STRONG DICHOTOMY
OF CARDINALITY
SH664

SAHARON SHELAH
Institute of Mathematics
The Hebrew University
Jerusalem, Israel
Rutgers University
Mathematics Department
New Brunswick, NJ USA

Abstract. We investigate strong dichotomical behaviour of the number of equivalence classes and related cardinal.

Saharon: compare with Journal proofs!

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§0 Introduction

§1 Countable Groups

[We present a result on a sequence of analytic equivalence relations on \( \mathcal{P}(\omega) \) and apply it to \( \aleph_0 \)-system of groups getting a strong dichotomy: being infinite implies cardinality continuum sharpening [GrSh 302a].]

§2 On \( \lambda \)-analytic equivalence relations

[We generalize theorems on the number of equivalence classes for analytic equivalent relations replacing \( \aleph_0 \) by \( \lambda \) regular, unfortunately this is only consistent. Noting that if we just add many Cohen subsets to \( \lambda \) we get something, but first the dichotomy is \( \leq \lambda^+ = 2^\lambda \) rather than \( \leq \lambda, = 2^\lambda \), second we assume much less.]

§3 On \( \lambda \)-systems of groups

[This relates to §2 as the application relates to the lemma in §1.]

§4 Back to the \( p \)-rank of Ext

[We show that we can put the problem in the title to the previous context, and show that in Easton model, §2 and §3 apply to every regular \( \lambda \).]

§5 Strong limit of countable cofinality

[We generalize the theorem on \( \aleph_0 \) systems of groups from §1, replace \( \aleph_0 \) by a strong limit uncountable cardinal of countable cofinality; this continues [GrSh 302a].]
A usual dichotomy is that in many cases, reasonably definable sets, satisfies the continuum hypothesis, i.e. if they are uncountable they have cardinality continuum. A strong dichotomy is when: if the cardinality is infinite it is continuum, as in [Sh 273]. We are interested in such phenomena when $\lambda = \aleph_0$ is replaced by $\lambda$ regular uncountable and also by $\lambda = \beth_\omega$ or more generally by strong limit of cofinality $\aleph_0$.

**Question:** Does the parallel of 1.2 holds for e.g. $\beth_\omega$ portion?

This continues Grossberg Shelah [GrSh 302], [GrSh 302a] and see history there. We also generalize results on the number of analytic equivalence relations, continuing Harrington Shelah [HrSh 152] and [Sh 202] and see history there.

On the connection to the rank of the $p$-torsion subgroup see [MRSh 314] and history there. See more [ShVs 719].

On $\operatorname{Ext}(G, \mathbb{Z})$, $\operatorname{rk}_p(\operatorname{Ext}(G, \mathbb{Z})$ see [EM].
Here we give a complete proof of a strengthening of the theorem of [GrSh 302a], for the case $\lambda = \aleph_0$ using a variant of [Sh 273].

1.1 Theorem. 1) Suppose

(A) $\lambda$ is $\aleph_0$. Let $(G_m, \pi_{m,n} : m \leq n < \omega)$ be an inverse system whose inverse limit is $G_\omega$ with $|G_n| < \lambda$. (So $\pi_{m,n}$ is a homomorphism from $G_n$ to $G_m$, $\alpha \leq \beta \leq \gamma \leq \omega \Rightarrow \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} = \pi_{\alpha,\gamma}$ and $\pi_{\alpha,\alpha}$ is the identity).

(B) Let $I$ be an index set. For every $t \in I$, let $\langle H^t_m, \pi^t_{m,n} : m \leq n < \omega \rangle$ be an inverse system of groups and $H^t_\omega$ with $\pi^t_{n,\omega}$ be the corresponding inverse limit and $H^t_m$ of cardinality $\leq \lambda$.

(C) Let for every $t \in I$, $\sigma^t_n : H^t_n \to G_n$ be a homomorphism such that all diagrams commute (i.e. $\pi^t_{m,n} \circ \sigma^t_n = \sigma^t_m \circ \pi^t_{m,n}$ for $m \leq n < \omega$), and let $\sigma^t_\omega$ be the induced homomorphism from $H^t_\omega$ into $G_\omega$.

(D) $I$ is countable

(E) For every $\mu < \lambda$ and $t \in I$ there is a sequence $\langle f_i \in G_\omega : i < \mu \rangle$ such that

$i < j \Rightarrow f_i f_j^{-1} \notin Rang(\sigma^t_\omega)$.

Then there is $\langle f_i \in G_\omega : i < 2^\lambda \rangle$ such that

$i \neq j \& t \in I \Rightarrow f_i f_j^{-1} \notin Rang(\sigma^t_\omega)$.

2) We can weaken in clause (A) to $(A)^-$ replacing $|G_n| < \lambda$ by $|G_n| \leq \lambda$, if we change clause (E) to

$(E)^*$ for every $t \in I, m < \omega$ there are $n, f$ such that $f$ is a member of $G_\omega$, $n < k < \omega \Rightarrow \pi^t_k,\omega(f) \notin Rang(\sigma^t_\omega)$ and $e_{G_n} = \pi^t_{n,\omega}(f)$.

We shall show below that 1.1 follows from 1.2.

1.2 Lemma. Assume for every $n < \omega, \varepsilon_n$ is an analytic two place transitive relation on $\mathcal{P}(\omega) = \{A : A \subseteq \omega^+\}$ which satisfies, for each $m < \omega$ for some infinite $Z_m \subseteq \omega$ we have

\[ \ast \]$_{m,Z_m}$ if $A, B \subset Z^+, n \in Z_m, n \notin B, A = B \cup \{n\}$, then $- (A \varepsilon_m B) \lor -(B \varepsilon_m A)$

\[ \ast\ast \] if $m < \omega, A \varepsilon_m B$ and $A'' \varepsilon_m B$ then $A' \varepsilon_m A''$.

Then there is a perfect subset $P$ of $\mathcal{P}(\omega)$ of pairwise $\varepsilon_m$-nonrelated $A \subseteq \omega$, simultaneously for all $n$, that is $A \neq B$ & $A \in P$ & $B \in P$ & $m < \omega \Rightarrow -(A \varepsilon_m B)$.\footnote{this is stronger, earlier $I$ was finite}
1.3 Remark. 1) The proof uses some knowledge of set theory and is close to [Sh 273, Lemma 1.3].
2) We say \( A, B \) are \( \mathcal{E} \)-related if \( A \mathcal{E} B \), and we say \( A, B \) are non-\( \mathcal{E} \)-related if \( \neg(A \mathcal{E} B) \).

**Proof.** Let \( r_m \in \omega^2 \) be the real parameter involved in a definition \( \varphi_m(x, y, r_m) \) of \( \mathcal{E}_m \). Let \( \vec{\varphi} = \langle \varphi_m : m < \omega \rangle, \vec{r} = \langle r_m : m < \omega \rangle, \mathcal{E} = \langle \mathcal{E}_m : m < \omega \rangle \). Let \( N \) be a countable elementary submodel of \( (\mathcal{H}((2^{\aleph_0})^+), \in) \) to which \( \vec{\varphi}, \vec{r}, \mathcal{E} \) belong. Now we shall show

\[
\text{(***) if } \langle A_1, A_2 \rangle \text{ is a pair of subsets of } \omega \text{ which is Cohen generic over } N \text{ [this means that it belongs to no first category subset of } \mathcal{P}(\omega) \times \mathcal{P}(\omega) \text{ which belongs to } N] \text{ then}
\]

\[
\begin{align*}
(\alpha) & \quad A_1, A_2 \text{ are } \mathcal{E}_m\text{-related in } N[A_1, A_2] \text{ if they are } \mathcal{E}_m\text{-related} \\
(\beta) & \quad A_1, A_2 \text{ are non-}\mathcal{E}_m\text{-related in } N[A_1, A_2].
\end{align*}
\]

**Proof of (***)**.

(\(\alpha\)) by the absoluteness criterions (Levy She"onfied)

(\(\beta\)) if not, then some finite information forces this, hence for some \( n \)

\[
\odot \quad \text{if } \langle A'_1, A'_2 \rangle \text{ is Cohen generic over } N \text{ and } A'_1 \cap \{0, 1, \ldots, n\} = A_1 \cap \{0, 1, \ldots, n\} \text{ and } A'_2 \cap \{0, 1, \ldots, n\} = A_2 \cap \{1, \ldots, n\} \text{ then } A'_1, A'_2 \text{ are } \mathcal{E}_m\text{-related in } N[A'_1, A'_2].
\]

Choose \( k \in Z_m \setminus \{0, 1, \ldots, n + 1\} \). Let \( A''_1 \) be \( A_1 \cup \{k\} \) if \( k \notin A_1 \) and \( A_1 \setminus \{k\} \) if \( k \in A_1 \).

Trivially also \( \langle A''_1, A_2 \rangle \) is Cohen generic over \( N \), hence by \(\odot\) above \( A''_1, A_2 \) are \( \mathcal{E}_m\text{-related in } N[A''_1, A_2] \). By (***)(\(\alpha\)) we know that really \( A''_1, A_2 \) are \( \mathcal{E}_m\text{-related}. \)

By (**) clearly \( A_1, A''_1 \) are \( \mathcal{E}_m\text{-related and also } A''_1, A_1 \) are \( \mathcal{E}_m\text{-related. But this contradicts the hypothesis (*\_m, Z_m). So (***) holds.}

We can easily find a perfect (nonempty) subset \( \mathbf{P} \) of \( \{A : A \subseteq \omega\} \) such that for any distinct \( A, B \in \mathbf{P}, (A, B) \) is Cohen generic over \( N \). So for each \( m \) for \( A \neq B \in \mathbf{P} \) we have \( N[A, B] \models "A, B \text{ are not } \mathcal{E}_m\text{-equivalent}" \) and by (***)(\(\alpha\)) clearly \( A, B \) are not \( \mathcal{E}_m\text{-equivalent. This finishes the proof.} \)

\[
\square_{1.2}
\]

\[
* \quad * \quad *
\]
1.4 Proof of 1.1. 1) Follows from part (2) as \((E) \Rightarrow (E)^+\) when the \(G_n\)'s are finite (use \((E)\) for \(\mu^* = |G_n| + 1\)).

2) Let \(k_n = n^2\) and we choose \(\langle f_n : n < \omega \rangle\) such that:

- \(a\) \(f_n \in G_\omega\)
- \(b\) \(k_n \leq i < k_{n+1} \Rightarrow e_{G_n} = \pi_{n, \omega}(f_i)\)
- \(c\) for every \(t \in I\), for arbitrarily large \(k\) we have \(\pi_{k+1, \omega}(f_k) \notin \text{Rang}(\sigma_{k+1}^t)\).

Clearly \(a\), \(b\) are straight for \(c\) use assumption \((E)^+\) and bookkeeping.

By induction on \(n\) for every \(\eta \in n^2\) we choose \(f_\eta \in G_\omega\) as follows: for \(n = 0\), \(f_\eta = e_{G_\omega}\), for \(\eta = \nu \setminus \langle 0 \rangle\), \(\nu \in n^2 + 2\) let \(f_\eta = f_\nu\) and for \(\eta = \check{\nu} \langle 1 \rangle\) let \(f_\eta = f_\eta f_{n-1}^{-1}\). Clearly \(m \leq n < \omega\) & \(\eta \in n^2 \Rightarrow \pi_{m, \omega}(f_\eta | m) = \pi_{m, \omega}(f_\eta)\).

Lastly, for \(A \subseteq \omega\), let \(\eta_A \in \omega^2\) be its characteristic function and \(g_A \in G_\omega\) be the unique \(f \in G_\omega\) satisfying \(m \leq n < \omega \Rightarrow \pi_{m, \omega}(f_\eta | n) = \pi_{m, \omega}(f_A)\). Let \(I = \{t_m : m < \omega\}\) (well we can add trivial \(H\)’s) and let \(\varepsilon_m\) be \(A \varepsilon_m \iff A \subseteq \omega\) & \(B \subseteq \omega\) & \(g_A^{-1} g_B \in \text{Rang}(\sigma_m^t)\). Clearly \(\varepsilon_m\) is an equivalence relation hence it satisfies condition \((**\) of 1.2. Lastly, let \(Z_m =: \{k : \pi_{k+1, \omega}(f_k) \notin \text{Rang}(\sigma_m^t)\}\). If \(A, B, m, k\) are as in \((*)\) of 1.2 then \(\pi_{k+1, \omega}(g_A^{-1} g_B) = \pi_{k+1, \omega}(f_k) \notin \text{Rang}(\sigma_{k+1})\). We have the assumptions of 1.2, hence get its conclusion. \(\square_{1.1}\)
\section*{§2 On $\lambda$-analytic equivalence relations}

\subsection*{2.1 Hypothesis.} $\lambda = \text{cf}(\lambda)$ is fixed.

\subsection*{2.2 Definition.} 1) A sequence of relations $\bar{R} = \langle R_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ on $\lambda^2$ (equivalently $\mathcal{P}(\lambda)$) i.e. a sequence of definitions of such relations in $(\mathcal{H}(\lambda^+), \varepsilon)$ and with parameters in $\mathcal{H}(\lambda^+)$ is called $\lambda$-w.c.a. sequence (weakly Cohen absolute) if: for any $A \subseteq \lambda$ we have

\begin{enumerate}[(a)]
\item there are $N, r$ such that:
\begin{enumerate}[(a)\text{.}]
\item $N$ is a transitive model
\item $N^\lambda \subseteq N, \lambda + 1 \subseteq N$, the sequence of the definitions of $\bar{R}$ (including the parameters) belongs to $N$
\item $A \in N$
\item $r \in \lambda^2$ is Cohen over $N$; that is generic for $(\lambda^2, \triangleleft)$ over $N$
\item $R_\varepsilon$ and $\neg R_\varepsilon$ are absolute from $N[r]$ to $V$ for each $\varepsilon < \varepsilon(*)$.
\end{enumerate}
\end{enumerate}

2) We say $\bar{R}$ is $(\lambda, \mu)$-w.c.a. if for $A \subseteq \lambda$ we can find $N, r_\alpha$ (for $\alpha < \mu$) satisfying clauses (a), (b), (c) from above and

\begin{enumerate}[(a)\text{.}]
\item for $\alpha \neq \beta < \mu$, $(r_\alpha, r_\beta)$ is a pair of Cohens over $N$
\item $R_\varepsilon$ and $\neg R_\varepsilon$ are absolute from $N[r_\alpha, r_\beta]$ to $V$ for each $\alpha \neq \beta < \mu$ and $\varepsilon < \varepsilon(*)$.
\end{enumerate}

3) We say $\lambda$ is $(\lambda, \mu)$-w.c.a. if every $\lambda$-analytic relation $R$ on $\lambda^2$ is $(\lambda, \mu)$-w.c.a. Analytic means that it has the form $R(X_1, \ldots, X_n) = (\exists Y_1, \ldots, Y_m \subseteq \lambda \times \lambda) \varphi(Y_1, \ldots, Y_m; X_1, \ldots, X_n)$

\subsection*{2.3 Claim.} Assume

\begin{enumerate}[(A)\text{.}]
\item $\varepsilon(*) \leq \lambda$ and $\langle \varepsilon_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is a $(\lambda, \mu)$-w.c.a. sequence, each $\varepsilon_\varepsilon$ an equivalence relation on $\mathcal{P}(\lambda)$, more exactly a definition of one and
\item if $\varepsilon < \varepsilon(*)$ and $A, B \subseteq \lambda$ and $\alpha \in A \setminus B \setminus \varepsilon, A = B \cup \{\alpha\}$, then $A, B$ are not $\varepsilon$-equivalent.
\end{enumerate}

Then there is a set $\mathcal{P} \subseteq \mathcal{P}(\lambda)$ of $\mu$-pairwise non-$\varepsilon$-equivalent members of $\mathcal{P}(\lambda)$ for all $\varepsilon < \varepsilon(*)$ simultaneously.

\subsection*{2.4 Remark.} If in 2.2 we ask that $\{r_\eta : \eta \in \lambda^2\}$ perfect (see 2.5 below), then we can demand that so is $\mathcal{P}$.
2.5 Definition. 1) $\mathcal{P} \subseteq \mathcal{P}(\lambda)$ is perfect if there is a $\lambda$-perfect tree $T \subseteq \lambda^+2$ (see below) such that $\mathcal{P} = \{ \{ \alpha < \lambda : \eta(\alpha) = 1 \} : \eta \in \lim_\lambda(T) \}$. 
2) $T$ is a $\lambda$-perfect tree if:

- (a) $T \subseteq \lambda^+2$ is non-empty
- (b) $\eta \in T$ & $\alpha < \ell g(\eta) \Rightarrow \eta \upharpoonright \alpha \in T$
- (c) if $\delta < \lambda$ is a limit ordinal, $\eta \in \delta^2$ and $(\forall \alpha < \delta)(\eta \upharpoonright \alpha \in T)$, then $\eta \in T$
- (d) if $\eta \in T, \ell g(\eta) < \alpha < \lambda$ then there is $\nu, \eta \lesssim \nu \in T \cap \alpha^2$
- (e) if $\eta \in T$ then there are $\prec$-incomparable $\nu_1, \nu_2 \in T$ such that $\eta \lesssim \nu_1 \& \eta \lesssim \nu_1$.

3) $\lim_{\delta}(T) = \{ \eta : \ell g(\eta) = \delta \text{ and } (\forall \alpha < \delta)(\eta \upharpoonright \alpha \in T) \}$.

Proof of 2.3.

Let $T^* = \lambda^+2$.

Let $N$ and $r_\alpha \in \lambda^2$ for $\alpha < \mu$ be as in Definition 2.2. We identify $r_\alpha$ with $\{ \gamma < \lambda : r_\alpha(\gamma) = 1 \}$.

By clause $(\varepsilon)'$ of Definition 2.2(2) clearly

$(*)_0$ if $\varepsilon < \varepsilon(\ast)$, and $\alpha \neq \beta < \mu$, then $\mathcal{E}_\varepsilon$ define an equivalence relation in $N[r_\alpha, r_\beta]$ on $\mathcal{P}(\lambda)^N[r_\alpha, r_\beta]$.

It is enough to prove assuming $\alpha \neq \beta < \mu$ and $\varepsilon < \varepsilon(\ast)$ that,

$(*)_1 \neg r_\alpha \mathcal{E}_\varepsilon r_\beta$.

By clause $(\varepsilon)'$ of Definition 2.2(2) it is enough to prove

$(*)_2 \ N[r_\alpha, r_\beta] \models \neg r_\alpha \mathcal{E}_\varepsilon r_\beta$.

Assume this fails, so $N[r_\alpha, r_\beta] \models r_\alpha \mathcal{E}_\varepsilon r_\beta$ then for some $i < \lambda$

$(r_\alpha \upharpoonright i, r_\beta \upharpoonright i) \Vdash (\lambda)^2 \times (\lambda)^2 \models \mathcal{T}_1 \mathcal{E}_\varepsilon r_\beta$.

and without loss of generality $i > \varepsilon$. Define $r \in \lambda^2$ by

$$r(j) = \begin{cases} r_\beta(j) & \text{if } j \neq i \\ 1 - r_\beta(j) & \text{if } j = i \end{cases}$$

So also $(r_\alpha, r)$ is a generic pair for $\lambda^+2 \times \lambda^+2$ over $N$ and $(r_\alpha \upharpoonright i, r \upharpoonright i) = (r_\alpha \upharpoonright i, r_\beta \upharpoonright i)$ hence by the forcing theorem

$(*)_3 \ N[r_\alpha, r] \models r_\alpha \mathcal{E}_\varepsilon r$. 
But \( r_\alpha, r_\beta, r \in N[r_\alpha, r_\beta] = N[r_\alpha, r] \). As we are assuming that \((*)_2\) fail (toward contradiction) we have \( N[r_\alpha, r_\beta] \models r_\alpha \in r_\beta \) and by \((*)_3\) and the previous sentence we have \( N[r_\alpha, r_\beta] \models r_\beta \in r \) hence \( V \models r_\beta \in r \), a contradiction to assumption (b). \( \Box_{2,3} \)

2.6 Definition. We call \( Q \) a pseudo \( \lambda \)-Cohen forcing if:

(a) \( Q \) is a nonempty subset of \( \{ p : p \) a partial function from \( \lambda \) to \( \{0, 1\} \} \)
(b) \( p \leq_Q q \Rightarrow p \subseteq q \)
(c) \( \mathcal{J}_i = \{ p \in Q : i \in \text{Dom}(p) \} \) is a dense subset for \( i < \lambda \)
(d) define \( F_i : \mathcal{J}_i \rightarrow \mathcal{J}_i \) by: \( \text{Dom}(F_i(p)) = \text{Dom}(F_i(p)) \) and

\[
(F_i(p))(j) = \begin{cases} p(j) & \text{if } j = i \\ 1 - p(j) & \text{if } j \neq i \end{cases}
\]

then \( F_i \) is an automorphism of \( (\mathcal{J}_i, <^Q \upharpoonright \mathcal{J}_i) \).

2.7 Claim. In 2.2, 2.5 we can replace \((\lambda^+ < 2, *)\) by \( Q \).

2.8 Observation: So if \( V \models G.C.H., P \) is Easton forcing, then in \( V^P \) for every regular \( \lambda \), for \( Q = ((\lambda^+ < 2)^V, *) \) we have: \( Q \) is pseudo \( \lambda \)-Cohen and in \( V^P \) we have \( \lambda \) is \( (\lambda, 2^\lambda) \)-w.c.a.

2.9 Discussion: But in fact \( \lambda \) being \( (\lambda, 2^\lambda) \)-w.c.a. is a weak condition.

We can generalize further using the following definition

2.10 Definition. 1) For \( r_0, r_1 \in \mathcal{L} \) we say \( (r_0, r_1) \) or \( r_0, r_1 \) is an \( \bar{R} \)-pseudo Cohen pair over \( N \) if \( (\bar{R} \text{ is a definition (in } (\mathcal{H}(\lambda^+), )), \text{ relation on } \mathcal{P}(\lambda) \text{ or } \lambda^2, \text{ the definition belongs to } N \text{ and}) \) for some forcing notion \( Q \in N \) and \( Q \)-names \( r_0, r_1 \)
and \( G \subseteq Q \) \( (G \in V) \) generic over \( N \) we have:

(a) \( r_0[G] = r_0 \) and \( r_1[G] = r_1 \)
(b) for every \( p \in G \), for every \( i < \lambda \) large enough and \( \ell(*) < 2 \) there is \( G' \subseteq Q \)
generic over \( N \) such that: \( p \in G \) and \( (r_\ell[G']) (j) = (r_\ell[G]) (j) \leftrightarrow (j, \ell) \neq (i, \ell(*) ) \)
(c) for \( \varepsilon < \varepsilon(*) \), \( R_\varepsilon \) is absolute from \( N[G] \) and from \( N[G'] \) to \( V \).
2) We say $\lambda$ is $\mu$-p.c.a for $\tilde{R}$ if for every $x \in \mathcal{H}(\lambda^+)$ there are $N, \langle r_i : i < \mu \rangle$ such that:

(a) $N$ is a transitive model of $ZFC^-$

(b) for $i \neq j < \mu$, $(r_i, r_j)$ is an $\tilde{R}$-pseudo Cohen pair over $N$.

3) We omit $\tilde{R}$ if this holds for any $\lambda$-sequence of $\sum_1^1$ formula in $\mathcal{H}(\lambda^+)$.

Clearly

2.11 Claim. 1) If $\lambda$ is $\mu$-p.c.a for $\mathcal{E}$, $\mathcal{E}$ an equivalence relation on $\mathcal{P}(\lambda)$ and $A \subseteq B \subseteq \lambda$ & $|B \setminus A| = 1 \Rightarrow \neg A \mathcal{E} B$, then $\mathcal{E}$ has $\geq \mu$ equivalence classes.

2) Similarly if $\mathcal{E} = \bigvee_{\varepsilon < \varepsilon(*)} \mathcal{E}_{\varepsilon, \varepsilon(*)} \leq \lambda$ and $\lambda$ is $\mu$-p.c.a. for $\langle \mathcal{E}_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ and $A \subseteq B \subseteq \lambda$ & $|B \setminus A| = |B \setminus A| = 1 \Rightarrow \neg A \mathcal{E}_\varepsilon B$, then there are $A_\alpha \subseteq \lambda$ for $\alpha < \mu$ such that $\varepsilon < \varepsilon(*)$ & $\alpha < \beta < \mu \Rightarrow \neg(A_\alpha \mathcal{E}_\varepsilon A_\beta)$. 
§3 On $\lambda$-systems of groups

3.1 Hypothesis. $\lambda = \text{cf}(\lambda)$.
We may wonder does 2.3 have any cases it covers?

3.2 Definition. 1) We say $\mathcal{Y} = (\bar{A}, \bar{K}, \bar{G}, \bar{D}, \bar{g}^*)$ is a $\lambda$-system if

(A) $\bar{A} = \langle A_i : i \leq \lambda \rangle$ is an increasing sequence of sets, $A = A_\lambda = \{ A_i : i < \lambda \}$

(B) $\bar{K} = \langle K_t : t \in A \rangle$ is a sequence of finite groups

(C) $\bar{G} = \langle G_i : i \leq \lambda \rangle$ is a sequence of groups, $G_i \subseteq \prod_{t \in A_i} K_t$, each $G_i$ is closed and $i < j \leq \lambda \Rightarrow G_i = \{ g \mid A_i : g \in G_j \}$ and $G_\lambda = \{ g \in \prod_{t \in A_\lambda} K_t : (\forall i < \lambda) (g \mid A_i \in G_i) \}$

(D) $\bar{D} = \langle D_\delta : \delta \leq \lambda \text{ (a limit ordinal)} \rangle$, $D_\delta$ an ultrafilter on $\delta$ such that $\alpha < \delta \Rightarrow [\alpha, \delta) \in D_\delta$

(E) $\bar{g}^* = \langle g_i^* : i < \lambda \rangle$, $g_i^* \in G_\lambda$ and $g_i^* \mid A_i = e_{G_i} = \langle e_{K_t} : t \in A_i \rangle$.

Of course, formally we should write $A_i^\mathcal{Y}, K_t^\mathcal{Y}, G_i^\mathcal{Y}, D_\delta^\mathcal{Y}, g_i^\mathcal{Y}$, etc., if clear from the context we shall not write this.

2) Let $\mathcal{Y}^-$ be the same omitting $D_\lambda$ and we call it a lean $\lambda$-system.

3.3 Definition. For a $\lambda$-system $\mathcal{Y}$ and $j \leq \lambda + 1$ we say $\bar{f} \in \text{cont}(j, \mathcal{Y})$ if:

(a) $\bar{f} = \langle f_i : i < j \rangle$

(b) $f_i \in G_\lambda$

(c) if $\delta < j$ is a limit ordinal then $f_\delta = \text{Lim}_{D_\delta}(\bar{f} \mid \delta)$ which means:

for every $t \in A$, $f_\delta(t) = \text{Lim}_{D_\delta} \langle f_i(t) : i < \delta \rangle$

which means $\{ i < \delta : f_\delta(t) = f_i(t) \} \in D_\delta$.

3.4 Fact: 1) If $\bar{f} \in \text{cont}(j, \mathcal{Y}), i < j$ then $\bar{f} \mid i \in \text{cont}(i, \mathcal{Y})$.

2) If $\bar{f} \in \text{cont}(j, \mathcal{Y})$ and $j < \lambda$ is non-limit, and $f_j \in G_\lambda$ then

$\bar{f}^*(f_j) \in \text{cont}(j + 1, \mathcal{Y})$. 
3) If $\bar{f} \in \text{cont}(j, \mathcal{Y})$ and $j$ is a limit ordinal $\leq \lambda$, then for some unique $f_j \in G_\lambda$ we have $\bar{f}^*(f_j) \in \text{cont}(j+1, \mathcal{Y})$.

4) If $j \leq \lambda + 1, f \in G$ then $\bar{f} = \langle f : i < j \rangle \in \text{cont}(j, \mathcal{Y})$.

5) If $\bar{f}, \bar{g} \in \text{cont}(j, \mathcal{Y})$, then $\langle f^*_i : i < j \rangle$ and $\langle f_i^{-1} : i < j \rangle$ belongs to $\text{cont}(j, \mathcal{Y})$.

Proof. Straight (for part (3) we use each $K_t$ is finite).

3.5 Definition. Let $\mathcal{Y}$ be a $\lambda$-system.

1) For $\bar{g} \in \mathcal{Y}(G_\lambda)$ and $j \leq \lambda$ we define $f_\bar{g} \in G_\lambda$ by induction on $j$ for all such $\bar{g}$ as follows:

   $i = 0$: $f_\bar{g} = e_G = \langle e_{K_t} : t \in A \rangle$

   $i = i + 1$: $f_\bar{g} = f_{\bar{g}|i}g_i$

   $i$ limit: $f_\bar{g} = \text{Lim}_{D_\delta}(f_{\bar{g}|i} : i < j)$

2) We say $\bar{g}$ is trivial on $X$ if $i \in X \cap \ell g(\bar{g}) \Rightarrow g_i = e_{G_\lambda}$.

3) For $\eta \in \lambda^+ \geq 2$ let $\bar{g}_\eta = \langle g^\eta_{i} : i < \ell g(\eta) \rangle$, where

   $$g^\eta_{i} = \begin{cases} g^*_i & \text{if } \eta(i) = 1 \\ e_{G_\lambda} & \text{if } \eta(i) = 0 \end{cases}$$

recall $g^*_i$ is part of $\mathcal{Y}$ (see Definition 3.2).

3.6 Claim. 1) If $i \leq j$ and $\bar{g}, \bar{g}', \bar{g}'' \in \mathcal{Y}(G_\lambda), \bar{g}' | i = \bar{g} | i, \bar{g}'$ is trivial on $[i, j)$, $\bar{g}'' | [i, j) = \bar{g} | [i, j)$ and $\bar{g}''$ is trivial on $i$, then:

   $$f_\bar{g} = f_{\bar{g}'} f_{\bar{g}''} \text{ and } f_{\bar{g}'} = f_{\bar{g}''|i}.$$

2) For $\eta \in \lambda^+, f(\bar{g}_\eta) = \text{Lim}(f_{\bar{g}_\eta|i} : i < \lambda)$ (i.e. any ultrafilter $D'_\lambda$ on $\lambda$ containing the co-bounded sets will do), so $\mathcal{Y}^-$, a lean $\lambda$-system, is enough.

Proof. Straight.
3.7 Claim. Let $\mathcal{Y}$ be a $\lambda$-system (or just a lean one), $H_\varepsilon$ a subgroup of $G_\lambda$ for $\varepsilon < \varepsilon(*) \leq \lambda$ and $\delta_\varepsilon$ the equivalence relation $[f'(f'')^{-1} \in H_\varepsilon]$ and assume: $\lambda > i \geq \varepsilon \Rightarrow g_i^* \notin H_\varepsilon$.

1. The assumption (B) of 2.3 holds with $f_A = f(\eta^v)$ when $A \subseteq \lambda, \eta \in \lambda^2, A = \{i : \eta(i) = 1\}$

2. if in addition $\bar{A}, \bar{K}, G \upharpoonright K, \bar{D}, \bar{g}^* \in \mathcal{H}(\lambda^+)$ and $\langle H_\varepsilon : \varepsilon < \varepsilon(*) \rangle$ is $(\lambda, \mu)$-w.c.a., then also assumption (A) of 3.3 holds (hence its conclusion).

Proof. Straight.

3.8 Claim. Assume

(A) $\mathcal{Y}$ a $\lambda$-system (or just a lean one), $A_i \subseteq \lambda^+, |A_i| \leq \lambda, G_i \in \mathcal{H}(\lambda^+)$

(i) $\varepsilon(*) \leq \lambda$,

(ii) $\tilde{H} = \langle H^\varepsilon_i : i \leq \lambda, \varepsilon < \varepsilon(*) \rangle$,

(iii) $\pi^\varepsilon_{i,j} : H^\varepsilon_j \rightarrow H^\varepsilon_i$ a homomorphism,

(iv) for $i_0 \leq i_1 \leq i_2$ we have $\pi^\varepsilon_{i_0,i_1} \circ \pi^\varepsilon_{i_1,i_2} = \pi^\varepsilon_{i_0,i_2}$,

(v) $\sigma^\varepsilon_i : H^\varepsilon_i \rightarrow G_i$,

(vi) $\sigma^\varepsilon_i \pi^\varepsilon_{i,j}(f) = (\sigma^\varepsilon_j(f)) \upharpoonright A_i$,

(vii) $H^\varepsilon_\lambda, \sigma^\varepsilon_\lambda$ is the inverse limit (with $\pi^\varepsilon_{i,\lambda}$) of $\langle H^\varepsilon_i, \pi^\varepsilon_{i,j}, \sigma^\varepsilon_i : i \leq j < \lambda \rangle$ and

(viii) $i < \lambda \Rightarrow H^\varepsilon_i \in \mathcal{H}(\lambda^+)$

Then

(\alpha) the assumptions of 3.7 holds

(\beta) if $\lambda$ is $(\lambda, \mu)$-w.c.a. then also the conclusion of 3.7, 2.3 holds so there are $h_\alpha \in G_\lambda$ for $\alpha \mu$ such that $\alpha \neq \beta < \mu \& \varepsilon < \varepsilon(*) \Rightarrow f_{\alpha}f_{\beta}^{-1} \notin H_\varepsilon$.

Proof. Straight.

* * *

We can go one more step in concretization.
3.9 Claim. 1) Assume

(a) \( L \) is an abelian group of cardinality \( \lambda \)
(b) \( p \) a prime number
(c) if \( L' \subseteq L, |L'| < \lambda \), then \( \text{Ext}_p(L',\mathbb{Z}) \neq 0 \)
(d) \( \lambda \) is \( \mu \)-w.c.a. \( \text{in } V \).

Then \( \mu \leq \tau_p(\text{Ext}(L,\mathbb{Z})) \), see definition below.
2) If (a), (b), (d) above, \( \mu > \lambda, \lambda \) strongly inaccessible then \( \tau_p(\text{Ext}(L,\mathbb{Z})) \not\in [\lambda, \mu) \).

3.10 Remark. 1) For an abelian group \( M \) let prime \( p \) and \( r_p(M) \) be the dimension of the subgroup of \( \{ x \in M : px = 0 \} \) as a vector space over \( \mathbb{Z}/p\mathbb{Z} \).
2) For an abelian group \( M \) let \( r_0(M) \) be \( \max\{|X| : X \subseteq M \setminus \text{Tor}(M) \} \) and is independent in \( M/\text{Tor}(M) \).

Proof. Without loss of generality \( L \) is \( \aleph_1 \)-free (so torsion free).
Without loss of generality the set of elements of \( G \) is \( \lambda \). Let \( A = A_\lambda = \lambda, L_\lambda = L, \)
for \( j < \lambda, A_j \) a proper initial segment of \( \lambda \) such that \( L_j = L \upharpoonright A_j \) is a pure subgroup of \( L \), increasing continuously with \( j \).
Let \( K_t = \mathbb{Z}/p\mathbb{Z}, G_i = \text{HOM}(L_i,\mathbb{Z}/p\mathbb{Z}) \).
Let \( \varepsilon(*) = 1, \varepsilon = 0 \); let \( H_i = \text{HOM}(L_i,\mathbb{Z}) \) and \( (\sigma^*_\varepsilon(f))(x) = f(x) + p\mathbb{Z}, M_\varepsilon = \text{Rang}(\sigma^*_\varepsilon) \) for \( i \leq j \) let \( \pi_{i,j} : G_j \rightarrow G_i \) is \( \pi_{i,j}(f) = f \upharpoonright G_i \). We know that \( \tau_p(\text{Ext}(G,\mathbb{Z})) \) is \( (G_\lambda : M_0) \). By assumption (d) for each \( i < \lambda \) we can choose \( g^*_i \in G_\lambda \setminus M_\varepsilon \) such that \( g^*_i \upharpoonright L_i \) is zero. The rest is left to the reader (using 3.8 using any lean \( \lambda \)-system \( \mathcal{Y} \) with \( G_i, K_t, \varepsilon(*), \pi_{i,j}, \sigma^*_\varepsilon \) as above (and \( D_\delta \) for limit ordinal \( < \lambda \), any ultrafilter as in 3.2). \( \square_{3.9} \)
§4 Back to the \( p \)-rank of \( \text{Ext} \)

For consistency of “no examples” see [MRSh 314].

**4.1 Definition.**
1) Let

\[ \Xi_{\mathbb{Z}} = \{ \lambda : \lambda = (\lambda_p : p < \omega \text{ prime or zero}) \text{ and for some abelian } (\aleph_1 \text{-free}) \text{ group } L, \lambda_p = r_p(\text{Ext}(G, \mathbb{Z})) \} . \]

2) For an abelian group \( G \) let \( \text{rk}(G) = \min\{ \text{rk}(G') : G/G' \text{ is free} \} \).

Clearly \( \Xi_{\mathbb{Z}} \) is closed under products. Let \( \mathbf{P} \) be the set of primes.

Remember that (see [Sh:f, AP], 2.7, 2.7A, 2.13(1),(2)).

**4.2 Fact:** In the Easton model if \( G \) is \( \aleph_1 \)-free not free, \( G' \subseteq G, |G'| < |G| \Rightarrow G/G' \) not free then \( r_0(\text{Ext}(G, \mathbb{Z})) = 2^{|G|} \).

**4.3 Fact:**
1) Assume \( \mu \) is a strong limit cardinal > \( \aleph_0 \), cf(\( \mu \)) = \( \aleph_0 \), \( \lambda = \mu^+, 2^\mu = \mu^+ \) and some \( Y \subseteq [\omega^\mu]^\lambda \) is \( \mu \)-free, (equivalently \( \mu^+ \)-free, see in proof).

Let \( \mathbf{P}_0, \mathbf{P}_1 \) be a partition of the set of primes.

Then for some \( \aleph_1 \)-free abelian group \( L, |L| = \mu^+, 2^\lambda = r_0(\text{Ext}(G, \mathbb{Z})) \) and \( p \in \mathbf{P}_1 \Rightarrow r_p(\text{Ext}(G, \mathbb{Z})) = 2^\lambda \) and \( p \in \mathbf{P}_0 \Rightarrow r_p(\text{Ext}(G, \mathbb{Z})) = 0 \).

**Remark.** On other cardinals see [MRSh 314], close to [MkSh 418, Th.12].

**Proof.**

For notational simplicity assume \( \mathbf{P}_0 \neq \emptyset \). Let \( Y = \{ \eta_i : i < \lambda \} \).

Let \( \text{pr} : \mu^2 \to \mu \) be a pairing function, so \( \text{pr}(\text{pr}_1(\alpha), \text{pr}_2(\alpha)) = \alpha \). Without loss of generality \( \eta_i(n) = \eta_j(m) \Rightarrow n = m \) & \( \eta_i \upharpoonright m = \eta_j \upharpoonright m \). Let \( L \) be \( \bigoplus_{\alpha < \lambda} \mathbb{Z} \times \alpha \). Let \( \langle (p_i, f_i) : i < \lambda^+ \rangle \) list the pairs \((p, f)\) where \( p \in \mathbf{P}_0 \) and \( f \in \text{HOM}(L, \mathbb{Z}/p\mathbb{Z}) \). We shall choose \((g_i, \nu_i, \rho_i)\) by induction on \( i < \lambda \) such that:

\( \exists(\alpha) \) \( g_i \in \text{HOM}(L, \mathbb{Z}) \)

\( (\beta) (\forall x \in L)[g_i(x)/p\mathbb{Z} = f_i(x)] \)

\( (\gamma) \rho_i, \nu_i \in \omega^\mu \) and \( \eta_i(n) = \text{pr}_1(\nu_i(n)) = \text{pr}_1(\rho_i(n)) \)

\( (\delta) (\forall j \leq i)(\exists n < \omega)(\forall m)[n \leq m < \omega \Rightarrow g_j(x_{\nu_i(m)}) = g_j(x_{\rho_i(m)})] \)

\( (\varepsilon) (\forall j < i)(\exists n < \omega) \) [for some sequence \( \langle b_m : m \in [n, \omega) \rangle \) of natural numbers we have \( n \leq m < \omega \Rightarrow ( \prod_{p \in \mathbf{P}_0 \cap m} p ) b_{m+1} = b_m + g_i(x_{\nu_j(m)}) - g_i(x_{\rho_j(m)}) ] \)

\( (\zeta) \nu_i(m) \neq \rho_i(m) \) for \( m < \omega \).
Arriving to $i$ first choose a function let $h_i : i \to \omega$ be such that $j < i \Rightarrow h_i(j) > p_j$ and \langle \{\eta_j : \ell \in [h_i(j), \omega) : j < i\} \rangle$ a sequence of pairwise disjoint sets (possible as $Y$ is $\mu^+$-free). Second choose $g_i$ such that clauses $(\alpha) + (\beta)$ holds with $n = h_i(j)$, this is possible as the choice of $h$ splits the problem, that is, the various cases of $(\epsilon)$ (one for each $j$) does not conflict. More specifically, first choose $g \mid \{x_{\alpha} : (\forall j < i)(\forall \ell)(h_i(j) \leq \ell < \omega \rightarrow \alpha \neq \eta_j(\ell))\}$ as required in $(\beta)$, possible as $L$ is free. Second by induction on $m \geq h_i(j)$ we choose $b_{m+1}$ such that \[ 0/p\mathbb{Z} = b_{m+1}/p_i\mathbb{Z} + f_i(x_{\nu_i(m)}) - f_i(x_{\rho_i(m)}) \] and then choose $g_i(x_{\nu_i(m)}), g(x_{\rho_i(m)})$ such that the $m$-th equation in clause $(\epsilon)$ for $j$ holds. Let $i = \bigcup_{n<\omega} A_n^i$ be such that

\[ A_n^i \subseteq A_{n+1}^i \text{ and } |A_n^i| < \mu. \]

Now choose by induction on $n, \rho_i(n), \nu_i(n)$ as distinct ordinals $\in \{\alpha < \mu^+ : \alpha \notin \{\nu_i(m), \rho_i(m) : m < \omega\}\}$ and $\rho_1(\alpha) = \eta_i(n) \} \}$ such that \langle \langle g_j(x_{\nu_i(\alpha)}) : j \in A_n^i \rangle \rangle = \langle \langle g_j(x_{\rho_i(m)}) : j \in A_n^i \rangle \rangle. \] So we have carried the induction. Let $G$ be generated by $L \cup \{g_i,n : i < \lambda, m < \omega\}$ freely except that (the equations of $L$ and) \( \prod_{p \in \mathcal{P} \cap \mathcal{M}} p \rangle g_i,n+1 = g_i,n + x_{\nu_i(n)} - x_{\rho_i(n)}. \]

Why is the abelian group as required?

\[ \boxed{1} \] $G$ is $\mu^+$-free

[Why? As $\langle \eta_{\alpha} : \alpha \in \mu^+ \rangle$ is and clause $(\gamma)$.]

\[ \boxed{2} \] if $p \in \mathcal{P}_0$, then $r_p(\text{Ext}(G, \mathbb{Z})) = 0.$

[Why? So let $f \in \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ and we should find $g \in \text{Hom}(G, \mathbb{Z})$ such that $f = g/p\mathbb{Z}$. Clearly for some $i < \mu^+$ we have $(p_i, f_i) = (p, f)$, now $g_i$ was chosen such that we can extend $g_i$ to a homomorphism $g_i,i$ from $G_i = \langle L \cup \{y_j,n : j < i, n < \omega\} \rangle$ to $\mathbb{Z}$ such that $g_i,i(x)/p\mathbb{Z} = f(x)$ and if $j < i$ we choose $n^{i,j}$ and $(b_{\alpha}^{n,j} : m \in [n^{i,j}, \omega])$ are as required in closed $(\epsilon)$, we let $g_i,i(y_{j,m}) = b_m$ for $m \in [n^{i,j}, \omega]$]. Lastly, we define by induction on $j \in [i, \mu^+]$ a homomorphism $g_i,j$ from $G_j$ into $\mathbb{Z}$ such that

\[ g_i,j(x)/p\mathbb{Z} = f(x) \]

for $x \in G_j$, $g_i,j$ is increasing with $j$. For $j = i$ this was done, for limit take union and for $j = \epsilon + 1$, by clause $(\delta)$ of $\boxed{2}$ we know that for some $n = n^{i,j}$ we have $m[n, \omega) \Rightarrow g_i,i(x_{\nu_i(m)}) = g_i,i(x_{\rho_i(n)})$, so for $m \in [n, \omega]$ we let $g_i,\epsilon+1(y_{\epsilon,n}) = 0$ and solve the equations to determine $g_i,\epsilon+1(y_{\epsilon,n})$ by downward induction.]

\[ \boxed{3} \] if $p \in \mathcal{P}_1$, then $r_p(\text{Ext}(G, \mathbb{Z})) = 2^\mu.$

[Why? Because every $h \in \text{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ has $> 1$ extensions to $h' \in \text{Hom}(G_{\alpha+1}, \mathbb{Z}/p\mathbb{Z})$ hence $\text{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ has cardinality $2^{\mu+} > 2^\mu$, whereas every $h \in \text{Hom}(L, \mathbb{Z})$ has at most one extension to $h^+ \in \text{Hom}(G, \mathbb{Z})$, so the result follows.]

\[ \boxed{4} \] $r_0(\text{Ext}(G, \mathbb{Z})) = 2^{2^\mu}$

[Why? Similar to $\boxed{3}$, i.e. using cardinality considerations.]
4.4 Question: Do we have compactness for singular for $\text{Ext}_p(G, \mathbb{Z}) = 0$?

4.5 Claim. [Omitted, see [Sh 724] and x.x.]

4.6 Question: If $\bar{\lambda} \in \Xi_\mathbb{Z}$ can we derive $\bar{\lambda}' \in \Xi_\mathbb{Z}$ by increasing some $\lambda_p$’s?

4.7 Fact: If $\bar{\lambda}^i = \langle \lambda^i_p : p \in P \cup \{0\} \rangle \in \Xi_\mathbb{Z}$ for $i < \alpha$ and $\lambda_p = \prod_{i<\alpha} \lambda^i_p$, then $\langle \lambda_p : p \in P \cup \{0\} \rangle \in \Xi_\mathbb{Z}$.

**Proof.** As if $G = \bigoplus_{i<\alpha} G_i$ then $\text{Ext}(G, \mathbb{Z}) = \prod_{i<\lambda} \text{Ext}(G, \mathbb{Z})$ hence $r_p(\text{Ext}(G, \mathbb{Z})) = \prod_{i<\alpha} r_p(\text{Ext}(G_i, \mathbb{Z}))$.

4.8 Concluding Remark: In [EkSh 505] the statement “there is a $W$-abelian group” is characterized.

We can similarly characterize “there is a separable group”. We have the same characterization for “there is a non-free abelian group” such that for some $p$, $r_p(\text{Ext}(G, \mathbb{Z})) = 0$.

**Question:** What can $P^* = \{p : p$ prime and $\bar{\lambda} \in \Xi_\mathbb{Z}$ & $\lambda_0 > 0 \Rightarrow \lambda_p > 0\}$ be (if $V = L$ it is $\emptyset$, in 4.5 it is $P$, are there other possibilities?)

4.9 Claim. If $\lambda$ is strong inaccessible or $\lambda = \mu^+$, $\mu$ strong limit singular of cofinality $\aleph_0$, $S \subseteq \{\delta < \lambda : cf(\delta) = \aleph_0\}$ is stationary not reflecting and $\diamondsuit^*_S$ and $P_0$ a set of primes, then there is a $\lambda$-free abelian group $G$ such that $r_0(\text{Ext}(G, \mathbb{Z})) = 2^\lambda = 0$ and: $p \in P_0 \Rightarrow r_p(\text{Ext}(G, \mathbb{Z})) = 2^\lambda$ and $p$ prime and $p / \notin P_0 \Rightarrow r_p(\text{Ext}(G, \mathbb{Z})) = 0$. 
§5 Strong limit of countable cofinality

We continue [GrSh 302] and [GrSh 302a].

5.1 Definition. 1) We say \( \mathcal{A} \) is a \((\lambda, I)\)-system if \( \mathcal{A} = (\lambda, I, G, \dot{H}, \pi, \bar{\sigma}) \) where \( G = \langle G_\alpha : \alpha \leq \omega \rangle, \dot{H} = \langle H^t : t \in I \rangle, H^t = \langle H^t_\alpha : \alpha \leq \omega \rangle, \pi = \langle \pi_{\alpha, \beta}, \pi_{\alpha, \beta, \gamma} : \alpha \leq \beta \leq \omega, t \in I \rangle, \bar{\sigma} = \langle \sigma^t_\alpha : t \in I, \alpha \leq \omega \rangle \) satisfies (we may write \( \lambda^{\mathcal{A}}, \pi_{\alpha, \beta}^{\mathcal{A}}, \bar{\sigma}^{\mathcal{A}} \), etc.)

\[
\begin{align*}
(A) & \quad \lambda \text{ is } \aleph_0 \text{ or generally a cardinal of cofinality } \aleph_0 \\
(B) & \quad \langle G_m, \pi_{m,n} : m \leq n < \omega \rangle \text{ is an inverse system of groups whose inverse limit is } G_\omega \text{ with } \pi_{n,w} \text{ such that } |G_n| \leq \lambda. \text{ (So } \pi_{m,n} \text{ is a homomorphism from } G_n \text{ to } G_m, \alpha \leq \beta \leq \gamma \leq \omega \Rightarrow \pi_{\alpha, \beta} \circ \pi_{\beta, \gamma} = \pi_{\alpha, \beta, \gamma} \text{ and } \pi_{\alpha, \alpha} \text{ is the identity).}
(C) & \quad I \text{ is an index set of cardinality } \leq \lambda. \text{ For every } t \in I \text{ we have } \\
& \quad \langle H^t_m, \pi^t_{m,n} : m \leq n < \omega \rangle \text{ is an inverse system of groups and } H^t_\omega \text{ with } \pi^t_{n,\omega} \text{ being the corresponding inverse limit } H^t_\omega \text{ with } \pi^t_{m,\omega} \text{ and } H^t_m \text{ has cardinality } \leq \lambda.
(D) \quad \text{for every } t \in I, \sigma^t_\alpha : H^t_n \to G_n \text{ is a homomorphism such that all diagrams commute (i.e. } \pi_{m,n} \circ \sigma^t_n = \pi^t_m \circ \pi^t_{m,n} \text{ for } m \leq n < \omega), \text{ and let } \sigma^t_\alpha \text{ be the induced homomorphism from } H^t_\omega \text{ into } G_\omega.
(E) & \quad G_0 = \{ e_{G_0}, H_0^t = \{ e_{H^t_0} \} \text{ (just for simplicity).}
\end{align*}
\]

2) We say \( \mathcal{A} \) is strict if \( |G_n| < \lambda, |H^t_n| < \lambda, |I| < \lambda \). Let \( \mathcal{E}_t \) be the following equivalence relation on \( \mathcal{G}_\omega : f \mathcal{E}_t g \iff fg^{-1} \in \text{ Rang}(\sigma^t_\omega) \).

3) Let \( \nu(\mathcal{A}) = \sup \{ \mu : \text{ for each } n < \omega, \text{ there is a sequence } \langle f_i : i < \mu \rangle \text{ such that } f_i \in G_\omega \text{ and } \mu \leq \lambda \Rightarrow \pi_{n,\omega}(f_i) = \pi_{n,\omega}(f_0) \text{ for } i < \mu \text{ and } i < j < \mu \text{ & } t \in I \Rightarrow f_i \mathcal{E}_t f_j \} \).

We write \( \nu(\mathcal{A}) =^{=} \mu \) to mean that the moreover the supremum is obtained. Let \( \nu^{+}(\mathcal{A}) \) be the first \( \mu \) such that for \( n = 0 \), there is no \( \langle f_i : i < \mu \rangle \) as above (so \( \nu(\mathcal{A}) \leq \nu^{+}(\mathcal{A}) \) and if \( \nu(\mathcal{A}) > \mu \) then \( \nu(\mathcal{A}) \leq \nu^{+}(\mathcal{A}) \) and \( \nu(\mathcal{A}) < \nu^{+}(\mathcal{A}) \) implies \( \nu(\mathcal{A}) \) is a limit cardinal and the supremum not obtained).

4) We say \( \mathcal{A} \) is an explicit \((\lambda, J)\)-system if: \( \mathcal{A} = (\lambda, J, G, \dot{H}, \pi, \bar{\sigma}) \) and

\[
\begin{align*}
(\alpha) & \quad \lambda = \langle \lambda_n : n < \omega \rangle, J = \langle J_n : n < \omega \rangle \\
(\beta) & \quad \lambda_n < \lambda_{n+1}, J_n \subseteq J_{n+1}, \\
(\gamma) & \quad \text{letting } \lambda^\mathcal{A} = \sum_{n<\omega} \lambda_n, I^\mathcal{A} = \bigcup_{n<\omega} J_n \text{ we have sys}(\mathcal{A}) =: (\lambda, I, G, \dot{H}, \pi, \bar{\sigma}) \text{ is a } \langle \lambda, I \rangle \text{-system} \\
(\delta) & \quad |J_n| \leq \lambda_n, |G_n| \leq \lambda_n, |H^t_n| < \lambda \text{ and } |H^t_n| \leq |H^t_{n+1}|.
\end{align*}
\]
5) We add in (4), full if
\((\varepsilon) \mid H_n^I \leq \lambda_n.\)

6) For an explicit \((\lambda, \mathbf{J})\)-system \(\mathcal{A}\) let \(\text{nu}^+ (\mathcal{A}) = \text{sup}\{\mu^+ : \text{for every } n < \omega \text{ there is a sequence } \langle f_i : i < \mu \rangle \text{ such that } f_i \in G, \text{ and } \mu \leq \lambda \Rightarrow \pi_{n,\omega}(p_i) = \pi_{n,\omega}(f_0) \text{ for } i < \mu \text{ and } i < j < \mu \text{ and } t \in J_n \Rightarrow \neg f_i \cap f_j \} \).

7) For a \(\lambda\)-system \(\mathcal{A}\), we define \(\text{nu}^+ (\mathcal{A})\) similarly, except we say: for some \(\bar{J} = \langle J_n : n < \omega \rangle\) such that \(I = \bigcup_{n < \omega} J_n, J_n \subseteq J_{n+1}.\)

5.2 Claim. 1) For any strict \((\lambda, \mathbf{I})\)-system \(\mathcal{A}\) there are \(\bar{\lambda}, \bar{J}\) and an explicit \((\bar{\lambda}, \bar{J})\)-system \(\mathcal{B}\) such that \(\text{sys}(\mathcal{B}) = \mathcal{A}\) so
\[\lambda = \sum_{n < \omega} \lambda_n, \mathbf{I} = \bigcup_{n < \omega} J_n, \text{ nu}(\mathcal{B}) = \text{nu}(\mathcal{A})\]
(and if in one side the supremum is obtained, so in the other).

2) For any \((\lambda, \mathbf{I})\)-system \(\mathcal{A}\) such that \(\lambda > 2^{\aleph_0}\) and \(\text{nu}^+ (\mathcal{A}) \geq \mu \geq \lambda\) and \(\text{cf}(\mu) \notin [\aleph_1, 2^{\aleph_0}]\) there is an explicit \((\bar{\lambda}, \bar{J})\)-system \(\mathcal{B}\) such that \(\lambda = \sum_{n < \omega} \lambda_n, \mathbf{I} = \bigcup_{n < \omega} J_n\) and \(\text{nu}^+ (\mathcal{A}) \geq \text{nu}^+ (\mathcal{B}) \geq \mu.\)

3) In part (2) if \(f : \text{Card} \cap \lambda \rightarrow \text{Card}\) is increasing we can demand \(\lambda_n \in \text{Rang}(f), f(\lambda_n) < \lambda_{n+1}\). So if \(\lambda\) is strong limit > \(\aleph_0\), then we can demand \(2^{\lambda_n} < \lambda_{n+1} = \text{cf}(\lambda_{n+1}).\)

4) As in (2), (3) above with \(\text{nu}^+\) instead of \(\text{nu}^+.\)

Proof. 1) Straight.

2) Let \(\bar{\lambda} = \langle \lambda_n : n < \omega \rangle\) be such that \(\lambda = \sum_{n < \omega} \lambda_n, 2^{\aleph_0} < \lambda_n < \lambda_{n+1}, \text{cf}(\lambda_n) = \lambda_n.\)

Let \(\langle G_n, \ell \leq \omega \rangle\) be increasing, \(G_n, \ell\) a subgroup of \(G_n\) of cardinality \(\leq \lambda_\ell\) and \(G_n = \bigcup_{\ell < \omega} G_n, \ell\). Let \(\langle H_n, \ell \leq \omega \rangle\) be an increasing sequence of subgroups of \(H_n^I\) with union \(H_n^I, |H_{n,\ell}^I| \leq \lambda_\ell\). Let \(\langle J_n : n < \omega \rangle\) be an increasing sequence of subsets of \(\mathbf{I}\) with union \(\mathbf{I}\) such that \(|J_n| \leq \lambda_n.\)

Without loss of generality \(\pi_{n,\omega}\) maps \(G_n, \ell\) into \(G_m, \ell\) and \(\pi^t_{m,\omega}\) maps \(H^t_n, \ell\) into \(H^t_m, \ell\) and \(\sigma^t_{n,\ell}\) maps \(H^t_{n,\ell}\) into \(G^t_{n,\ell}\) (why? just close the witness).

Now for every increasing \(\eta \in \omega\) we let
\[G^\eta_n = \{ g \in G : \text{ for every } n < \omega \text{ we have } \pi_{n,\omega}(g) \in G_{n,\eta(n)} \}.\]
Clearly

\((*)_1(\alpha)\) \(G^n_\omega\) is a subgroup of \(G_\omega\)

\((\beta)\) \(\{G^n_\omega : \eta \in \omega^\omega\) increasing\} is directed, i.e. if \((\forall n < \omega)\eta(n) \leq \nu(n)\) where \(\eta, \nu \in \omega^\omega\) then \(G^n_\omega \subseteq G^n_\omega\)

\((\gamma)\) \(G_\omega = \cup\{G^n_\omega : \eta \in \omega^\omega\) (increasing)\}.

First assume \(\text{cf}(\mu) \neq \aleph_0\) so as \(\text{cf}(\mu) > 2^\aleph_0\) for some \(\eta \in \omega^\omega\), strictly increasing, we have

\((*)_2\) \(\mu \leq \sup\{|X|^+: X \subseteq G_\omega, \eta \text{ and } t \in I \& \ f \neq g \in X \Rightarrow fg^{-1} \notin \sigma^t_\omega(H^t_\omega)\}\).

However, as \(\lambda \leq \mu\), \(\text{cf}(\lambda) = \aleph_0\), \(\text{cf}(\mu) > 2^\aleph_0\) clearly \(\mu > \lambda\); also if \(X_1, X_2\) are as in \((*)_2\) then for some \(X \subseteq X_2\) we have \(|X| \leq |X_1| + |I|\) and \(X_1 \cup (X_2 \setminus X_2)\) is as required there. So we can choose \(\eta \in \omega^\omega\), increasing such that

\((*)_3\) there is \(X \subseteq G^n_\omega\) of cardinality \(\mu\) such that \(t \in I \& \ f \neq g \in X \Rightarrow fg^{-1} \notin \sigma^t_\omega(H^t_\omega)\).

Second assume \(\text{cf}(\mu) = \aleph_0\), so let \(\mu = \sum_{n<\omega} \mu_n, \mu_n < \mu_{n+1}\), and without loss of generality \(\lambda_n < \mu_n = \text{cf}(\mu_n)\) and \(\mu > \lambda \Rightarrow \mu_n > \lambda\). If \(\mu > \lambda\), for each \(n\) there is a witness \(\langle f^n_\alpha : \alpha < \mu_n\rangle\) to \(\text{cf}(\alpha) > \mu_n\), so \(f^n_\alpha \in G^n_\omega\) and as \(\mu_n > \lambda \geq |G^n_\alpha|\), without loss of generality \(\pi_{n,\omega}(f^n_\alpha) = \pi_{n,\omega}(f^n_\beta)\) so as we can replace \(f^n_\alpha\) by \(f^n_{\alpha+1}(f^n_\beta)^{\alpha+1}\), without loss of generality \(m \leq n \Rightarrow \pi_{m,\omega}(f^n_\alpha) = e_G\). For each \(\alpha\) let \(\eta^n_\alpha \in \omega^\omega\) be increasing be such that \(\pi_{n,\omega}(f^n_\alpha) \in G_{n,\eta_n(n)}\). As \(2^{\aleph_0} < \text{cf}(\mu_n) = \mu_n\), for some increasing \(\eta_n \in \omega^\omega\) we have \((\exists \mu_n \alpha < \mu_n), \eta^n_\alpha = \eta_n\).

So, hence without loss of generality \(\alpha < \mu \Rightarrow \eta^n_\alpha = \eta_n\). Let \(\eta \in \omega^\omega\) be \(\eta(n) = \text{Max}\{\eta_n(n) : m \leq n\}\). So we have \(n < \omega \& \ \alpha < \mu_n \Rightarrow \pi_{n,\omega}(f^n_\alpha) \in G_n\).

\((*)_4\) for every \(n < \omega\) and \(\mu'_0 < \mu\) (in fact even \(\mu_i = n\)) there are \(f_\alpha \in G^n_\omega\) for \(\alpha < \mu'\) such that \(\mu \leq \lambda \Rightarrow \pi_{n,\omega}(f_\alpha) = e_G\) and \(\alpha < \beta < \mu' \& \ t \in I \Rightarrow fg^{-1} \notin \sigma^t_\omega(H^t_\omega)\).

Lastly, if \(\mu = \lambda\), so \(\text{cf}(\mu) = \aleph_0\) the proof is as in the case \(\mu > \lambda \& \ \text{cf}(\mu) = \aleph_0\), except that \(\pi_{n,\omega}(f^n_\alpha) = \pi_{n,\omega}(f^n_\beta)\) holds by the choice of \(\langle f^n_\alpha : \alpha < \mu_n\rangle\) instead of by “without loss of generality”.

For each \(t \in J_n\) and strictly increasing \(\nu \in \omega^\omega\) let \(H^{(t,\nu)}_n\) be the subgroup \(\{g \in H^n_\omega : \) for every \(n < \omega\) we have \(\sigma_{n,\omega}(g) \in H^{t}_{n,\nu(n)}\}\). So let \(J'_n = \{(t, \nu) : t \in J\) and \(\nu \in \omega^\omega\) increasing\}.

We define \(G^n_{\omega,\zeta}\), a subgroup of \(G_{n,\eta(n)}\), decreasing with \(\zeta\) by induction on \(\zeta\):
ζ = 0: \( G_{n,ζ}^n = G_{n,η(n)} \)

ζ = ε + 1: \( G_{n,ζ}^n = \{ x : x \in G_{n,ε}^n \text{ and } x \in \text{Rang}(π_{n,n+1} \upharpoonright G_{n+1,ε}^n) \text{ and } n > 0 \Rightarrow π_{n-1,n}(x) \in G_{n-1,η(n-1),ε} \} \)

ζ limit: \( G_{n,ζ}^n = \bigcap_{\varepsilon < ζ} G_{n,ε}^n \).

Let \( G_{n,ζ}^n = \bigcap_{ζ < λ^+} G_{n,η(n),ζ}^n \), \( π_{m,n}^n = π_{m,n} \upharpoonright G_{n}^n \). Easily \( \langle G_{n}^n, π_{m,n}^n : m ≤ n < ω \rangle \) is directed with limit \( G_ω^n \) with \( π_{n,ω} = π_{n,ω} \upharpoonright G_ω^n \).

Define \( H_{n,ζ}^{(t,ν)} : π_{m,n,ζ}^n \) (for any ζ), \( H_{n,ζ}^{(t,ν)} : π_{m,n,ζ}^n \) parallel to \( G_{n}^n, π_{m,n}^n \) but such that \( σ_α^t \) maps \( H_{α}^{(t,ν)} \) into \( G_α^n \) (note: element of \( H_{α}^{(t,ν)} \) not mapped to \( G_α^n \) are irrelevant).

Let \( σ_ω^{(t,ν)} : H_ω^{(t,ν)} \to G_ω^n \) be \( σ_ω^t \upharpoonright H_ω^{(t,ν)} \) and \( σ_ω^{(t,σ)} = σ_ω^t \upharpoonright H_ω^{(t,ν)} \).

We have defined actually \( \mathcal{B} = \langle λ^{B}, J^{B}, G, H, \bar{π}^{B}, \bar{σ}^{B} \rangle \) where \( \bar{λ}^{B} = \langle λ_n : n < ω \rangle, \bar{J}^{B} = \langle J'_n : n < ω \rangle, \bar{G}^{B} = \langle G_{α}^n : α ≤ ω \rangle \),

\[
\bar{H}^{B} = \left\{ (H_α^t : α ≤ ω) : x \in \bigcup_{n} J'_n \right\},
\]

\[
\bar{π}^{B} = \left\langle π_{α,β}^n : α ≤ β ≤ ω \right\rangle \cap \left\{ π_{α,β}^{(t,ν)} : α ≤ β ≤ ω : (t, ν) \in \bigcup_{n} J'_n \right\}
\]

and

\[
\bar{σ}^{B} = \left\langle σ_{α}^{(t,ν)} : α ≤ ω \right\rangle : (t, ν) \in \bigcup_{n < ω} J'_n \right\}.
\]

We have almost finished. Still \( G_{n}^n \) may be of cardinality > \( λ_n \) but note that for \( k : ω \to ω \) non-decreasing with limit \( ω, \langle G_{n}^n : n < ω \rangle \) can be replaced by \( \langle G_{k(n)} : n < ω \rangle \).

By the definition of \( \mathcal{B}, G_ω^n \) is a subgroup of \( G_ω^n \) and for each \( t \in I \) for some \( n, t \in J_n \) and \( H_t^f \cap G_ω^n = \bigcup_{n < ω} H_{t,η}^f \) hence for \( f, g \in G_ω^n \subseteq G_ω^n \) we have

\[
f \circ g \Leftrightarrow f \circ g^{-1} \in H_t^f \Leftrightarrow (\exists h \in H_t^f)(f g^{-1} = h) \Leftrightarrow (\exists h)(h = (h_n : n < ω) \& h_n = π_{n,n+1}^f(σ_{h_{n+1}}) \cap \bigcap_{n < ω} f g^{-1} \upharpoonright n = σ_{n,n}^t(\bar{h}_{n}) \Leftrightarrow (\exists h)(h = (h_n : n < ω) \& h_n = H_{n,n+1}^f(σ_{h_{n+1}}) \cap \bigcap_{n < ω} f g^{-1} \upharpoonright n = σ_{n}^t(\bar{h}_{n}) \Leftrightarrow 2
\]

\[
\bigvee_{h \in ω} (\exists h)(h = (h_n : n < ω) \& \bigwedge_{n \in ω} h_n \in H_{n,n+1}^f) \& \bigwedge_{n \in ω} h_n \in H_{n,n+1}^f \& \bigwedge_{n \in ω} h_n = π_{n,n+1}^f(h_{n+1}) \&
\]

\( ^2 \)for each ζ separately, by induction on T
\[ \bigwedge_{n<\omega} f g^{-1} = \sigma_n^{t,\varphi}(h_n) \iff \bigvee_{\nu \in \omega} (\exists \tilde{h})(\tilde{h} = (h_n : n < \omega) \& \bigwedge_n h_n \in H_n^{t,\varphi} \& \bigwedge_n h_n = \pi_{n,n+1}(h_{n+1}) \& \bigwedge_{n<\omega} \pi_n^{t,\varphi}(f g^{-1}) = \sigma_n^{t,\varphi}(h_n) \bigvee_{\nu \in \omega} f g^{-1} \in H_{(t,\nu)}^{\varphi} \iff \bigvee_{\nu \in \omega} f \& (t,\nu)g; \text{ so clearly } \mu^+(\varphi) \leq \mu^+(\mathcal{A}). \] But also \( \mu^+(\varphi) > \mu \) by the choice of \( \eta \), i.e. by \((*)_3.3)\), \(4)\) Easy.

For the rest of this section we adopt:

5.3 Convention. 1) \( \mathcal{A} \) is an explicit \((\lambda, J)\)-system, so below \( \text{rk}_t(g, f) \) should be written as \( \text{rk}_t(g, f, \mathcal{A}) \), etc.
2) \( \lambda = \sum_{n<\omega} \lambda_n, \lambda_n = \lambda_n^{t,\varphi}, J_n = J_n^{t,\varphi}, I = I^{t,\varphi} = \bigcup_{n<\omega} J_n, G_\alpha = G_\alpha^{t,\varphi}, \) etc.
3) \( k_t(n) = \text{Max}\{m : m \leq n, |H_m| \leq \lambda_n\} \) so \( k_t : \omega \rightarrow \omega \) is non-decreasing converging to \( \infty \).

For the reader’s convenience we repeat 5.5 - 5.8 from [GrSh 302a].

5.4 Definition. 1) For \( g \in H_\alpha^t \) let \( \text{lev}(g) = \alpha \) (without loss of generality this is well defined).
2) For \( \alpha \leq \beta \leq \omega, g \in H_\beta^t \) let \( g \upharpoonright H_\alpha^t = \pi_{\alpha,\beta}(g) \) and we say \( g \upharpoonright H_\alpha^t \) is below \( g \) and \( g \) is above \( g \upharpoonright H_\alpha^t \) or extend \( g \) up \( H_\beta^t \).
3) For \( \alpha \leq \beta \leq \omega, f \in G_\beta \) let \( f \upharpoonright G_\alpha = \pi_{\alpha,\beta}(f) \).

We will now describe the rank function used in the proof of the main theorem.

5.5 Definition. 1) For \( g \in H_n^t, f \in G_\omega \) we say that \( (g, f) \) is a nice \( t \)-pair if \( \sigma_n^{t,\varphi}(g) = f \upharpoonright G_n \).
2) Define, for \( t \in I \), a ranking function \( \text{rk}_t(g, f) \) for any nice \( t \)-pair. First by induction on the ordinal \( \alpha \) (we can fix \( f \in G_\omega \)), we define when \( \text{rk}_t(g, f) \geq \alpha \) simultaneously for all \( n < \omega, g \in H_n^t \):

\( a) \ \text{rk}_t(g, f) \geq 0 \iff (g, f) \) is a nice \( t \)-pair
\( b) \ \text{rk}_t(g, f) \geq \delta \) for a limit ordinal \( \delta \iff \) for every \( \beta < \delta \) we have \( \text{rk}_t(g, f) \geq \beta \)
\( c) \ \text{rk}_t(g, f) \geq \beta + 1 \iff \) (g, f) is a nice \( t \)-pair, and letting \( n = \text{lev}(g) \) there exists \( g' \in H_{n+1}^t \) extending \( g \) such that \( \text{rk}_t(g', f) \geq \beta \)
\( d) \ \text{rk}_t(g, f) \geq -1 \).

3) For \( \alpha \) an ordinal or \(-1 \) (stipulating \(-1 < \alpha < \infty \) for any ordinal \( \alpha \)) we let \( \text{rk}_t(g, f) = \alpha \iff \text{rk}_t(g, f) \geq \alpha \) and it is false that \( \text{rk}_t(g, f) \geq \alpha + 1 \).
4) \( \text{rk}_t(g, f) = \infty \iff \) for every ordinal \( \alpha \) we have \( \text{rk}_t(g, f) \geq \alpha \).

The following two claims give the principal properties of \( \text{rk}_t(g, f) \).
5.6 Claim. Let \((g, f)\) be a nice \(t\)-pair.

1) The following statements are equivalent:

(a) \(rk_t(g, f) = \infty\)

(b) there exists \(g' \in H^t_\omega\) extending \(g\) such that \(\sigma_t^\omega(g') = f\).

2) If \(rk_t(g, f) < \infty\), then \(rk_t(g, f) < \mu^+\) where \(\mu = \sum_{n < \omega} 2^{\lambda_n}\) (for \(\lambda\) strong limit, \(\mu = \lambda\)).

3) If \(g'\) is a proper extension of \(g\) and \((g', f)\) is also a nice \(t\)-pair then

(\(\alpha\)) \(rk_t(g', f) \leq rk_t(g, f)\) and

(\(\beta\)) if \(0 \leq rk_t(g, f) < \infty\) then the inequality is strict.

4) For \(f_1, f_2 \in G^\omega_\omega, n < \omega\) and \(t \in \bigcup_{n < \omega} J_n\) we have \(f_1 E_t f_2\) iff \(rk_t(g, f_1^{-1} f_2) = \infty\) for some \(g \in H^\omega_n\).

Proof.

1) Statement \((a) \Rightarrow (b)\).

Let \(n\) be the value such that \(g \in H^t_n\). If we will be able to choose \(g_k \in H^t_k\) for \(k < \omega, k \geq n\) such that

(i) \(g_n = g\)

(ii) \(g_k\) is below \(g_{k+1}\) that is \(\pi^t_{k,k+1}(g_{k+1}) = g_k\) and

(iii) \(rk_t(g_k, f) = \infty\),

then clearly we will be done since \(g' = \lim_{k \to \omega} g_k\) is as required. The definition is by induction on \(k \geq n\).

For \(k = n\) let \(g_0 = g\).

For \(k \geq n\), suppose \(g_k\) is defined. By \((iii)\) we have \(rk_t(g_k, f) = \infty\), hence for every ordinal \(\alpha\), \(rk_t(g_k, f) > \alpha\) hence there is \(g^\alpha \in H^t_{k+1}\) extending \(g\) such that \(rk_t(g^\alpha, f) \geq \alpha\). Hence there exists \(g^* \in H^t_{k+1}\) extending \(g_k\) such that \(\{\alpha : g^\alpha = g^*\}\) is unbounded hence \(rk_t(g^*, f) = \infty\), and let \(g_{k+1} = g^*\).

Statement \((b) \Rightarrow (a)\).

Since \(g\) is below \(g'\), it is enough to prove by induction on \(\alpha\) that for every \(k \geq n\) when \(g_k = g' \upharpoonright H^t_k\) we have that \(rk_t(g, f) \geq \alpha\).

For \(\alpha = 0\), since \(\sigma^\omega_t(g') = f \upharpoonright G_n\) clearly for every \(k\) we have \(\sigma^t_k(g_k) = f \upharpoonright G_k\) so \((g_k, f)\) is a nice \(t\)-pair.
For limit $\alpha$, by the induction hypothesis for every $\beta < \alpha$ and every $k$ we have $\text{rk}_t(g_k, f) \geq \beta$, hence by Definition 5.5(2)(b), $\text{rk}_t(g_k, f) \geq \alpha$.

For $\alpha = \beta + 1$, by the induction hypothesis for every $k \geq n$ we have $\text{rk}_t(g_k, f) \geq \beta$. Let $k_0 \geq n$ be given. Since $g_{k_0}$ is below $g_{k_0+1}$ and $\text{rk}_t(g_{k_0+1}, f) \geq \beta$, Definition 5.5(2)(c) implies that $\text{rk}_t(g_{k_0}, f) \geq \beta + 1$; i.e. for every $k \geq n$ we have $\text{rk}_t(g_k, f) \geq \alpha$. So we are done.

2) Let $g \in H^t_n$ and $f \in G_\omega$ be given. It is enough to prove that if $\text{rk}_t(g, f) \geq \mu^+$ then $\text{rk}_t(g, f) = \infty$. Using part (1) it is enough to find $g' \in H^t_\omega$ such that $g$ is below $g'$ and $f = \sigma^t_\omega(g')$.

We choose by induction on $k < \omega, g_k \in H^t_{n+k}$ such that $g_k$ is below $g_{k+1}$ and $\text{rk}_t(g_k, f) \geq \mu^+$. For $k = 0$ let $g_k = g$. For $k+1$, for every $\alpha < \mu^+$, as $\text{rk}_t(g_k, f) > \alpha$ by 5.5(2)(c) there is $g_{k,\alpha} \in G^t_{n+k+1}$ extending $g_k$ such that $\text{rk}_t(g_{k,\alpha}, f) \geq \alpha$. But the number of possible $g_{k,\alpha}$ is $\leq |H^t_{n+k+1}| \leq 2^\lambda n^{+k+1} < \mu^+$ hence there are a function $g$ and a set $S \subseteq \mu^+$ of cardinality $\mu^+$ such that $\alpha \in S \implies g_{k,\alpha} = g$. Then take $g_{k+1} = g$.

3) Immediate from the definition.

4) Check the definitions.

□ 5.6

5.7 Lemma. 1) Let $(g, f)$ be a nice t-pair. Then we have $\text{rk}(g, f) \leq \text{rk}(g^{-1}, f^{-1})$.

2) For every nice t-pair $(g, f)$ we have $\text{rk}(g, f) = \text{rk}(g^{-1}, f^{-1})$.

Proof. 1) By induction on $\alpha$ prove that $\text{rk}(g, f) \geq \alpha \implies \text{rk}(g^{-1}, f^{-1}) \geq \alpha$ (see more details in the proof of Lemma 5.8).

2) Apply part (1) twice. □ 5.7

5.8 Lemma. 1) Let $n < \omega$ be fixed, and let $(g_1, f_1), (g_2, f_2)$ be nice t-pairs with $g_1 \in H^t_n(\ell = 1, 2)$. Then $(g_1, g_2, f_1, f_2)$ is a nice pair and $\text{rk}_t(g_1, g_2, f_1, f_2) \geq \text{Min}\{\text{rk}_t(g_\ell, f_\ell) : \ell = 1, 2\}$.

2) Let $n, (f_1, g_1)$ and $(f_2, g_2)$ be as above. If $\text{rk}_t(g_1, f_1) \neq \text{rk}_t(g_2, f_2)$, then $\text{rk}_t(g_1, g_2, f_1, f_2) = \text{Min}\{\text{rk}_t(g_\ell, f_\ell) : \ell = 1, 2\}$.

Proof. 1) It is easy to show that the pair $(g_1, f_2, f_1, f_2)$ is t-nice. We show by induction on $\alpha$ simultaneously for all $n < \omega$ and every $g_1, g_2 \in H^t_n$ that $\text{Min}\{\text{rk}_t(g_\ell, f_\ell) : \ell = 1, 2\} \geq \alpha$ implies that $\text{rk}(g_1, g_2, f_1, f_2) \geq \alpha$.

When $\alpha = 0$ or $\alpha$ is a limit ordinal this should be clear. Suppose $\alpha = \beta + 1$ and that $\text{rk}(g_\ell, f_\ell) \geq \beta + 1$ for $\ell = 1, 2$; by the definition of rank for $\ell = 1, 2$ there exists $g_{\ell}' \in H^t_{n+1}$ extending $g_\ell$ such that $(g_{\ell}', f_\ell)$ is a nice pair and $\text{rk}_t(g_{\ell}', f_\ell) \geq \beta$. By the induction assumption $\text{rk}_t(g_{\ell}', g_2, f_1, f_2) \geq \beta$ and clearly $(g_1 g_2) \cap n = g_1 g_2$. Hence $g_1 g_2$ is as required in the definition of $\text{rk}_t(g_1, g_2, f_1, f_2) \geq \beta + 1$. 

□ 5.8
2) Suppose without loss of generality that \( \text{rk}(g_1, f_1) < \text{rk}(g_2, f_2) \), let \( \alpha_1 = \text{rk}(g_1, f_1) \) and let \( \alpha_2 = \text{rk}(g_2, f_2) \). By part (1), \( \text{rk}(g_1g_2, f_1f_2) \geq \alpha_1 \), by Proposition 5.7, \( \text{rk}(g_2^{-1}, f_2^{-1}) = \alpha_2 > \alpha_1 \). So we have

\[
\alpha_1 = \text{rk}(g_1, f_1) = \text{rk}(g_1g_2g_2^{-1}, f_1f_2f_2^{-1}) \\
\geq \min\{\text{rk}(g_1g_2, f_1f_2), \text{rk}(g_2^{-1}, f_2^{-1})\} \\
= \min\{\text{rk}(g_1g_2, f_1f_2), \alpha_2\} \geq \min\{\alpha_1, \alpha_2\} = \alpha_1.
\]

Hence the conclusion follows. \( \square \)

5.9 Theorem. Assume \( (\mathcal{A} \text{ is an explicit } \lambda\text{-system and}) \)

(a) \( \lambda \) is strong limit \( \lambda > cf(\lambda) = \aleph_0 \)

(b) \( \text{nu} (\mathcal{A}) \geq \lambda \text{ or just } \text{nu}^+ (\mathcal{A}) \geq \lambda. \)

Then \( \text{nu} (\mathcal{A}) = + 2^{\lambda}. \)

The proof is broken into parts.

5.10 Fact: We can choose by induction on \( n, (f_{n,i} : i < \lambda_n) \) such that

(a) \( f_{n,i} \in G_\omega \) and \( f_{n,i} \upharpoonright G_{n+1} = e G_{n+1} \)

(b) \( i < j < \lambda_n \) \& \( t \in J_n \Rightarrow f_{n,i} \circ _t f_{n,j} \)

(c) \( \text{rk}(g, f_{n,i}f_{n,j}^{-1}) < \infty \) for any \( t \in J_n, k \leq n, g \in H_k^t \) and \( i \neq j < \lambda_n \)

(d) if \( f^* \) belongs to the subgroup \( K_n \) of \( G_\omega \) generated by the \( \{f_{m,j} : m < n, j < \lambda_m\} \) and \( t \in J_n, g \in \bigcup_{m \leq k_t(n)} H_k^t \), then for every \( i_0 < i_1 < i_2 < i_3 < \lambda_n \) each of the following statements have the same truth value, (i.e. the truth value does not depend on \( (i_0, i_1, i_2, i_3) \))

(i) \( \text{rk}(g, f_{n,i}f_{n,j}^{-1}f_{n,i_0}f_{n,i_0}^{-1}) < \infty \)

(ii) \( \text{rk}(g, f_{n,j}f_{n,i_0}f_{n,i_0}^{-1}f_{n,i_0}f_{n,i_j}^{-1}) < \infty \)

(iii) \( \text{rk}(e_{H_t^t, n, i_j}f_{n,j}f_{n,i_0}^{-1}) < \text{rk}(g, f^*) \)

(iv) \( \text{rk}(e_{H_t^t, n, i_0}f_{n,i_0}^{-1}f_{n,i_0}f_{n,i_j}^{-1}) > \text{rk}(g, f^*) \)

(v) \( \text{rk}(g, f^*) < \text{rk}(g, f_{n,i_0}f_{n,i_0}^{-1}f_{n,i_0}f_{n,i_j}^{-1}) \)

(vi) \( \text{rk}(g, f^*) < \text{rk}(g, f_{n,i_0}f_{n,i_0}^{-1}f_{n,i_0}f_{n,i_0}^{-1}) \)

(vii) \( \text{rk}(g, f_{i_0}f_{i_1}^{-1}) < \infty \)

(viii) \( \text{rk}(g, f_{i_1}f_{i_0}^{-1}) < \infty \)
(ε) for each t ∈ Jn one of the following occurs:

(a) for i₀ < i₁ ≤ i₂ < i₃ < λₙ we have
\[ \text{rk}_t(e_{H_{k_t(n)}}, f_{n,i_0}^{-1}) < \text{rk}(e_{H_{k_t(n)}}, f_{n,i_2}^{-1}) \]

(b) for some γᵗₙ for every i < j < λₙ we have
\[ γᵗₙ = \text{rk}_t(e_{H_{k_t(n)}}, f_{n,i}^{-1}) \]

Proof. We can satisfy clauses (α), (β) by the definitions and clause (γ) follows. Now clause (δ) is straight by Erdős Rado Theorem applied to a higher n.
For clause (ε) notice the transitivity of the order and of equality and “there is no decreasing sequence of ordinals of length ω”.

5.11 Notation. For α ≤ ω let \( T_α = \times_{k<α} λ_k \), \( T = \bigcup_{n<ω} T_n \) (note: by the partial order \( \triangleleft \), \( T \) is a tree; treeness will be used).

5.12 Definition. Now by induction on \( n < ω \), for every \( η ∈ \times_{m<n} λ_m \) we define \( f_η \in G_ω \) as follows:

for \( n = 0 \):
\[ f_η = f_\triangleleft = e_{G_ω} \]

for \( n = m + 1 \):
\[ f_η = f_{m,3η(m)+1} f_{m,3η(m)}^{-1} f_η|m. \]

5.13 Fact. 1) For \( η ∈ T_ω \) and \( m ≤ n < ω \) we have
\[ f_η|n \upharpoonright G_{m+1} = f_η|m \upharpoonright G_{m+1} \]

2) \( η ∈ \times_{m<n} λ_m ⇒ f_η ∈ K_n \) and \( K_n ⊆ K_{n+1} \).

Proof. As \( π_{m,ω} \) is a homomorphism it is enough to prove \( (f_η|n(f_η|m)^{-1}) \upharpoonright G_{m+1} = e_{G_{m+1}} \), hence it is enough to prove \( m ≤ k < ω ⇒ (f_η|k f_η|k+1)^{-1} \upharpoonright G_{m+1} = e_{G_{m+1}} \) (of course, \( k < n \) is enough). Now this statement follows from \( k < ω ⇒ f_η|k f_η|k+1)^{-1} \upharpoonright G_{k+1} = e_{G_{k+1}} \), which by Definition 5.12 means \( f_{k,3η(k)+1} f_{k,3η(k)}^{-1} \upharpoonright G_{k+1} = e_{G_{k+1}} \), which follows from \( ζ < λ_k ⇒ f_{k,η(ζ)} \upharpoonright G_{k+1} = e_{G_{k+1}} \) which holds by clause (α) above. □
5.14 Definition. For \( \eta \in T_\omega \) we have \( f_\eta \in G_\omega \) is well defined as the inverse limit of \( \langle f_\eta \rangle \mid G_n : n < \omega \), so \( n < \omega \rightarrow f_\eta \mid G_n = f_\eta \mid n \). This being well defined follows by 5.13 and \( G^\omega \) being an inverse limit.

5.15 Proposition. Let \( \eta, \nu \in T_\omega \) be such that \( (\forall \infty n)(\eta(n) \neq \nu(n)), \eta(n) > 0, \nu(n) > 0 \). If \( t \in I \), then \( f_\eta f_\nu^{-1} \notin \sigma^\omega_I(H^1_\omega) \).

Proof. Suppose toward contradiction that for some \( t \in H^t_\omega \) we have \( \sigma^\omega_I(g) = f_\eta f_\nu^{-1} \). Let \( k < \omega \) be large enough such that \( t \in J_k \), \( (\forall \ell) [k \leq \ell < \omega \rightarrow \eta(\ell) \neq \nu(\ell)] \). Let \( \xi^\ell = \text{rk}_t(g \upharpoonright H^t_{k_1(\ell)}, f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1}) \) and \( \zeta^\ell = \text{rk}_t(g \upharpoonright H^t_{k_1(\ell)}, f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1}) \) (the difference between the two is the use of \( k_1(\ell) \) vis \( k_2(\ell) \)). Clearly

\[
\begin{align*}
(+) & \quad f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1} = (f_\ell, 3\eta(\ell) + 1, f_\nu[\ell(\ell)]^{-1})(f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1}) (f_\ell, 3\nu(\ell) f_\ell[\ell(\ell)]^{-1})
\end{align*}
\]

[Why? Algebraic computations and Definition 5.12.] Next we claim that

\[
(+) \quad \xi^\ell < \infty \text{ for } \ell \geq k \ (\ell < \omega).
\]

Why?

Case 1: \( \eta(\ell) < \nu(\ell) \).

Assume toward contradiction \( \xi^\ell = \infty \), but by clause (\( \gamma \)) of 5.10 above

\[
\text{rk}_t(e_{H^t_{k_1(\ell)}, f_\ell, 3\eta(\ell) + 2 f_\ell, 3\nu(\ell) + 1}) < \infty = \xi^\ell, \text{ hence by 5.8(2)}.
\]

Now (by the choice of \( f_\eta[\ell(\ell)], f_\nu[\ell(\ell)] \) that is Definition 5.12 that is \( (+) \), algebraic computation and the previous inequality) we have

\[
\begin{align*}
\infty > \text{rk}_t(g \upharpoonright H^t_{k_1(\ell)}, f_\ell, 3\eta(\ell) + 2 f_\ell, 3\nu(\ell) + 1 f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1}) &= \text{rk}_t(g \upharpoonright H^t_{k_1(\ell)}, f_\ell, 3\eta(\ell) + 2 f_\ell, 3\nu(\ell) + 1 f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1})(f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1})(f_\ell, 3\nu(\ell) f_\ell[\ell(\ell)]^{-1}).
\end{align*}
\]

This and the assumption \( \xi^\ell = \infty \) gives a contradiction to (\( \delta \)) of 5.10 (for

\[
\begin{align*}
n = \ell \text{ and } f^* = f_\eta[\ell(\ell)] f_\nu[\ell(\ell)]^{-1} \in K_\ell \text{ (see 5.13(1)) and } (i_0, i_1, i_2, i_3) \text{ being } (3\eta(\ell), 3\eta(\ell) + 2, 3\nu(\ell), 3\nu(\ell) + 1) \text{ and being } (3\eta(\ell), 3\eta(\ell) + 1, 3\nu(\ell), 3\nu(\ell) + 1); \text{ the contradiction is}
\end{align*}
\]
that for the first quadruple we get rank $< \infty$ by the previous inequality by the last
inequality, for the second quadruple we get equality as we are temporarily assuming
$\xi_\ell = \omega$, the definition of $\xi_\ell$ and $(*)_1$.

Case 2: $\nu(\ell) > \eta(\ell)$.

Similar using $(\delta)(ii)$ of 5.10 instead of $(\delta)(i)$ of 5.10 (using $\eta(\ell) > 0$).
So we have proved $(*)_2$.

$$(*)_3 \quad \xi_{\ell+1} \leq \zeta_\ell$$ for $\ell > k$.

Why? Assume toward contradiction that $\xi_{\ell+1} > \zeta_\ell$.

Let $f^* = f_{\eta(\ell+1)}f_{\nu(\ell+1)}^{-1}$, so $\zeta_\ell = \text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f^*)$ and using the choice of $\xi_{\ell+1}$
and $(*)_1$ we have $\xi_{\ell+1} = \text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\nu(\ell+1)}^{-1})f_{\ell+1,3\nu(\ell+1)+1}$.

If $\zeta_\ell < \text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\nu(\ell+1)}^{-1})$ then by 5.10$(\delta)(iii)$ also
$\zeta_\ell < \text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1}f_{\ell+1,3\nu(\ell+1)}^{-1})$ hence using twice 5.8$(2)$ we have first
$\zeta_\ell = \text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\nu(\ell+1)}^{-1}f^*)$ and second (using also 5.7$(2)$)
we have $\zeta_\ell = \text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\eta(\ell+1)}^{-1}f^*f_{\ell+1,3\nu(\ell+1)}^{-1}f_{\ell+1,3\nu(\ell+1)+1})$, so by the second statement in the previous paragraph (on $\xi_{\ell+1}$) we get $\zeta_\ell = \xi_{\ell+1}$
contradicting our temporary assumption toward contradiction $-(*)_3$; so we have
$\zeta_\ell \geq \text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\eta(\ell+1)}^{-1})$.

Also if $\text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\eta(\ell+1)}^{-1}) \neq \text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1}f_{\ell+1,3\nu(\ell+1)}^{-1})$
then $\zeta_\ell$ is not equal to at least one of them hence by 5.10$(\delta)(iii) + (iv)$ also $\zeta_\ell$ is
not equal to those two ordinals so similarly to the previous sentence, 5.8$(2)$ gives
$\xi_{\ell+1} = \text{Min}\{\text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\eta(\ell+1)}^{-1}),$
$\text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\eta(\ell+1)}^{-1})\}$ which is $\leq \zeta_\ell$ so $\xi_{\ell+1} \leq
\zeta_\ell$, contradicting our assumption toward contradiction, $-(*)_3$.

Together the case left (inside the proof of $(*)_3$, remember 5.7) is:

$$\boxtimes \quad \zeta_\ell = \text{rk}_t(g \upharpoonright H_{k_t(\ell+1)}^{t(\ell+1)}, f^*) \geq \text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\eta(\ell+1)+1}f_{\ell+1,3\eta(\ell+1)}^{-1}) =
\text{rk}_t(e_{H_{k_t(\ell+1)}^{t(\ell+1)}}, f_{\ell+1,3\nu(\ell+1)+1}f_{\ell+1,3\nu(\ell+1)}^{-1}).$$

So in clause 5.10$(\varepsilon)$, for $n = \ell + 1$, case (b) holds, call this constant value $\varepsilon_\ell$.
As, toward contradiction we are assuming $\xi_{\ell+1} > \zeta_\ell$ during the proof of $(*)_3$; so by
$\boxtimes, \xi_{\ell+1} > \zeta_\ell \geq \varepsilon_\ell$ hence we get, by computation and by 5.8 that if $\eta(\ell + 1) > \nu(\ell +$
1) then \( \text{rk}_t (g \upharpoonright H^t_{k_i (\ell + 1)}; f_{\ell+1, 3\eta (\ell+1)+2} f_{\ell+1, 3\eta (\ell+1)+1} f^* f_{\ell+1, 3\nu (\ell+1)} f^{-1}_{\ell+1, 3\nu (\ell+1)+1}) = \text{rk}_t (e H^t_{k_i (\ell + 1)}; f_{\ell+1, 3\eta (\ell+1)+2} f_{\ell+1, 3\eta (\ell+1)+1} f^* f_{\ell+1, 3\nu (\ell+1)} f^{-1}_{\ell+1, 3\nu (\ell+1)+1}) \) by (b) of 5.10(\( \varepsilon \)) proved above the later is \( \varepsilon^\ell \leq \xi^\ell < \xi^{\ell+1} = \text{rk}_t (g \upharpoonright H^t_{k_i (\ell + 1)}; f_{\ell+1, 3\eta (\ell+1)+1} f^* f_{\ell+1, 3\nu (\ell+1)} f^{-1}_{\ell+1, 3\nu (\ell+1)+1}) \) contradiction to 5.10(\( \delta \))(\( \nu \)) for the two quadruples \((3\nu (\ell + 1), 3\nu (\ell + 1) + 1, 3\eta (\ell + 1), 3\eta (\ell + 1) + 2)\) and \((3\nu (\ell + 1), 3\nu (\ell + 1) + 1, 3\eta (\ell + 1), 3\eta (\ell + 1) + 1)\) and \( n = \ell + 1 \) if \( \eta (\ell + 1) < \nu (\ell + 1) \) we use similarly \( f_{\ell+1, 3\nu (\ell+1)+2} f_{\ell+1, 3\nu (\ell+1)+1} \). So \((*)_3\) holds.

\[(*)_4 \quad \xi^\ell \leq \xi^\ell \]

[Why? Look at their definitions, as \( g \upharpoonright H^t_{k_i (\ell + 1)} \) is above \( g \upharpoonright H^t_{k_i (\ell)} \). Now if \( k_t (\ell), k_t (\ell + 1) \) are equal trivial otherwise use 5.6(3).]

\[(*)_5 \quad \text{if } k_t (\ell + 1) > k_t (\ell) \text{ then } \xi^\ell < \xi^\ell \text{ (so } \xi^\ell > 0) \]

[Why? Like \((*)_4\).]

\[(*)_6 \quad \xi^\ell \geq \xi^{\ell+1} \text{ and if } k_t (\ell + 1) > k_t (\ell) \text{ then } \xi^\ell > \xi^{\ell+1} \]

[Why? By \((*)_3 + (*)_4\) the first phrase, and \((*)_3 + (*)_5\) for the second phrase.]

So \( \langle \xi^\ell : \ell \in [k, \omega) \rangle \) is non-increasing, and not eventually constant sequence of ordinals, contradiction.

\(\square_5.15\)

**Proof of 5.9.** Obvious as we can find \( T' \subseteq T \), a subtree with \( \lambda^{\aleph_0} \) \( \omega \)-branches such that \( \eta \neq \nu \in \lim(T') \Rightarrow (\forall \ell) \eta (\ell) \neq \nu (\ell) \) and \( \eta \in \lim(T') \) & \( n < \omega \Rightarrow \eta (n) > 0 \). Now \( \langle f_\eta : \eta \in \lim(T') \rangle \) is as required by 5.15.

5.16 Conclusion: If \( \mathcal{A} \) is a \( (\lambda, \mathbb{I}) \)-system, and \( \lambda \) is an uncountable strong limit of cofinality \( \aleph_0 \) and \( \text{nu}(\mathcal{A}) \geq \lambda \) (or just \( \text{nu}^+(\mathcal{A}) \geq \lambda \)), then \( \text{nu}(\mathcal{A}) =^{+} 2^\lambda \).

**Proof.** So we assume \( \lambda > \aleph_0 \) hence \( \lambda > 2^{\aleph_0} \) and trivially \( \text{nu}^+(\mathcal{A}) \geq \text{nu}(\mathcal{A}) \geq \lambda \). We apply 5.2(2) to \( \mathcal{A} \) and \( \mu = \lambda \) (so \( \text{cf}(\mu) = \aleph_0 \)) and get an explicit \( (\lambda, \mathbb{J}) \)-system \( \mathcal{B} \) such that \( \mu \leq \text{nu}^+(\mathcal{B}) \leq \text{nu}(\mathcal{A}) \) hence by 5.9 we have \( \text{nu}(\mathcal{B}) =^{+} 2^\lambda \) hence by the choice of \( \mathcal{B} \) also \( \text{nu}(\mathcal{A}) =^{+} 2^\lambda \). The proof for \( \text{nu}^+(\mathcal{A}) \geq \lambda \) is similar.

\(\square_5.16\)

5.17 Concluding Remarks. Can we weaken condition \( (E)^+ \) in Theorem 1.1(2)? Can we use rank?
REFERENCES.


