}(\mu) \ne \aleph_0$ so as $\mathrm{cf}(\mu) > 2^{\aleph_0}$ for some $\eta \in {}^\omega\omega$, strictly increasing, we have

$(*)_2$ $\mu \le \sup\{|X|^+ : X \subseteq G_{\omega,\eta}$ and $t \in \mathbf{I}$ & $f \ne g \in X \Rightarrow fg^{-1} \notin \sigma^t_\omega(H^t_\omega)\}$.

However, as $\lambda \le \mu$, $\mathrm{cf}(\lambda) = \aleph_0$, $\mathrm{cf}(\mu) > 2^{\aleph_0}$ clearly $\mu > \lambda$; also if $X_1, X_2$ are as in $(*)_2$ then for some $X \subseteq X_2$ we have $|X| \le |X_1| + |\mathbf{I}|$ and $X_1 \cup (X_2 \backslash X_2)$ is as required there. So we can choose $\eta \in {}^\omega\omega$, increasing such that

$(*)_3$ there is $X \subseteq G^\eta_\omega$ of cardinality $\mu$ such that $t \in \mathbf{I}$ & $f \ne g \in X \Rightarrow fg^{-1} \notin \sigma^t_\omega(H^t_\omega)$.

Second assume $\mathrm{cf}(\mu) = \aleph_0$, so let $\mu = \sum\limits_{n<\omega} \mu_n, \mu_n < \mu_{n+1}$, and without loss of generality $\lambda_n < \mu_n = \mathrm{cf}(\mu_n)$ and $\mu > \lambda \Rightarrow \mu_n > \lambda$. If $\mu > \lambda$, for each $n$ there is a witness $\langle f^n_\alpha : \alpha < \mu_n\rangle$ to $\mathrm{nu}^+(\mathscr{A}) > \mu_n$, so $f^n_\alpha \in G^{\mathscr{A}}_\omega$ and as $\mu_n > \lambda \ge |G^{\mathscr{A}}_n|$, without loss of generality $\pi_{n,\omega}(f^n_\alpha) = \pi_{n,\omega}(f^0_\alpha)$ so as we can replace $f^n_\alpha$ by $f^n_{\alpha+1}(f^n_0)^{+1}$, without loss of generality $m \le n \Rightarrow \pi_{m,\omega}(f^n_\alpha) = e_G$. For each $\alpha$ let $\eta^n_\alpha \in {}^\omega\omega$ be increasing be such that $\pi_{n,\omega}(f^n_\alpha) \in G_{n,\eta_\alpha(n)}$. As $2^{\aleph_0} <$ $\mathrm{cf}(\mu_n) = \mu_n$, for some increasing $\eta_n \in {}^\omega\omega$ we have $(\exists^{\mu_n}\alpha < \mu_n), \eta^n_\alpha = \eta_n$. So, hence without loss of generality $\alpha < \mu \Rightarrow \eta^n_\alpha = \eta_n$. Let $\eta \in {}^\omega\omega$ be $\eta(n) =$ $\mathrm{Max}\{\eta_n(n) : m \le n\}$. So we have $n < \omega$ & $\alpha < \mu_n \Rightarrow \pi_{n,\omega}(f^n_\alpha) \in G_n$. So

$(*)_4$ for every $n < \omega$ and $\mu'_0 < \mu$ (in fact even $\mu_i = n$) there are $f_\alpha \in G^\eta_\omega$ for $\alpha < \mu'$ such that $\mu \le \lambda \Rightarrow \pi_{n,\omega}(f_\alpha) = e_{G_n}$ and $\alpha < \beta < \mu'$ & $t \in \mathbf{I} \Rightarrow fg^{-1} \notin \sigma^t_\omega(H^t_\omega)$.

Lastly, if $\mu = \lambda$, so $\mathrm{cf}(\mu) = \aleph_0$ the proof is as in the case $\mu > \lambda$ & $\mathrm{cf}(\mu) = \aleph_0$, except that $\pi_{n,\omega}(f^n_\alpha) = \pi_{n,o}(f^n_0)$ holds by the choice of $\langle f^n_\alpha : \alpha < \mu_n\rangle$ instead of by "without loss of generality".

For each $t \in \mathbf{J}_n$ and strictly increasing $\nu \in {}^\omega\omega$ let $H^{(t,\nu)}_\omega$ be the subgroup $\{g \in H^t_\omega :$ for every $n < \omega$ we have $\sigma_{n,\omega}(g) \in H^t_{n,\nu(n)}\}$. So let $\mathbf{J}'_n = \{(t,\nu) : t \in \mathbf{J}$ and $\nu \in {}^\omega\omega$ increasing$\}$.

We define $G^\eta_{n,\zeta}$, a subgroup of $G_{n,\eta(n)}$, decreasing with $\zeta$ by induction on $\zeta$:

$\underline{\zeta = 0}$: $G^\eta_{n,\zeta} = G_{n,\eta(n)}$

$\underline{\zeta = \varepsilon + 1}$: $G^\eta_{n,\zeta} = \{x : x \in G^\eta_{n,\varepsilon} \text{ and } x \in \text{ Rang}(\pi_{n,n+1} \restriction G^\eta_{n+1,\varepsilon})$
$\text{and } n > 0 \Rightarrow \pi_{n-1,n}(x) \in G_{n-1,\eta(n-1),\varepsilon}\}$

$\underline{\zeta \text{ limit}}$: $G^\eta_{n,\zeta} = \bigcap\limits_{\varepsilon<\zeta} G^\eta_{n,\varepsilon}$.

Let $G^\eta_n = \bigcap\limits_{\zeta<\lambda^+} G^\eta_{n,\eta(n),\zeta}, \pi^\eta_{m,n} = \pi_{m,n} \restriction G^\eta_n$. Easily $\langle G^\eta_n, \pi^\eta_{m,n} : m \le n < \omega\rangle$ is directed with limit $G^\eta_\omega$ with $\pi^\eta_{n,\omega} = \pi_{n,\omega} \restriction G^\eta_\omega$.

Define $H^{(t,\nu)}_{n,\zeta}, \pi^{(t,\nu)}_{m,n,\zeta}$ (for any $\zeta$), $H^{(t,\nu)}_n, \pi^{(t,\nu)}_{m,n}$ parallely to $G^\eta_n, \pi^\eta_{m,n}$ but such that $\sigma^t_\alpha$ maps $H^{(t,\nu)}_\alpha$ into $G^\eta_\alpha$ (note: element of $H^{(t,\nu)}_\alpha$ not mapped to $G^\eta_\alpha$ are irrelevant). Let $\sigma^{(t,\nu)}_\omega : H^{(t,\nu)}_\omega \to G^\eta_\omega$ be $\sigma^t_\omega \restriction H^{(t,\nu)}_\omega$ and $\sigma^{(t,\sigma)}_n = \sigma^t_n \restriction H^{(t,\nu)}_n$.

We have defined actually $\mathscr{B} = (\bar\lambda^{\mathscr{B}}, \bar{\mathbf{J}}^{\mathscr{B}}, \bar G, \bar H, \bar\pi^{\mathscr{B}}, \bar\sigma^{\mathscr{B}})$ where
$\bar\lambda^{\mathscr{B}} = \langle\lambda_n : n<\omega\rangle, \mathbf{J}^{\mathscr{B}} = \langle\mathbf{J}'_n : n<\omega\rangle, \bar G^{\mathscr{B}} = \langle G^\eta_\alpha : \alpha \le \omega\rangle$,

$$\bar H^{\mathscr{B}} = \left\langle \langle H^x_\alpha : \alpha \le \omega\rangle : x \in \bigcup_n \mathbf{J}'_n \right\rangle,$$

$$\bar\pi^{\mathscr{B}} = \langle \pi^\eta_{\alpha,\beta} : \alpha \le \beta \le \omega\rangle\hat{\ } \left\langle \langle \pi^{(t,\nu)}_{\alpha,\beta} : \alpha \le \beta \le \omega\rangle : (t,\nu) \in \bigcup_n \mathbf{J}'_n \right\rangle \text{ and}$$

$$\bar\sigma^{\mathscr{B}} = \left\langle \langle \sigma^{(t,\nu)}_\alpha : \alpha \le \omega\rangle : (t,\nu) \in \bigcup_{n<\omega} \mathbf{J}'_n \right\rangle.$$

We have almost finished. Still $G^\eta_n$ may be of cardinality $> \lambda_n$ but note that for $k : \omega \to \omega$ non-decreasing with limit $\omega, \langle G^\eta_n : n < \omega\rangle$ can be replaced by $\langle G_{k(n)} : n < \omega\rangle$.

By the definition of $\mathscr{B}, G^{\mathscr{B}}_\omega$ is a subgroup of $G^{\mathscr{A}}_\omega$ and for each $t \in \mathbf{I}$ for some $n, t \in \mathbf{J}_n$ and $H^{\mathscr{A}}_t \cap G^{\mathscr{B}}_\omega = \bigcup\limits_{\eta \in {}^\omega\omega} H^{\mathscr{B}}_{(t,\eta)}$ hence for $f, g \in G^{\mathscr{B}}_\omega \subseteq G^{\mathscr{A}}_\omega$ we have $f \mathscr{E}_t g \Leftrightarrow fg^{-1} \in H^{\mathscr{A}}_t \Leftrightarrow -(\exists h \in H^{\mathscr{A}}_t)(fg^{-1} = h) \Leftrightarrow (\exists \bar h)(\bar h = \langle h_n : n < \omega\rangle\ \&\ h_n = \pi^{t,\mathscr{A}}_{n,n+1}(\sigma h_{n+1}) \cap \bigwedge\limits_{n<\omega} fg^{-1} \restriction n = \sigma^{t,\mathscr{A}}_n(h_n)) \Leftrightarrow -(\exists \bar h) \bigvee\limits_{\nu\in\omega} (\bar h) = \langle h_n : n < \omega\rangle\ \&\ h_n \in H^{t,\mathscr{A}}_{n,\nu(n)}\ \&\ \bigwedge\limits_n = \pi^{t,\mathscr{A}}_{n,n+1}(h_{n+1})\ \&\ \bigwedge\limits_{n<\omega} fg^{-1} \restriction n = \sigma^{t,\mathscr{A}}_n(h_n)) \Leftrightarrow$[2] $\bigvee\limits_{\nu \in {}^\omega\omega} (\exists \bar h)(\bar h = \langle h_n : n < \omega\rangle\ \&\ \bigwedge\limits_n h_n \in H^{t,\mathscr{A}}_{n,\zeta}\ \&\ \bigwedge\limits_n h_n = \pi^{t,\mathscr{A}}_{n,n+1}(h_{n+1})\ \&$

[2]for each $\zeta$ separately, by induction on $T$

$\bigwedge\limits_{n<\omega} fg^{-1} = \sigma^{t,\mathscr{A}}_n(h_n) \Leftrightarrow \bigvee\limits_{\nu\in{}^\omega\omega} (\exists\bar h)(\bar h = \langle h_n : n<\omega\rangle \ \& \ \bigwedge\limits_n h_n \in H^{t,\mathscr{B}}_n \ \& \ \bigwedge\limits_n h_n = \pi^{t,\mathscr{B}}_{n,n+1}(h_{n+1}) \ \& \ \bigwedge\limits_{n<\omega} \pi^{\mathscr{B}}_{n,\omega} fg^{-1}) = \sigma^{t,\mathscr{B}}_n(h_n) \bigvee\limits_{\nu\in{}^\omega\omega} fg^{-1} \in H^{\mathscr{B}}_{(t,\nu)} \Leftrightarrow \bigvee\limits_{\nu\in{}^\omega\omega} f\mathscr{E}_{(t,\nu)}g$; so clearly $\mathrm{nu}^+(\mathscr{B}) \le \mathrm{nu}^+(\mathscr{A})$. But also $\mathrm{nu}^+(\mathscr{B}) > \mu$ by the choice of $\eta$, i.e. by $(*)_3$.
3), 4) Easy. $\square_{5.2}$

For the rest of this section we adopt:

*5.3 Convention.* 1) $\mathscr{A}$ is an explicit $(\bar\lambda, \mathbf{\bar J})$-system, so below $\mathrm{rk}_t(g,f)$ should be written as $\mathrm{rk}_t(g,f,\mathscr{A})$, etc.
2) $\lambda = \sum\limits_{n<\omega} \lambda_n, \lambda_n = \lambda^{\mathscr{A}}_n, \mathbf{J}_n = \mathbf{J}^{\mathscr{A}}_n, \mathbf{I} = \mathbf{I}^{\mathscr{A}} = \bigcup\limits_{n<\omega} \mathbf{J}_n, G_\alpha = G^{\mathscr{A}}_\alpha$, etc.
3) $k_t(n) = \mathrm{Max}\{m : m \le n, |H^t_m| \le \lambda_n\}$ so $k_t : \omega \to \omega$ is non-decreasing converging to $\infty$.
For the reader's convenience we repeat 5.5 - 5.8 from [GrSh 302a].

**5.4 Definition.** 1) For $g \in H^t_\alpha$ let $\mathrm{lev}(g) = \alpha$ (without loss of generality this is well defined).
2) For $\alpha \le \beta \le \omega, g \in H^t_\beta$ let $g \restriction H^t_\alpha = \pi^t_{\alpha,\beta}(g)$ and we say $g \restriction H^t_\alpha$ is below $g$ and $g$ is above $g \restriction H^t_\alpha$ or extend $g \restriction H^t_\alpha$.
3) For $\alpha \le \beta \le \omega, f \in G_\beta$ let $f \restriction G_\alpha = \pi_{\alpha,\beta}(f)$.
We will now describe the rank function used in the proof of the main theorem.

**5.5 Definition.** 1) For $g \in H^t_n, f \in G_\omega$ we say that $(g,f)$ is a nice $t$-pair if $\sigma^t_n(g) = f \restriction G_n$.
2) Define, for $t \in \mathbf{I}$, a ranking function $\mathrm{rk}_t(g,f)$ for any nice $t$-pair. First by induction on the ordinal $\alpha$ (we can fix $f \in G_\omega$), we define when $\mathrm{rk}_t(g,f) \ge \alpha$ simultaneously for all $n < \omega, g \in H^t_n$

(*a*) $\mathrm{rk}_t(g,f) \ge 0$ <u>iff</u> $(g,f)$ is a nice $t$-pair
(*b*) $\mathrm{rk}_t(g,f) \ge \delta$ for a limit ordinal $\delta$ <u>iff</u> for every $\beta < \delta$ we have $\mathrm{rk}_t(g,f) \ge \beta$
(*c*) $\mathrm{rk}_t(g,f) \ge \beta+1$ <u>iff</u> $(g,f)$ is a nice $t$-pair, and letting $n = \mathrm{lev}(g)$ there exists $g' \in H^t_{n+1}$ extending $g$ such that $\mathrm{rk}_t(g',f) \ge \beta$
(*d*) $\mathrm{rk}_t(g,f) \ge -1$.

3) For $\alpha$ an ordinal or $-1$ (stipulating $-1 < \alpha < \infty$ for any ordinal $\alpha$) we let $\mathrm{rk}_t(g,f) = \alpha$ <u>iff</u> $\mathrm{rk}_t(g,f) \ge \alpha$ and it is false that $\mathrm{rk}_t(g,f) \ge \alpha+1$.
4) $\mathrm{rk}_t(g,f) = \infty$ iff for every ordinal $\alpha$ we have $\mathrm{rk}_t(g,f) \ge \alpha$.
The following two claims give the principal properties of $\mathrm{rk}_t(g,f)$.

**5.6 Claim.** *Let $(g, f)$ be a nice $t$-pair.*
*1) The following statements are equivalent:*

(a) $rk_t(g, f) = \infty$
(b) *there exists* $g' \in H^t_\omega$ *extending* $g$ *such that* $\sigma^t_\omega(g') = f$.

*2) If* $rk_t(g, f) < \infty$, *<u>then</u>* $rk_t(g, f) < \mu^+$ *where* $\mu = \sum_{n<\omega} 2^{\lambda_n}$ *(for* $\lambda$ *strong limit,* $\mu = \lambda$*).*
*3) If* $g'$ *is a proper extension of* $g$ *and* $(g', f)$ *is also a nice* $t$*-pair <u>then</u>*

($\alpha$) $rk_t(g', f) \leq rk_t(g, f)$ *and*
($\beta$) *if* $0 \leq rk_t(g, f) < \infty$ *<u>then</u> the inequality is strict.*

*4) For* $f_1, f_2 \in G^{\mathscr{A}}_\omega, n < \omega$ *and* $t \in \bigcup_{n<\omega} \mathbf{J}_n$ *we have* $f_1 \mathscr{E}_t f_2$ *iff* $rk_t(g, f_1 f_2^{-1}) = \infty$ *for some* $g \in H^{\mathscr{A}}_n$.

*Proof.*
1) <u>Statement (a) $\Rightarrow$ (b)</u>.
Let $n$ be the value such that $g \in H^t_n$. If we will be able to choose $g_k \in H^t_k$ for $k < \omega, k \geq n$ such that

(i) $g_n = g$
(ii) $g_k$ is below $g_{k+1}$ that is $\pi^t_{k,k+1}(g_{k+1}) = g_k$ and
(iii) $\mathrm{rk}_t(g_k, f) = \infty$,

<u>then</u> clearly we will be done since $g' =: \lim_k g_k$ is as required. The definition is by induction on $k \geq n$.
For $k = n$ let $g_0 = g$.
For $k \geq n$, suppose $g_k$ is defined. By (iii) we have $\mathrm{rk}_t(g_k, f) = \infty$, hence for every ordinal $\alpha$, $\mathrm{rk}_t(g, f) > \alpha$ hence there is $g^\alpha \in H^t_{k+1}$ extending $g$ such that $\mathrm{rk}_t(g^\alpha, f) \geq \alpha$. Hence there exists $g^* \in H^t_{k+1}$ extending $g_k$ such that $\{\alpha : g^\alpha = g^*\}$ is unbounded hence $\mathrm{rk}_t(g^*, f) = \infty$, and let $g_{k+1} =: g^*$.

<u>Statement (b) $\Rightarrow$ (a)</u>.
Since $g$ is below $g'$, it is enough to prove by induction on $\alpha$ that for every $k \geq n$ when $g_k =: g' \restriction H^t_k$ we have that $\mathrm{rk}_t(g, f) \geq \alpha$.
For $\alpha = 0$, since $\sigma^t_\omega(g') = f \restriction G_n$ clearly for every $k$ we have $\sigma^t_k(g_k) = f \restriction G_k$ so $(g_k, f)$ is a nice $t$-pair.

For limit $\alpha$, by the induction hypothesis for every $\beta < \alpha$ and every $k$ we have $\mathrm{rk}_t(g_k, f) \geq \beta$, hence by Definition 5.5(2)(b), $\mathrm{rk}_t(g_k, f) \geq \alpha$.

For $\alpha = \beta+1$, by the induction hypothesis for every $k \geq n$ we have $\mathrm{rk}_t(g_k, f) \geq \beta$. Let $k_0 \geq n$ be given. Since $g_{k_0}$ is below $g_{k_0+1}$ and $\mathrm{rk}_t(g_{k_0+1}, f) \geq \beta$, Definition 5.5(2)(c) implies that $\mathrm{rk}_t(g_{k_0}, f) \geq \beta+1$; i.e. for every $k \geq n$ we have $\mathrm{rk}_t(g_k, f) \geq \alpha$. So we are done.

2) Let $g \in H^t_n$ and $f \in G_\omega$ be given. It is enough to prove that if $\mathrm{rk}_t(g, f) \geq \mu^+$ then $\mathrm{rk}_t(g, f) = \infty$. Using part (1) it is enough to find $g' \in H^t_\omega$ such that $g$ is below $g'$ and $f = \sigma^t_\omega(g')$.

We choose by induction on $k < \omega, g_k \in H^t_{n+k}$ such that $g_k$ is below $g_{k+1}$ and $\mathrm{rk}_t(g_k, f) \geq \mu^+$. For $k = 0$ let $g_k = g$. For $k+1$, for every $\alpha < \mu^+$, as $\mathrm{rk}_t(g_k, f) > \alpha$ by 5.5(2)(c) there is $g_{k,\alpha} \in G_{n+k+1}$ extending $g_k$ such that $\mathrm{rk}_t(g_{k,\alpha}, f) \geq \alpha$. But the number of possible $g_{k,\alpha}$ is $\leq |H^t_{n+k+1}| \leq 2^{\lambda_{n+k+1}} < \mu^+$ hence there are a function $g$ and a set $S \subseteq \mu^+$ of cardinality $\mu^+$ such that $\alpha \in S \Rightarrow g_{k,\alpha} = g$. Then take $g_{k+1} = g$.

3) Immediate from the definition.

4) Check the definitions. $\square_{5.6}$

**5.7 Lemma.** *1) Let $(g, f)$ be a nice $t$-pair. <u>Then</u> we have $rk(g, f) \leq rk(g^{-1}, f^{-1})$.*
*2) For every nice $t$-pair $(g, f)$ we have $rk(g, f) = rk(g^{-1}, f^{-1})$.*

*Proof.* 1) By induction on $\alpha$ prove that $\mathrm{rk}(g, f) \geq \alpha \Rightarrow \mathrm{rk}(g^{-1}, f^{-1}) \geq \alpha$ (see more details in the proof of Lemma 5.8).

2) Apply part (1) twice. $\square_{5.7}$

**5.8 Lemma.** *1) Let $n < \omega$ be fixed, and let $(g_1, f_1), (g_2, f_2)$ be nice $t$-pairs with $g_\ell \in H^t_n (\ell = 1, 2)$. <u>Then</u> $(g_1 g_2, f_1 f_2)$ is a nice pair and $rk_t(g_1 g_2, f_1 f_2) \geq Min\{rk_t(g_\ell, f_\ell) : \ell = 1, 2\}$.*
*2) Let $n, (f_1, g_1)$ and $(f_2, g_2)$ be as above. If $rk_t(g_1, f_1) \neq rk_t(g_2, f_2)$, <u>then</u> $rk_t(g_1 g_2, f_1 f_2) = Min\{rk_t(g_\ell, f_\ell) : \ell = 1, 2\}$.*

*Proof.* 1) It is easy to show that the pair $(g_1 f_2, f_1, f_2)$ is $t$-nice. We show by induction on $\alpha$ simultaneously for all $n < \omega$ and every $g_1, g_2 \in H^t_n$ that $\mathrm{Min}\{\mathrm{rk}(g_\ell, f_\ell) : \ell = 1, 2\} \geq \alpha$ implies that $\mathrm{rk}(g_1 g_2, f_1 f_2) \geq \alpha$.

When $\alpha = 0$ or $\alpha$ is a limit ordinal this should be clear. Suppose $\alpha = \beta + 1$ and that $\mathrm{rk}(g_\ell, f_\ell) \geq \beta + 1$ for $\ell = 1, 2$; by the definition of rank for $\ell = 1, 2$ there exists $g'_\ell \in H^t_{n+1}$ extending $g_\ell$ such that $(g'_\ell, f_\ell)$ is a nice pair and $\mathrm{rk}_t(g'_\ell, f_\ell) \geq \beta$. By the induction assumption $\mathrm{rk}_t(g'_1 g'_2, f_1 f_2) \geq \beta$ and clearly $(g'_1 g'_2) \restriction n = g_1 g_2$. Hence $g'_1 g'_2$ is as required in the definition of $\mathrm{rk}_t(g_1 g_2, f_1 f_2) \geq \beta + 1$.

2) Suppose without loss of generality that $\mathrm{rk}(g_1, f_1) < \mathrm{rk}(g_2, f_2)$, let $\alpha_1 = \mathrm{rk}(g_1, f_1)$ and let $\alpha_2 = \mathrm{rk}_t(g_2, f_2)$. By part (1), $\mathrm{rk}_t(g_1g_2, f_1f_2) \geq \alpha_1$, by Proposition 5.7, $\mathrm{rk}_t(g_2^{-1}, f_2^{-1}) = \alpha_2 > \alpha_1$. So we have

$$\begin{aligned}\alpha_1 = \mathrm{rk}_t(g_1, f_1) = &\ \mathrm{rk}_t(g_1g_2g_2^{-1}, f_1f_2f_2^{-1}) \\ \geq &\ \mathrm{Min}\{\mathrm{rk}_t(g_1g_2, f_1f_2), \mathrm{rk}_t(g_2^{-1}, f_2^{-1})\} \\ = &\ \mathrm{Min}\{\mathrm{rk}_t(g_1g_2, f_1f_2), \alpha_2\} \geq \mathrm{Min}\{\alpha_1, \alpha_2\} =\geq \alpha_1.\end{aligned}$$

Hence the conclusion follows. $\square_{5.8}$

**5.9 Theorem.** *Assume ($\mathscr{A}$ is an explicit $\lambda$-system and)*

- *(a) $\lambda$ is strong limit $\lambda >$ $cf(\lambda) = \aleph_0$*
- *(b) $nu(\mathscr{A}) \geq \lambda$ or just $nu_*^+(\mathscr{A}) \geq \lambda$.*

*Then $nu(\mathscr{A}) =^+ 2^\lambda$.*

The proof is broken into parts.
5.10 Fact: We can choose by induction on $n, \langle f_{n,i} : i < \lambda_n\rangle$ such that

- ($\alpha$) $f_{n,i} \in G_\omega$ and $f_{n,i} \restriction G_{n+1} = e_{G_{n+1}}$
- ($\beta$) $i < j < \lambda_n$ & $t \in \mathbf{J}_n \Rightarrow \neg f_{n,i} \mathscr{E}_t f_{n,j}$
- ($\gamma$) $\mathrm{rk}_t(g, f_{n,i}f_{n,j}^{-1}) < \infty$ for any $t \in \mathbf{J}_n, k \leq n, g \in H_k^t$ and $i \neq j < \lambda_n$
- ($\delta$) if $f^*$ belongs to the subgroup $K_n$ of $G_\omega$ generated by the $\{f_{m,j} : m < n, j < \lambda_m\}$ and $t \in \mathbf{J}_n, g \in \bigcup\limits_{m \leq k_t(n)} H^t_{k_t(n)}$, then for every $i_0 < i_1 < i_2 < i_3 < \lambda_n$ each of the following statements have the same truth value, (i.e. the truth value does not depend on $(i_0, i_1, i_2, i_3)$)
  - ($i$) $\mathrm{rk}_t(g, f_{n,i_1}f_{n,i_0}^{-1}f^*f_{n,i_2}f_{n,i_3}^{-1}) < \infty$
  - ($ii$) $\mathrm{rk}_t(g, f_{n,i_3}f_{n,i_2}^{-1}f^*f_{n,i_0}f_{n,i_1}^{-1}) < \infty$
  - ($iii$) $\mathrm{rk}_t(e_{H^t_{k_t(n)}}, f_{n,i_1}f_{n,i_0}^{-1}) < \mathrm{rk}_t(g, f^*)$
  - ($iv$) $\mathrm{rk}_t(e_{H^t_{k_t(n)}}, f_{n,i_1}f_{n,i_0}^{-1}) > \mathrm{rk}_t(g, f^*)$
  - ($v$) $\mathrm{rk}_t(g, f^*) < \mathrm{rk}_t(g, f_{n,i_0}f_{n,i_1}^{-1}f^*f_{n,i_2}f_{n,i_3}^{-1})$
  - ($vi$) $\mathrm{rk}_t(g, f^*) < \mathrm{rk}_t(g, f_{n,i_2}f_{n,i_3}^{-1}f^*f_{n,i_0}f_{n,i_1}^{-1})$
  - ($vii$) $\mathrm{rk}_t(g, f_{i_0}f_{i_1}^{-1}) < \infty$
  - ($viii$) $\mathrm{rk}_t(g, f_{i_1}f_{i_0}^{-1}) < \infty$

$(\varepsilon)$ for each $t \in \mathbf{J}_n$ one of the following occurs:

$(a)$ for $i_0 < i_1 \leq i_2 < i_3 < \lambda_n$ we have
$\mathrm{rk}_t(e_{H^t_{k_t(n)}}, f_{n,i_0} f^{-1}_{n,i_1}) < \mathrm{rk}(e_{H^t_{k_t(n)}}, f_{n,i_2} f^{-1}_{n,i_3})$

$(b)$ for some $\gamma^n_t$ for every $i < j < \lambda_n$ we have
$\gamma^n_t = \mathrm{rk}_t(e_{H^t_{k_t(n)}}, f_{n,i} f^{-1}_{n,j})$.

*Proof.* We can satisfy clauses $(\alpha), (\beta)$ by the definitions and clause $(\gamma)$ follows. Now clause $(\delta)$ is straight by Erdös Rado Theorem applied to a higher $n$.
For clause $(\varepsilon)$ notice the transitivity of the order and of equality and "there is no decreasing sequence of ordinals of length $\omega$". $\square_{5.10}$

*5.11 Notation.* For $\alpha \leq \omega$ let $T_\alpha = \times_{k<\alpha} \lambda_k, T =: \bigcup\limits_{n<\omega} T_n$ (note: by the partial order $\triangleleft, T$ is a tree; treeness will be used).

**5.12 Definition.** Now by induction on $n < \omega$, for every $\eta \in \times_{m<n} \lambda_m$ we define $f_\eta \in G_\omega$ as follows:

<u>for $n = 0$</u>: $f_\eta = f_{<>} = e_{G_\omega}$
<u>for $n = m+1$</u>: $f_\eta = f_{m,3\eta(m)+1} f^{-1}_{m,3\eta(m)} f_{\eta \restriction m}$.

*5.13 Fact.* 1) For $\eta \in T_\omega$ and $m \leq n < \omega$ we have

$$f_{\eta \restriction n} \restriction G_{m+1} = f_{\eta \restriction m} \restriction G_{m+1}$$

2) $\eta \in \times_{m<n} \lambda_m \Rightarrow f_\eta \in K_n$ and $K_n \subseteq K_{n+1}$.

*Proof.* As $\pi_{m,\omega}$ is a homomorphism it is enough to prove $(f_{\eta \restriction n} (f_{\eta \restriction m})^{-1}) \restriction G_{m+1} = e_{G_{m+1}}$, hence it is enough to prove $m \leq k < \omega \Rightarrow (f_{\eta \restriction k} f^{-1}_{\eta \restriction (k+1)}) \restriction G_{m+1} = e_{G_{m+1}}$ (of course, $k < n$ is enough). Now this statement follows from $k < \omega \Rightarrow f_{\eta \restriction k} f^{-1}_{\eta \restriction (k+1)} \restriction G_{k+1} = e_{G_{k+1}}$, which by Definition 5.12 means $f_{k,3\eta(k)+1} f^{-1}_{k,3\eta(k)} \restriction G_{k+1} = e_{G_{k+1}}$ which follows from $\zeta < \lambda_k \Rightarrow f_{k,\eta(\zeta)} \restriction G_{k+1} = e_{G_{k+1}}$ which holds by clause $(\alpha)$ above. $\square_{5.13}$

**5.14 Definition.** For $\eta \in T_\omega$ we have $f_\eta \in G_\omega$ is well defined as the inverse limit of $\langle f_{\eta \restriction n} \restriction G_n : n < \omega \rangle$, so $n < \omega \rightarrow f_\eta \restriction G_n = f_{\eta \restriction n}$. This being well defined follows by 5.13 and $G^\omega$ being an inverse limit.

*5.15 Proposition.* Let $\eta, \nu \in T_\omega$ be such that $(\forall^\infty n)(\eta(n) \neq \nu(n)), \eta(n) > 0, \nu(n) > 0$. If $t \in \mathbf{I}$, <u>then</u> $f_\eta f_\nu^{-1} \notin \sigma^t_\omega(H^t_\omega)$.

*Proof.* Suppose toward contradiction that for some $g \in H^t_\omega$ we have $\sigma^t_\omega(g) = f_\eta f_\nu^{-1}$. Let $k < \omega$ be large enough such that $t \in \mathbf{J}_k, (\forall \ell)[k \leq \ell < \omega \rightarrow \eta(\ell) \neq \nu(\ell)]$. Let $\xi^\ell = \mathrm{rk}_t(g \restriction H^t_{k_t(\ell)}, f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)})$ and $\zeta^\ell = \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)})$ (the difference between the two is the use of $k_t(\ell)$ vis $k_t(\ell+1)$). Clearly

$$(*)_1 \;\; f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)} = (f_{\ell, 3\eta(\ell)+1} f^{-1}_{\ell, 3\eta(\ell)})(f_{\eta \restriction \ell} f^{-1}_{\nu \restriction \ell}) f_{\ell, 3\nu(\ell)} f^{-1}_{\ell, 3\nu(\ell)+1}$$

[Why? Algebraic computations and Definition 5.12.] Next we claim that

$$(*)_2 \;\; \xi^\ell < \infty \text{ for } \ell \geq k \; (\ell < \omega).$$

Why?
<u>Case 1</u>: $\eta(\ell) < \nu(\ell)$.

Assume toward contradiction $\xi^\ell = \infty$, but by clause $(\gamma)$ of 5.10 above $\mathrm{rk}_t(e_{H^t_{k_t(\ell)}}, f_{\ell, 3\eta(\ell)+2} f^{-1}_{\ell, 3\eta(\ell)+1}) < \infty = \xi^\ell$, hence by 5.8(2).

$$\begin{aligned} \mathrm{rk}_t(g \restriction H^t_{k_t(\ell)}, f_{\ell, 3\eta(\ell)+2} f^{-1}_{\ell, 3\eta(\ell)+1} f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)}) = \; & \mathrm{Min}\{\mathrm{rk}_t(e_{H^t_{k_t(\ell)}}, f_{\ell, 2(\eta(\ell)+2} f^{-1}_{\ell, 2\eta(\ell)+1}), \\ & \mathrm{rk}_t(g \restriction H^t_{k_t(\ell)}, f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)})\} = \\ & \mathrm{rk}_t(e_{H^t_{k_t(\ell)}}, f_{\ell, 2\eta(\ell)+2} f^{-1}_{\ell, 2\eta(\ell)+1}) < \infty. \end{aligned}$$

Now (by the choice of $f_{\eta \restriction (\ell+1)}, f_{\nu \restriction (\ell+1)}$ that is Definition 5.12 that is $(*)_1$, algebraic computation and the previous inequality) we have

$$\begin{aligned} \infty > \; & \mathrm{rk}_t(g \restriction H^t_{k_t(\ell)}, f_{\ell, 3\eta(\ell)+2} f^{-1}_{\ell, 3\eta(\ell)+1} f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)}) = \\ & \mathrm{rk}_t(g \restriction H^t_{k_t(\ell)}, (f_{\ell, 3\eta(\ell)+2} f^{-1}_{\ell, 3\eta(\ell)})(f_{\eta \restriction \ell} f^{-1}_{\nu \restriction \ell})(f_{\ell, 3\nu(\ell)} f^{-1}_{\ell, 3\nu(\ell)+1})). \end{aligned}$$

This and the assumption $\xi_\ell = \infty$ gives a contradiction to $(\delta)(i)$ of 5.10 (for $n = \ell$ and $f^* = f_{\eta, \ell} f^{-1}_{\nu \restriction \ell} \in K_\ell$ (see 5.13(1)) and $(i_0, i_1, i_2, i_3)$ being $(3\eta(\ell), 3\eta(\ell)+2, 3\nu(\ell), 3\nu(\ell)+1)$ and being $(3\eta(\ell), 3\eta(\ell)+1, 3\nu(\ell), 3\nu(\ell)+1)$; the contradiction is that for the first quadruple we get rank $< \infty$ by the previous inequality by the last inequality, for the second quadruple we get equality as we are temporarily assuming $\xi_\ell = \omega$, the definition of $\xi_\ell$ and $(*)_1$).

Case 2: $\nu(\ell) > \eta(\ell)$.

Similar using $(\delta)(ii)$ of 5.10 instead of $(\delta)(i)$ of 5.10 (using $\eta(\ell) > 0$).
So we have proved $(*)_2$.

$(*)_3$ $\xi^{\ell+1} \le \zeta^\ell$ for $\ell > k$.

Why? Assume toward contradiction that $\xi^{\ell+1} > \zeta^\ell$.
Let $f^* = f_{\eta \restriction (\ell+1)} f^{-1}_{\nu \restriction (\ell+1)}$, so $\zeta^\ell = \ \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f^*)$ and using the choice of $\xi^{\ell+1}$ and $(*)_1$ we have $\xi^{\ell+1} = \ \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f_{(\ell+1),3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)} f^* f_{\ell+1,3\nu(\ell+1)} f^{-1}_{\ell+1,3\nu(\ell+1)+1})$.

If $\zeta^\ell < \ \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)})$ then by $5.10(\delta)(iii)$ also $\zeta^\ell < \ \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1} f^{-1}_{\ell+1,3\nu(\ell+1)})$ hence using twice 5.8(2) we have first $\zeta^\ell = \ \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)} f^*)$ and second (using also 5.7(2)) we have $\zeta^\ell = \ \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)} f^* f_{\ell+1,3\nu(\ell+1)} f^{-1}_{\ell+1,3\nu(\ell+1)+1})$, so by the second statement in the previous paragraph (on $\xi^{\ell+1}$) we get $\zeta_\ell = \xi^{\ell+1}$ contradicting our temporary assumption toward contradiction $\neg(*)_3$; so we have $\zeta^\ell \ge \ \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)})$.

Also if $\mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)}) \ne \ \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1} f^{-1}_{\ell+1,3\nu(\ell+1)})$ then $\zeta^\ell$ is not equal to at least one of them hence by $5.10(\delta)(iii) + (iv)$ also $\zeta^\ell$ is not equal to those two ordinals so similarly to the previous sentence, 5.8(2) gives[3] $\xi^{\ell+1} = \ \mathrm{Min}\{\mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)})$, $\mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f^*), \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1} f^{-1}_{\ell+1,3\nu(\ell+1)})\}$ which is $\le \zeta^\ell$ so $\xi^{\ell+1} \le \zeta^\ell$, contradicting our assumption toward contradiction, $\neg(*)_3$.

Together the case left (inside the proof of $(*)_3$, remember 5.7) is:

$$\boxtimes\ \zeta^\ell = \ \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f^*) \ge \ \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)}) = \mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1} f^{-1}_{\ell+1,3\nu(\ell+1)}).$$

So in clause $5.10(\varepsilon)$, for $n = \ell + 1$, case (b) holds, call this constant value $\varepsilon^\ell$.
As, toward contradiction we are assuming $\xi^{\ell+1} > \zeta^\ell$ during the proof of $(*)_3$; so by $\boxtimes, \xi^{\ell+1} > \zeta^\ell \ge \varepsilon^\ell$ hence we get, by computation and by 5.8 that if $\eta(\ell+1) > \nu(\ell +$

[3]as the three are pairwise non equal

1) then $\mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+2} f^{-1}_{\ell+1,3\eta(\ell+1)} f^* f_{\ell+1,3\nu(\ell+1)} f^{-1}_{\ell+1,3\nu(\ell+1)+1}) =$
$\mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}(g \restriction H^t_{k_t(\ell+1)}), (f_{\ell+1,3\eta(\ell)+2} f^{-1}_{\ell+1,3\eta(\ell+1)+1})(f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell+1)} f^* f_{\ell+1,3\nu(\ell)+1} f^{-1}_{\ell+1,3}$
$\mathrm{rk}_t(e_{H^t_{k_t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+2} f^{-1}_{\ell+1,3\eta(\ell+1)})$ but by (b) of 5.10($\varepsilon$) proved above the later is $\varepsilon^\ell \le \zeta^\ell < \xi^{\ell+1} = \mathrm{rk}_t(g \restriction H^t_{k_t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1} f^{-1}_{\ell+1,3\eta(\ell)} f^* f_{\ell+1,3\nu(\ell+1)} f^{-1}_{\ell+1,3\nu(\ell+1)+1})$ contradiction to 5.10($\delta$)($v$) for the two quadruples $(3\nu(\ell+1), 3\nu(\ell+1)+1, 3\eta(\ell+1), 3\eta(\ell+1)+2)$ and $(3\nu(\ell+1), 3\nu(\ell+1)+1, 3\eta(\ell+1), 3\eta(\ell+1)+1)$ and $n = \ell+1$. If $\eta(\ell+1) < \nu(\ell+1)$ we use similarly $f_{\ell+1,3\nu(\ell+1)+2} f^{-1}_{\ell+1,3\nu(\ell+1)}$. So $(*)_3$ holds.

$(*)_4$ $\zeta^\ell \le \xi^\ell$
[Why? Look at their definitions, as $g \restriction H^t_{k_t(\ell+1)}$ is above $g \restriction H^t_{k_t(\ell)}$. Now if $k_t(\ell), k_t(\ell+1)$ are equal trivial otherwise use 5.6(3).]

$(*)_5$ if $k_t(\ell+1) > k_t(\ell)$ then $\zeta^\ell < \xi^\ell$ (so $\xi^\ell > 0$)
[Why? Like $(*)_4$.]

$(*)_6$ $\xi^\ell \ge \xi^{\ell+1}$ and if $k_t(\ell+1) > k_t(\ell)$ then $\xi^\ell > \xi^{\ell+1}$
[Why? By $(*)_3 + (*)_4$ the first phrase, and $(*)_3 + (*)_5$ for the second phrase.]

So $\langle \xi^\ell : \ell \in [k, \omega) \rangle$ is non-increasing, and not eventually constant sequence of ordinals, contradiction.

$\square_{5.15}$

*Proof of 5.9.* Obvious as we can find $T' \subseteq T$, a subtree with $\lambda^{\aleph_0}$ $\omega$-branches such that $\eta \ne \nu \in \lim(T') \Rightarrow (\forall^\infty \ell)\eta(\ell) \ne \nu(\ell)$ and $\eta \in \lim(T')$ & $n < \omega \Rightarrow \eta(n) > 0$. Now $\langle f_\eta : \eta \in \lim(T') \rangle$ is as required by 5.15.

<u>5.16 Conclusion</u>: If $\mathscr{A}$ is a $(\lambda, \mathbf{I})$-system, and $\lambda$ is an uncountable strong limit of cofinality $\aleph_0$ and $\mathrm{nu}(\mathscr{A}) \ge \lambda$ (or just $\mathrm{nu}^+_*(\mathscr{A}) \ge \lambda$), <u>then</u> $\mathrm{nu}(\mathscr{A}) =^+ 2^\lambda$.

*Proof.* So we assume $\lambda > \aleph_0$ hence $\lambda > 2^{\aleph_0}$ and trivially $\mathrm{nu}^+(\mathscr{A}) \ge \mathrm{nu}(\mathscr{A}) \ge \lambda$. We apply 5.2(2) to $\mathscr{A}$ and $\mu = \lambda$ (so $\mathrm{cf}(\mu) = \aleph_0$) and get an explicit $(\lambda, \bar{\mathbf{J}})$-system $\mathscr{B}$ such that $\mu \le \mathrm{nu}^+(\mathscr{B}) \le \mathrm{nu}(\mathscr{A})$ hence by 5.9 we have $\mathrm{nu}(\mathscr{B}) =^+ 2^\lambda$ hence by the choice of $\mathscr{B}$ also $\mathrm{nu}(\mathscr{A}) =^+ 2^\lambda$. The proof for $\mathrm{nu}^+_*(\mathscr{A}) \ge \lambda$ is similar. $\square_{5.16}$

*5.17 Concluding Remarks.* Can we weaken condition $(E)^+$ in Theorem 1.1(2)? Can we use rank?

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