THE COVERING NUMBERS OF MYCIELSKI IDEALS ARE ALL EQUAL

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Abstract. The Mycielski ideal $\mathfrak{M}_k$ is defined to consist of all sets $A \subseteq {}^{\omega}k$ such that $\{f \restriction X : f \in A\} \neq {}^{X}k$ for all $X \in [\omega]^{\aleph_0}$. It will be shown that the covering numbers for these ideals are all equal. However, the covering numbers of the closely associated Rosłanowski ideals will be shown to be consistently different.

1. Introduction

In [6] J. Mycielski defined a class of ideals which have been studied in various contexts by several authors [7, 11, 8, 10, 5, 1, 9, 2, 4, 3]. This paper is devoted to examining the covering numbers of these ideals as well as those of a closely related class of ideals. It will be shown that, while the covering number of the Mycielski ideals is independent of their dimension, the covering number of the related ideals is very closely related to their dimension.

Definition 1.1. The Mycielski ideal $\mathfrak{M}_k$ is defined to consist of all sets $A \subseteq {}^{\omega}k$ such that for all $X \in [\omega]^{\aleph_0}$

$$\{f \restriction X : f \in A\} \neq {}^{X}k \tag{1.1}$$

A function Φ on $[\omega]^{\aleph_0}$ will be said to witness that $A \in \mathfrak{M}_k$ if $\Phi(X) \in {}^{X}k \setminus \{f \restriction X : f \in A\}$ for each $X \in [\omega]^{\aleph_0}$.

Notice that if $A \in \mathfrak{M}_k$ and X is an infinite subset of ω then not only is there some $g \in {}^{X}k$ such that for all $f \in A$ there is some $x \in X$ such that $f(x) \neq g(x)$ but, in fact, there some $g \in {}^{X}k$ such that for all $f \in A$ there are infinitely many $x \in X$ such that $f(x) \neq g(x)$. The next definition will generalize this version of the Mycielski ideals.

Definition 1.2. Let $\mathbb{PF}_k$ denote the set of all functions $f : X \to k$ where X is a coinfinite subset of ω. The Rosłonowski ideal $\mathfrak{R}_k$ is defined to consist of all sets $A \subseteq {}^{\omega}k$ such that for all $g \in \mathbb{PF}_k$ there is

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an extension $g' \supseteq^* g$ such that $g' \in \mathbb{PF}_k$ and $g' \not\subseteq^* f$ for all $f \in A$. A function Φ on $\mathbb{PF}_k$ will be said to witness that $A \in R(k)$ if $g \subseteq \Phi(g) \in \mathbb{PF}_k$ for each $g \in \mathbb{PF}_k$ and $\Phi(g) \not\subseteq^* f$ for all $f \in A$.

It is worth noting that neither of these ideals has a simple definition. Indeed, since the definition given is Π^1_2 many of the usual arguments which apply to Borel ideals must be applied with great care, if at all, in this context. For an alternate approach to finding a nice base for the Mycielski ideals see [10].

The covering numbers of the ideals $\mathfrak{R}_k$ have a connection to gaps in $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$. Indeed, the assertion that $\text{cov}(\mathfrak{R}_2) = \aleph_1$ can be interpreted as saying there are many Hausdorff gaps. To see this, suppose that $\{A_\xi\}_{\xi\in\omega_1}$ is a cover of 2^ω by sets in $\mathfrak{R}_2$ witnessed by $\{\Phi_\xi\}_{\xi\in\omega_1}$. If $\{f_\xi\}_{\xi\in\omega_1}$ is any $\subseteq^*$-increasing sequence in $\mathbb{PF}_k$ such that $f_{\xi+1} = \Phi_\xi(f_x i)$ then $\{(f_\xi^{-1}\{0\}, f_\xi^{-1}\{1\})\}_{\xi\in\omega_1}$ is a Hausdorff gap. Hence a large tree all of whose branches are Hausdorff gaps can be constructed using $\text{cov}(\mathfrak{R}_2) = \aleph_1$. It will be shown that similar assertions for $\text{cov}(\mathfrak{R}_n) = \aleph_1$ are not equivalent to $\text{cov}(\mathfrak{R}_2) = \aleph_1$ for $n > 2$.

2. Equality and inequality

Theorem 2.1. *If m and n are integers greater than 1 then* $\text{cov}(\mathfrak{M}_k) = \text{cov}(\mathfrak{M}_n)$.

Proof. To begin, notice that if Φ witnesses that $A \in \mathfrak{M}_k$ then

$$\{f \in {}^\omega k + 1 : (\forall X \in [\omega]^{\aleph_0}) f \restriction X \neq \Phi(X)\}$$

belongs to $\mathfrak{M}_{k+1}$. It follows that $\text{cov}(\mathfrak{M}_k) \geq \text{cov}(\mathfrak{M}_{k+1})$. It therefore suffices to show that $\text{cov}(\mathfrak{M}_{k^2}) \geq \text{cov}(\mathfrak{M}_k)$ for each $n \geq 2$.

To this end, let $\beta : \omega \to [\omega]^2$ be a bijection and let $\beta_s(n)$ be the smallest member of $\beta(n)$ and $\beta_g(n)$ be the greatest member of $\beta(n)$. Define a relation $\equiv_\beta$ on partial functions from ω to k and partial functions from ω to k^2 by $f \equiv_\beta g$ if and only if the following conditions (2.1) and (2.2) hold:

$$(\forall \{n,m\} \in [\text{domain}(g)]^2)\beta(n) \cap \beta(m) = \emptyset \tag{2.1}$$

$$(\forall n \in \text{domain}(g)) g(n) = kf(\beta_s(n)) + f(\beta_g(n)) \tag{2.2}$$

Now suppose that $\mathcal{A}$ is a cover of ${}^\omega(k^2)$ by sets in $\mathfrak{M}_{k^2}$ and that Φ_A witnesses that $A \in \mathfrak{M}_{k^2}$ for each $A \in \mathcal{A}$. Now, for $A \in \mathcal{A}$ define

$$A^* = \{f \in {}^\omega k : (\forall X \in [\omega]^{\aleph_0})(\forall Z \in [\omega]^{\aleph_0}) f \restriction X \not\equiv_\beta \Phi(Z)\} \tag{2.3}$$

It will be shown that $\{A^* : A \in \mathcal{A}\}$ is a cover of ${}^\omega k$ by sets in the ideal $\mathfrak{M}_k$.

To see that each $A^* \in \mathfrak{M}_k$ let $A \in \mathfrak{M}_{k^2}$ and $X \in [\omega]^{\aleph_0}$. Let $\{\{e_i, d_i\}\}_{i\in\omega}$ be disjoint pairs from X such that $e_i < d_i$ for all i. Let $Z = \{\beta^{-1}(\{e_i, d_i\})\}_{i\in\omega}$ and define $h : \bigcup_{i\in\omega}\{e_i, d_i\} \to k$ such that $\Phi_A(Z) = kh(e_i) + h(d_i)$ for all i. It follows that no member of A^* extends h.

To see that $\{A^* : A \in \mathcal{A}\}$ is a cover of ${}^\omega k$ let $f \in {}^\omega k$. Let $g : \omega \to k^2$ be defined such that $g(n) = kf(\beta_s(n)) + f(\beta_g(n))$. Then there is some $A \in \mathcal{A}$ such that $g \in A$. It is easy to check that $f \in A^*$. □

Proposition 1. *If $i \geq j$ then $\mathrm{cov}(\mathfrak{R}_i) \leq \mathrm{cov}(\mathfrak{R}_j)$.*

Proof. Let $\bigcup_{\zeta\in\kappa} A_\zeta$ be a cover of ${}^\omega j$ by sets in $\mathfrak{R}_j$. Let $\Phi_\zeta : \mathbb{PF}_j\times \to \mathbb{PF}_j$ witness that A_ζ belongs to $\mathfrak{R}_i$. Define $S : \mathbb{PF}_i \to \mathbb{PF}_j$ by

$$S(f)(m) = \begin{cases} f(m) & \text{if } f(m) \in j \\ j-1 & \text{if } f(m) \notin j \end{cases}$$

and then let $\Psi_\zeta : \mathbb{PF}_i\times \to \mathbb{PF}_i$ be defined by

$$\Psi_\zeta(f)(m) = \begin{cases} \Phi_\zeta(S(f))(m) & \text{if } m \notin \mathrm{domain}(f) \\ f(m) & \text{if } m \in \mathrm{domain}(f) \end{cases}$$

Let $B_\zeta = \{f \in {}^\omega i : (\forall g \in \mathbb{PF}_i)(\Psi_\zeta(g) \not\subseteq^* f)\}$ and note that if $f \in {}^\omega i \setminus \bigcup_{\zeta\in\kappa} B_\zeta$ then $S(f) \in {}^\omega j \setminus \bigcup_{\zeta\in\kappa} A_\zeta$. □

3. Covering Numbers of Many Rosłonowski Ideals may be Different

In this section it will be shown that any combination of values for the cardinal invariants $\mathrm{cov}(\mathfrak{R}_k)$ is consistent so long as it does not violate the basic monotonicity result of Proposition 1.

Theorem 3.1. *Let κ be a non-increasing function from $\omega\setminus 2$ to the uncountable regular cardinals. It is consistent, relative to the consistency of set theory itself, that $\mathrm{cov}(\mathfrak{R}_i) = \kappa(i)$ for each $i \geq 2$.*

The basic idea of the construction is that a finite support iteration of length $\kappa(2)$ of countable chain condition partial orders will be constructed. At successor stages, Cohen reals will be added and these will be used to construct trees which will provide an upper bound on $\mathrm{cov}(\mathfrak{R}_i)$. At the typical limit stage an approximation to a function witnessing that $\mathrm{cov}(\mathfrak{R}_i)$ is small will have been trapped. A tower of partial functions with respect to $\subseteq^*$ will be constructed and a new function will be added to the top of this tower. This new function will prevent the approximation from witnessing that $\mathrm{cov}(\mathfrak{R}_i)$ is small. The countable chain condition of this tower forcing is not an obstacle since this will follow from the genericity of the construction. More care will have to be taken to preserve the key property of the trees which guarantee an upper bound on the covering numbers. The remainder of this section will supply the details.

Let V be a model where there the following hold:

- $2^\lambda \leq \kappa(2)$ for each $\lambda < \kappa(2)$
- There is a $\square_{\kappa(2)}$ sequence — in other words, there is family $\{C_\gamma : \gamma \in \kappa(2)$ and γ is a limit$\}$ such that
  - each C_γ is closed and unbounded in γ
  - $|C_\gamma| = \mathrm{cof}(\gamma)$ for each γ
  - if δ is a limit point of some C_γ then $C_\delta = C_\gamma \cap \delta$
- The following version of $\diamondsuit$ holds: There is a sequence $\{D_\alpha\}_{\alpha\in\kappa(2)}$ such that for each $X \subseteq \kappa(2)$, each closed unbounded set $C \subseteq \kappa(2)$, each cardinal $\lambda \in \kappa(2)$ and each $\mu \in \kappa(2)$ there is some $\gamma \in \kappa(2)$ such that
  - the order type of C_γ is λ
  - $C_\gamma \subseteq C \setminus \mu$
  - $D_\zeta = X \cap \zeta$ for each $\zeta \in C_\gamma$ which is a limit of C_γ.

This can be obtained by a strategically closed forcing which is outlined in the appendix.

The first step is to define a finite support interation of countable chain condition partial orders $\{\mathbb{Q}_\alpha\}_{\alpha\in\kappa(2)}$. The iteration of $\{\mathbb{Q}_\alpha\}_{\alpha\in\eta}$ will be denoted by $\mathbb{P}_\eta$. Before proceeding, using the cardinal arithmetic hypothesis, let all sets of hereditary cardinality less than $\kappa(2)$ be enumerated by $\{F_\eta\}_{\eta\in\kappa(2)}$.

If $\alpha = \beta + 2$ then $\mathbb{Q}_\alpha$ is simply Cohen forcing for adding a generic function $c_\alpha : \omega \to \omega$. Defined simultaneously with $\mathbb{P}_\alpha$ will be trees $T^\alpha_j \subseteq \Omega_j = {}^{\kappa(j)}\kappa(2)$ and functions Θ^α_j with domain T^α_j such that, for each $j \geq 2$

- if $\beta \in \alpha$ then $T^\beta_j \subseteq T^\alpha_j$
- if $\beta \in \alpha$ then $\Theta^\beta_j \subseteq \Theta^\alpha_j$
- if $\xi \in T^\alpha_j$ then $1 \Vdash_{\mathbb{P}_\alpha}$ "$\Theta^\alpha_j(\xi) \in \mathbb{PF}_j$"
- if ξ and ξ' belong to T^α_j and $\xi \subseteq \xi'$ then $1 \Vdash_{\mathbb{P}_\alpha}$ "$\Theta^\alpha_j(\xi) \subseteq^* \Theta^\alpha_j(\xi')$"
- if ξ and ξ' are distinct elements of T^α_j of the same height then
$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}|\{n \in \omega : \Theta^\alpha_j(\xi)(n) \neq \Theta^\alpha_j(\xi')(n)\}| = \aleph_0\text{”}$$
- if α is a limit then $T^\alpha_j = \bigcup_{\beta\in\alpha} T^\beta_j$ and $\Theta^\alpha_j = \bigcup_{\beta\in\alpha} \Theta^\beta_j$
- if $\alpha = \beta + i$ where $i \in \{1, 2\}$ and β is a limit then $T^\alpha_j = T^\beta_j$ and $\Theta^\alpha_j = \Theta^\beta_j$

Notice that by the induction hypothesis, if $F \in \mathbb{PF}_j \cup {}^\omega j$ and $B^\alpha_j(F)$ is defined to be $\{\xi \in T^\alpha_j : \Theta^\alpha_j \subseteq^* F\}$ then $B^\alpha_j(F)$ forms a chain in T^α_j. The following additional induction hypothesis will play a crucial role in the construction:

$$(\forall j \geq 2)(\forall F \in \mathbb{PF}_j)(|B^\alpha_j(F)| < \kappa(j)) \tag{3.1}$$

If $\alpha = \beta + 3$ then let $\varphi(j, \alpha)$ be the least ordinal such that $F_{\varphi(j,\alpha)}$ is a $\mathbb{P}_{\beta+2}$-name for an element of $\mathbb{PF}_j$ which does not appear in the range of $\Theta^{\beta+2}_j$. (Such an ordinal must exist because α is a successor and, hence, many new reals have been added at the previous stage.) Given a generic extension by $\mathbb{P}_\alpha$, let F^α_j be the interpretation of $F_{\varphi(j,\alpha)}$ in this extension. Let $\bar{\xi}$ be the lexicographically least member of $\Omega_j \setminus T^\alpha_j$ which extends each member of $B^{\beta+2}_j(F^\alpha_j)$ and let $T^\alpha_j = T^{\beta+1}_j \cup \{\bar{\xi}\}$. Note that by 3.1 the sequence $\bar{\xi}$ belongs to Ω_j. Define $\Theta^\alpha_j(\bar{\xi})$ by

$$\Theta^\alpha_j(\bar{\xi})(n) = \begin{cases} F^\alpha_j(i) & \text{if } i \in \text{domain}(F^\alpha_j) \\ c_\alpha(i) & \text{if } i \in \omega \setminus \text{domain}(F^\alpha_j) \text{ and } c_\alpha(i) < j \\ \text{undefined} & \text{if } i \in \omega \setminus \text{domain}(F^\alpha_j) \text{ and } c_\alpha(i) \geq j \end{cases}$$

Notice that this definition will satisfy the induction hypotheses because of the genericity of c_α. Observe also, that adding a Cohen real does no harm to the induction hypothesis 3.1.

The next step is to define $\mathbb{Q}_\alpha$ when α is a limit or the successor of a limit ordinal.

Definition 3.1. If β is an ordinal and $\mathcal{H} = \{h_\mu\}_{\mu\in\beta} \subseteq \mathbb{PF}_k$ is such that $h_\mu \subseteq^* h_\nu$ whenever $\mu \leq \nu$ then the partial order $\mathbb{Q}(\mathcal{H})$ is defined to be the set of all functions $f \in \mathbb{PF}_k$ such that there is some $\mu \in \beta$ such that $f \subseteq^* h_\mu$ ordered under inclusion. If G is a filter on $\mathbb{Q}(\mathcal{H})$ then define $f_G = \cup G$ and note that if G is a sufficiently generic filter then $f_G : \omega \to k$.

Observe that if $X \subseteq \beta$ is a cofinal set then $\mathbb{Q}(\{h_\mu\}_{\mu\in X})$ is a dense subset of $\mathbb{Q}(\{h_\mu\}_{\mu\in\beta})A$. This fact will be used in the sequel without further mention. The function f_G is intended to be used to extend the given chain and obtain a new partial order extending the given one. However, since f_G is a total function, it will be necessary to cut it down to obtain a member of $\mathbb{PF}_k$. The following partial order is designed to do this.

Definition 3.2. If $\mathbb{Q}(\mathcal{H})$ is as in Definition 3.1 and G is a filter on $\mathbb{Q}(\mathcal{H})$ then define

$$\mathbb{S}(G) = \{(a, p) \in [\omega]^{<\aleph_0} \times G : a \cap \text{domain}(p) = \emptyset\}$$

ordered under coordinatewise inclusion. If $H \subseteq \mathbb{S}(G)$ is a filter then define $A_H = \bigcup_{(a,p)\in H} a$ and define $f_{G,H} = f_G \restriction (\omega \setminus A_H)$.

Observe that $\mathbb{S}(G)$ is σ-centred regardless of the cofinality of $\mathcal{H}$. Hence $\mathbb{Q}(\mathcal{H}) * \mathbb{S}(G)$ has the countable chain condition so long as $\mathbb{Q}(\mathcal{H})$ does. Furthemore, $\mathbb{Q}(\mathcal{H}) \subseteq \mathbb{Q}(\{f_{G,H}\})$. The main question to be addressed is: Do dense sets in $\mathbb{Q}(\mathcal{H})$ remain dense in $\mathbb{Q}(\{f_{G,H}\})$? The next pair of lemmas provide some information on this.

Lemma 1. *If $\mathcal{H} \subseteq \mathbb{PF}_k$, $p \in \mathbb{Q}(\mathcal{H})$, $g : l \to k$, $a \in [\omega \setminus \mathrm{domain}(p)]^{<\aleph_0}$ and D is a dense subset of $\mathbb{Q}(\mathcal{H})$ then there is $p' \supseteq p$ such that $a \cap \mathrm{domain}(p') = \emptyset$ and $(p' \restriction (\omega \setminus l) \cup \theta) \cup g \in D$ for each $\theta : a \setminus l \to k$.*

Proof. This is part of thestandard fusion argument for tree-like forcing. □

Lemma 2. *Let $\mathcal{H} = \{h_\mu\}_{\mu\in\beta} \subseteq \mathbb{PF}_k$ be such that $h_\mu \subseteq^* h_\nu$ whenever $\mu \leq \nu$ and let G be $\mathbb{Q}(\mathcal{H})$-generic over the model V. Suppose also that H is $\mathbb{S}(G)$ generic over $V[G]$. If $D \subseteq \mathbb{Q}(\mathcal{H})$ is predense then it remains so in $\mathbb{Q}(\mathcal{F})$ for any family $\mathcal{F} \subseteq \mathbb{PF}_k$ such that $f_{G,H} \in \mathcal{F}$.*

Proof. ¿From Lemma 1 it follows that for each dense $D \subseteq \mathbb{Q}(\mathcal{H})$ and each $g : l \to k$ the set

$$D_g = \{(a,p) \in \mathbb{S}(G) : (\forall\theta(a \setminus l) \to k)((p \restriction (\omega \setminus l) \cup \theta) \cup g \in D\}$$

is dense in $\mathbb{S}(G)$. Hence, given $f \supseteq^* f_{G,H}$ choose $l \in \omega$ such that $f \restriction (\omega\setminus l) \supseteq f_{G,H} \restriction (\omega\setminus l)$. It may, without loss of genrality, be assumed that $l \subseteq \mathrm{domain}(f)$ and so it is possible to let $g = f \restriction l$. Now choose $(a,p) \in D_g \cap H$. Let $\theta = f \restriction (a \setminus l)$ and, using the definition of D_g, conclude that $(p \restriction (\omega\setminus l) \cup \theta) \cup g \in D$. Since $p \restriction (\omega\setminus l) \subseteq f_{G,H} \restriction (\omega\setminus l) \subseteq f$ it follows that $(p \restriction (\omega \setminus l) \cup \theta) \cup g \subseteq f$ and hence, f extends an element of D. □

Whenever α is a limit ordinal of cofinality $\kappa(j)$, the partial order $\mathbb{Q}_\alpha$ will be defined to be of the form $\mathbb{Q}(\mathcal{H}_\alpha)$ where $\mathcal{H}_\alpha \subseteq \mathbb{PF}_J$ for some $J < \kappa(j)$ is an increasing tower with respect to $\subseteq^*$ which has cofinality $\kappa(j)$. Moreover, in this case, $\mathbb{Q}_{\alpha+1}$ will always be of the form $\mathbb{S}(G)$ where G is the generic filter on $\mathbb{Q}(\mathcal{H}_\alpha)$. Keeping this in mind, let H be the generic filter on $\mathbb{S}(G)$ and define $H_\alpha = f_{G,H} \in \mathbb{PF}_J$. The only point which requires elaboration is how to choose $\mathcal{H}_\alpha$.

There are three cases to consider. Before proceeding, recall that if C is a set of ordinals then C' denotes the Cantor-Bendixon derived set of C with respect to the order topology; in other words, C' is the set of points in C which are limits of C. Suppose that for each $\xi \in \alpha'$ a family $\{H^\xi_\gamma\}_{\gamma\in C_\xi}$ has been defined. To begin, suppose that the following statement fails:

$$(\forall\eta \in C'_\alpha)(\forall\bar\eta \in C'_\eta)(\forall\xi \in C_{\bar\eta})(H^\eta_\xi = H^{\bar\eta}_\xi) \tag{3.2}$$

and there is some $J < j$ such that $\{H^\eta_\xi\}_{\xi\in C_\eta} \subseteq \mathbb{PF}_J$ is an increasing tower with respect to $\subseteq^*$ for each $\eta \in C_\alpha$. In this case let $\mathcal{H}_\alpha$ be any increasing countable family; in other words, $\mathbb{Q}_\alpha$ and $\mathbb{Q}_{\alpha+1}$ will both be Cohen forcing. If the statement holds then, for $\xi \in C_\alpha$, let $H^\alpha_\xi = H^\eta_\xi$ for some (any) $\eta \in C'_\alpha \setminus \xi$. There are two remaining cases. First, suppose that C'_α is cofinal in α. In this case $\mathcal{H}_\alpha = \{H^\alpha_\gamma\}_{\gamma\in C_\alpha}$. The second case arises if C'_α is not cofinal in α. Let $\mu(\alpha)$ be the largest limit of C_α or, if no such limit exists, let $\mu(\alpha) = 0$. Suppose also that D_α, as given by the $\diamondsuit$-sequence, is a $\mathbb{P}_\alpha$-name and $J < j$

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}D_\alpha = \{\Phi^\alpha_\xi\}_{\xi\in\lambda} \text{ and } \Phi^\alpha_\xi : \mathbb{PF}_J \to \mathbb{PF}_J \text{ witnesses that } \operatorname{cov}(\mathfrak{R}_J) \leq \lambda\text{”}$$

for some $\lambda < \kappa(j)$ Let $\{\gamma_n\}_{n\in\omega}$ enumerate $C_\alpha\setminus\mu(\alpha)$ in increasing order. In this case, let $H^\alpha_{\mu(\alpha)} = H_{\mu(\alpha)}$ and choose $H^\alpha_{\gamma_n}$ to be a $\mathbb{P}_\alpha$ name such that

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}H^\alpha_{\gamma_n} = \Phi^\alpha_{\rho(n)}(H^\alpha_{\gamma_n})\text{”}$$

where $\rho(n)$ is the order type of $C_\alpha \cap (\mu(\alpha) + \gamma_n)$. Let $\mathcal{H}_\alpha = \{H^\alpha_\eta\}_{\eta\in C_\alpha}$.

Lemma 3. *The partial order $\mathbb{P}_{\kappa(2)}$ has the countable chain condition.*

Proof. Proceed by induction to show that

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}\mathbb{Q}_\alpha \text{ has the countable chain condition”}$$

for each α. The countable chain condition for $\mathbb{Q}(\mathcal{H})$ is problematic only when the cofinality of β is uncountable. Indeed, if $\operatorname{cof}(\beta) = \omega$ or $\operatorname{cof}(\beta) = 1$ then $\mathbb{Q}(\mathcal{H})$ is σ-centred. If $A \subseteq \mathbb{Q}(\{H^\alpha_\gamma\}_{\gamma\in C_\alpha})$ is a maximal antichain then, using the fact that C_α is closed and unbounded, it is possible to find some $\gamma \in C_\alpha$ such that $A \cap \mathbb{Q}(\{H^\alpha_\eta\}_{\eta\in C_\gamma})$ is a maximal antichain. By the induction hypothesis, it follows that $A\cap\mathbb{Q}(\{H^\alpha_\eta\}_{\eta\in C_\gamma})$ is countable. By Lemma 2 it follows that $A \cap \mathbb{Q}(\{H^\alpha_\eta\}_{\eta\in C_\gamma})$ is also maximal in $\mathbb{Q}_\alpha = \mathbb{Q}(\{H_\eta\}_{\eta\in C_\alpha})$. □

Before proceeding some notation will be introduced.

Definition 3.3. Suppose that $\mathbb{P} \subseteq \mathbb{P}'$ and that X is $\mathbb{P}'$ name. The $\mathbb{P}$-name $X \restriction \mathbb{P}$ is defined by induction on the rank of the inductive definition of names. If X is of the form $X \subseteq \mathbb{P}' \times Z$ where Z is a ground model set then $X \restriction \mathbb{P} = X \cap \mathbb{P} \times Z$. In general, $X \restriction \mathbb{P} = \{(p, A \restriction \mathbb{P}) : (p, A) \in X\}$.

Lemma 4. *If G is $\mathbb{P}_{\kappa(2)}$ generic over V then* $\operatorname{cov}(\mathfrak{R}_j) > \kappa(j)$ *in $V[G]$ for $j \geq 2$.*

Proof. If $\text{cov}(\mathfrak{R}_j) \leq \kappa(j)$ then let $\Phi_\xi : \mathbb{PF}_j \to \mathbb{PF}_j$ be such that $\{\Phi_\xi\}_{\xi\in\lambda}$ witness this fact for some $\lambda < \kappa(j)$. Let $\tilde{\Phi}_\xi$ be a name for Φ_ξ and suppose that

$$1 \Vdash_{\mathbb{P}_{\kappa(2)}} \text{“}\{\tilde{\Phi}_\xi\}_{\xi\in\lambda} \text{ witnesses that } \text{cov}(\mathfrak{R}_j) \leq \lambda\text{”}$$

Let C be a closed unbounded set in $\kappa(2)$ such that for each $\alpha \in C$ the restricted names $\tilde{\Phi}_\xi \restriction \mathbb{P}_\alpha$ satisfy that

$$1 \Vdash_{\mathbb{P}_\alpha} \text{“}\{\tilde{\Phi}_\xi \restriction \mathbb{P}_\alpha\}_{\xi\in\lambda} \text{ witnesses that } \text{cov}(\mathfrak{R}_j) \leq \lambda\text{”}$$

Find some γ such that $\text{cof}(\gamma) = \lambda$, $C_\gamma \subseteq C \setminus \sup(\text{domain}(p))$ and $D_\eta = \{\tilde{\Phi}_\xi \restriction \mathbb{P}_\eta\}_{\xi\in\lambda}$ for each $\eta \in C_\gamma$. It follows directly from the construction of $\mathbb{P}_{\kappa(2)}$ that $\{H_\rho\}_{\rho\in C_\lambda}$ is an increasing sequence in $\mathbb{PF}_j$. Moreover, the construction at isolated limit ordinals guarantees that $H^\gamma_{\rho+1} \supseteq^* \Phi_\xi(H^\gamma_\rho)$ where ξ is the order type of $\rho \cap C_\gamma$ for each $\rho \in C_\gamma$. This, together with the fact that the order type of C_γ is λ, yields that $f = f_{G\cap\mathbb{Q}(\{H^\gamma_\eta\}_{\eta\in C_\gamma})}$ extends each $\Phi_\xi(H^\gamma_\rho)$ where ξ is the order type of $\rho \cap C_\gamma$. Hence f does not belong to any of the members of the ideal $\mathfrak{R}_j$ defined by the witnesses Φ_ξ. □

Lemma 5. *If G is $\mathbb{P}_{\kappa(2)}$ generic over V then* $\text{cov}(\mathfrak{R}_i) \leq \aleph_{m-j}$ *in $V[G]$ provided that $i \geq k_j$.*

Proof. This follows directly from the induction hypothesis 3.1. In $V[G]$, for each $\alpha \in \kappa(j)$, let E_α be the set of all $f : \omega \to k_j$ such that there is some $\sigma \in T^{\kappa(2)}_{k_j}$ such that the length of σ is at least α and $\Theta^{\kappa(2)}_{k_j}(\sigma) \subseteq^* f$. It is easily verified that $\bigcup_{\alpha\in\kappa(j)} = {}^\omega k_j$. The monotonicity esatblished in Proposition 1 yields the lemma. □

Hence, in order to finish the proof of Theorem 3.1, it suffices to show that 3.1 holds. The first thing to notice is that it suffices to show that the induction hypothesis holds at a single stage for any particular name for a function since Cohen genericity will handle the rest. The point of the next three lemmas is a stronger version of this assertion

Lemma 6. *Let G be $\mathbb{P}_{\kappa(2)}$ generic over V and $J < j$. If $\alpha \in \beta \in \kappa(2)$ and T is a J-branching subtree of ${}^\omega\omega$ which belongs to $V[G \cap \mathbb{P}_\alpha]$ then for any $\xi \in T^\beta_j \setminus T^\alpha_j$ there are infinitely many integers i such that there is some $i' > i$ so that*

$$\Theta^\beta_j(\xi) \restriction (i' \setminus i) \neq b \restriction (i' \setminus i)$$

for any $b \in \overline{T}$.

Proof. Recall that a tree T is said to be J-branching of height n if $T \subseteq \bigcup_{k\leq n} {}^k\omega$ and no node has more than J successors. The following fact is easily proved by induction on n: If $\{T_i\}_{i\in n}$ is a family of J-branching trees of height n then $\bigcup_{i\in n} T_i \not\supseteq {}^n(J+1)$. A direct corollary of this fact is that if $T \subseteq {}^{\underline{\omega}}\omega$ is a J-branching tree and $n \in \omega$ then there is a function $f : (i+J^i)\setminus i \to J+1$ such that $f \neq b \restriction ((i+J^i)\setminus i)$ for any $b \in \overline{T}$. This fact will be used with Cohen genericity to obtain the desired conclusion.

Before this can be done however, let T and G be given and let i be an arbitrary integer. Let A denote the domain of the interpretation of $F_{\varphi(j,\beta)}$ in $V[G\cap \mathbb{P}_\beta]$. Define a tree $T(i)$ in $V[G\cap\mathbb{P}_\beta]$ by $T(i) = \{t \in T : t \restriction (A\setminus i) \subseteq F_{\varphi(j,\beta)}\}$ and let ψ_i be the order preserving bijection from ω to $\omega\setminus(A\cup i)$. Define $T^*(i) = \{t\circ\psi : t \in T(i)\}$ and notice that $T^*(i)$ is a J-branching tree. Using this and the Cohen genericity of c_β it is possible to apply the observation of the previous paragraph to conclude that there are infinitely many integers i such that $c_\beta\circ\psi\restriction((i+J^i)\setminus i) \neq b\restriction((i+J^i)\setminus i)$ for any $b \in \overline{T^*(i)}$. Given any such i let $i' = i+J^i+|A\cap(\psi(J^i)|$. It follows that $c_\beta \restriction (i'\setminus i) \neq b\restriction(i'\setminus i)$ for any $b\in\overline{T}$. □

Definition 3.4. If $\mathcal{H} \subseteq \mathbb{PF}_k$, $g : l \to k$ and f is a $\mathbb{Q}(\mathcal{H})$-name such that $p \Vdash_{\mathbb{Q}(\mathcal{H})}$ "$f \in {}^\omega\omega$" then a finite subset $a \subseteq \omega$ will be said to k-approximate f with respect to p and g if

- $\tau : \bigcup_{m\leq |a|} {}^{m\cap a\setminus l}k \to \omega$
- $a \cap \mathrm{domain}(p) = \emptyset$
- for each $\theta : a\setminus l \to k$
$$g\cup\theta\cup p\restriction(\omega\setminus l) \Vdash_{\mathbb{Q}(\mathcal{H})} \text{“}\tau(\theta\restriction j) = f(\theta\restriction j)\text{”}$$
for each $j \leq |a|$.

Let f be a $\mathbb{Q}(\mathcal{H})$-name for a function from ω to ω and let $\overline{a} \subseteq a \in [\omega]^{<\aleph_0}$. If G is a generic filter on $\mathbb{Q}(\mathcal{H})$ define $D(f,\overline{a},g)$ to be the set of all $(a',p)\in\mathbb{S}(G)$ such that $a\subseteq a'$ and there exists $\overline{a'}\subseteq a'$ such that $\overline{a}$ is a proper subset of $\overline{a'}$ and $\overline{a'}$ k-approximates f with respect to p and g.

It is worth observing that if $a\subseteq\omega$ k-approximates f with respect to p and g then the function τ witnessing this fact is uniquely defined. Henceforth, this function wil be denoted by $\tau(a,f,g,p)$.

Lemma 7. *If $\mathcal{H}\subseteq\mathbb{PF}_k$, $g : l\to k$ and $p\Vdash_{\mathbb{Q}(\mathcal{H})}$ "$f\in{}^\omega\omega$" then, for any finite subset $a\subseteq\omega$ and any G which is a generic filter on $\mathbb{Q}(\mathcal{H})$, $D(f,a,g)$ is dense in $\mathbb{S}(G)$ below p provided that a k-approximates f with respect to p and g.*

Proof. This is a standard argument based on enumerating all possible $\theta : a\to k$ and finding a decreasing sequence of appropriate extensions. □

Lemma 8. *If it is given that*

- $\mathrm{cof}(\alpha) = \kappa(j)$
- G *is* $\mathbb{P}_{\alpha+1}$ *generic over* V
- $f \in {}^{\omega}\omega$ *in* $V[G]$

then there is a J*-branching tree* $T \subseteq {}^{\underset{\smile}{\omega}}\omega$ *in* $V[G \cap \mathbb{P}_\alpha]$ *such that* $f \in \overline{T}$ *and* $J < j$.

Proof. Let $\mathbb{Q}_\alpha = \mathbb{Q}(\{H_\eta\}_{\eta\in\alpha})$. Using the countable chain condition of $\mathbb{Q}_\alpha$ and the uncountable cofinality of α it is possible to find a limit ordinal $\beta \in C_\alpha$ such that f is a $\mathbb{Q}(\{H_\eta\}_{\eta\in\beta})$-name and the name f belongs to $V[G\cap\mathbb{P}_\beta]$. Notice that $\mathrm{cof}(\alpha) = \kappa(j)$ implies that $\{H_\eta\}_{\eta\in\alpha} \subseteq \mathbb{PF}_J$ for some $J < j$.

Let H be $\mathbb{S}(G \cap \mathbb{Q}_\alpha)$ generic over $V[G \cap \mathbb{P}_\alpha * \mathbb{Q}(\{H_\eta\}_{\eta\in\alpha})]$ and let $H_\alpha = f_{G\cap\mathbb{Q}_\alpha,H}$. If $h \in \mathbb{PF}_J$ and $h \supseteq^* H_\alpha$ let $l \in \omega$ be such that $h \supseteq H_\alpha \restriction (\omega \setminus l)$ and let $g = h \restriction l$. Now use Lemma 7 to conclude that there is an infinite chain $\{a^g_i\}_{i\in\omega}$ such that for each i there is some $p^g_i \in G$ such that a_i J-approximates f with respect to g and p^g_i. Let $A^g = \bigcup_{i\in\omega} a^g_i$ and $\tau^g = \bigcup_{i\in\omega} \tau(a^g_i, f, g, p^g_i)$. Given $m \in \omega$ it is possible to extend h to h' such that $a_m \subseteq \mathrm{domain}(h')$. Let $\theta = h' \restriction a_m$ and observe that

$$g \cup p^g_m \restriction (\omega \setminus a_m) \cup \theta \Vdash_{\mathbb{Q}(\{H_\eta\}_{\eta\in\alpha})} \text{“}f(|\theta \restriction m|) = \tau^g_m(\theta \restriction m)\text{”}$$

for each $n \leq |a_m|$. Hence, since H_α extends each p^g_m, it follows that h' forces f to belong to the j-branching tree determined by τ^g. The desired result now follows directly from Lemma 2. □

The countable chain condition guarantees that the induction hypothesis 3.1 will hold at limit stages of uncountable cofinality, provided that it holds at all previous stages. The argument at limit stages of countable cofinality rquires that a bit more care must be taken, but nothing particular about the forcing is used.

Lemma 9. *The induction hypothesis 3.1 holds at limits of countable cofinality, provided that it holds at all previous stages.*

Proof. Let α have countable cofinality and suppose that G is $\mathbb{P}_\alpha$ generic over V. If F is a function from ω to j in $V[G]$ then notice is that, if $B^\alpha_j(F)$ has length $\kappa(j)$ then, by the countable cofinality of α, there is some $\beta \in \alpha$ such that there is a cofinal subset $B \subseteq B^\beta_j(F)$. This determines the branch through T^β_j in $V[G \cap \mathbb{P}_\beta]$. Hence, it suffices to show that if $B \subseteq T^\beta_j$ is a branch of length $\kappa(j)$ in $V[G \cap \mathbb{P}_\beta]$ and F is in $V[G \cap \mathbb{P}_\alpha]$ then $B \not\subseteq B^\beta_j(F)$.

To this end, let B be a $\mathbb{P}_\beta$ name for a long branch through T^β_j and F a $\mathbb{P}_\alpha$-name. Let $\{\beta_n\}_{n\in\omega}$ be a sequence of ordinals cofinal in α such that $\beta_n > \beta$ for each n. For any $p \in \mathbb{P}_\alpha$ define $F_p = \{(i,j) : p \Vdash_{\mathbb{P}_\alpha}$ "$F(i) = j$"$\}$. It will first be shown that for each $n \in \omega$ the set

$$D(n) = \{q \in \mathbb{P}_{\beta_n} : (\exists \sigma \in B)(\forall r \leq q)(F_r \not\supseteq^* \Theta^\beta_j(\sigma))\}$$

is dense in $\mathbb{P}_{\beta_n}$. To see that this is so, suppose that $q \in \mathbb{P}_{\beta_n}$ is such that for each $\sigma \in B$ and $\bar{q} \leq q$ there is some $r \leq \bar{q}$ such that $F_r \supseteq^* \Theta^\beta_j(\sigma)$. Then let $\bar{F}$ to be the $\mathbb{P}_{\beta_n}$-name defined by $p \Vdash_{\mathbb{P}_{\beta_n}}$ "$\bar{F}(i) = j$" if and only if $p \Vdash_{\mathbb{P}_\alpha}$ "$F(i) = j$". It follows that $q \Vdash_{\mathbb{P}_{\beta_n}}$ "$\bar{F} \supseteq^* \Theta^\beta_j(\sigma)$" for each $\sigma \in B$ contradicting the induction hypothesis.

Using the density of each $D(n)$, let $\mathcal{A}_n \subseteq D(n)$ be a maximal antichain and, for each $q \in \mathcal{A}_n$, let $\sigma^n_q \in B$ witness that $q \in D(n)$. Let $\sigma \in B$ be such that $\sigma \supseteq \sigma^n_q$ for each $n \in \omega$ and $q \in \mathcal{A}_n$. Now suppose that $p \in \mathbb{P}_\alpha$ is such that $p \Vdash_{\mathbb{P}_\alpha}$ "$F \cup (\Theta^\beta_j(\sigma) \restriction m) \supseteq \Theta^\beta_j(\sigma)$". Let n be such that $p \in \mathbb{P}_{\beta_n}$ and choose $q \in \mathcal{A}_n$ such that there is some $r \in \mathbb{P}_{\beta_n}$ such that $r \leq q$ and $r \leq p$. Since $q \in D(n)$ it follows that $F_r \not\supseteq^* \Theta^\beta_j(\sigma^n_q) \subseteq^* \Theta^\beta_j(\sigma)$. Hence, there is some $i > m$ in the domain of $\Theta^\beta_j(\sigma)$ such that either $r \Vdash_{\mathbb{P}_\alpha}$ "$F(i) \neq \Theta^\beta_j(\sigma)(i)$" or r does not decide a value for $F(i)$. The first case directly contradicts that $r \leq p$ and, in the second case, it is possible to extend r to r' such that $r' \Vdash_{\mathbb{P}_\alpha}$ "$F(i) \neq \Theta^\beta_j(\sigma)(i)$". This agina yields a contradiction. □

It remains to consider successor ordinals. If $\alpha = \beta + 1$ and β itself is a successor, then $\mathbb{Q}_\alpha$ is σ-centred and, hence, a standard argument shows that it preserves the induction hypothesis. If β is a limit of countable confinality, then $\mathbb{Q}_\alpha$ is also σ-centred. So the only problem may arise whent β is a limit of uncountable cofinality.

Lemma 10. *Suppose that α is a limit ordinal of uncountable cofinality. Given that each preceding stage satisfies the induction hypothesis 3.1, the partial order $\mathbb{P}_{\alpha+1}$ will also satisfy the induction hypothesis.*

Proof. Let G be $\mathbb{P}_\alpha$ generic over V and argue in $V[G]$. There are two types of branches which might provide difficulties. To begin, consider branches which occur at some stage before α. Let B be a branch through T^β_j of length $\kappa(j)$ in $V[G \cap \mathbb{P}_\beta]$ and let F be a $\mathbb{Q}_\alpha$-name for a function from ω to j such that

$$1 \Vdash_{\mathbb{Q}_\alpha} \text{"}(\forall \sigma \in B)(F \supseteq^* \Theta^\beta_j(\sigma))\text{"}$$

If $\kappa(j) > \mathrm{cof}(\alpha)$ then $\mathbb{Q}_\alpha$ has a dense subset of cardinality $\mathrm{cof}(\alpha)$ and a pigeonhole argument shows that there is some $M \in \omega$ and a single condition $q \in \mathbb{Q}_\alpha$ such that the set of $\sigma \in B$ such that $q \Vdash_{\mathbb{Q}_\alpha}$ "$F \cup \Theta^\beta_j(\sigma) \restriction M \supseteq \Theta^\beta_j(\sigma)$" is cofinal in B. On the other hand, if $\kappa(j) < \operatorname{cof}(\alpha)$ then $\mathbb{Q}_\alpha$ has $\kappa(j)$ as a precalibre. In this case it is possible to find $\{q_\sigma\}_{\sigma\in B'}$ a centred subset of $\mathbb{Q}_\alpha$ and $M \in \omega$ such that $q_\sigma \Vdash_{\mathbb{Q}_\alpha}$ "$F \cup \Theta^\beta_j(\sigma) \restriction M \supseteq \Theta^\beta_j(\sigma)$" for each $\sigma \in B'$ and, furthermore, B' is a cofinal subset of B. In either case a contradiction is obtained since it follows that $\mathbb{P}_\alpha$ violates the induction hypothesis. Hence, it may be assumed that $\kappa(j) = \operatorname{cof}(\alpha)$. Using the countable chain condition of $\mathbb{Q}_\alpha$, let $\xi \in C_\alpha \setminus \beta$ be such that $F \restriction \mathbb{Q}(\{h_\eta\}_{\eta\in\xi})$ is a $\mathbb{Q}(\{h_\eta\}_{\eta\in\xi})$-name. Since the cardinality of $\mathbb{Q}(\{h_\eta\}_{\eta\in\xi})$ is less than that of B, it follows that $1 \Vdash_{\mathbb{Q}(\{h_\eta\}_{\eta\in\xi})}$ "$F \not\supseteq^* \Theta^\beta_j(\sigma)$)" for some fixed $\sigma \in B$. Now use Lemma 2 to conclude that the dense sets witnessing this remain dense in $\mathbb{Q}_\alpha$.

The second possibility is that a cofinal branch is added to T^α_j. To see that this can not happen, suppose that $1 \Vdash_{\mathbb{Q}_\alpha}$ "$F : \omega \to j$" Then, by Lemma 8, there is some J-branching tree T such that $J < j$ and $1 \Vdash_{\mathbb{Q}_\alpha}$ "$F \in \overline{T}$". Since α has uncountable cofinality and the iterands all have the countable chain condition, it follows that if G is a generic set for $\mathbb{P}_{\kappa(2)}$ then there is some $\beta \in \alpha$ such that T belongs to $V[G \cap \mathbb{P}_\beta]$. Choose $\sigma \in B^\alpha_j(F) \setminus T^\beta_j$. Now use Lemma 6 to obtain a contradiction. □

4. Appendix

A brief note regarding the consistency of the required combination of □ and ◊ may be helpful to some readers. To obtain the required initial model, begin with a model where λ is regular and $2^\kappa \leq \lambda$ for $\kappa < \lambda$. Let $\mathbb{P}$ be the partial order consisting of initial segments of the required □ and ◊ sequence. To be precise, $p \in \mathbb{P}$ if and only if p is a function defined on some $\alpha \in \lambda$ such that

- $p(\eta) = (C_\eta, D_\eta)$ for each $\eta \in \alpha$
- $D_\eta \subseteq \eta$
- $C_\eta \subseteq \eta$ is closed and unbounded in η
- if $\eta \in \alpha$ and $\xi \in C'_\eta$ then $C_\xi = C_\eta \cap \xi$.

This partial order has size λ and is stategically λ-closed.

References

[1] Marek Balcerzak. Typical properties of continuous functions via the Vietoris topology. *Real Anal. Exchange*, 18(2):532–536, 1992/93.
[2] Marek Balcerzak and Andrzej Rosłanowski. On Mycielski ideals. *Proc. Amer. Math. Soc.*, 110(1):243–250, 1990.
[3] J. Cichoń, A. Rosłanowski, J. Steprāns, and B. Węglorz. Combinatorial properties of the ideal $\mathfrak{p}_2$. *J. Symbolic Logic*, 58(1):42–54, 1993.

[4] Jacek Hejduk. Convergence with respect to the Mycielski σ-ideal. *Demonstratio Math.*, 22(1):43–50, 1989.
[5] Shizuo Kamo. Some remarks about Mycielski ideals. *Colloq. Math.*, 65(2):291–299, 1993.
[6] Jan Mycielski. Some new ideals of sets on the real line. *Colloq. Math.*, 20:71–76, 1969.
[7] Szymon Plewik. On some problem of A. Rosłanowski. *Colloq. Math.*, 69(2):297–298, 1995.
[8] A. Rosłanowski. Mycielski ideals generated by uncountable systems. *Colloq. Math.*, 66(2):187–200, 1994.
[9] Andrzej Rosłanowski. On game ideals. *Colloq. Math.*, 59(2):159–168, 1990.
[10] Kenneth Schilling. A category base for Mycielski's ideals. *Real Anal. Exchange*, 19(1):98–105, 1993/94.
[11] James D. Sharp and Simon Thomas. Uniformization problems and the cofinality of the infinite symmetric group. *Notre Dame J. Formal Logic*, 35(3):328–345, 1994.

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