ON WHAT I DO NOT UNDERSTAND (AND HAVE SOMETHING TO SAY): PART I

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Abstract. This is a non-standard paper, containing some problems in set theory I have in various degrees been interested in. Sometimes with a discussion on what I have to say; sometimes, of what makes them interesting to me, sometimes the problems are presented with a discussion of how I have tried to solve them, and sometimes with failed tries, anecdote and opinion. So the discussion is quite personal, in other words, egocentric and somewhat accidental. As we discuss many problems, history and side references are erratic, usually kept at a minimum ("see ..." means: see the references there and possibly the paper itself).

The base were lectures in Rutgers Fall '97 and reflect my knowledge then. The other half, [122], concentrating on model theory, will subsequently appear. I thank Andreas Blass and Andrzej Rosłanowski for many helpful comments.

Contents

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1. Cardinal problems and pcf

Here, we deal with cardinal arithmetic as I understand it (see [161] or [166]), maybe better called cofinality arithmetic (see definitions below). What should be our questions? Wrong questions usually have no interesting answers or none at all.
Probably the most popular[1] question is:

Problem 1.1. Is $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_1}$?

Recall:

Definition 1.2. Let $\mathfrak{a}$ be a set of regular cardinals (usually $|\mathfrak{a}| < \min(\mathfrak{a})$). We define

(1) $\mathrm{pcf}(\mathfrak{a}) = \{\mathrm{cf}(\prod \mathfrak{a}/D) : D$ is an ultrafilter on $\mathfrak{a}$ $\}$.
(2) $\mathrm{cf}(\prod \mathfrak{a}) = \min\{|F| : F \subseteq \prod \mathfrak{a}$ and $(\forall g \in \prod \mathfrak{a})(\exists f \in F)(g \le f)\}$.
(3) For a filter D on $\mathfrak{a}$, $\mathrm{tcf}(\prod \mathfrak{a}/D) = \lambda$ means that in $\prod \mathfrak{a}/D$ there is an increasing cofinal sequence of length λ.
(4) For a singular cardinal μ and a cardinal θ such that $\mathrm{cf}(\mu) \le \theta < \mu$ let

$$\mathrm{pp}_\theta(\mu) = \sup\big\{\mathrm{tcf}(\textstyle\prod \mathfrak{a}/I) : \ \mathfrak{a} \subseteq \mathrm{Reg} \cap \mu,\ |\mathfrak{a}| < \min(\mathfrak{a}),\ \sup(\mathfrak{a}) = \mu,\ I \text{ an ideal on } \mathfrak{a} \text{ such that } J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq I, \text{ and } |\mathfrak{a}| \le \theta\big\},$$

were for a set A of ordinals with no last element, J^{bd}_A is the ideal of bounded subsets of A.
(5) Let $\mathrm{pp}(\mu) = \mathrm{pp}_{\mathrm{cf}(\mu)}(\mu)$.
(6) We define similarly $\mathrm{pp}_\Gamma(\mu)$ for a family (equivalently: a property) Γ of ideals; e.g., $\Gamma(\theta,\tau) =$ the family of $(<\tau)$-complete ideals on a cardinal $<\theta$, $\Gamma(\theta) = \Gamma(\theta^+,\theta)$.

Definition 1.3. (1) For a partial order P,

$$\mathrm{cf}(P) = \min\{|Q| : Q \subseteq P \text{ and } (\forall p \in P)(\exists q \in Q)(p \le q)\}.$$

(2) For cardinals $\lambda, \mu, \theta, \sigma$,

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma) = \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq [\lambda]^{<\mu} \text{ and any } a \in [\lambda]^{<\theta} \text{ is included in the union of } < \sigma \text{ members of } \mathcal{A}\ \}.$$

Problem 1.1 is for me the right form of

Question 1.4. (1) Assume $\aleph_\omega$ is strong limit. Is $2^{\aleph_\omega} < \aleph_{\omega_1}$?
(2) Assume $2^{\aleph_0} < \aleph_\omega$. Is $(\aleph_\omega)^{\aleph_0} < \aleph_{\omega_1}$?

Why do I think 1.1 is a better form? Because we know that:

[1]that is, most people who are aware of this direction, will mention it, and probably many have tried it to some extent

$(*)_1$ If $\aleph_\omega$ is strong limit, then $2^{\aleph_\omega} = (\aleph_\omega)^{\aleph_0}$ (classical cardinal arithmetic).
$(*)_2$ $\mathrm{pp}(\aleph_\omega) = \mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ (see [98]),
$(*)_3$ $\aleph_\omega^{\aleph_0} = 2^{\aleph_0} + \mathrm{cf}([\aleph_\omega]^{\aleph_0}, \subseteq)$ (trivial).

So the three versions are equivalent and say the same thing when they say something at all, but Problem 1.1 is always meaningful.

To present what I think are central problems, we can start from what I called the solution of the "Hilbert's first problem", see [178] (though without being seconded).

Theorem 1.5. *For $\lambda \geq \beth_\omega$, there are $\kappa < \beth_\omega$ and $\mathcal{P} \subseteq [\lambda]^{<\beth_\omega}$, $|\mathcal{P}| = \lambda$ such that every $A \in [\lambda]^{<\beth_\omega}$ is equal to the union of $< \kappa$ members of $\mathcal{P}$.*

So $\mathcal{P}$ is "very dense". E.g., if $c : [\lambda]^n \longrightarrow \beth_n$ *then* for some $B_m \in \mathcal{P}$ (for $m < \omega$), the restrictions $c \restriction [B_m]^n$ are constant and $|B_m| = \beth_m$. We can replace $\beth_\omega$ by any strong limit cardinal $> \aleph_0$.

In [103, §8] the following application of 1.5 to the theory of Boolean Algebras is proved:

Theorem 1.6. *If B is a c.c.c. Boolean algebra and $\mu = \mu^{\beth_\omega} \leq |B| \leq 2^\mu$,* then *$B$ is μ-linked, i.e., $B \backslash \{0\}$ is the union of μ sets of pairwise compatible elements.*

(See also [139], [141] and Hajnal, Juhász and Szentmiklossy [55].)

We also have the following application:

Theorem 1.7 (See [167]). *If X is a topological space (not necessarily T_2) with λ points, $\mu \leq \lambda < 2^\mu$ and $> \lambda$ open sets and μ is strong limit of cofinality $\aleph_0$,* then *X has $\geq 2^\mu$ open sets.*

Another connection to the general topology is the following

Definition 1.8. For topological spaces X, Y and a cardinal θ, write $X \rightarrow (Y)^1_\theta$ iff for every partition $\langle X_i : i < \theta \rangle$ of X into θ parts, X has a closed subspace Y' homeomorphic to Y which is included in one part of the partition.

Arhangel'skii asked whether for every compact Hausdorff space X,

$$X \nrightarrow (\text{Cantor discontinuum})^1_2.$$

Arhangel'skii's problem $+\neg$CH is sandwiched between two pcf statements of which we really do not know whether they are true. If, for simplicity $2^{\aleph_0} \geq \aleph_3$, then e.g.:

$(*)_1$ if for no $\mathfrak{a}$, $\mathfrak{a} \subseteq \mathrm{Reg}$, where Reg is the class of regular cardinals, $|\mathfrak{a}| \geq \aleph_2$, $\prod \mathfrak{a}/[\mathfrak{a}]^{\leq \aleph_0}$ is $\sup(\mathfrak{a})$–directed, *then* the answer is: for every Hausdorff space X, we have $X \nrightarrow (\text{Cantor discontinuum})^1_2$ and more.

$(*)_2$ If for some $\mathfrak{a} \subseteq \mathrm{Reg} \setminus 2^{<\kappa}$, $|\mathfrak{a}| = 2^{\aleph_0} \leq \kappa$ and $\prod \mathfrak{a}/[\mathfrak{a}]^{\leq \aleph_0}$ is $\sup(\mathfrak{a})$–directed, *then* in some forcing extension there exists a zero-dimensional Hausdorff space X such that $X \to (\text{Cantor discontinuum})^1_2$.
The Stone-Čech compactification of this space gives a negative answer to Arhangel'skii's question.

(On the problem, see [178] and more in [99].)

However, we can start from inside pcf theory.

Problem 1.9. Is $\mathrm{pcf}(\mathfrak{a})$ countable for each countable set of cardinals?

This seems to me more basic than 1.1, but yet 1.1 is weaker. I think it is better to look at the battlefield between independence by forcing from large cardinals and proofs in ZFC (I would tend to say between the armies of Satan and God but the armies are not disjoint).

The advances in pcf theory show us ZFC is more powerful than expected before. I will try to give a line of statements on which both known methods fail – so far.

Conjecture 1.10. If $\mathfrak{a}$ is a set of regular cardinals $> |\mathfrak{a}|$, *then* for no inaccessible λ the intersection $\lambda \cap \mathrm{pcf}(\mathfrak{a})$ is unbounded in λ.

Conjecture 1.11. For every $\mu \geq \aleph_\omega$, for every $\aleph_n < \aleph_\omega$ large enough there is no $\lambda < \mu$ of cofinality $\aleph_n$ such that $\mathrm{pp}_{\Gamma(\aleph_n)}(\lambda) > \mu$ (or replace $\aleph_n < \aleph_\omega$ by $\aleph_\alpha < \aleph_{\omega^2}$ or even $\aleph_\alpha < \aleph_{\omega_1}$, or whatever).

Conjecture 1.12. (A) It is consistent, for any uncountable θ (e.g., $\aleph_1$), that for some λ

$$\theta \leq |\{\mu < \lambda : \mathrm{cf}(\mu) = \aleph_0,\ \mathrm{pp}(\mu) > \lambda\}|.$$

(B) It is consistent that for some λ, the set

$$\{\mu < \lambda : \mathrm{cf}(\mu) > \aleph_0,\ \mathrm{pp}_{\aleph_1\text{–complete}}(\mu) > \lambda\}$$

is infinite.

Those three conjectures seem to be fundamental. Note that having ZFC-provable answer in 1.10, 1.11, but independent answer for 1.12 are conscious choices. For all of those problems, present methods of independence fail, and in addition they are known to require higher consistency strength. Of course, we can concentrate on other variants; e.g., in 1.12(B) use θ instead of $\aleph_0$.

Other problems tend to be sandwiched between those, or at least those more basic problems are embedded into them. E.g., 1.11 implies that in 1.5 we can replace $\beth_\omega$ by $\aleph_\omega$ if we replace equal by included (or demand $\lambda \geq \sum\limits_{n<\omega} 2^{\aleph_n}$) and this implies $|\mathfrak{a}| \leq \aleph_0 \Rightarrow |\mathrm{pcf}(\mathfrak{a})| \leq \aleph_\omega$, while e.g. $|\mathfrak{a}| \leq \aleph_{\omega n} \Rightarrow |\mathrm{pcf}(\mathfrak{a})| \leq \aleph_{\omega n+\omega}$ implies the analog of 1.5 for $\aleph_{\omega^2}$, see [178], [123].

See [123] for more on the ZFC side; it is very helpful in preventing futile attempts to force.

Note that $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1}$, *implies* that for some countable $\mathfrak{a}$, $\mathrm{pcf}(\mathfrak{a})$ is uncountable, which *implies* that clause (A) from Conjecture 1.12 holds. Also $\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_2}$ *implies* that for some countable $\mathfrak{a}, |\mathrm{pcf}(\mathfrak{a})| \geq \aleph_2$ which *implies* that clause (B) of Conjecture 1.12 fails).

So there is no point to try to prove $\mathrm{CON}(\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1})$ before having the consistency of 1.12(A) and, thus, $\mathrm{CON}(\mathrm{pp}(\aleph_\omega) > \aleph_{\omega_1})$ is a more specialized case. (Also if we look at the earlier history of consistency proofs – clearly there is no point to start with Problem 1.1).

In Conjecture 1.10 the situation (which we say is impossible) may look bizarre, as $\mathrm{pcf}(\mathfrak{a})$ is extremely large. Of course, much better is $|\mathrm{pcf}(\mathfrak{a})| <$ "first inaccessible $> |\mathfrak{a}|$" and even $|\mathrm{pcf}(\mathfrak{a})| \leq |\mathfrak{a}|^{+\omega}$, which follows from Conjecture 1.11. Of course, replacing in 1.10, "λ inaccessible" by "λ-Mahlo" is still a very important conjecture while getting $\mathrm{pcf}(\mathfrak{a}) <$ "the first fixed point $> |\mathfrak{a}|$" is much better, so why from all variants of 1.10, those we have just mentioned and others, "the accumulation inaccessible" was chosen? The point is that it implies

$(*)$ $\mathrm{cf}(\prod \mathrm{pcf}(\mathfrak{a})) = \mathrm{cf}(\prod \mathfrak{a})$,
if $\mathfrak{a}$ is a set of regular cardinals $> |\mathfrak{a}|$

(see [166, Ch.VIII,§3], [97]; note that in the notation of [97], conjecture 1.10 says that $\mathrm{pcf}(\mathfrak{a}) \in J^*_{\mathfrak{a}}$.) If there is a failure of Conjecture 1.10 then consistently $(*)$ fails. We can force by $(< \lambda)$–complete forcing iterating adding $f \in \prod(\mathrm{pcf}(\mathfrak{a}) \cap \lambda)$ dominating the old product (or for any μ, just adding μ many λ–Cohen functions, i.e., forcing with

$$\{f : f \text{ is a partial function from } \mu \text{ to } \lambda,\ \mathrm{Dom}(f)| < \lambda\}).$$

So 1.10 denotes the significant dividing line between chaos and order.

Concerning the last conjecture 1.12, maybe the proofs in Gitik and Shelah [45] are relevant. There we force for hypermeasurable cardinals $\kappa_0 < \kappa_1 < \ldots < \kappa_n$ with a forcing which makes each κ_i hypermeasurable indestructible under reasonable forcing notions, including those which may add new Prikry sequences of ordinals $> \kappa_\ell$ of length $< \kappa_\ell$. (So in this case supercompact cannot serve, unlike in many proofs which do with hypermeasurable cardinals what is relatively easy to do with supercompact cardinals.) Let $\lambda = \lambda^{<\lambda} > \kappa_n$, $\theta_\ell < \lambda$. Then we blow up 2^{κ_n} to λ, change $\mathrm{cf}(\kappa_n)$ to θ_n; blow up $2^{\kappa_{n-1}}$ to λ, change $\mathrm{cf}(\kappa_{n-1})$ to σ_{n-1}, etc. The point is that when we arrive to κ_i the forcing so far is fairly "κ_i-complete for pure extensions", etc, so does not destroy "κ_i is λ-hypermeasurable". So for Conjecture 1.12 we (fix the desired cofinality θ and we) need to do it not n times but θ^* times (θ^* = inverse order of θ) so we need "anti-well-founded iteration".

In other words, we have $\langle \kappa_i : i < \theta \rangle$ increasing; κ_i is λ-hypermeasurable indestructible (necessarily in a strong way), and $\lambda > \sum\limits_{j<\theta} \kappa_j$.

First try:

We may try to define by induction on $i < \theta$, $\lessdot$–decreasing sequences $\bar{\mathbb{P}}^i = \langle \mathbb{P}^i_j : j \leq i \rangle$ of forcing notions such that $|\mathbb{P}^i_0| = \lambda_i$,

$$\Vdash_{\mathbb{P}^i_0} \text{“ } \mathrm{cf}(\kappa_j) = \aleph_0 \text{ for } j < i \text{ ”}$$

(or whatever fixed value, but $\aleph_0$ is surely easier), $\mathbb{P}^i_j$ is κ^+_j-c.c., purely κ_j-complete, $\mathbb{P}^i_0$ makes $\mathrm{pp}(\kappa_j) = \lambda$ for every $j < i$.

In successor stage - no problem: $i = j + 1$ and

$$\begin{array}{rcl} \mathbb{P}^{i+1}_{i+1} & = & (\text{blowing up } 2^\kappa \text{ to } \lambda \text{ changing } \mathrm{cf}(\kappa_j) \text{ to } \aleph_0) \\ \mathbb{P}^{i+1}_j & = & \mathbb{P}^{i+1}_{i+1} * \mathbb{P}^i_j. \end{array}$$

Not good enough: $\mathbb{P}^{i+1}_i$ changes the definition of: "blowing up 2^{κ_j} to λ" as there are more ω–sequences. So we should correct ourselves to $|\mathbb{P}^i_j| = \sum\limits_{\zeta<i} \kappa_\zeta$:

$$\mathbb{P}^i_j \text{ blows up } 2^{\kappa_j} \text{ to essentially } \sum_{\zeta<i} \kappa_\zeta.$$

So we have to prove the forcing notions extend as they should. If $\bar{\mathbb{P}}^i$ is defined, there is no problem to choose an appropriate $\mathbb{P}^{i+1}_{i+1}$. Now for each $j \leq i$ separately we would like to choose $\mathbb{P}^{i+1}_j$ to be a $\lessdot$-extension of $\mathbb{P}^i_j$ and of $\mathbb{P}^{i+1}_{i+1}$, but we have to do it for all $j \leq i$ together. The limit case seems harder.

* * *

Why, in 1.12(A), do we have $\theta \geq \aleph_1$? Moti Gitik shows consistency for $\theta = \aleph_0$ by known methods.

Audience Question: How dare you conjecture ZFC can show 1.10, 1.11?

For Conjecture 1.12 I have a scenario for an independence proof (outlined above). For 1.10 and 1.11 the statements imply there is quite a complicated pcf structure you necessarily drag with you. So it is reasonable to assume that if we shall know enough theorems on the pcf structure we shall get a contradiction. Of course, those arguments are not decisive.

* * *

Traditionally, remnants of GCH have strongly influenced the research on cardinal arithmetic, so e.g. people concentrate on the strong limit case, see [166, AG], [98]; probably also it was clear what to do and easier. On the other hand, [166] aims to get "exponentiation-free theorems", so we put forward:

Thesis 1.13. "Everything" is expressibly by cases of pp_J (and 2^κ for κ regular).

E.g. in [100, §2] this is done to the tree power of λ,

$$\lambda^{\langle\kappa,\mathrm{tr}\rangle} = \sup\{|\lim_\kappa(T)| : T \text{ a tree with } \leq \lambda \text{ nodes and } \kappa \text{ levels}\},$$

where $\lim_\kappa(T)$ is the set of κ–branches of T (well, using $\kappa^{\langle\kappa,\mathrm{tr}\rangle}$ for regular κ, which is malleable by forcing, a relative of 2^κ for κ regular).

But maybe there are also forcing proofs by which we can get interesting situations say below the continuum, whose strong limit counterparts are false, or have bigger consistency strength, or at least are harder to prove. The known forcing proofs may be open to such variations, e.g., when we add many Prikry sequences to one κ we may have the order between them such that every condition decides little about it. The following problem may be relevant to 1.13, and anyhow is a central one.

Problem 1.14. For a singular cardinal $\mu > \theta = \mathrm{cf}(\mu)$, is

$$\mathrm{cov}(\mu, \mu, \theta^+, \theta) = \mathrm{pp}_{\Gamma(\theta^+,\theta)}(\mu) \ ?$$

Note that other cases of cov can be reduced to those above. Now, this is almost proved: it holds when $\theta = \mathrm{cf}(\mu) > \aleph_0$. Furthermore, if μ is strong limit, $\aleph_0 = \mathrm{cf}(\mu)$ and the two expressions in 1.14 are not equal, both are quite large above μ as in Conjecture 1.12. Also, e.g., for a club of $\delta < \omega_1$

$$\mathrm{cov}(\beth_\delta, \beth_\delta, \aleph_1, \aleph_0) = \mathrm{pp}(\beth_\delta),$$

(see [98], the "$\beth_{\omega_1}$" can be weakened to strong limit in cov sense). But

Question 1.15. Can we force that there is $\mu < 2^{\aleph_0}$ such that $\mathrm{cf}(\mu) = \aleph_0$ and $\mathrm{cov}(\mu, \mu, \aleph_1, \aleph_0) > \mathrm{pp}(\mu)$?

[Why $< 2^{\aleph_0}$? As blowing up the continuum does not change the situation, proving the consistency for $\mu < 2^{\aleph_0}$ can be only easier. But for $\mu < 2^{\aleph_0}$ maybe it is even consistent that

$$\mathrm{cov}(\mu, \mu, \aleph_1, \aleph_0) > \mu^+ = \mathrm{pp}(\mu),$$

that is, by our present ignorance, it is even possible that the behaviour below the continuum is different than above it.]

Note that all cases of λ^κ can be reduced to cases of 2^θ, θ regular, and $\mathrm{cf}([\mu]^{\leq\theta}, \subseteq)$ where $\mu > \mathrm{cf}(\theta) = \theta \geq \mathrm{cf}(\mu)$.
Why? If κ is regular, $\lambda \leq 2^\kappa$ then $\lambda^\kappa = 2^\kappa$. If κ is regular and $\lambda > 2^\kappa$ then $\lambda^\kappa = \mathrm{cf}([\lambda]^{\leq\kappa}, \subseteq)$. So assume κ is singular and let $\sigma = \mathrm{cf}(\kappa)$ and $\kappa = \sum\limits_{i<\sigma} \kappa_i$, where each κ_i is regular and $\sigma < \kappa_i < \kappa$, so $\lambda^\kappa = \lambda^{\sum\{\kappa_i : i<\sigma\}} = \prod\limits_{i<\sigma} \lambda^{\kappa_i}$. Thus,

if $\lambda \le 2^\kappa$ then

$$\lambda^\kappa = 2^\kappa = \prod_{i<\sigma} 2^{\kappa_i} = (\sum_{i<\sigma} 2^{\kappa_i})^\sigma = \mathrm{cf}([\sum_{i<\sigma} 2^{\kappa_i}]^\sigma, \subseteq).$$

Lastly, if $\lambda > 2^\kappa$,

$$\lambda^\kappa = \prod_{i<\sigma} \lambda^{\kappa_i} = (\sum_{i<\sigma} \lambda^{\kappa_i})^\sigma = (\max_{i<\sigma} \lambda^{\kappa_i})^\sigma = \max_{i<\sigma} \lambda^{\kappa_i} = \max_{i<\sigma} \mathrm{cf}([\lambda]^{\kappa_i}, \subseteq)$$

(on the third equality see Hajnal and Hamburger [54], [151, 2.11(4),p. 164]).

If the answer to 1.14 is yes, then we can reduce all cases of λ^κ and of cov to statements on cases of pp.

Problem 1.16. If $\mathrm{cf}(\mu) = \aleph_0$, is $\mathrm{pp}(\mu)$ equal to $\mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\omega}(\mu)$, where

$$\mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\omega}(\mu) = \sup\{\lambda : \text{ for some increasing sequence } \langle\lambda_n : n<\omega\rangle \text{ of regular cardinals converging to } \mu \text{ we have } \lambda = \mathrm{tcf}(\prod_{n<\omega} \lambda_n / J^{\mathrm{bd}}_\omega)\} \ ?$$

A variant is: except when $\mathrm{pp}^{\mathrm{cr}}_{J^{\mathrm{bd}}_\omega}(\mu)$ has cofinality $\aleph_0$ and $\mathrm{pp}(\mu)$ is its successor.

By pcf calculus, if $\mathrm{pp}(\mu) < \mu^{+\omega_1}$ then this is true. Similarly, if $\theta < \mu_0 < \mu$ and

$$(\forall \mu')([\mathrm{cf}(\mu') \le \theta \ \&\ \mu' \in (\mu_0, \mu) \quad \Rightarrow \quad \mathrm{pp}(\mu') < \mu^{+\theta^+}]$$

then $\mathrm{pp}(\mu) = \mathrm{pp}_\theta(\mu)$ and see [170, 6.5]. Also, by [98, Part C], e.g., for a club of $\delta < \omega_1$, $\mu = \beth_\delta$ satisfies the conclusion.

* * *

On pcf for set theories with weak versions of Choice (say DC_κ, the dependent choice of length κ) see [175].

Problem 1.17. Develop combinatorial set theory generally and, in particular, pcf theory using only little choice (say DC_κ).

Inner model theory and descriptive set theory are not hampered by lack of choice, and much was done on variants of the axiom of choice. [175] may be a beginning of combinatorial set theory, and pcf in particular; i.e., it is enough to show that there are interesting theorems. In particular

Question 1.18. (a) Does DC_κ for κ large enough imply the existence of a proper class of regular cardinals?

(b) Does DC_κ for κ large enough imply that for a class of λ, $\mathcal{P}(\lambda)$ is not the union of $< \lambda$ sets, each of cardinality $\le \lambda$?

See more in [175]. Gitik [41] had proved

$$\mathrm{CON}\big((\forall\delta)(\mathrm{cf}(\delta) \leq \aleph_0)\big)$$

relative to suitable large cardinals. Woodin asked if

$$\mathrm{CON}\left(\mathrm{DC}_{\aleph_0} + (\forall\delta)(\mathrm{cf}(\delta) \leq \aleph_1)\right).$$

Specker asked if, consistently, for every λ, for some $\langle A_n : n < \omega\rangle$ we have $\mathcal{P}(\lambda) = \bigcup\limits_n A_n$, $|A_n| \leq \lambda$.

* * *

On how the problem of the existence of universal objects is connected to pcf see Kojman and Shelah [69], and [173], [117]. The following conjecture will simplify the answers:

Conjecture 1.19. For every limit of limit cardinals μ, for arbitrarily large regular $\lambda < \mu$, we have

$$(\forall^* \mu_1 < \mu)[\mathrm{cf}(\mu_1) = \lambda \quad\Rightarrow\quad \mathrm{pp}_{\Gamma(\lambda)}(\mu_1) < \mu],$$

where $\forall^*$ means "for every large enough".

After we learned that, on the one hand, $2^{\aleph_\omega}$ ($\aleph_\omega$ strong limit) has a bound (in fact, every $2^{\aleph_\delta}$, if $\aleph_\delta$ is strong limit $> |\delta|$, in [142, Ch.XIII]), and on the other hand there are bounds for 2^μ, μ the ω_1-th fix point (when μ is strong limit or less), it becomes natural to ask:

Conjecture 1.20. If $\aleph_\delta$ is the first fix point (i.e., the first such that $\aleph_\delta = \delta$, so it has cofinality $\aleph_0$), *then* $\mathrm{pp}(\aleph_\delta) < (2^{\aleph_0})^+$–th fix point.
(Even assuming GCH below $\aleph_\delta$ and proving just $\mathrm{pp}(\aleph_\delta) <$ "the first inaccessible" is good, but "$< \omega_4$-th fix point" is better, and "$< \omega_1$-th fix point" is best, but seems pointless to ask as long as 1.1 is open).

Note that we almost know: if $\aleph_\delta$ is the ω_1-th fix point (strong limit), then $\mathrm{pp}(\aleph_\delta) < \omega_4$–th fix point, we know it if the answer to 1.10 is yes, see [163] and see [166, Ch.V].

* * *

Traditionally we have asked: "can we find all the laws of cardinal arithmetic?" This had been accomplished for regular cardinals, and we prefer

Problem 1.21. Find all the rules of the pcf calculus or at least find more (or show that the set of rules is inherently too complicated).

Note: if for simplicity $|\mathrm{pcf}(\mathfrak{a})| < \min(\mathfrak{a})$, then on $\mathrm{pcf}(\mathfrak{a})$ the pcf structure is naturally a compact topology: $\mathfrak{b}$ is closed iff $\mathfrak{b} = \mathrm{pcf}(\mathfrak{b})$, and the theorem on existence of generators $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ says that the topology is a particularly nice one. If 1.10 holds this is true whenever $|\mathfrak{a}| < \min(\mathfrak{a})$ (see [97]).

There may well be some "global phenomena". Also there may be special behaviour near

$$\min\{\lambda : \text{for some } A \subseteq \lambda, \text{ there is no indiscernible class for } K[A]\},$$

as above it the covering theorem (Dodd and Jensen [27]) shows that cardinal arithmetic is trivial. On the other hand, on the behaviour below it, see [166, Ch.V].

An extreme case of our non-understanding concerning global behavior is:

Question 1.22. Is it possible that:

if $\mathfrak{a}$ is a set of odd [even] regular cardinals $> |\mathfrak{a}|$,
then every $\theta \in \mathrm{pcf}(\mathfrak{a})$ is odd [even]?

(where $\aleph_{2\alpha}$ is even and $\aleph_{2\alpha+1}$ is odd).

Instead of looking more on $\mathrm{pp}(\aleph_\omega)$ we may ask if the best result was derived from the known laws of cardinal arithmetic.

Question 1.23. Let $\ell < 4$. Can there be $\delta \in [\omega_\ell, \omega_{\ell+1})$ and a closure operation $c\ell$ on $\mathcal{P}(\delta+1)$ such that all the rules used in the proof of $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$ hold? (see Jech and Shelah [59]).

Question 1.24. (1) Characterize the possible sequences

$$\langle J_{<\theta}[\{\aleph_n : n \in [1,\omega)\}] : \ \theta \in \mathrm{pcf}\{\aleph_n : \ n \in [1,\omega)\}\rangle.$$

(2) For every ordinal γ characterize the possible $\langle J_{<\theta}[\mathfrak{a}] : \ \theta \in \mathrm{pcf}(\mathfrak{a})\rangle$ up to isomorphism when $\mathrm{otp}(\mathfrak{a}) = \gamma$.

[For $\mathfrak{a}'$, $\mathfrak{a}''$ we have an isomorphism if there there is a one–to–one order preserving $f : \mathfrak{a}' \longrightarrow \mathfrak{a}''$ such that $\{J_{<\theta}[\mathfrak{a}''] : \theta \in \mathrm{pcf}(\mathfrak{a}'')\} = \big\{\{f[\mathfrak{b}] : \mathfrak{b} \in J_{<\theta}[\mathfrak{a}']\} : \theta \in \mathrm{pcf}(\mathfrak{a}')\big\}$.]

* * *

I feel that

Thesis 1.25. Proving a theorem from ZFC + "cardinal arithmetic assumptions" is a "semi ZFC result".

This view makes proofs from cases of the failure of the SCH related to the thesis below more interesting.

Thesis 1.26. Assumptions on the failure of GCH (and even more so, of SCH) are good assumptions, practical ones, in the sense that from them you can deduce theorems.

Traditionally this is how instances of GCH were treated (with large supporting evidence). Clearly 1.26 may be supported by positive evidence (though hard to refute), whereas 1.25 remains a matter of taste. So Magidor would stress looking at "existence of a large cardinal" as semi-ZFC axioms (unlike some randomly chosen consistent theorems), which seems to mean in our terminology that we will look at consequences of it as semi-ZFC theorems. Jensen stresses that showing ψ holds in a universe with structure is much better than mere consistency (so the fine structure in $\mathbf{L}$ was the only one we know of at one time, but e.g. K is no less good than $\mathbf{L}$; the statement in [108] was inaccurate).

I agree with both, just to a lesser degree. Kojman criticized 1.25 saying cases of failure of SCH are large cardinal assumption in disguise; and I agree that $2^\lambda > \aleph_{\lambda+4}$ is a weaker assumption than $2^{\beth_\omega} > \beth_\omega^+$, but I still stick to 1.25. We may hope to really resolve problems by partitioning to cases according to what the cardinal arithmetic is.

* * *

Discussion: The following should be obvious, but I have found that mentioning them explicitly is helpful. Assume, e.g., that $\mathrm{cf}(\mu) = \aleph_0$, $\mathrm{pp}(\mu) > \mu^{+\omega_n}$, $n > 0$ (ω_n for simplicity) and let

$(*)_{\mu,n}$ for stationarily many $\delta < \omega_n$ of cofinality $\aleph_0$, $\mathrm{pp}_{\aleph_n}(\mu^{+\delta}) < \mu^{+\omega_n}$ (a "soft" assumption, see [166, IX,§4]).

Then we can find pairwise disjoint countable $\mathfrak{a}_i \subseteq \mathrm{Reg} \cap \mu$ unbounded in μ and $\alpha_i < \omega_n$ successor, strictly increasing and such that

$$\mu^{+\alpha_i} = \max\ \mathrm{pcf}(\mathfrak{a}_i), \quad \mu^{+\alpha_i} \notin \mathrm{pcf}(\bigcup_{j \neq i} \mathfrak{a}_j),$$

moreover $\mu' < \mu \ \Rightarrow\ \mu^{\alpha_i} = \max \mathrm{pcf}(\mathfrak{a}_i \setminus \mu')$.
[Why? We can find, by the assumption and Fodor Lemma, $\alpha^* < \omega_n$ such that

$$\alpha \in [\alpha^*, \omega_n) \quad \Rightarrow \quad \max\ \mathrm{pcf}\{\mu^{+\beta+1} : \beta \in (\alpha_0, \alpha)\} < \mu^{+\omega_n}.$$

By the assumption $\mathrm{pp}(\mu) \geq \mu^{+\omega_n}$, there is $\mathfrak{a} \subseteq \mu \backslash \omega_n$, $|\mathfrak{a}| = \aleph_n$ such that $\alpha < \omega_n \ \Rightarrow\ \mu^{+\alpha+1} \in \mathrm{pcf}(\mathfrak{a})$. First assume $2^{\aleph_n} < \mu$, so without loss of generality $\min(\mathfrak{a}) > 2^{\aleph_n}$, and we have a smooth closed generating sequence $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a}) \rangle$ for $\mathrm{pcf}(\mathfrak{a})$ (so $\mathfrak{b}_\lambda \subseteq \mathrm{pcf}(\mathfrak{a})$, etc.). Now choose by induction on $i < \omega_n$ pairs $(\alpha_i, \mathfrak{b}'_i)$ as follows. If $\langle \alpha_j : j < i \rangle$ has been defined, we know that

$$\max\ \mathrm{pcf}\{\mu^{+\beta} : \beta \text{ successor},\ \alpha^* \leq \beta \leq (\alpha^* + 2) \cup \bigcup_{j<i} \alpha_j\} < \mu^{+\omega_n},$$

and hence we can find $m_i < \omega$ and successor ordinals

$$\gamma^i_\ell \in [\alpha^*, (\alpha^*+2) \cup \bigcup_{j<i} \alpha_j] \qquad (\text{for } \ell < m_i)$$

such that

$$\{\mu^{+\beta} : \beta \text{ a successor, } \alpha^* \le \beta \le (\alpha^*+1) \cup \bigcup_{j<i} \alpha_j\} \subseteq \bigcup_{\ell<m_i} \mathfrak{b}_{\gamma^i_\ell}.$$

Let $\alpha_i < \omega_n$ be the minimal successor such that

$$\mu^{+\alpha_i} > \max\ \mathrm{pcf}\{\mu^{+\beta} : \beta \text{ a successor, } \alpha^* \le \beta \le (\alpha^*+1) \cup \bigcup_{j<i} \alpha_j\},$$

and let $\mathfrak{a}_i = \mathfrak{b}_{\alpha_i} \setminus \bigcup\limits_{\ell<m_i} \mathfrak{b}_{\gamma_\ell}$. If $\neg(2^{\aleph_\omega} < \mu)$ use the end of [170, §6].]

If, weakening $(*)_{\mu,n}$, we assume that for some $\alpha^* < \omega_n$ we have

$$\delta > \alpha^* \ \&\ \delta < \omega_n \text{ is limit } \quad \Rightarrow \quad \mathrm{pp}(\mu^{+\delta}) < \mu^{+\omega_n},$$

then we can get the same conclusion. Of course, omitting $(*)_{\mu,n}$ if $2^{\aleph_0} < \omega_n$, by the Δ–system lemma, we can get $\langle(\mathfrak{a}_i, \alpha_i) : i < \omega_n\rangle$ as above but demanding only $i \ne j \ \Rightarrow\ \mu^{+\alpha_i} \notin \mathrm{pcf}(\mathfrak{a}_j)$. Of course, we cannot let $\alpha_i = i+1$, as e.g. for some infinite $A \subseteq \omega$, $\mu^{+\omega+1} = \mathrm{tcf}(\prod\limits_{n\in A} \mu^{+n}/J^{\mathrm{bd}}_A)$, and hence $\mu^{+\omega+1} \in \mathrm{pcf}(\bigcup\limits_{n\in A} \mathfrak{a}_n)$.

Another remark: Even if $\mathrm{pcf}(\mathfrak{a})$ is large and $\mathfrak{a}$ is countable, we can find a c.c.c. forcing notion $\mathbb{Q}$ such that in $\mathbf{V}^{\mathbb{Q}}$ we can find $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a}) \setminus \mathfrak{a}\rangle$ satisfying: $\mathfrak{b}_\lambda \subseteq \mathfrak{a}$ has order type ω and $\prod \mathfrak{b}_\lambda / J^{\mathrm{bd}}_{\mathfrak{b}_\lambda}$ has true cofinality λ. [Why? If $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ is a generating sequence, let $\mathbb{Q}$ force for each λ an ω–sequence $\subseteq \mathfrak{b}_\lambda$, almost disjoint to $\mathfrak{b}_{\lambda_1}$ for $\lambda_1 < \lambda$.] Such forcing does not change the pcf structure (in fact, if $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ is a generating sequence for $\mathfrak{a}$, $\mathbb{Q}$ is a $\min(\mathfrak{a})$–c.c. forcing notion, *then* $\langle \mathfrak{b}_\lambda : \lambda \in \mathrm{pcf}(\mathfrak{a})\rangle$ is still a generating sequence for $\mathfrak{a}$, witnessed by the same $\langle f^*_\alpha : \alpha < \lambda\rangle$).

Question 1.27. For a regular cardinal θ, can we find an increasing sequence $\langle \lambda_i : i < \theta\rangle$ of regular cardinals such that for some successor λ and $f_\alpha \in \prod\limits_{i<\theta} \lambda_i$ for $\alpha < \lambda$ we have:

- $(*)$ if C_i is a club of λ_i for $i < \theta$, then for every large enough $\alpha < \lambda$ for every large enough $i < \theta$ we have $f_\alpha(i) \in C_i$.

By [127, §6] an approximation to this holds: if μ is a strong limit singular cardinal, $\mathrm{pp}(\mu) =^+ 2^\mu$ and $\lambda = 2^\mu = \mathrm{cf}(2^\mu)$ then the answer is yes, i.e. $(*)$ holds true, but 2^μ may be a limit cardinal (if 2^μ is singular, a related statement holds).

Question 1.28. Assume $\kappa = \mathrm{cf}(\kappa)$, $\langle \mu_i : i \le \kappa \rangle$ is an increasing continuous sequence of strong limit cardinals, for nonlimit i, $\mathrm{cf}(\mu_i) = \aleph_0$ and $\prod\limits_{i<\kappa} \mu_i^{+n}/J_\kappa^{\mathrm{bd}}$ has true cofinality μ^{+n}. Can we find an interesting colouring theorem on μ^{+n}? (The point is that for $n \ge 2$, we can have both a colouring as μ^{+n} is a successor of regulars (as in [158], [172]) and using a witness to $\mathrm{tcf}(\mu_i^{+n}/J_\kappa^{\mathrm{bd}}) = \mu^{+n}$ as in [103], [127].) The question is whether combining we shall get something startling.

Question 1.29. (1) Are there non-metrizable first countable Hausdorff topological spaces which are $\aleph_2$-metrizable (i.e., the induced topology on any $\le \aleph_1$ points is metrizable)?
(2) Are there non-collectionwise Hausdorff, first countable Hausdorff topological spaces which are $\aleph_1$–collectionwise Hausdorff?

See [126]. Concerning hopes to answer yes note that if SCH fails (or just $\mathrm{cf}(\mu) = \aleph_0$, $\mathrm{pp}(\mu) > \mu^+$) then there are examples (see [126, §1]), so we are allowed to assume $2^{\beth_\omega} = \beth_\omega^+$, etc.

Question 1.30. Let D be an ultrafilter on κ and $\mathrm{Spc}(D) = \{\prod\limits_{i<\kappa} \lambda_i/D,\ \lambda_i \ge 2^\kappa$ for $i < \kappa\}$. Is $\mathrm{Spc}(D)$ equal to $\{\mu : 2^\kappa \le \mu = \mu^{<\mathrm{reg}(D)}\}$?
(Where $\mathrm{reg}(D) = \sup\{\theta :$ for some $A_i \in D$, $i < \theta$, for every $\alpha < \kappa$, the number of $i < \theta$ such that $\alpha \in A_i$ is finite $\}$.)

See on this [100] where some information is gained.

Question 1.31. For which $\lambda \ge \mu$ we can find an almost disjoint family $\mathcal{A} \subseteq [\lambda]^{\aleph_0}$ such that
$$(\forall X \in [\lambda]^\mu)(\exists A \in \mathcal{A})(A \subseteq^* X)\ ?$$
At least when $\lambda \ge \mu = \beth_\omega$? (See [178], [99]).

Question 1.32. Is it consistent that for some strong limit singular cardinal μ, for no regular $\lambda \in [\mu, 2^\mu]$ do we have a c.c.c. Boolean Algebra which is not λ–Knaster?

On related ZFC constructions see [103], [127]; see also §6 here.

Question 1.33. Are all the assumptions in the result of [128] (see below) necessary? In particular, are assumptions (a), (b), (c) below sufficient?

Theorem 1.34 (See [128])**.** *Assume that*

(a) $\mathbf{V}$ *is our universe of sets,* $\mathbf{W}$ *is another model of ZFC (i.e., a transitive class of* $\mathbf{V}$ *containing all the ordinals),*
(b) κ *is a regular cardinal in* $\mathbf{V}$*,*
(c) $(\mathbf{W}, \mathbf{V})$ *has* κ*–covering (that is, every set of* $< \kappa$ *ordinals from* $\mathbf{V}$ *is included in a set of* $< \kappa$ *ordinals from* $\mathbf{W}$*),*

(d) *the successor of* κ *in* $\mathbf{V}$ *is the same as its successor in* $\mathbf{W}$*, call it* κ^+*,*
(e) $(\mathbf{W}, \mathbf{V})$ *has* κ^+*–covering.*

Then $(\mathbf{W}, \mathbf{V})$ *has the strong* κ*–covering (that is, for every structure* M *with universe an ordinal* α *and a countable vocabulary, and a set* X *from* $\mathbf{V}$ *of cardinality* $< \kappa$*, there is a set* Y *from* $\mathbf{W}$ *of cardinality* $< \kappa$ *including* X *which is the universe of an elementary submodel of* M*).*

2. The quest for the test: on the theory of Iterated Forcing for the continuum

On the subject see [176], and recent papers, too, but this section is hampered by some works in progress.

The issue is:

Problem 2.1. (a) Assuming we know something about each iterand $\underset{\sim}{\mathbb{Q}}_i$, what can we say about $\mathbb{P}_\alpha$, where $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \alpha,\ j < \alpha \rangle$ is an iteration (which may be FS (finite support), or CS (countable support) or RSC (revised countable support) and more) ?
(b) Find more useful ways to iterate (say, new "supports").

So "c.c.c. is preserved by FS iteration", "properness is preserved by CS iteration" can be seen as prototypes. But also many times: "adding no Cohen real over $\mathbf{V}$", "adding no dominating real over $\mathbf{V}$", etc., and, very natural, "adding no new real".

Note that this is not the same as having forcing axioms, e.g., having (the very important) MA does not discard the interest in FS iterations of c.c.c. forcing. The point is that in many questions you want to add reals for some purpose (which appear as generic sets for some forcing notions), but not another (e.g., a well-ordering of ω of order type ω_1). Also considering an axiom speaking on forcing notions with some property, when considering a candidate, a forcing notion $\mathbb{P}$, during an iteration we may force that it will not satisfy the property, discard it instead "honestly" forcing with it.

What we get by iterations as above can be phrased as having some axioms, but we have many combinations of adding reals of kinds A, B, and C while preserving properties Pr_1, Pr_2, in other words practically one preservation theorem may be used in many such contexts.

In fact, some of the most intriguing problems are fine distinctions: adding solution to one kind, but not to a close variant, e.g., the old problem:

Question 2.2. CON($\mathfrak{p} < \mathfrak{t}$) ?
(Note that if $\mathfrak{p} < \mathfrak{t}$ then $2^{\aleph_0} \geq \aleph_3$. See 3.7).

With FS iteration, all values of the continuum were similar, except $\aleph_1$ (well, also there is a distinction between regular and singular).

In fact, the advances in proper forcing make us "rich in forcing" for $2^{\aleph_1} = \aleph_2$, making the higher values more mysterious. (So in [176, Ch.VII,VIII] we separate according to the size of the $\underset{\sim}{\mathbb{Q}}_i$'s and whether we add reals, but we concentrate on the length ω_2). So, because we know much more how to force to get $2^{\aleph_0} = \aleph_2$, the independence results on the problems of the interrelation of cardinal invariants of the continuum have, mostly dealt with relationships of two cardinals, as their values are $\in \{\aleph_1, 2^{\aleph_0}\}$. Thus, having only two possible values $\{\aleph_1, \aleph_2\}$ among any three, two are equal; the Pigeonhole Principle acts against us. As we are rich in our knowledge to force for $2^{\aleph_0} = \aleph_2$, naturally we are quite poor concerning ZFC results. If we try for cardinal invariants $\mathbf{c}_1, \mathbf{c}_2$ to prove they consistently are $\aleph_1, \aleph_2$, respectively, much of our way exists (quoting existing preservation theorems) and we can look at the peculiarities of those invariants which may be still intractable. We are not poor concerning forcing for $2^{\aleph_0} = \aleph_1$ (and are rich in ZFC). But for $2^{\aleph_0} \geq \aleph_3$ we are totally lost: very poor in both directions. We would like to have iteration theory for length $\geq \omega_3$. I tend to think good test problems will be important in developing such iterations.

In some senses, most suitable is

Problem 2.3. Investigate cardinal invariants of the continuum showing ≥ 3 may have prescribed order.

Of course, the lack of forcing ability does not stop you from proving hopeful ZFC theorems about them, if true. Now I think there are some, but:

Thesis 2.4. They are camouflaged by the independent statements.

[Yes, I really believe there are interesting restrictions.] However, once we prove 90 percent of the problems are independent we will know where to look (as in hindsight occurs in cardinal arithmetic). So cardinal invariants from this perspective are excellent excuses to find iteration theorems. Mainly for $2^{\aleph_0} \geq \aleph_3$, but, of course, there is more to be said on $2^{\aleph_0} = \aleph_1$ (though not for 2.3), and even $2^{\aleph_0} = \aleph_2$.

Without good test problems you are in danger of imitating the king who painted the target after shooting the arrow. Let us consider some additional well known problems:

Question 2.5 (See Just, Mathias, Prikry and Simon [65])**.** Is there a filter D on ω such that:

(a) every co-finite subset of ω belongs to D,
(b) D is a P–filter (i.e., if $A_n \in D$ for $n < \omega$, then for some $A \in D$, $n < \omega \;\Rightarrow\; A \subseteq^* A_n$),
(c) D is not feeble, i.e., if $0 = n_0 < n_1 < \ldots$, then for some $A \in D$ for infinitely many $i < \omega$ we have $[n_i, n_{i+1}) \cap A = \emptyset$.

Question 2.6 (See Garcia–Ferreira and Just [40])**.** Is there an almost disjoint family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ (i.e., $(\forall A \neq B \in \mathcal{A})[|A \cap B| < \aleph_0]$) of cardinality $\mathfrak{b}$ satisfying the following condition:

if $A_n \in \mathcal{A}$ are pairwise distinct and $h : \omega \longrightarrow \omega$
then for some $B \in \mathcal{A}$ we have $(\exists^\infty n)(A_n \cap B \nsubseteq h(n))$?

If not, then $2^{\aleph_0} > \aleph_\omega$; on both questions see the discussions after 2.13.

Question 2.7 (See van Mill [194, Problem 4, p.563], Miller [83, Problem 9.1])**.**

$$\text{CON(no } P\text{-point and no } Q\text{-point) ?}$$

If so, $2^{\aleph_0} \geq \aleph_3$. [Why? Mathias [78] showed that if $\mathfrak{d}$ (the minimal size of a dominating family) is $\aleph_1$, then there is a Q–point. Ketonen [68] showed that $\mathfrak{d} = 2^{\aleph_0}$ implies the existence of P–points.]

Question 2.8. CON(${}^\omega(\omega + 1)$ with box product topology is not paracompact)?

If so, $2^{\aleph_0} \geq \aleph_3$. See on this Williams [196].

Question 2.9 (See Miller [83, Problem 16.3])**.**

$$\text{CON(Borel Conjecture and Dual Borel Conjecture) ?}$$

(See 3.3).

Question 2.10. CON(cf(cov(meagre)) $<$ additivity(meagre))?
(See before 2.14).

Problem 2.11. (1) CON(every function $f : {}^\omega 2 \longrightarrow {}^\omega 2$ is continuous when restricted to some non-null set)?
[Here "null" means of Lebesgue measure zero.]

(2) Similarly for other natural ideals. This in particular means if $\mathbb{Q}$ is a nicely defined forcing notion (see §5 below, e.g., Souslin c.c.c.), $\underset{\sim}{\eta}$ a $\mathbb{Q}$-name of a real, $A \subseteq {}^\omega 2$ is called $(\mathbb{Q}, \underset{\sim}{\eta})$–positive if for every countable $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ to which $\mathbb{Q}$, $\underset{\sim}{\eta}$ belong, some $\eta \in A$ is $\underset{\sim}{\eta}[G]^N$ for some $G \subseteq \mathbb{Q}^N$ generic over N; so the question for such $\mathbb{Q}$ is "CON(every $f : {}^\omega 2 \longrightarrow {}^\omega 2$ has a continuous restriction to some $(\mathbb{Q}, \underset{\sim}{\eta})$–positive set A)?"

(3) Is the following consistent:
if $A \subseteq {}^\omega 2$ is non-null, $f : A \longrightarrow {}^\omega 2$ *then* for some positive $B \subseteq A$, $f \restriction B$ is continuous. Similarly for general ideals as in part (2).

(4) If $A \subseteq {}^\omega 2 \times {}^\omega 2$ is not equivalent to a Borel set modulo one ideal I_1 (as described in part (2) above), *then* for some continuous $f : {}^\omega 2 \longrightarrow {}^\omega 2$, the set $\{\eta \in {}^\omega 2 : (\eta, f(\eta)) \in A\}$ is not equivalent to a Borel set modulo another ideal I_0 for suitable pairs (I_0, I_1).

See Fremlin [37], Ciesielski [20, Theorem 3.13, Problem 5]; [169] shows "yes" for (2) for non meagre, Ciesielski and Shelah [21] prove "yes" for (4) for non meagre, on work in progress see Rosłanowski and Shelah [88, §2]. With Juris Steprāns we have had some discussions on trying to use the oracle cc to the case of non-meagre ideal in (2). See 3.8.

Note: Mathematicians who are not set theorists generally consider "null" as senior to "meagre", that is as a more important case; set theorists inversely, as set-theoretically Cohen reals are much more manageable than random reals and have generalizations, relatives, etc. Particularly, in FS iterations, we get Cohen reals "for free" (in the limit), which kills our chances for many things and until now we have nothing parallel for random reals (but see [88]).

Judah suggests:

Question 2.12 (V = L). Find a forcing making $\mathfrak{d} = \aleph_3$ but not adding Cohen reals.

I am skeptical whether this is a good test question, as you may make $\mathfrak{d} = \aleph_3 = \mathfrak{b}$ by c.c.c. forcing, then add $\aleph_1$ random reals $\langle \nu_i : i < \omega_1 \rangle$ by a measure algebra; so over $\mathbf{L}[\langle \nu_i : i < \omega_1 \rangle]$ we have such a forcing. But certainly "not adding Cohen" is important, as many problems are resolved if $\mathrm{cov}(\text{meagre}) = 2^{\aleph_0}$.

There is a basic question for us:

Problem 2.13. Is there an iteration theorem solving all the problems described above or at least for all cases involving large continuum not adding Cohen reals?

I suspect not, and the answers will be ramified.

Let us review some problems. Now, Problems 2.5, 2.6 are for $2^{\aleph_0} > \aleph_\omega$, as: in 2.5, if $\mathrm{cf}([\mathfrak{d}]^{\aleph_0}, \subseteq) = \mathfrak{d}$ then there is such a filter (see [65]), and also in 2.6, if $\mathrm{cf}([\mathfrak{b}]^{\aleph_0}, \subseteq) = \mathfrak{b}$ there is a solution (see Just, Mathias, Prikry and Simon [65]).

It may well be that the solution will look like: let μ be a strong limit singular cardinal with $2^\mu \geq \mu^{++} > \mu^+$ and we use FS iteration of length μ^{++}. This will be great, but probably does not increase our knowledge of iterations. If on the other hand along the way we will add new ω-sequences say to μ (say $\mathrm{cf}(\mu) = \aleph_0$) and necessarily we use more complicated iteration, then it will involve better understanding of iterations, probably new ones.

We can ([176, Ch.XIV]) iterate up to "large" κ, and for many $\alpha < \kappa$, α strongly inaccessible, we have $\mathbb{Q}_\alpha$ change its cofinality to $\aleph_0$. Sounds nice, but no target yet.

* * *

We may note that "FS iterations of c.c.c. forcing notions" is not dead. Concerning 2.3 and 2.10, there are recent indications that FS iteration of c.c.c. still can be exploited even in cases for which for a long time we thought new supports are needed. We can iterate with FS, $\langle \mathbb{P}_i, \underaccent{\tilde}{\mathbb{Q}}_i : i < \alpha \rangle$, where $\underaccent{\tilde}{\mathbb{Q}}_i$ is (partially random or is Cohen) adding a generic real r_i, $\underaccent{\tilde}{\mathbb{Q}}_i$ is Cohen forcing or random forcing over $\mathbf{V}[\langle r_j : j \in A_i \rangle]$, where $A_i \subseteq i$ and each $\underaccent{\tilde}{\mathbb{Q}}_i$ is reasonably understood; but we do not require $j \in A_i \;\Rightarrow\; A_j \subseteq A_i$ (so called transitive memory). It is not so immediate to understand this sort of iterations, e.g., can the iteration add a dominating real?

It appears that if the A_i's are sufficiently closed, it will not, see [105], more generally look at [130]. There we prove:

CON($\exists$ non-null A such that the null ideal restricted to A is $\aleph_1$-saturated).

Clearly we should use a measurable cardinal κ, a normal ultrafilter $\mathcal{D}$ on κ in $\mathbf{V}$ and we add κ random reals $\langle r_\zeta : \zeta < \kappa \rangle$, but how do we make

$$A \in \mathcal{D}^{\mathbf{V}} \quad \Rightarrow \quad \{r_\zeta : \zeta \in \kappa \setminus A\} \text{ is null ?}$$

Cohen forcing does the job, but unfortunately too strongly. (In the case with non meagre, by Komjath [71], there are no problems in this respect). The solution is that we use FS iteration, but first we add 2^κ Cohens, that is $\mathbb{Q}_i$ is Cohen forcing for $i < 2^\kappa$; only, then we add the (somewhat) random reals:

$$\underaccent{\tilde}{\mathbb{Q}}_{\lambda+\zeta} \text{ for } \zeta < \kappa \text{ is random forcing over } \langle r_{\lambda+\xi} : \xi < \zeta \rangle,\ \langle r_i : i \in A_\zeta \subseteq \lambda \rangle.$$

We use $\{r_i : i < \lambda\}$ such that: r_i makes $\{r_{\lambda+\zeta} : \zeta \text{ satisfies } i \in A_\zeta\}$ null for $i < 2^{\aleph_0}$. So we need: for $A \in \mathcal{D}^{\mathbf{V}}$ for some $i < \lambda$ for every $\xi < \kappa$, $\xi \in \kappa \setminus A \;\Leftrightarrow\; i \notin A_\xi$.

This works for specially chosen A_ζ's.

Problem 2.14. (1) Can you make this into a general method?
(2) Can you deal with n or even κ kinds of "reals" (getting interesting results)?

What does this mean? This means that we use FS iteration $\langle \mathbb{P}_i, \underaccent{\tilde}{\mathbb{Q}}_j : i \leq \delta,\ j < \delta \rangle$ and $h : \alpha \longrightarrow \beta$, for $\zeta < \beta$, $\mathbb{R}_\zeta$ is a nep c.c.c. forcing notion in $\mathbf{V}$, (on nep see §5 and more see [115]; e.g., $\mathbb{R}_\zeta$ is Cohen, random, or as in [90], [89] or whatever), and $\underaccent{\tilde}{s}_\zeta \in {}^\omega 2$ is a generic real for $\mathbb{R}_\zeta$, and $\underaccent{\tilde}{\mathbb{Q}}_i$ is $\mathbb{R}_{h(\zeta)}$ as interpreted in $\mathbf{V}[\langle \underaccent{\tilde}{r}_j : j \in A_i \rangle]$ and $\underaccent{\tilde}{r}_j$ is $\underaccent{\tilde}{s}_{h(j)}$ there, and $A_i \subseteq i$. So the idea is that $0 = \delta_0 < \delta_i < \ldots < \delta_n = \delta$ and $j \in [\delta_\ell, \delta_{\ell+1}) \;\Rightarrow\; h(j) = \ell$.

* * *

In [109] we use $\aleph_\varepsilon$–support. This is less than $(< \aleph_1)$–support (i.e., countable support). This looks quite special, but

Problem 2.15. Can we make a general (interesting) theorem?

We can note that long FS iterations not only add Cohen reals, they also add, e.g., $\aleph_2$–Cohens, i.e. generics for $\{f : f$ a finite function from ω_2 to $\{0,1\}\ \}$. So we may like to iterate, allowing to add Cohen reals but not $\aleph_2$–Cohens in the sense above. This is done in [109], but the family of allowable iterands can be probably widened.

* * *

If we agree that preservation theorems are worthwhile, then after not collapsing $\aleph_1$, probably the most natural case is adding no reals. Now, whereas properness seems to me both naturally clear and covers considerable ground for not collapsing $\aleph_1$ and there are reasonable preservation theorems for "proper $+X$" for many natural properties X (e.g., adding no dominating reals, see [176, Ch.VI,§1,§2,§3], [176, Ch.XVIII,§2]), the situation with NNR (no new real) is inherently more complicated. In the early seventies when I heard on Jensen's CON(GCH + SH), I thought it would be easy to derive an axiom; some years later this materialized as Abraham, Devlin and Shelah [6], but reality is not as nice as dreams. One obstacle is the weak diamond, see Devlin and Shelah [26], more in [142, Ch.XIV,§1], [176, AP,§1], [113]. For a time the iteration theorem in [142, Ch.V,§5,§7,Ch.VIII,§4] seemed satisfactory to me. [There we use two demands. The first was $\mathbb{D}$–completeness (this is a "medicine" against the weak diamond, and $\mathbb{D}$ is a completeness system, $\aleph_1$–complete in [142, Ch V, §5,7], that is any countably many demands are compatible, and 2–complete in [142, Ch VIII, §4], that is any two demands are compatible). The second demand was α–properness for each countable ordinal α (or relativized version, see [142, Ch VIII, §4], [176, Ch VIII, §4]).] But [148, §1] (better [176, Ch.XVIII, §1]) gives on the one hand very nice and easy forcing notions not adding reals (running away from club guessing sequences) which are not covered as they fail $(<\omega_1)$–properness and on the other hand, shows by a not so nice example that generally you cannot just omit the $(<\omega_1)$–properness demand and promise an iteration theorem covering them. The problem concerning that forcing was resolved (promised in [148], carried out in a different way in lectures in MSRI '89 = [176, Ch.XVIII,§2]), but resulted in a dichotomy: we can get by forcing CON(ZFC + CH + SH) and we can get by forcing CON(ZFC + CH + no club guessing), but can we have both? More generally, can we have two other such contradictory statements (more generally for such results see Shelah and Zapletal [185]).

Question 2.16. Can we have two statements of the form[2]

$$(\forall x \in \mathcal{H}(\aleph_2))(\exists y \in \mathcal{H}(\aleph_2))\varphi$$

each consistent with

$$\begin{aligned}\mathrm{CH} + [\mathrm{Axiom}(\mathbb{Q} \text{ is } & (<\omega_1)\text{–proper and } \mathbb{D}\text{–complete} \\ & \text{for some simple 2-completeness system } \mathbb{D})]\end{aligned}$$

but not simultaneously?
(We may change the axiom used, we may speak directly on the iteration; we may deal with CS and proper or with RCS and semi-proper, etc.)

Note: possible failure of iteration does not prove a ZFC consequence, we may have freedom in the iteration only in some stages (like c.c.c. productive under MA).

This leaves me in bad shape: the iteration theorems seem not good enough, but the test problem (of getting both) does not seem so good. Now, [120] deals with NNR solving the specific dichotomy (and really satisfies the [148] promise circumvented in [176, Ch.XVIII, §2]) but left 2.15 open.

Eisworth suggested to me (motived by Abraham and Todorčević [4])

Question 2.17. Is the following consistent with ZFC+CH

(∗) if $A_\alpha \in [\omega_1]^{\aleph_0}$ and $\alpha < \beta \;\Rightarrow\; A_\alpha \subseteq A_\beta$ mod finite, and for every stationary $S \subseteq \omega_1$ the set $\bigcup\{[A_\alpha]^{<\aleph_0} : \alpha \in S\}$ contains $[E]^{<\aleph_0}$ for some club E of ω_1,
then for some club C of ω_1 we have

$$(\forall\alpha < \omega_1)(\exists\beta < \omega_1)(C \cap \alpha \subseteq A_\beta).$$

For long, an exciting problem for me has been

Problem 2.18.
(1) Can we find a consequence of ZFC + CH which "stands behind" the "club objection to NNR", e.g. it implies the failure of CH + Axiom($\mathbb{Q}$ proper $\mathbb{D}$–complete for some single 2-completeness system) ?
(2) Similarly for other limitations on iteration theorems?

Question 2.19. Is "CH$+\mathcal{D}_{\omega_1}$ is $\aleph_2$–saturated" consistent, where $\mathcal{D}_{\omega_1}$ is the club filter on ω_1?
Recall that a filter D on a set A is λ–saturated if there are no $A_i \in D^+$ for $i < \lambda$ such that $i < j \;\Rightarrow\; A_i \cap A_j = \emptyset$ mod D.

See [176, Ch.XVI]. Woodin proved that if there is a measurable cardinal then no. So we may look at $L[A]$, $A \subseteq \kappa$ codes $\mathcal{H}(\chi)$, κ large and try to collapse it to ω_2.

[2] $\mathcal{H}(\lambda)$ is the family of sets with transitive closure of cardinality $< \lambda$

Note that by [113], if $\mathrm{CH} + \mathcal{D}_{\omega_1}$ is $\aleph_2$–saturated, then essentially we have the weak diamond for three colours (or any finite number).

* * *

Baumgartner [12] asked

Question 2.20. Is it consistent that $2^{\aleph_0} > \aleph_2$ and any two $\aleph_2$–dense subsets of $\mathbb{R}$ of cardinality $\aleph_2$ (that is, any interval has $\aleph_2$ points) are isomorphic (as linear orders).

I think it is more reasonable to try

Question 2.21. Is it consistent that: $2^{\aleph_0} > \lambda \geq \aleph_2$ and there are no two far subsets $A \in [\mathbb{R}]^\lambda$, where

Definition 2.22. The (linear orders) I, J are θ–far if there is no linear order of cardinality θ embedded into both. If θ is omitted, we mean $\min\{|I|, |J|\}$.

On OCA$'$ (i.e., $\mathrm{OCA}'_{\aleph_1,\aleph_0}$, see the definition below) see Abraham, Rubin and Shelah [2], continued for $\mathrm{OCA}''_{\aleph_1,\aleph_0}$ (and its variants) in Todorčević [193], Veličković [195]; on a parallel for subsets of the plain which follows from MA, see Steprāns and Watson [191].

Question 2.23. (1) Is $\mathrm{OCA}'_{\aleph_2}$ consistent? Is $\mathrm{OCA}''_{\aleph_2}$ consistent?
(2) The parallel problems for ${}^\kappa 2$ and λ, even for $\lambda = \kappa^+$, $\kappa > \aleph_0$, where

Definition 2.24. (1) $\mathrm{OCA}'_{\lambda,\kappa}$ means $\lambda \leq 2^\kappa$ and: for any $A \in [{}^\kappa 2]^\lambda$ and an open symmetric set $\mathcal{U} \subseteq {}^\kappa 2 \times {}^\kappa 2$ there is $B \subseteq A$ of cardinality λ such that $\{(a,b) : a \neq b \text{ are from } B\}$ is included in $\mathcal{U}$ or is disjoint to $\mathcal{U}$ (we use the space ${}^\kappa 2$ for simplicity).
(2) $\mathrm{OCA}''_{\lambda,\kappa}$ is defined similarly only we have $B_i \subseteq A$ for $i < \kappa$, $A = \bigcup\limits_{i<\kappa} B_i$, each B_i as in part (1).
(3) If we omit λ we mean $\lambda = \kappa^+$, if in addition we omit κ, we mean $\kappa = \aleph_0$.

3. Case studies for iterated forcing for the reals

The following was suggested during the lecture on §2 by Juhász who was in the audience:

Question 3.1. Does CH imply that there is an S–space of cardinality $\aleph_2$, where S-space is defined as being regular, hereditarily separable, not Lindelöf?

Eisworth prefers the variant:
Does CH imply the existence of a locally compact S–space?

This problem looks important, but it is not clear to me if it is relevant to developing iteration theorems, though an existence proof may be related to the weak diamond, consistency to NNR iterations.

The same goes for the well known:

Question 3.2. CON($\mathfrak{d} < \mathfrak{a}$) ?

This definitely seems not to be connected to the iteration problem. It seems to me that a good test problem for our purpose in §2 should have one step clear but the iteration problematic, whereas for those two problems the situation is the inverse.

Note: by existing iteration theorems to get the consistency of $\mathfrak{d} < \mathfrak{a} + 2^{\aleph_0} = \aleph_2$ it is enough to show

(∗) for any MAD family $\{A_i : i < i^*\} \subseteq [\omega]^{\aleph_0}$, there is an ${}^\omega\omega$–bounding proper forcing notion $\mathbb{Q}$ of cardinality $\aleph_1$ adding $\underset{\sim}{A} \in [\omega]^{\aleph_0}$ almost disjoint to each A_i.

You are allowed to assume CH (start with $\mathbf{V} \models 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ and use CS iteration of such forcing notions); even $\diamondsuit_{\aleph_1}$ (if $\mathbf{V} \models \diamondsuit_{\aleph_1}$). We can weaken $|\mathbb{Q}| = \aleph_1$ to "$\mathbb{Q}$ satisfies $\aleph_2$–pic" (this is a strong form of $\aleph_2$–c.c. good for iterating proper forcing, see [176, Ch.VIII,2.1,p.409]). If you agree to use large cardinals, it is okay to assume in (∗) that an appropriate forcing axiom holds and not restrict $|\mathbb{Q}|$, and as we can first collapse $2^{\aleph_0}$ to $\aleph_1$, we can get $\diamondsuit_{\aleph_1}$ for "free". I idly thought to use free forcing for the problem, ([176, Ch.IX]), but no illumination resulted.

We can try in another way: start with a universe with a forcing axiom (say MA) and force by some $\mathbb{P}$, which makes $\mathfrak{d} = \aleph_1$, but $\mathbb{P}$ is understood well enough and we can show that $\mathfrak{a}$ is still large (just as adding a Cohen real to a model of MA preserves some consequences of MA (see Roitman [85], Judah and Shelah [60]]). So clearly FS iteration will not do.

I think that a more interesting way is to consider, assuming CH,

$$\begin{aligned} K_{\omega_1} = \{(\bar{\mathbb{P}}, \underset{\sim}{\bar{r}}) :\ & \bar{\mathbb{P}} = \langle \mathbb{P}_i : i < \omega_1 \rangle \text{ is } \lessdot\text{–increasing, } |\mathbb{P}_i| \leq \aleph_1, \\ & \underset{\sim}{\bar{r}} = \langle \underset{\sim}{r}_i : i < \omega_1 \rangle,\ \underset{\sim}{r}_i \text{ is a } \mathbb{P}_{i+1}\text{–name,} \\ & \Vdash_{\mathbb{P}_{i+1}} \text{“ } \underset{\sim}{r}_i \in {}^\omega\omega \text{ dominates } ({}^\omega\omega)^{\mathbf{V}^{\mathbb{P}_i]}} \text{ ”}\} \end{aligned}$$

ordered naturally, and for a generic enough ω_2–limit $\langle (\bar{\mathbb{P}}^\zeta, \underset{\sim}{\bar{r}}^\zeta) : \zeta < \omega_2 \rangle$ we may use $\bigcup\limits_{\substack{i<\omega_1,\\ \zeta<\omega_2}} \mathbb{P}^\zeta_i$. Another way to try is the non-Cohen Oracle [116]. The difference is small. Also the "$\omega_2 + \omega_1$–length mix finite/countable pure support iteration" seems similar.

I have just heard about CON($\mathfrak{u} < \mathfrak{a}$) being an old problem, clearly related to CON($\mathfrak{d} < \mathfrak{a}$). I do not see much difference at present.

Another direction is to develop the historic $\aleph_\varepsilon$–support iteration from [109].

Discussion 3.3. Concerning 2.9, I had not really considered it (except when Judah spoke to me about it) but just before the lecture, Bartoszyński reminded me of it (see [7]). Now, "the" proof of CON(Borel conjecture) is by CS iteration of Laver forcing (see Laver [76]), whereas the consistency proof of the dual is adding many Cohen reals (see Carlson [19]). So in a (hopeful) iteration proving consistency we have two kinds of assignments. We are given, say in stage α, in $\mathbf{V}^{\mathbb{P}_\alpha}$ a set $A = \{\eta_i : i < \omega_1\} \subseteq {}^\omega 2$, and we should make it not of strong measure zero, so we should add an increasing sequence $\bar n = \langle n_\ell : \ell < \omega\rangle$ of natural numbers such that for no $\bar\nu = \langle \nu_\ell : \ell < \omega\rangle \in \prod\limits_{\ell<\omega} {}^{(n_\ell)}2$ do we have $(\forall i < \omega_1)(\exists^\infty \ell)(\nu_\ell \lhd \eta_i)$. Now, even if we define $\mathbb{Q}_\alpha$ to add such $\bar n$, we have to preserve it later, so it is easier to preserve, for some family $F \subseteq \prod\limits_{\ell<\omega} 2^{n_\ell}$, the demand

$$(\forall f \in F)\neg(\exists \langle \nu_{\ell,k} : \ell < \omega, k < f(\ell)\rangle)(\forall i < \omega_1)(\exists^\infty \ell)(\exists k < f(\ell))(\nu_{\ell,k} \lhd \eta_i).$$

The second kind of assignment which we may have in stage α is the following. In $\mathbf{V}^{\mathbb{P}_\alpha}$, we are given $A = \{\eta_i : i < \omega_1\} \subseteq {}^\omega 2$ and we should make it non-strongly meagre, so we should add, by $\mathbb{Q}_\alpha$, a subtree $T_\alpha \subseteq {}^{\omega>}2$ (i.e., $\langle\rangle \in T_\alpha$, $\eta \in T_\alpha$ & $\nu \lhd \eta \Rightarrow \nu \in T_\alpha$, $\eta \in T_\alpha \Rightarrow (\exists \ell < 2)(\eta^\frown\langle \ell\rangle \in T_\alpha)$) of positive measure (i.e., $0 < \inf\{|T_\alpha \cap {}^n 2|/2^n : n < \omega\}$) such that $(\forall \eta \in {}^\omega 2)(\exists i < \omega_1)[\eta \oplus \eta_i \notin \bigcup\limits_n (T_\alpha)^{[n]}]$, where

$$\begin{aligned}(T_\alpha)^{[n]} = \{\nu : \quad & \text{for some } \rho \in T_\alpha \text{ we have } \ell g(\nu) = \ell g(\rho)\\ & \text{and } (\forall \ell)(n \le \ell < \ell g(\nu) \Rightarrow \rho(\ell) = \nu(\ell))\}.\end{aligned}$$

Again we have to preserve this.

A way to deal with such preservation problems is to generalize "oracle c.c.c." (see [176, Ch.IV]) replacing Cohen by other things. To explain this, it seems reasonable to look at the "oracle for random" (or even sequence of c.c.c. Souslin forcing, from [116]). This evolves to: for iterations of length $\le \omega_2$ of forcing notions of cardinality $\aleph_1$, prove that we can preserve the following condition on $\mathbb{P} = \mathbb{P}_\alpha$ for some $\langle M_\delta, M^+_\delta, r_\delta : \delta \in S\rangle$, $S \subseteq \omega_1$ stationary such that $\langle M_\delta : \delta \in S\rangle$ is an oracle, i.e., a $\diamondsuit^*$–sequence and $M_\delta \models \delta = \omega_1$, $M_\delta \models \mathrm{ZFC}^-_*$, $M^+_\delta \models$ "$\mathrm{ZFC}^-_* + M_\delta$ is countable" and r_δ is random over M^+_δ. Now without loss of generality, $\mathbb{P} \subseteq \omega_1$ and

$$\begin{aligned}\{\delta \in S : \quad & \mathbb{P} \restriction \delta \in M_\delta, \text{ and for every } p \in \mathbb{P} \cap \delta,\\ & \text{for some } q \text{ we have } p \le q \in \mathbb{P} \text{ and}\\ & q \Vdash \text{“ } r_\delta \text{ is random over } M^+_\delta[\underset{\sim}{G}_{\mathbb{P}} \cap \delta] \text{ ” }\} \in \mathcal{D}_{\omega_1} \restriction S\end{aligned}$$

(so this is like the oracle c.c.c. ([176, Ch.IV]), but the support is not countable so on other stationary $S_1 \subseteq \omega_1 \setminus S$ we may have different behaviour). Of course, we use "small" S so that we have "space" for more demands, see [116]. But trying to explain it (to Rosłanowski) it seemed the proof is too simple, so we can go back to good old CS and just preserving an appropriate property, a watered-down relative in the nep family ([115]).

We mainly try to combine the two iterations (of Cohen and of Laver forcing notions):

Definition 3.4. A forcing notion $\mathbb{Q}$ is 1–e.l.c. if the following condition is satisfied:

> whenever χ is large enough, $M_0 \prec M_1 \prec (\mathcal{H}(\chi), \in)$, $\mathbb{Q} \in M_0$, $M_0 \in M_1$ and M_0, M_1 are countable and $p \in \mathbb{Q} \cap M_0$, *then* for some condition $q \in \mathbb{Q}$ stronger than p we have
>
> $$q \Vdash \text{"for every } \mathcal{I} \in M_1 \text{ such that } \mathcal{I} \cap M_0 \text{ is predense in } \mathbb{Q}^{M_0} \text{ we have } \underset{\sim}{G}_{\mathbb{Q}} \cap \mathcal{I} \neq \emptyset\text{".}$$

(Note that $q \Vdash$" $M_0[\underset{\sim}{G}_{\mathbb{Q}} \cap M_0]$ is a generic extension of M_0 for a forcing notion which M_1 thinks is countable ".)

Note: e.l.c. stands for *elementary locally Cohen.* This is, of course, close to Cohen, or more accurately is another way to present strongly proper. But we also seem to need Laver forcing (or a close relative of it), but it is far from being strongly proper. Still it satisfies the parallel if we demand "$\mathcal{I} \subseteq Q^{M_1}$ is predense under pure extensions", i.e., with the same trunk. This approach seems to me promising but it is not clear what it delivers.

We may consider a more general definition (and natural preservation):

Definition 3.5. Let Pr be a property. A forcing notion $\mathbb{Q}$ with generic $\underset{\sim}{X} \subseteq \alpha_{\mathbb{Q}}$ (i.e. $\mathbf{V}[\underset{\sim}{G}_{\mathbb{Q}}] = \mathbf{V}[\underset{\sim}{X}[\underset{\sim}{G}_{\mathbb{Q}}]]$, $\alpha_{\mathbb{Q}}$ an ordinal) is called e.l.–Pr forcing if:

> for χ large enough, if $\mathbb{Q}, \underset{\sim}{X} \in M_1 \prec M_2 \prec (\mathcal{H}(\chi), \in)$, M_1, M_2 countable, $M_1 \in M_2$, $p \in \mathbb{Q} \cap M_1$,
> *then* we can find $q, \mathbb{Q}'$ such that
> (a) $p \leq q \in \mathbb{Q}$,
> (b) $\mathbb{Q}' \in M_2$ is a forcing notion with $\underset{\sim}{X}' \subseteq \alpha_{\mathbb{Q}}$ generic,
> (c) $M_2 \models \Pr(\mathbb{Q}', M_1, p)$,
> (d) $q \Vdash$" $\underset{\sim}{X} \restriction M_2$ is a $\mathbb{Q}'$-generic over p and for some set $G' \subseteq (\mathbb{Q}')^{M_2}$, generic over M_2 we have $\underset{\sim}{X} \restriction M_2 = \underset{\sim}{X}'[G']$ ".

This seems to me interesting but though Laver forcing satisfies some relatives of those properties it does not seem to be enough.

Note: this definition tells us that generically for many countable models $M \prec (\mathcal{H}(\chi), \in)$, we have some $q \in \underset{\sim}{G}_{\mathbb{Q}}$ which is almost $(M, \mathbb{Q})$–generic, but not quite. The "almost" is because this holds for another forcing $\mathbb{Q}'$. So when the whole universe is extended generically for $\mathbb{Q}$, M "fakes" and is instead extended generically for $\mathbb{Q}'$. So for preservation in iteration it is not natural to demand $M_2 \prec (\mathcal{H}(\chi), \in)$, but rather to proceed as in [115], this will be n.e.l.–Pr.

We may wonder (considering 2.9) whether we can replace Laver forcing in the proof of the consistency of the Borel conjecture, by a forcing notion not adding a dominating real. So a related question to 2.9 is

Question 3.6.

$$\mathrm{CON}(\mathfrak{b} = \aleph_1 + \text{ Borel Conjecture})\ ?$$

It is most natural to iterate, one basic step will be $\mathbb{Q}$, adding an increasing sequence $\langle \underset{\sim}{n}_i : i < \omega \rangle$ such that on the one hand:

(a) no old non-dominated family $\subseteq {}^{\omega}\omega$ is dominated (or at least some particular old family remains undominated),

while on the other hand

(b) for any uncountable $A \subseteq {}^{\omega}2$, from $\mathbf{V}$, we have:

$$\Vdash_{\mathbb{Q}} \text{" for no } \underset{\sim}{\eta}_i \in {}^{n_i}2,\ (i < \omega) \text{ do we have } (\forall \nu \in A)(\exists^\infty i)(\underset{\sim}{\eta}_i \lhd \nu) \text{ "},$$

or at least

(b)$'$ like (b) for one A given by bookkeeping.

(To preserve we need to strengthen the statement, replacing $\langle \underset{\sim}{\eta}_i : i < \omega \rangle$ by a thin enough tree.) The $\underset{\sim}{\eta}_i$ should "grow" fast enough, so naturally we think of forcing notions as in Rosłanowski and Shelah [92], [88], which proved easily checked sufficient conditions for what we desire. It is natural to look for forcing notions in the "neighborhood" of Blass and Shelah [15]. But what should be the norm?

* * *

Discussion 3.7. Concerning $\mathfrak{p} < \mathfrak{t}$, I have made quite a few failed tries. Some try to use long iterations ($\geq \aleph_{\omega+1}$) or a new support. But also I thought that Blass and Shelah [16] would be a reasonable starting point, the point is how to extend $\aleph_1$-generated filters to a good enough P-point.

That is, trying to force $\mathfrak{p} = \aleph_2$, $\mathfrak{t} = \aleph_3 = \mathfrak{c}$ start, say, with $\mathbf{V} = \mathbf{L}$ and use a FS iteration $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \omega_3, j < \omega_3 \rangle$, where $\underset{\sim}{\mathbb{Q}}_i$ is a Cohen forcing adding $\underset{\sim}{r}_i \in {}^{\omega}2$ for some i's, and $\underset{\sim}{\mathbb{Q}}_i$ is shooting an ω–sequence through a P-point filter (or ultrafilter) on ω for some $i \geq \omega_2$. The point is that when we have to find a $\leq^*$–lower bound to the downward directed $\mathcal{A} \in [\mathcal{P}(\omega)]^{\aleph_1}$, we extend it to a P-point, possibly also for the ω_2–towers we have to do this. It is natural to try to preserve, for $\alpha \in [\omega_2, \omega_3)$, the statement:

in $\mathbf{V}^{\mathbb{P}_\alpha}$, noting that $\mathcal{H}(\aleph_1)$ has cardinality $\aleph_2$, if $\mathcal{H}(\aleph_1) = \bigcup\limits_{\alpha<\omega_2} M_\alpha$, M_α increasing continuous, $\|M_\alpha\| < \aleph_2$,

then the following set is $= \emptyset \mod \mathcal{D}_{\omega_2} + S^2_1$:

$$\big\{\delta : \text{ if some } a \in M_\delta \cap [\omega]^{\aleph_0} \text{ is almost included in } r_i^{-1}(\{1\}) \text{ for many } i<\delta, \text{ then } a \text{ is almost disjoint to } r_\delta^{-1}(\{1\})\ \big\}.$$

* * *

Discussion 3.8. Concerning 2.11 consider the problem "every $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on a non-null set".

We can try to use a forcing notion which looks locally random (like the forcing for "non meager set" of [169] looked locally like the Cohen forcing notion) or a mixture of random and quite bounding ones. Such forcing notions are considered in [88], do they help for "every function $f : {}^\omega 2 \longrightarrow {}^\omega 2$ is continuous on a non-null set"?

How can we try to prove the consistency of "for every non-meagre $A \subseteq {}^\omega 2$ and $f : {}^\omega 2 \longrightarrow {}^\omega 2$ for some non-meagre $B \subseteq A$, $f \restriction B$ is continuous"? We may use CS or even FS iteration of length ω_2, (with $\mathbf{V} \models \mathrm{GCH} + \diamondsuit_{\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}} + S_\alpha \subseteq \omega_1$ $(\alpha < \omega_2)$ increasing $\mod \mathcal{D}_{\omega_1}$).

In Stage α we have $\bar{r}^\alpha = \langle \underset{\sim}{r}^\alpha_i : i \in S_\alpha \rangle$ such that $\bar{r}^\alpha \in N \prec (\mathcal{H}(\chi), \in) \Rightarrow \underset{\sim}{r}^\alpha_{N\cap\omega_1}$ is forced to be Cohen over N and $\beta < \alpha \Rightarrow \{i \in S_\beta : \underset{\sim}{r}^\beta_i \neq \underset{\sim}{r}^\alpha_i\}$ is not stationary.

Sometimes in Stage α, bookkeeping gives us a $\mathbb{P}_\alpha$-name $\underset{\sim}{A}_\alpha$ of a non-meagre subset of ${}^\omega 2$ and we choose $\bar{r}^{\alpha+1}$ such that $\bar{r}^{\alpha+1} \restriction S_\alpha = \bar{r}^\alpha$ and $\{r^{\alpha+1}_i : i \in S_{\alpha+1} \setminus S_\alpha\}$ is a non-meagre subset of $\underset{\sim}{A}_\alpha$.

Sometimes in Stage α, bookkeeping gives us a stationary subset S^α of S_α (from $\mathbf{V}$) and a $\mathbb{P}_\alpha$–name $\underset{\sim}{f}_\alpha$ of a function from $\{\underset{\sim}{r}^\alpha_i : i \in S^\alpha\}$ to ${}^\omega 2$ and we try to choose $\bar{r}^{\alpha+1}$ such that: $\bar{r}^{\alpha+1} \restriction S_\alpha = \bar{r}^\alpha$, $\{\underset{\sim}{r}^{\alpha+1}_i : i \in S_{\alpha+1} \setminus S_\alpha\} \subseteq \{\underset{\sim}{r}^\alpha_i : i \in S^\alpha\}$ and $\underset{\sim}{f}_\alpha \restriction \{\underset{\sim}{r}^{\alpha+1}_i : i \in S_{\alpha+1} \setminus S_\alpha\}$ is continuous. So the aim is that in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$, every non-meagre $A \subseteq {}^\omega 2$ contains a subset of the form $\{\underset{\sim}{r}^\alpha_i : i \in S_{\alpha+1} \setminus S_\alpha\}$ and

$$S' \subseteq S_\alpha \ \&\ \alpha < \omega_1 \ \&\ S' \text{ stationary } \Rightarrow \{r^\alpha_i : i \in S'\} \text{ is non-meagre.}$$

We may try to define iterations for forcing related to measure: we can use CS, or try to imitate the measure algebra, there are various ways to interpret it. If each $\mathbb{Q}_i$ is as in [92], so each condition has possible pair: a norm $\in \omega$ and real $r \in (0, 1)$ and using those we define what is a condition in the iteration. See [88].

More on preservation (for not necessarily c.c.c. ones) commutativity, associativity, generic sets and countable for pure/finite for a pure support iterations see [116]. Remember that an automorphism F of $\mathcal{P}(\omega)$/finite is called trivial if it is induced by a permutation f of $\mathbb{Z}$ (the integers, where $\omega = \{n \in \mathbb{Z} : n \geq 0\}$) such that $\{n \in \mathbb{Z} : n < 0 \ \Leftrightarrow \ f(n) \geq 0\}$ is finite.

Question 3.9. What can AUT($\mathcal{P}(\omega)$/finite) be? Is it consistent that AUT($\mathcal{P}(\omega)$/finite) is not the group of trivial automorphisms of $\mathcal{P}(\omega)$/finite, but is of cardinality continuum (or even is generated by adding one automorphism to the subgroup of the trivial automorphisms of $\mathcal{P}(\omega)$/finite) ?

It is reasonable try to combine Shelah and Steprāns [183] and the later part of the proof in [176, Ch.IV,§6] (from being "locally trivial" to being trivial).

* * *

Discussion 3.10. Next we deal with the variants of OCA and isomorphisms or farness of sets of $\aleph_2$ reals, i.e. 2.21, 2.24.

Concerning 2.21 (on " there are no far $A, B \in [{}^{\omega}2]^{\lambda}$ "), assume that for $\ell = 1, 2$ we have $A_\ell = \{\eta^\ell_\alpha : \alpha < \lambda\} \subseteq {}^{\omega}2$ with no repetitions. Considering Baumgartner [11] and Abraham, Rubin and Shelah [2], it is natural to try to find $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ such that:

(a) f_α is a partial, countable, non-empty function from ω_2 to ω_2, (for the present aim, $\mathrm{Dom}(f_\alpha)$ a singleton in (a) and $\gamma = 1$ in (b) are fine, so we assume so),

(b) for some $\gamma = \gamma^* \leq \omega$, the sequence

$$\langle \bigcup_{n<\gamma} (\mathrm{Dom}(f_{\gamma\alpha+n}) \cup \mathrm{Rang}(f_{\gamma\alpha+n})) : \alpha < \lambda\rangle$$

is a sequence of pairwise disjoint sets.

We let $\hat{f}_\alpha = \{(\eta^1_i, \eta^2_{f_\alpha(i)}) : i \in \mathrm{Dom}(f_\alpha)\}$, so $\hat{f}_\alpha(\eta)$ is well defined iff $i \in \mathrm{Dom}(f_\alpha)$, $\eta = \eta^1_i$, and then $\hat{f}_\alpha(\eta^1_i) = \eta^2_{f_\alpha(i)}$.

It is natural to try the following forcing notion

$$\begin{aligned} \mathbb{Q}_{\bar{f}} = \big\{g :\ & g \text{ is a finite 1-to-1 order preserving function from } A_1 \text{ to } A_2, \\ & \text{which has the form } \bigcup_{\ell=1}^{n} f_{\alpha_\ell},\ \alpha_\ell < \omega_2, \text{ and such that} \\ & \ell_1 \neq \ell_2 \quad \Rightarrow \quad \neg(\exists \alpha)(\{\alpha_{\ell_1}, \alpha_{\ell_2}\} \subseteq [\gamma^*\alpha, \gamma^*\alpha + \gamma^*])\big\}. \end{aligned}$$

The order is the inclusion.

It is enough to have "$\mathbb{Q}_{\bar{f}}$ satisfies the c.c.c." (for some $\bar{f}$ as above): clearly $\bigcup\{g : g \in \underset{\sim}{G}_{\mathbb{Q}_{\bar{f}}}\}$ is an order preserving function from some $A'_1 \subseteq A_1$ into A_2; but does it have cardinality λ? Essentially yes, as e.g. if $\mathrm{cf}(\lambda) > \aleph_0$ then some $p \in \mathbb{Q}_{\bar{f}}$ forces this. For this it is enough to have: if $u \in [\lambda]^{\aleph_1}$ then $\mathbb{Q}_{\bar{f}\restriction u}$

satisfies the c.c.c. and $\lambda = \aleph_2$, it is enough to check for $u = \alpha \in [\omega_1, \omega_2)$. So as in [2], it is enough that

(c) if $n < \omega$ and $C \subseteq {}^{2n}({}^{\omega}2)$ is closed and

$$(\eta_0, \eta_1, \dots, \eta_{2n-1}) \in C \quad \Rightarrow \quad (\forall \ell{<}m{<}n)(\eta_{2\ell} <_{\ell ex} \eta_{2m} \Leftrightarrow \eta_{2\ell+1} <_{\ell ex} \eta_{2m+1})$$

and there are $p_\zeta = \{(\eta_{\zeta,\ell}, \nu_{\zeta,\ell}) : \ell \leq n\} \in \mathbb{Q}_{\bar{f}}$ for $\zeta < \omega_1$, $\ell < n$ with

$$(\zeta_1, \ell_1) \neq (\zeta_2, \ell_2) \quad \Rightarrow \quad \eta_{\zeta_1,\ell_1} \neq \eta_{\zeta_2,\ell_2} \text{ and } \nu_{\zeta_1,\ell_1} \neq \nu_{\zeta_2,\ell_2},$$

hence

$$(\forall \zeta < \omega_1)(\forall \ell < m < n)(\eta_{\zeta,\ell} <_{\ell ex} \eta_{\zeta,m} \Leftrightarrow \nu_{\zeta,\ell} <_{\ell ex} \nu_{\zeta,m})$$

(if $\gamma^* = 1$, and each $\mathrm{Dom}(f_\alpha)$ is a singleton, $p_\zeta \in \mathbb{Q}_{\bar{f}}$ means, in addition only $\{(\eta_{\zeta,\ell}, \nu_{\zeta,\ell})\} = f_{\alpha_{\zeta,\ell}}$ for some $\alpha_{\zeta,\ell}$),
then there are $(\eta'_0, \eta'_1, \dots, \eta'_{2n-1}), (\eta''_0, \eta''_1, \dots, \eta''_{2n-1}) \in C$ such that $\eta'_\ell \neq \eta''_\ell$ and $\eta'_{2\ell} <_{\ell ex} \eta''_{2\ell} \Leftrightarrow \eta'_{2\ell+1} <_{\ell ex} \eta''_{2\ell}$.

Any counterexample to clause (c) induces a continuous partial function for which we get dependencies. If $\mathbf{V} \models$ CH, and $\mathbb{P}$ is adding Cohen reals, then in $\mathbf{V}^{\mathbb{P}}$ this holds, so it is natural to try to retain during the iteration similarly to this case. For 2.20 we can use similar, but somewhat more involved forcing notion as in [2].

There are similar considerations on OCA. We may consider trying to get negative ZFC results, so Kojman and Shelah [69] seems to me a reasonable starting point (of course, the problem there is different).

* * *

Baumgartner [13] defines

Definition 3.11. (1) For a (non-principal) ultrafilter D on ω, and a countable ordinal δ, we say D is a δ–ultrafilter if:
 $(*)$ for every function f from ω to ω_1, for some $A \in D$ we have $\mathrm{otp}(f(A)) < \delta$.
(2) We say that D is a weak δ–ultrafilter if:
 $(*)^-$ for every function f from ω into δ, for some $A \in D$ we have $\mathrm{otp}(f(A)) < \delta$.
(3) We say that D is a NWD if for every function f from ω to $\mathbb{R}$, for some $A \in D$, $f(A)$ is a nowhere dense subset of $\mathbb{R}$.
(4) For an ideal I, D is an I–filter if for any $f : \mathrm{Dom}(D) \longrightarrow \mathrm{Dom}(I)$ and $A \in D^+$ there is $B \subseteq A$, $B \in D^+$ such that $f(B) \in I$.

Then he asked whether such ultrafilters exist (if CH yes, so):

Question 3.12. Prove the consistency of "there is no δ–ultrafilter on ω".

It seemed the solution of the related "CON(there is no NWD-ultrafilter)" in [177] should give this, but it did not, and it is not clear if the question 3.12 is harder (the NWD eluded me several times, but when solved, the solution seems a straightforward generalization of CON(no P-point), which was also a priori the natural starting point).

A nice feature of P-points is that "D generates a P-point ultrafilter on ω" is preserved in limit for CS iterations, so P-points generated by $\aleph_1 < 2^{\aleph_0}$ sets are gotten naturally. Are they the only ones? Of course, by c.c.c. forcing P you may have ultrafilters on ω generated by $< 2^{\aleph_0}$ sets, and forcing by a subforcing $\mathbb{Q} \lessdot \mathbb{P}$, in $\mathbf{V}^{\mathbb{Q}}$ we get an ultrafilter preserved (see more in Brendle and Shelah [18]); but we have no understanding, though the suggestion in 3.2 may help.

Question 3.13. Are ultrafilters D as defined below in 3.21 preserved in limit stages of CS iterations? This means that:

if $\bar{\mathbb{Q}} = \langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \leq \delta, j < \delta \rangle$ is a CS iteration of proper forcing notions and D is an ultrafilter as above in $\mathbf{V}$,
then $(\forall i < \delta)(\boxtimes_i) \quad \Rightarrow \quad (\boxtimes_\delta)$, where
$(\boxtimes_\beta)$ $\Vdash_{\mathbb{P}_\beta}$ "in $\mathbf{V}^{\mathbb{P}_\beta}$ the filter on ω that the family D generates in $\mathbf{V}^{\mathbb{P}_\beta}$ is an ultrafilter".

Less nice, but still good, is to prove the preservation of "D generates an ultrafilter +Pr" (where Pr is some additional property like: $({}^{\omega}\omega)^{\mathbf{V}}$ is dominating).

The following 3.14 — 3.22 suggest an approach to question 3.13.

Definition 3.14. (1) Let

$$\begin{aligned}\mathcal{T} = \big\{t :\ & t \subseteq {}^{\omega>}\omega,\ t \text{ has a } \lhd\text{–minimal element } \mathrm{rt}(t),\\ & t \text{ is closed under initial segments of length } \geq \ell g(\mathrm{rt}(t)),\\ & \text{for } \eta \in t,\ \mathrm{Suc}_t(\eta) = \{\eta^\frown\langle\ell\rangle : \eta^\frown\langle\ell\rangle \in t\} \text{ is empty or infinite}\big\}.\end{aligned}$$

For $t \in \mathcal{T}$ let $h_t : t \longrightarrow \omega_1 \cup \{\infty\}$ be defined by

$$h_t(\eta) = \bigcup \big\{h_t(\nu) + 1 : \nu \in \mathrm{Suc}_t(\eta)\big\}.$$

(So $h_t(\eta) = \infty$ iff there is an ω-branch through η).
We say that t is standard if

$$\eta \in t \;\&\; \beta < h_t(\eta) \quad \Rightarrow \quad (\forall^\infty \nu \in \mathrm{Suc}_t(\eta))(\beta \leq h_t(\nu)).$$

If not said otherwise, every t is standard. Note:

$$s \in \mathrm{sub}(t) \;\&\; t \text{ is standard} \quad \Rightarrow \quad s \text{ is standard},$$

where on sub(t) see part (8) below.

(2) For an ordinal $\alpha < \omega_1$, let

$$\begin{array}{lcl}\mathcal{T}_\alpha & = & \big\{t \in \mathcal{T} : \mathrm{Rang}(h_t) \subseteq \omega_1 \text{ and } h_t(\mathrm{rt}(t)) = \alpha\big\},\\ \mathcal{T}_{<\alpha} & = & \bigcup\limits_{\beta<\alpha} \mathcal{T}_\beta.\end{array}$$

(3) For $t \in \mathcal{T}$ let

$$\mathcal{A}_t = \Big\{\bar{A} : \begin{array}{l}\bar{A} = \langle A_\eta : \eta \in t\rangle,\ A_\eta \in [\omega]^{\aleph_0},\ \text{and if } \eta \in t \text{ is not maximal,}\\ \text{then } \langle A_\nu : \nu \in \mathrm{Suc}_T(\eta)\rangle \text{ is a sequence of pairwise disjoint}\\ \hfill \text{subsets of } A_\eta\end{array}\Big\},$$

$$\begin{array}{lcl}\mathcal{A}_\alpha & = & \bigcup\big\{\mathcal{A}_t : t \in \mathcal{T}_\alpha\big\},\\ \mathcal{A}_{<\alpha} & = & \bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta.\end{array}$$

(4) For $t \in \mathcal{T}$, $\eta \in t$ let $t^{[\eta]} = \{\nu : \eta \trianglelefteq \nu \in t\} \in \mathcal{T}$.
(5) For $t \in \mathcal{T}$ let $\lim(t) = \{\eta \in {}^\omega\omega : (\forall \ell < \omega)(\ell \geq \ell g(\mathrm{rt}(t)) \ \Rightarrow\ \eta \restriction \ell \in t)\}$.
(6) $\max(t) = \{\eta \in t : \mathrm{Suc}_t(\eta) = \emptyset\}$.
(7) We say that y is a front of $t \in \mathcal{T}$ *if*: $y \subseteq t$, $(\forall \eta, \nu \in y)(\neg \eta \vartriangleleft \nu)$ and

$$(\forall \eta)[(\eta \in \lim(t) \vee \eta \in \max(t)) \ \Rightarrow\ (\exists \ell \leq \ell g(\eta))(\eta \restriction \ell \in y)].$$

We let $\mathrm{fr}(t) = \{y : y \text{ is a front of } t\}$.
(8) For $t \in \mathcal{T}$ let $\mathrm{sub}(t) = \{s \in \mathcal{T} : s \subseteq t \text{ and } \max(s) = \max(t) \cap s\}$. Clearly for a standard t and $s \in \mathrm{sub}(t)$ we have $h_s = h_t \restriction s$. Let

$$\mathrm{sub}^-(t) = \{s \in \mathcal{T} : s \subseteq t,\ s \in \mathrm{sub}(t) \text{ and } \mathrm{rt}(s) = \mathrm{rt}(t)\}.$$

(9) For $s, t \in \mathcal{T}$ and a set $\mathcal{Y}$ of fronts of t let $\mathrm{FT}^0_{\mathcal{Y}}(t, s)$ be the set of embeddings $f : s \longrightarrow t$ (i.e., f is one–to–one, $\mathrm{Dom}(f) = s$, $\mathrm{Rang}(f) \subseteq t$, $f(\mathrm{rt}(s)) = \mathrm{rt}(t)$ and $(\forall \eta, \nu \in t)(\eta \vartriangleleft \nu \ \Rightarrow\ f(\eta) \vartriangleleft f(\nu))$) which respect each $y \in \mathcal{Y}$, i.e., $\{\eta : f(\eta) \in y\}$ is a front of s for every $y \in \mathcal{Y}$.
(10) If $\mathcal{Y}$ is the set of all fronts of t we may omit it.
(11) $\mathrm{FT}^1(t, s)$ is the set of $f \in \mathrm{FT}^0(t, s)$ such that

$$\eta \in \mathrm{Suc}_s(\mathrm{rt}(s)) \quad \Rightarrow \quad h_s(\eta) = h_t(f(\eta)).$$

(12) $\mathrm{FT}^2(t, s)$ is the set of all $f : s \longrightarrow t$ such that:

$$\begin{array}{l}\eta \vartriangleleft \nu \ \Leftrightarrow\ f(\eta) \vartriangleleft f(\nu),\\ \{f(\eta) : \eta \in \mathrm{Suc}_s(\mathrm{rt}(s))\} \text{ is a front of } t, \quad \text{and}\\ \text{for } \eta \in \mathrm{Suc}_s(\mathrm{rt}(s)), \quad f \restriction s^{[\eta]} \text{ is one-to-one onto } t^{[f(\eta)]}.\end{array}$$

$\mathrm{FT}^3(t, s)$ is the set of all $f : s \longrightarrow t$ such that:

$$\begin{array}{ll}\eta \vartriangleleft \nu \ \Leftrightarrow\ f(\eta) \vartriangleleft f(\nu), & \text{for } \eta, \nu \in s,\\ \eta \in s \ \&\ \nu \in \mathrm{Suc}_t(\eta) \quad \Rightarrow \quad f(\nu) \in \mathrm{Suc}_t(f(\eta)).\end{array}$$

(13) For an ideal I let

$$\mathcal{ST} = \mathcal{ST}_I = \big\{(t,g) : \ t \in \mathcal{T} \text{ and } g : t \setminus \{\mathrm{rt}(t)\} \longrightarrow \mathrm{Dom}(I) \text{ are such that } \\ \eta \in t \setminus \max(t) \quad \Rightarrow \quad \{g(\nu) : \nu \in \mathrm{Suc}_T(\eta)\} \in I^+\big\}.$$

We usually omit I if it is clear from the context (here it is fixed).

(14) For $(t^1,g^1),(t^2,g^2) \in \mathcal{ST}$ for $\ell = 0,1,2$ let $\mathcal{FT}^\ell((t^1,g^1),(t^2,g^2))$ be the set of $f \in \mathrm{FT}^\ell(t^1,t^2)$ such that: $g^2 = g^1 \circ f$.

(15) Let

$$\begin{array}{lcl} \mathcal{TT}_t & = & \big\{(t,\bar{A},g) : \bar{A} \in \mathcal{A}_t \text{ and } (t,g) \in \mathcal{ST}\big\}, \\ \mathcal{TT} & = & \bigcup\limits_t \mathcal{TT}_t, \\ \mathcal{TT}_\alpha & = & \big\{(t,\bar{A},g) : t \in \mathcal{T}_\alpha \text{ and } (t,\bar{A},g) \in \mathcal{TT}_t\big\}, \\ \mathcal{TT}_{<\alpha} & = & \bigcup\limits_{\beta<\alpha} \mathcal{TT}_\beta. \end{array}$$

(16) Let $\mathcal{FT}^\ell((t^1,\bar{A}^1,g^1),(t^2,\bar{A}^2,g^2))$ be the set of $f \in \mathcal{FT}^\ell((t^1,g^1),(t^2,g^2))$ such that $\eta \in t^2 \quad \Rightarrow \quad A^1_{f(\eta)} \subseteq^* A^2_\eta$.

(17) If we omit ℓ (in $\mathrm{FT}^\ell, \mathcal{FT}^\ell$) we mean $\ell = 0$.

Definition 3.15. We define a partial order on $\mathcal{TT}_{<\alpha}$:

$$(t^1,\bar{A}^1,g^1) \le^\ell (t^2,\bar{A}^2,g^2) \quad \text{if and only if} \quad \mathcal{FT}^\ell((t^1,\bar{A}^1,g^1),(t^2,\bar{A}^2,g^2)) \neq \emptyset.$$

Observation 3.16. $\le^\ell$ really is a partially order of $\mathcal{TT}$.

Definition 3.17. (1) For $t \in \mathcal{T}$, let

$$\mathrm{fsub}(t) = \{s \in \mathcal{T} : \ \text{for some finite } w \subseteq \mathrm{Suc}_t(\mathrm{rt}(t)) \text{ we have} \\ s = \{\eta \in t : \neg(\exists \nu \in w)(\nu \trianglelefteq \eta)\}\}.$$

(2) For $(t^1,\bar{A}^1,g^1),(t^2,\bar{A}^2,g^2) \in \mathcal{TT}$ we define $\mathcal{FT}^\ell_*((t^1,\bar{A}^1,g^1),(t^2,\bar{A}^2,g^2))$ as the set of all f such that for some $t^3 \in \mathrm{fsub}(t^2)$ we have

$$f \in \mathcal{FT}^\ell((t^1,\bar{A}^1,g^1),(t^3,\bar{A}^2 \restriction t^3, g^2 \restriction t^3)).$$

(3) $(t^1,\bar{A}^1,g^1) \le^\ell_* (t^2,\bar{A}^2,g^2)$ if and only if $\mathcal{FT}^\ell_*((t^1,\bar{A}^1,g^1),(t^2,\bar{A}^2,g^2))$ is not empty.

(4) Let

$$\mathrm{ssub}(t) = \{s \in \mathcal{T} : \ s \subseteq t, \ \mathrm{rt}(s) = \mathrm{rt}(t) \text{ and for every } \eta \in s \\ \mathrm{Suc}_t(\eta) \setminus \mathrm{Suc}_s(\eta) \text{ is finite }\}.$$

Let $\mathcal{FT}^\ell_\otimes$ be defined like $\mathcal{FT}^\ell$ using $\mathrm{ssub}(t)$ and $\le^\ell_\otimes$ be defined like $\le^\ell_*$ but using $\mathcal{FT}^\ell_\otimes$.

Fact 3.18. (1) *$<^\ell_*$ is a partial order of $\mathcal{TT}$ such that: $<^\ell$ is a subset of $<^\ell_*$.*

(2) *Any $<^\ell_*$–increasing chain of length ω in $\mathcal{TT}_{\le\alpha}$ has an upper bound in $\mathcal{TT}_{\le\alpha}$.*

(3) $\mathcal{TT}_{\le\omega_1} = \mathcal{TT}_{<\omega_1}$.

Proposition 3.19. (1) *If* $(t, \bar{A}, g) \in \mathcal{TT}_{\le\alpha}$, *and* $B \subseteq \omega$,
then *for some* $(t', \bar{A}', g') \in \mathcal{TT}_{\le\alpha}$ *we have:*
(α) $(t, \bar{A}, g) \le^1 (t', \bar{A}', g')$, *in fact* $t' \in \mathrm{sub}(t)$,
(β) $\bigcup\{A_\eta : \eta \in t \setminus \{\mathrm{rt}(t)\}\}$ *is a subset of* B *or is disjoint to* B.
(2) *Similarly for* $\mathcal{TT}$.
(3) *Similarly* $\le^3$.

Proof. (1) By induction on α.
(2) Similarly. (Let

$$Z_B = \{\eta \in t : \begin{array}{l} \text{there is } (t', \bar{A}', g') \text{ such that} \\ (t^{[\eta]}, \bar{A} \restriction t^{[\eta]}, g \restriction t^{[\eta]}) \le^1 (t', \bar{A}', g') \\ \text{and } \bigcup\{A_\eta : \eta \in t \setminus \{\mathrm{rt}(t)\}\} \text{ is a subset of } B\ \}. \end{array}$$

If $\mathrm{rt}(t) \in Z_B \cup Z_{\omega\setminus B}$ we are done; if not then we can find $(t', \bar{A}', g')$ satisfying (α) and such that $t \cap (Z_B \cup Z_{\omega\setminus B})$; an easy contradiction.)
(3) Similarly. $\square$

Proposition 3.20. *If* $(t, \bar{A}, g) \in \mathcal{TT}_{\le\alpha}$ *and* E *is an equivalence relation on* ω, *then for some* $(t, \bar{A}', g')$ *and front* y *of* t' *we have:*
(α) $(t, \bar{A}, g) \le^1 (t', \bar{A}', g')$,
(β) *for* $\eta \in y$, A'_η *is included in one* E*–equivalence class,*
(γ) *for* $\eta \ne \nu$ *from* y, *the* E*–equivalence classes in which* A'_η, A'_ν *are included, are distinct (hence disjoint),*
(δ) *on* $A'_{\mathrm{rt}(t')}$ *we have:* E *is either trivial or refines* $\langle A'_\eta : \eta \in \mathrm{Suc}_{t'}(\mathrm{rt}(t'))\rangle$,
(ε) $A'_\eta = \bigcup\{A'_\nu : \nu \in \mathrm{Suc}_{t'}(\eta)\}$ *for* $\eta \in t'$.

Proof. By induction on α. (In clause (δ), the first possibility holds if the y from (γ) is $\{\mathrm{rt}(t)\}$, otherwise the second possibility holds.) $\square$

Proposition 3.21 (CH). *Let* I *be an ideal such that* $|\mathrm{Dom}(I)| \le \aleph_1$ *and*

$$(\forall X \in I^+)(\exists Y \in I^+)(Y \subseteq X \ \&\ |Y| = \aleph_0).$$

There is a $\le^1_*$*–increasing sequence* $(\langle t^\zeta, \bar{A}^1, g^\zeta\rangle : \zeta < \omega_1)$ *of members of* $\mathcal{TT}_{\le\omega_1}$ *such that* $\{\bigcup_{\eta\in t^\zeta} A^1_\eta : \zeta < \omega_1\}$ *generates a non-principal ultrafilter* D *on* ω *which is a* Q*-point, and for every equivalence relation* E *on* ω, *for some* ζ, $\langle A^\zeta_\eta : \eta \in \mathrm{Suc}_{t^\zeta}(rt(t^\zeta))\rangle$ *refines* $E \restriction A^\zeta_{\mathrm{rt}(t^\zeta)}$.

Remark: This construction gives an ultrafilter D on ω, a Q–point such that

$$D' \le_{\mathrm{RK}} D \ \Rightarrow\ D' \text{ is not NWD}.$$

Proposition 3.22. *In 3.21, if in addition an ideal* I' *satisfies*

(∗) $|\mathrm{Dom}(I')| \le \aleph_1$ *and if* $(t, \bar A, g) \in \mathcal{TT}_{<\omega_1}$ *and* $g' : \omega \longrightarrow \mathrm{Dom}(I')$ then *for some* $(t', \bar A', g') \ge (t, \bar A, g)$ *we have* $g'[A'_{\langle\rangle}] \in I'$,

then *we can demand that* D *is an* I'*-ultrafilter (see 3.11(4)).*

∗ ∗ ∗

There are many problems on the σ-versions of cardinal invariants, and I think for some the method of [105], [130] is relevant, e.g.

Question 3.23 (See Brendle and Shelah [18])**.** Does $\chi_\sigma(D) = \chi(D)$ for all ultrafilters D on ω? Recall that

$$\begin{aligned}\chi(D) = \min\{|\mathcal{A}| : \quad & \mathcal{A} \subseteq D \text{ and for every } A \in D, \\ & \text{for some } B \in \mathcal{A} \text{ we have } B \subseteq A\} \\ \chi_\sigma(D) = \min\{|\mathcal{A}| : \quad & \mathcal{A} \subseteq {}^\omega D \text{ is such that for every } \bar A \in {}^\omega D, \text{ for some} \\ & \bar B \in \mathcal{A} \text{ we have } (\forall n < \omega)(\exists m < \omega)(B_m \subseteq^* A_n)\}.\end{aligned}$$

So a reasonable scenario to prove the consistency of a negative answer runs as follows: let, e.g., $\mu = \aleph_\omega$. We use FS iteration of c.c.c. forcing notions, $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \le \delta^*, j < \delta^* \rangle$. We have $\mathbb{P}_i$-names $\underset{\sim}{D}^i_u, \underset{\sim}{A}_{u,\gamma}$ (for $\gamma < \gamma^i_u$ and $u \in [\mu]^{<\aleph_0}$) such that:

- $\underset{\sim}{D}^i_u$ is the filter on ω generated by $\{\underset{\sim}{A}_{u,\gamma} : \gamma < \gamma^i_u\}$ and the co-bounded sets,
- γ^i_u are increasing with i, $\underset{\sim}{D}^i_u \subseteq \underset{\sim}{D}^i_v$ if $u \subseteq v$, and $\underset{\sim}{D}^j_u \subseteq \underset{\sim}{D}^i_u$ for $j < i$.

To simplify we decide:

(∗) if $j < i$, $\Vdash_{\mathbb{P}_j}$ " $\underset{\sim}{A} \subseteq \omega$ and $u \in [\mu]^{<\aleph_0}$ and $\underset{\sim}{A} \subseteq^* \underset{\sim}{A}_{u,\gamma}$ for every $\gamma < \gamma^j_u$ "

then $\Vdash_{\mathbb{P}_i}$ " for every $v \in [\mu]^{<\aleph_0}$, $\underset{\sim}{A} \notin \underset{\sim}{D}^i_v$ ".

Also for the following it seems reasonable to try to be influenced by [105], [130].

Question 3.24 (See Brendle and Shelah [18])**.** . Can $\pi\chi(D)$ be singular, where

$$\begin{aligned}\pi\chi(D) = \min\{|\mathcal{A}| : \quad & \mathcal{A} \subseteq [\omega]^{\aleph_0}, \text{ and for every } B \in D \\ & \text{for some } A \in \mathcal{A} \text{ we have } A \subseteq^* B\}.\end{aligned}$$

4. Nicely defined forcing notions

Rosłanowski and Shelah [92], and [115], [176] relate as algebraic three dimensional varieties relate to manifolds in $\mathbb{R}^n$ and these, in turn, relate to general topology. In [115] (on nep and snep) and in Judah and Shelah [58] (on Souslin forcings) we deal with forcing notions defined in an absolute enough way; in [92] (more in [88], [89]) with forcing notions defined in an explicit way (say as tress and generally by creatures), in [176] we deal with forcing notions related to the continuum.

Our problem with speaking about [92], [90], [88] and [89] is that much work is in progress, still orthogonal to it is the question whether in the main theorems of [92], all the assumptions are needed. That is, within the framework of condition trees or ω-sequences of creatures, are the demands on the norms necessary? This is dealt with for the conditions for properness in [92], showing necessity but there are still gaps remaining.

Question 4.1. Are the sufficient conditions for properness in [92, §2] necessary? The test case (chosen in [92]) is

$$\mathbb{Q} = \{\langle w_n : n < \omega\rangle : w_n \subseteq 2^n, w_n \neq \emptyset \text{ and } \lim_{n\to\omega} |w_n|\} = \infty$$

ordered by $\bar{w} \leq \bar{w}' \ \Leftrightarrow\ (\forall n \in \omega)(w'_n \subseteq w_n)$.

Though properness is the main thing and there we look for counterexamples only for properness, it is interesting to know:

Question 4.2. Concerning other theorems of [92], are they sharp?

There are more specialized problems, probably solvable in this context.

Question 4.3. Is there an ${}^\omega\omega$–bounding forcing notion adding a perfect set of random reals?

It seems this should not be hard if true.

The following problems (raised by Komjath and Steprāns respectively) seem to me a matter of choosing the right variant of [92] or [88] and having the right finite combinatorics.

Question 4.4.
(1) Can each $A \in [{}^\omega 2]^{\aleph_1}$ be null while the union of some $\aleph_1$ lines in $\mathbb{R}\times\mathbb{R}$ is not null?
(2) For reals $0 < a_0 < a_1 \leq 1$, is it consistent with ZFC that: for $\ell < 2$,

$$\ell = 0 \text{ iff some } A \in [\mathbb{R}]^{\aleph_1} \text{ has positive Hausdorff capacity for } a_\ell \text{ ?}$$

* * *

I suppose that the feeling that the Cohen forcing notion and the random real forcing notion occupy a special place is old; probably more in the version speaking on the ideal of null sets and the ideal of meagre sets. I feel the former version is more interesting. For me this translates to

Problem 4.5. Among Souslin c.c.c. forcing notions, are Cohen forcing and random forcing special?

Some progress was made in [168].

Theorem 4.6. *If a Souslin c.c.c. forcing notion $\mathbb{Q}$ adds $\eta \in {}^\omega\omega$ not dominated by any old $\nu \in {}^\omega\omega$,* then *forcing with $\mathbb{Q}$ adds a Cohen real.*
(The "Souslin" is needed for enough absoluteness, so with the existence of large cardinals we can allow a larger family).

So the Cohen forcing notion is the minimal one among Souslin c.c.c. forcing notions adding an undominated real, so it is natural to conjecture:

Problem 4.7. Show that any Souslin c.c.c. forcing notion adding a real adds a Cohen real or adds a random real.

This really will show that Cohen and random are special.

In a sense the realm of Souslin c.c.c. forcing notion can be looked as being divided between the ${}^\omega\omega$-bounding (with Random forcing as prototype) and those forcing notions adding an undominated real (with Cohen forcing as prototype); we can further distinguish those adding a dominating real.

However, the situation is very unbalanced: among Souslin c.c.c. forcing notions adding an undominated real we have many examples and a $<^*$-minimal one, Cohen, (see Definition 4.8 below).

On the other side we have no idea what occurs among the ${}^\omega\omega$–bounding ones: probably random real is the unique one, but it is not out of the question that there is a plethora (adding one or many randoms is an irrelevant distinction; we can even order $(\mathbb{Q}, \underset{\sim}{r})$, $\underset{\sim}{r}$ a $\mathbb{Q}$–name of a real such that the order depends only on the subforcing $\underset{\sim}{r}$ generates).

Definition 4.8. Let $\mathbb{Q}_1, \mathbb{Q}_2$ be definitions of forcing notions (absolute enough) say as in [115], or Souslin.

(1) $\mathbb{Q}_1 \leq^*_0 \mathbb{Q}_2$ if forcing with $\mathbb{Q}_2$ adds a generic for $\mathbb{Q}_1$ and we let $<^*$ mean $<^*_0$.
(2) $\mathbb{Q}_1 \leq^*_1 \mathbb{Q}_2$ means: for some n, if we force by iteration n times of $\mathbb{Q}_2$, we add a generic for $\mathbb{Q}_1$.
(3) $\mathbb{Q}_1 \leq^*_{2,\mathrm{fs}} \mathbb{Q}_2$ is defined similarly using FS iteration of length $< \omega_1$.
(4) $\mathbb{Q}_1 \leq^*_{2,\mathrm{cs}} \mathbb{Q}_2$ is defined similarly using CS of length $< \omega_1$.

Note: 4.7 is on the interval between the *control measure problem* (see Fremlin [37]) and von Neumann question which says: is any complete c.c.c. Boolean Algebra which as a forcing is ${}^\omega\omega$–bounding, a measure algebra. Another way to express the thought that Cohen and random are special was Kunen's conjecture, see Kunen [73], Kechris and Solecki [67], Solecki [188], [187] and Rosłanowski and Shelah [90].

It is natural to investigate the partial orders from 4.8. So,

Problem 4.9. Investigate the quasi order $\leq^*$ (and its variants) for $\mathbb{Q}$ which are nep (see [115]) or which are c.c.c. $\aleph_0$–nep or which are c.c.c. $\aleph_0$–snep.

[We may concentrate on those with a generic real (those are the main interest for 4.13(1), (2) below).]

An example is (and probably not hard):

Question 4.10. Prove that dominating real forcing (i.e., the Hechler forcing notion) is $\leq^*$–minimal among Souslin c.c.c. forcing notions adding a dominating real.
[For $\leq^*_1$ this is easy.]

Looking more serious are

Question 4.11. Can you characterize the $<^*$–minimal $\mathbb{Q}$, which add a Cohen real but are not equivalent to the Cohen forcing (hopefully there is one or at least there are only few).

Question 4.12. Can you characterize the $<^*$–minimal $\mathbb{Q}$ among the non–minimal $\mathbb{Q}$ which add a dominating real but are stronger than the Hechler forcing notion?

A positive solution of 4.7 would also show that the only symmetric Souslin c.c.c. forcing notions are the Cohen forcing and the random forcing (by [115, §9]).

* * *

Why should we be interested in Souslin proper or in nep forcing or better yet, why am I? The reason has been iteration theorems; when you are interested in iterating some very special forcing notions, the proof of their properness gives more, e.g., the existence of generic conditions over models occurs also for countable models of versions of ZFC which are not necessarily $\prec (\mathcal{H}(\chi), \in)$. Moreover, some things are preserved by iterations and this is helpful for specific problems which is the point of Judah and Shelah [58]. [In [58] this was phrased using descriptive set theory, getting Souslin proper. However, this does not cover the Sacks forcing notion, the Laver forcing notion, etc., which was accomplished by nep.]

Needless to say, I think iteration theorems for forcing are important and interesting (otherwise, normally I would not have written a book on the subject - see §2).

Another basic reason is that the family of nep forcing notions forms a natural class. Now, while I feel that general sets are much more basic and interesting then families of definable ones, and so prefer $\mathcal{P}(\mathcal{P}(\omega))$ to the family of projective sets, certainly they are interesting and natural.

Another reason is "large" ideals. Let I be a κ-complete ideal on λ. Gitik and Shelah [44] start by proving that $\mathcal{P}(\kappa)/I$ cannot be (the Boolean algebra which up to isomorphism is equivalent to) the Cohen forcing or random real forcing, an old question which Fremlin promoted (see [38]), which comes from asking: can the classical result of Solovay [189] (saying that consistently $2^{\aleph_0}$ is real valued measurable, now the Maharam type there was large) be improved to get small Maharam type.

But then [44] turns to:

Problem 4.13. (1) Prove that $\mathcal{P}(\kappa)/I$ cannot be a Souslin c.c.c. forcing generated by the name of one real $\underset{\sim}{\eta}$ (where I is a κ-complete ideal on κ or at least $\aleph_1$-complete).

(2) Similarly for Souslin proper (or weaken the definability demand - natural as the existence of the ideal implies more absoluteness).

(3) Even reasonable subclasses or cases are interesting.

Problem 4.14. Similarly, we can ask about a σ^+–complete ideal I on κ such that $\mathcal{P}(\kappa)/I$ has a dense subset isomorphic to a partial order defined in $(\mathcal{H}_{<\sigma}(\kappa'), \in)$ with parameters.

In Gitik and Shelah [44], [46], [42], in addition to information on adding not too many random or Cohen reals, and (toward 4.13) to general criteria for impossibility, we consider more specific cases (see then [186]). The problems lead us to properties of definable forcing notions like symmetry. The theorems on Cohen and random reals use the symmetry (i.e, the Fubini theorem), but other properties pop up naturally, e.g., for Souslin c.c.c. forcing $\mathbb{Q}$ with a dominating real $\underset{\sim}{\eta}$ as generic, to show impossibility it suffices to show: $\Vdash_{\mathbb{Q}}$ " $\mathfrak{b} = \aleph_1$ " (by [44]). Maybe the work on the ideals is done and we just need to verify that always at least one of the criteria applies (at least for large subclasses). Now [168], [115, §8,§9] comes to my mind.

Considerations like this lead to questions like

Question 4.15. Find sufficient conditions on $\mathbb{Q}$ for "$\mathbb{Q} * \underset{\sim}{\text{Random}}/\text{Random}$ adds no random real".
(This question is chosen since it is also interesting because if the condition is reasonable enough, it suffices for proving CON(cov*(null) < non(meagre)), see Bartoszyński, Rosłanowski and Shelah [9], [8].)

Question 4.16. Investigate commuting pairs (see [115]).

For such considerations I had felt that a peculiar property of Cohen forcing and random forcing is their satisfying: "being a maximal antichain is a Borel property"; this leads to

Definition 4.17. A forcing notion $\mathbb{Q}$ is very Souslin c.c.c. if it is Souslin c.c.c. and also the notion of "$\{r_n : n < \omega\}$ is a maximal antichain" is Σ^1_1.

We hope this will turn out to be a good dividing line of the Souslin c.c.c. forcing (so helping to prove theorems). This is because I suspect the answer to the following is yes.

Question 4.18. Prove: If $\mathbb{Q}$ is a Souslin c.c.c. forcing notion, say with generic real $\underset{\sim}{r}$ and it is not very Souslin c.c.c. above any $p \in \mathbb{Q}$ *then* $\Vdash_{\mathbb{Q}}$" $\mathfrak{b} = \aleph_1$". (This should help 4.13 by [44, §4]). See on this [116].

As in [115] we can define (restricting κ to be $\aleph_0$ for simplicity)

Definition 4.19. (1) A forcing notion $\mathbb{Q}$ is ω–nw–nep if there is a sequence $\bar{\varphi} = \langle \varphi_0, \varphi_1, \varphi_2 \rangle$ of Σ^1_1 definitions such that:

(a) the set of members of $\mathbb{Q}$ and $\leq^{\mathbb{Q}}$ are Σ^1_1 sets (of reals) defined by φ_0, φ_1, respectively,

(b) if N is an $(\bar{\varphi}, \omega)$–nw–candidate (that is, a model of ZFC^-_*, suitable version of ZFC, not necessarily well-founded but with standard ω), and with the real parameters of the φ_ℓ's, and $p \in \mathbb{Q}^N = \{x : N \models \varphi_0(x)\}$, *then* for some $q \in \mathbb{Q}$, we have

  (α) $p \leq q$,

  (β) q is $(N, \mathbb{Q})$–generic, which means that for every $\mathcal{I} \in \mathrm{pd}(N, \mathbb{Q}) = \{\mathcal{J} \in N : N \models$ "$\mathcal{J}$ is predense in $\mathbb{Q}$"$\}$, for some list $\langle r_n : n < \omega \rangle$ of $\{x : N \models$" $x \in \mathcal{I}$ "$\}$ we have $\{r_n : n < \omega\}$ is predense above q,

  (γ) moreover $\varphi_2(q, \langle r_n : n < \omega \rangle)$ holds,

(c) if $\varphi_2(q', \langle r'_n : n < \omega \rangle)$ then the set $\{r'_n : n < \omega\}$ is predense above q'.

(2) Omitting the "nw" or writing just "w" means we allow only well-founded candidates.

(3) [On **Ur** see below]. We say $\underset{\sim}{r} \subseteq \mathbf{Ur}$ (or $\underset{\sim}{r} \subseteq \mathcal{H}_{<\aleph_1}(\mathbf{Ur})$ in the w-case) is generic for $\mathbb{Q}$ if: $\underset{\sim}{r}$ is a $\mathbb{Q}$-name and "$a \in \underset{\sim}{r}$" is determined by the truth value $\varphi_{\underset{\sim}{r},a}[\underset{\sim}{G}_{\mathbb{Q}}]$ and the sequence $\langle \varphi_{\underset{\sim}{r},a} : a \rangle$ is definable in $\mathfrak{B}_{\mathbb{Q}}$ (see [116]).

Now, there are more examples of ω–nw–nep forcing notions in addition to Cohen and random: say the Sacks forcing notion. But about the Laver forcing notion we should beware; note that we can guarantee that for every $\mathcal{I} \in \mathrm{pd}(\mathbb{Q}, N)$, for some front X of q we have $\eta \in X \Rightarrow q^{[\eta]}$ is above a member of $\mathcal{I}$, but being a front is not absolute from $(\bar{\varphi}, \omega)$–nw–candidates, as they are not necessarily well-founded. In fact, we can easily craft a counterexample: assume $N \models$ "α a countable ordinal", but from the outside not well ordered. There are $f, T \in N$ (so actually $f = f_\alpha$, $t = T_\alpha$) such that

$$\begin{array}{ll} N \models \text{“} & T \subseteq {}^{\omega>}\omega \text{ is closed under initial segments,} \quad \langle\rangle \in T, \\ & f : T \longrightarrow \alpha + 1, \qquad f(\langle\rangle) = \alpha, \\ & f(\eta) > 0 \quad \Rightarrow \quad (\forall n)(\eta^\frown\langle n\rangle \in T), \\ & f(\eta) = 0 \quad \Rightarrow \quad (\forall n)(\eta^\frown\langle n\rangle \notin T), \\ & f(\eta) = \beta + 1 \quad \Rightarrow \quad (\forall n < \omega)[\eta^\frown\langle n\rangle \in T \ \& \ f(\eta^\frown\langle n\rangle) = \beta], \\ & \text{if } f(\eta) = \delta \text{ is limit} \\ & \text{then } \langle f(\eta^\frown\langle n\rangle) : n < \omega\rangle \text{ is strictly increasing with limit } \delta \text{ ”.} \end{array}$$

Let

$$\begin{array}{lcl} \mathcal{I}_0 & = & \{({}^{\omega>}\omega)^{[\eta]} : \eta \in T, \ f(\eta) = 0\}, \\ \mathcal{I}_{n+1} & = & \{({}^{\omega>}\omega)^{[\eta^\frown\rho]} : \eta \in \mathcal{I}_n, \ \rho \in T, \ f(\rho) = 0\}. \end{array}$$

Clearly, above no q is every $\mathcal{I}_n$ predense.

Still,

Theorem 4.20. *The ${}^\omega\omega$-bounding and almost ${}^\omega\omega$–bounding forcing notions covered by* [92] *and* [88] *are all ω–nw–nep.*

What about iterations? Now, for unifying the treatment of finite support and countable support we revise our definition to have two quasi-orders $\leq^{\mathbb{Q}}, \leq^{\mathbb{Q}}_{\rm pr}$ such that $p \leq_{\rm pr} q \ \Rightarrow \ p \leq q$. Hence in Definition 4.19 we add $\varphi_{1,2}$ (absolute just like φ_1) serving as a definition of $p \leq_{\rm pr} q$. The support is countable but finite for the apure cases, i.e., only for finitely many α ,$\neg(\emptyset_\alpha \leq_{\rm pr} p(\alpha))$. First assume the length is $\alpha^* < \omega_1$, so we can use a parameter coding a well ordering on ω with this order type. We should repeat the proof in [115] in order to prove preservation in this case, but we better not use the $L_{\omega_1,\omega}$–completeness as there, as we have problem with well–foundedness. So we just demand: elements in $\mathbb{P}_\alpha$ have depth $< \omega\alpha$ (or so).

What about long iterations? It seems to me, at least now, better (and fit to [115], too) to use a set of urelements **Ur**; let $\mathfrak{C}$ [and $\mathfrak{B}$] be models with universes **Ur** [or $\subseteq$ **Ur**] and $\mathcal{S} \subseteq [\mathbf{Ur}]^{\leq\aleph_0}$ be unbounded (usually stationary, if $a \in \mathcal{S}$ then $\mathfrak{B} \restriction a \subseteq \mathfrak{B}$ and $\mathfrak{C} \restriction a \subseteq \mathfrak{C}$), and anyhow the family of nw–candidates should be $< (\aleph_1)$–directed and uord $\subseteq$ **Ur** is a well ordered set (which will serve as the length of the iteration). Now, a candidate will be a countable model $N \subseteq (\mathcal{H}(\chi), \in)$ of ZFC$^-_*$ (where $\mathcal{H}(\chi)$ includes the urelements), $N \cap \mathbf{Ur} \in S$, where **Ur** and the relations of $\mathfrak{B}$ and $\mathfrak{C}$ are the considered relations. We define an nw–candidate similarly but now $\in^N$ is a relation on N and N is not necessarily well-founded (but the order type of the well ordered ordinals of N is $>$ otp($N \cap$ uord). In Definition 4.19 we demand, of course, that φ_ℓ are upward absolute from the nw–candidates. Now we can use uord as the index set for iteration (instead of the true ordinals) and there are no problems.

This set-up looks like a nice army with no enemy yet, but this seems to me a natural dividing line among the nep forcing notions and therefore reasonable for our interest in those forcing notions per se. I hope it will help, particularly with 4.13 (and even more so by the c.c.c. version see 4.22 below).

More on preservation (for not necessarily c.c.c. forcing notions), commutativity, associativity of generic sets, and countable for pure/finite for apure support iterations see [116].

A restricted version of the large continuum is

Problem 4.21. Can we have long ($> \omega_2$) iterations of ${}^\omega\omega$–bounding forcing notions (or at least nw–nep ones) not collapsing cardinals and not adding Cohen reals?

What about the c.c.c. (nw-nep) ones (or even very Souslin c.c.c.) ones?

Definition 4.22. $\mathbb{Q}$ is c.c.c.–nw–nep *if* it is a pair $\langle \varphi_0, \varphi_1 \rangle$ of formulas such that

(α) $\mathbb{Q} = \bigcup \{\mathbb{Q}^N : N \text{ is an nw-candidate}\} \subseteq {}^\omega\text{uord}$, similarly $\leq_Q = \bigcup \{\leq^N_{\mathbb{Q}} : N \text{ is an nw-candidate}\}$, recall that in N, $\mathbb{Q}^N, \leq^N_{\mathbb{Q}}$ are defined by the appropriate φ_ℓ,

(β) all φ_ℓ are upward absolute among nw-candidates,

(γ) if $N \models$" $\mathcal{I} \subseteq \mathbb{Q}$ is predense ", *then* $\mathcal{I}^{\mathbb{Q}}$ is really predense.

Proposition 4.23. (1) *The Cohen and random forcing notions are c.c.c.*-nw-*nep.*
(2) *The class of c.c.c.*-nw-*nep is closed under FS iterations.*
(3) *This class is also closed under subforcings.*

And I am curious to know:

Problem 4.24. Does 4.23 exhaust all c.c.c-nw-nep forcing notions (at least those with a generic real)? Can we give a full characterization of such forcing notions?

* * *

Being interested in classifying nep c.c.c. forcing notions, we may consider sweetness; the discussion below is in fact an introduction to [89]. Sweetness phenomenons are when we can build homogeneous forcing notions (as in [146, §7,§8]), sour phenomenons are strong negations (as in [146, §6]).

Problem 4.25. (1) For which $(\mathbb{Q}, \underset{\sim}{\eta})$, nep c.c.c. forcing notions, is it consistent that:
(a) there is a $(\mathbb{Q}, \underset{\sim}{\eta})$–generic real over $\mathbf{L}[A]$ for every $A \subseteq \omega_1$, and

(b) for every subset $B \in \mathbf{L}[\mathbb{R}]$ of ${}^{\omega}2$ for some $A \subseteq \omega$, for a dense set of $p \in (\mathbb{Q}, \underset{\sim}{\eta})$: for some truth value $\mathbf{t}$, *if* η is a $(\mathbb{Q}^{\mathbf{L}[A]}_{\geq p}, \underset{\sim}{\eta})$–generic real *then* $\eta \in B \ \Leftrightarrow\ \mathbf{t}$.

So for this question, random reals are complicated (see [146, §6]), whereas Cohen real and universal-meagre one (and dominating = Hechler reals) are low, see Judah and Shelah [61].

More generally,

Problem 4.26. Let $(\mathbb{Q}, \underset{\sim}{\eta})$ be a nep c.c.c. forcing notion, and κ be a cardinal number. Let $I^{\kappa}_{(\mathbb{Q},\eta)}$ be the κ–complete ideal generated by sets of the form

$$A_N = \{\eta : \eta \text{ is not } (\mathbb{Q}, \underset{\sim}{\eta})\text{–generic over } N\}$$

for countable models $N \prec (\mathcal{H}(\chi), \in)$ to which $(\mathbb{Q}, \underset{\sim}{\eta})$ belongs.
What is the consistency strength of " every projective set, or even every set from $\mathbf{L}[\mathbb{R}]$, is equal to a Borel set modulo $I^{\kappa}_{(\mathbb{Q},\underset{\sim}{\eta})}$ " ?

We hope for a strong dichotomy phenomena, i.e., if the answer above is negative for $(\mathbb{Q}, \underset{\sim}{\eta})$ (so the sweetness fails), then a strong negation holds, so we call such phenomena sourness.

Definition 4.27. Let $\mathbb{Q}_1, \mathbb{Q}_2$ be nep c.c.c. forcing notions definable in $\mathbf{L}$ (or $\mathbf{L}[r]$). We say that $\mathbb{Q}_1, \mathbb{Q}_2$ are explicitly sour over Cohen *if* we can find $\mathbb{Q}_\ell$-names $\underset{\sim}{\eta}_\ell$ of Cohen reals (for $\ell = 1, 2$) such that

if $G_\ell \subseteq \mathbb{Q}_\ell$ is generic over $\mathbf{L}$ (or $\mathbf{L}[r]$) for $\ell = 1, 2$
then $\underset{\sim}{\eta}_1[G_1] \neq \underset{\sim}{\eta}_2[G_2]$.

We should note that there may be homogeneity for wrong reasons, i.e., maybe when we force, very few $(\mathbb{Q}, \underset{\sim}{\eta})$–generic reals over $\mathbf{L}$ are added and then homogeneity holds for "degenerated" reasons; we may call such cases sacharin. For more on this direction see [88].

Speaking about the class of sweet forcing notions we should mention the following problem.

Problem 4.28. For any cardinal κ and a large cardinal property (or consistency strength) we may ask the following.

(1) Is there a widest class $\mathcal{K}$ of absolute enough forcing notions such that for some forcing notion $\mathbb{P}$ we have
  (a) $\Vdash_{\mathbb{P}} \kappa = \aleph_1$,
  (b) $\mathbb{P}$ is homogeneous for complete subforcings from $\mathcal{K}$, moreover
  (c) if $\mathbb{P}^* \lessdot \mathbb{P}$, $\mathbb{P}^*$ has a generic real then $\mathbb{P}/\underset{\sim}{G}_{\mathbb{P}^*}$ satisfies (b) in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}^*}]$.
(2) If not, at least give a wide enough such class.

(3) Are there two classes $\mathcal{K}_1, \mathcal{K}_2$ as above, such that there is no class with the respective property including $\mathcal{K}_1 \cup \mathcal{K}_2$? Or even that the consistency strength of the (now) obvious conclusion is higher than the given one?

Now, the variants of sweetness try to deal with the case of $\kappa = \aleph_1$ and the consistency strength ZFC (see Rosłanowski and Shelah [89, §3]); the theory of determinacy is applicable to the case $\kappa = \aleph_1$ and maximal consistency strength (see Woodin [197]), and [125] intends to deal with the case of "ZFC + κ is strongly inaccessible" (and no further consistency strength assumptions).

Question 4.29. Is there a sweet forcing notion (see [146, §7]), preferably natural one, such that it cannot be completely embedded into the forcing notion constructed in [146, §7] (it was gotten by composing $\mathcal{UM}$, amalgamating and direct limits), or at least not "below $\mathcal{UM}$" (in the sense of $\leq^*_1$, see 4.8) ?

* * *

Let me now mention free iterations:

In nep (also above) we can replace CS by free limit as in [142, Ch.IX]. (This was my third proof for preservation of properness (the first was like [116, §2], the second was like the one in [142, Ch.III]; this third proof looks very natural but no real reason for replacing CS iteration by it has appeared).

In particular

Proposition 4.30. *Definition of $L_{\omega_1,\omega}$–free iteration is absolute enough, so we have our φ_2 (we ignore the* nw*).*

Preservation (of reasonable properties) by CS iterations of proper forcing (or variants) seems to me a worthwhile subject. For ${}^\omega\omega$–bounding the situation is nice, "proper $+{}^\omega\omega$–bounding" is preserved, and analogous results hold for a large family of properties even, e.g., "D generates a P-point ultrafilter on ω". But some properties do not fit, though still are preserved in limits (see [176, Ch.XVIII,§3]). For example, we shall consider below the case of "$A \subseteq {}^\omega 2$ is non-null ". The simple preservation is: in the existence of generic conditions we can preserve " a given old $\eta \in {}^\omega 2$ is random over the model $N[\underset{\sim}{G}_{\mathbb{P}}]$ ".

Question 4.31. Assume that

(a) $\bar{\mathbb{Q}}$ is a CS iteration, or a $L_{\omega_1,\omega}$–free iteration,
(b) each $\mathbb{Q}_\alpha$ is proper, or a nep forcing notion,

(c) each $\mathbb{Q}_\alpha$ is "non-null for $(\mathbb{S}, \underset{\sim}{r})$–preserving"[3] (if each $\mathbb{Q}_\alpha$ is nep and $\underset{\sim}{r}$ is, e.g., random, then by [115] this is equivalent to not making old ${}^\omega 2$ null).

Does $\mathbb{P}_\alpha$ have the property from clause (c)? (so we have four versions of the question, as for clause (a) and for clause (b) we can choose the first or the second possibilities).

Assume:

$(*)$ each $\underset{\sim}{\mathbb{Q}}_i$ has a generic $\underset{\sim}{r}_i \subseteq \underset{\sim}{\alpha}_i$.

(Hence $\mathbb{P}_\alpha$ have this property as it is preserved).

We assume knowledge of free iterations (see [176, IX,§1,§2]); in short, if $\mathbb{P}_n \lessdot \mathbb{P}_{n+1}$ for $n < \omega$, let $\mathbb{P}_\omega$ be

$\{\psi :$ ψ is a sentence in the $L_{\omega_1,\omega}$ propositional calculus with the set of propositional variables $\bigcup\{\mathbb{P}_n : n < \omega\}$ such that in some forcing extension of $\mathbf{V}$ there is $G \subseteq \bigcup\{\mathbb{P}_n : n < \omega\}$ satisfying
(a) for each $n < \omega$, $G \cap \mathbb{P}_n$ is generic over $\mathbf{V}$,
(b) looking at G as assigning truth values to members of $\bigcup\{\mathbb{P}_n : n < \omega\}$, it assigns the value truth to ψ $\}$.

The order of $\mathbb{P}_\omega$ is the natural one.

We deal with an $L_{\omega_1,\omega}$–free iteration $\bar{\mathbb{Q}}$ such that

$$\Vdash_{\mathbb{P}_i} \text{“ } \underset{\sim}{\mathbb{Q}}_i \text{ is nep and } ({}^\omega 2)^{\mathbf{V}^{\mathbb{P}_i}} \text{ not null in } ({}^\omega 2)^{\mathbf{V}^{\mathbb{P}_i * \underset{\sim}{\mathbb{Q}}_i}} \text{ ”}.$$

This is quite a wide case. What does it mean for a successor $\alpha = \ell g(\bar{\mathbb{Q}})$? Say, $\alpha = 2$ so we know that in $\mathbf{V}^{\mathbb{Q}_0}$, $({}^\omega 2)^{\mathbf{V}}$ is not null, and in $\mathbf{V}^{\mathbb{Q}_0 * \underset{\sim}{\mathbb{Q}}_1}$, $({}^\omega 2)^{\mathbf{V}^{\mathbb{Q}_0}}$ is not null. But for nep forcing notions preserving the non-nullity of ${}^\omega 2$ implies preserving the non-nullity of any old non-null set by [115, §7]. There is no problem with successor stages. So now assume $\delta = \ell g(\bar{\mathbb{Q}})$ is a limit ordinal of countable cofinality.

As $\mathbb{P}_\delta$ is nep it is enough to assume

$(**)$ $p_0 \Vdash_{P_\delta}$ "$\underset{\sim}{T} \subseteq {}^{\omega >}2$ is a subtree and $\mathrm{Leb}\big(\lim(\underset{\sim}{T})\big) > 0$ ", and $\delta = \bigcup\limits_{n<\omega} i_n$, $i_n < i_{n+1}$, and without loss of generality $\underset{\sim}{T} \cap {}^{n\geq}2$ is a hereditarily countable $\mathbb{P}_{i_n}$–name,

and to find an old $\eta \in {}^\omega 2$ such that $p_0 \nVdash_{\mathbb{P}_\delta}$ " $\eta \notin \lim(\underset{\sim}{T})$ ". Now, let N be a $\bar{\mathbb{Q}}$–candidate to which $\{\underset{\sim}{T}, p_0, \langle i_n : n < \omega\rangle\}$ belongs (without loss of generality $N \prec (\mathcal{H}(\chi), \in)$, with χ large enough). Let $G_0 \subseteq \mathrm{Levy}(\aleph_0, (2^{|\mathbb{P}_\delta|})^N)$ be generic over N. Let $p_1 \in N[G_0]$ be such that $p_1 \in (\mathbb{P}_\delta)^{N[G_0]}$, $N[G_0] \models$" $p_0 \leq p_1$ and p_1 is explicitly $(N, \mathbb{P}_\delta)$–generic ". Let $G_1 = G_{1,\delta} \subseteq \mathbb{P}_\delta^{N[G_0]}$ be

[3] $\mathbb{S}$ is a nep forcing notion, $\underset{\sim}{r}$ is a hereditarily countable $\mathbb{S}$–name, so $(\mathbb{S}, \underset{\sim}{r})$ induces an ideal on ${}^\omega 2$, $\mathbb{S}$ may be random and then the ideal is the ideal of null sets.

generic over $N[G_0]$ such that $p_1 \in G_{1,\delta}$ and let $G_{1,i_n} = G_1 \cap \mathbb{P}_{i_n}^{N[G_0]}$. Let s be a random real over $N[G_0][G_1]$ (if we replace random by other nep-explicitly demand even more models), hence over N too. Clearly we can choose such $G_0, G_{1,\delta}$ in $\mathbf{V}$. So $N[s]$ is a class (= definable) of $N[G_0][G_1][s]$, and clearly $N[s]$ is a $\bar{\mathbb{Q}}$–candidate. Also, there is $s' \in \lim(\underset{\sim}{T}[G_1])$, $s' \equiv s$ (i.e., $s' \in {}^{\omega}2$ and the set $\{\ell < \omega : s'(\ell) \neq s(\ell)\}$ is finite). Let us define ψ as

$$\psi = p \ \& \bigwedge_{n<\omega} \big[s' \restriction n \in (\underset{\sim}{T} \cap {}^{n}2)[\underset{\sim}{G}_{\mathbb{P}_{i_n}}]\big]$$

(note that, by the assumption, there is $\langle \psi_{n,\eta} : n < \omega,\ \eta \in {}^{n}2\rangle$, $\psi_{n,\eta}$ an $L_{\omega_1,\omega}$–sentence for $\mathbb{P}_{i_n}$, i.e. using (countably many) variables $q \in \mathbb{P}_{i_n}$, such that

$$p_0 \Vdash \text{``}\ \eta \in \underset{\sim}{T} \cap {}^{n}2 \ \text{ iff } \ \psi_{n,\eta} \in \underset{\sim}{G}_{P_{i_n}}\ \text{''}$$

for $n < \omega$, $\eta \in {}^{n}2$, so, up to equivalence,

$$\psi = p \ \& \bigwedge_{n} \psi_{n,s'\restriction n}$$

(recall that $\underset{\sim}{T} \cap {}^{n}2$ is a $\mathbb{P}_{i_n}$–name). The problem is whether $\psi \in \mathbb{P}_\delta$. Now, $\psi \in N[s]$, so by absoluteness to show $\psi \in \mathbb{P}_\delta$ it suffices to show that this holds in $N[s]$.

So we need

$$N[s] \models \text{``}\ \psi \ \& \bigwedge_{n} \Psi_{\mathbb{P}_{i_n}} \text{ has a Boolean-valued model }\text{''},$$

where $\Psi_{\mathbb{P}_{i_n}} = (\bigwedge\{\bigvee_{q\in\mathcal{I}} q : \mathcal{I} \subseteq \mathbb{P}_{i_n} \text{ is predense }\})^{N[s]}$. But $N[G_1][G_2][s]$ is a generic extension of $N[s]$, so it is enough to prove it there; so there is no problem. In fact for the case of "non-null" the answer is yes (for CS iterations use [176, Ch.XVIII,3.8, pp 912–916]).

On the other hand, in full generality the answer to 4.31 is no; note that life is harder if we want to preserve positiveness for $I_{(\mathbb{Q},\underset{\sim}{\eta})}$, where $\mathbb{Q}$ is (nep but) not c.c.c., on this and more see [116].

Another possible direction is

Problem 4.32. (1) Is there an interesting theory of nicely definable forcing notions in $\mathcal{H}(\theta)$ or $\mathcal{H}_{<\theta}(\sigma)$?

(2) Similarly for the theory of iterations (see later).

(3) Generalize [92], replacing $\aleph_0$ by κ. Probably require $\kappa = \kappa^{<\kappa} > \aleph_0$ or even that κ is strongly inaccessible.

E.g., let for simplicity κ be strongly inaccessible, D^* a normal filter on κ. For a cardinal $\theta < \kappa$ let a θ–creature $\mathfrak{c}$ consist of $(R^{\mathfrak{c}}, \mathrm{pos}^{\mathfrak{c}}, \mathrm{val}^{\mathfrak{c}})$, where: $R^{\mathfrak{c}}$ is a θ^+–complete forcing notion, $\mathrm{pos}^{\mathfrak{c}}$ is a non-empty set, $\mathrm{val}^{\mathfrak{c}}$ is a function from $R^{\mathfrak{c}}$ to $\mathcal{P}(\mathrm{pos}^{\mathfrak{c}}) \setminus \{\emptyset\}$ such that $R^{\mathfrak{c}} \models$"$x \leq y$" $\Rightarrow$ $\mathrm{val}^{\mathfrak{c}}(y) \subseteq \mathrm{val}^{\mathfrak{c}}(x)$.

A κ–normed tree is $(T^*, \bar{\mathfrak{c}}, \bar{\theta})$, where $T^* \subseteq {}^{\kappa>}\mathrm{Ord}$ is a subtree, usually closed under increasing sequences of length $< \kappa$, $\bar{\mathfrak{c}} = \langle \mathfrak{c}_\eta : \eta \in T^* \rangle$, $\mathfrak{c}_\eta$ is a θ_η–creature, $\bar{\theta} = \langle \theta_\eta : \eta \in T^* \rangle$ and $\mathrm{pos}^{\mathfrak{c}_\eta} = \mathrm{Suc}_{T^*}(\eta)$.

We can consider

$$\begin{array}{ll} \mathbb{Q}^1_{(T^*,\bar{\mathfrak{c}}^*,\bar{\theta}),D^*} = & \\ \mathbb{Q}^1 = \big\{(T, \bar{r}) : \text{(a)} & T \text{ is a subtree of } T^*,\ (<\kappa)\text{-closed}, \\ \quad\quad\quad\quad\quad\ \text{(b)} & \bar{r} = \langle r_\eta : \eta \in \mathrm{sp}(T)) \rangle,\ r_\eta \in R^{\mathfrak{c}_\eta} \text{ and } \mathrm{val}^{\mathfrak{c}_\eta}(r_\eta) = \mathrm{Suc}_T(\eta), \\ \quad\quad\quad\quad\quad\ \text{(c)} & \text{for every } \eta \in \lim_\kappa(T), \text{ we have} \\ & \{\zeta{<}\kappa : \eta \restriction \zeta \in \mathrm{sp}(T)\} \in D^* \quad \big\} \end{array}$$

with the natural order, where $\mathrm{sp}(T) = \{\eta \in T : (\exists^{\geq 2} x)[\eta^\frown\langle x \rangle \in T]\}$. Trivially, forcing with $\mathbb{Q}^1$ adds neither bounded subsets of κ nor sequences of ordinals of length $< \kappa$. For this forcing notion, if $T^*_\alpha =: \{\eta \in T^* : \ell g(\eta) = \alpha\}$ has cardinality $< \kappa$ for each $\alpha < \kappa$, and $(\forall \eta \in \lim_\kappa(T^*))[\{\zeta < \kappa : \theta_{\eta\restriction\zeta} > \zeta\} \in D^*]$ (or there is $A \in D^*$ such that $\zeta \in A$ & $\eta \in T$ & $\ell g(\eta) = \zeta \quad \Rightarrow \theta_\eta > \zeta$), *then* forcing with $\mathbb{Q}^1_{(T^*,\bar{\mathfrak{c}}^*,\bar{\theta}),D^*}$ does not collapse κ^+. Also, this forcing is $(<\kappa)$–complete ${}^\kappa\kappa$–bounding and $\mathbb{Q}^1$–names of $\underaccent{\tilde}{\tau} : \kappa \longrightarrow \mathrm{Ord}$ can be read continuously on T for a dense set of $(T, \bar{r}) \in \mathbb{Q}^1$. Moreover, this property is preserved by $(\leq \kappa)$–support iterations (by [110, §1], or directly).

We can allow gluing (i.e., putting together with η many nodes above, creating a new forcing notion, i.e. creatures, see [92, 3.3(2),§6.3]).

We may consider $\bar{\mathfrak{c}} = \langle \bar{\mathfrak{c}}_\zeta : \zeta < \kappa \rangle$, $\mathfrak{c}_\zeta$ a θ_ζ–creature and for some club E of κ, $\zeta \in E$ & $\xi \geq \zeta \quad \Rightarrow \quad \theta_\xi > \zeta$, and consider

$$\mathbb{Q}^2_{\bar{\mathfrak{c}}} = \Big\{(w, \bar{r}) : \text{for some } \zeta,\ w \in \prod_{\varepsilon<\zeta} \mathrm{pos}^{\mathfrak{c}_\varepsilon},\ \bar{r} = \langle r_\varepsilon : \varepsilon \in [\zeta, \kappa] \rangle,\ r_\varepsilon \in R^{\mathfrak{c}_\varepsilon}\Big\}$$

with the natural order. If $\zeta < \kappa \ \Rightarrow\ |R^{\mathfrak{c}_\zeta}| + |\mathrm{pos}^{\mathfrak{c}_\zeta}| < \kappa$ and

(α) $\kappa = \sup\{\zeta : \text{for } \xi \in [\zeta, \kappa),\ R^{\mathfrak{c}_\xi} \text{ is } |\prod_{\varepsilon<\zeta} \mathrm{pos}^{\mathfrak{c}_\varepsilon}|^+\text{–complete}\}$, or

(β) $\diamondsuit_\kappa$

then forcing does not collapse κ^+ and is $(<\kappa)$-complete but the "read continuously" is problematic for case (β).

Again, we may allow the forcing notion to be omittory (see [92, 2.1.1]), i.e., allow

$$\mathbb{Q}^3_{\bar{\mathfrak{c}}} = \{(w, \bar{r}) : \begin{array}{l} \text{for some bounded } u \subseteq \kappa \text{ and } A \in D, \\ w \in \prod_{\varepsilon \in u} \mathrm{pos}^{\mathfrak{c}_\varepsilon}, \text{ and } \bar{r} = \langle r_\alpha : \alpha \in A \rangle\} \end{array}$$

and/or a combination of creatures (i.e., the function Σ) and/or we may allow memory (=the object which $\mathfrak{c}_\alpha$ produces depends on the earlier $\mathfrak{c}_\beta$'s) and/or gluing.

Note that natural nice enough $\mathfrak{c}$'s make us regain "read continuously" and its parallels.

All those generalize naturally. But if in $\mathbb{Q}^1_{(T^*,\bar{\mathfrak{c}},\theta),D^*}$ we allow D^* to be the co-bounded filter, the proofs fail. There is more to be said and done.

Of course, we can now carry out generalizations of various independence results on cardinal invariants, e.g. between variants of (f,g)–bounding and slaloms (= corsets), see [162], [176, Ch.VI,§2], Goldstern and Shelah [48], Rosłanowski and Shelah [88]. We may consider in the tree version for every limit $\delta \in S$ ($S \subseteq \kappa$ is a stationary set) to omit tops for one or just $\leq |\delta|$ branches in a condition $p = (T, \bar{r})$, provided for each $\eta \in T$ we have $\langle |\{\rho : \eta \vartriangleleft \rho \in T,\ \ell g(\rho) = \ell g(\eta) + \alpha\}| : \alpha < \kappa\rangle$ goes to κ.

Recall that whereas Cohen forcing and many others have κ-parallels, not so with random real forcing.

Problem 4.33. (1) Prove that there is no reasonable parallel, say no κ^+–c.c. forcing notion such that any new member of ${}^\kappa\kappa$ is bounded by some old one.

(2) Similarly as far as the parallel of [168] is concerned.

Concerning nep forcing notions, we may define when "$\mathbb{Q}$ is semi-nep" as follows: for some nep forcing notion $\mathbb{Q}^*$ we have

(a) $\mathbb{Q} \subseteq \mathbb{Q}^*$ (that is the set of elements of $\mathbb{Q}$ is a subset of the set of elements of $\mathbb{Q}^*$ and the order of $\mathbb{Q}$ is the order of $\mathbb{Q}^*$ restricted to $\mathbb{Q}$),

(b) if N is a $\mathbb{Q}$–candidate, $\mathbb{S} \in N$, $\mathbb{S}^N = \mathbb{Q}^* \cap N$, $p \in \mathbb{S}$, $p \in N$ then for some q, $p \leq q$, q is $(N, \mathbb{S})$–generic (in an explicit way).

5. To prove or to force, this is the question

On many things we confidently "know" that they are independent, "just" a proof is needed (for many others we know that forcing will not help by absoluteness). The rest we may think are decidable, but actually we are not sure. More fall in the middle; our intuitions do not give an answer or worse: they give an answer which oscillates in time. In the mid-seventies, I was interested in (see Abraham and Shelah [3]):

Question 5.1. Is there an Aronszajn tree T and a function $c : T \longrightarrow \{\text{red, green}\}$ (or more colours) such that for any uncountable set $A \subseteq T$, all colors appear on the set $\{\eta \cap \nu : \eta, \nu \in A \text{ are incomparable}\}$, where $\eta \cap \nu$ is the maximal common lower bound?

Why? Baumgartner [12] proved that among uncountable real linear orders (e.g., with density $\aleph_0$) there may be a minimal one under embeddability. This follows from

$$\text{CON(if } A, B \in [\mathbb{R}]^{\aleph_1} \text{ are } \aleph_1\text{–dense, then they are isomorphic)}.$$

So we may ask: among uncountable linear orders, can there be finitely many such that any other embeds one of them? (call such a family a base).

A base should contain a real order and ω_1 and ω_1^*. Any linear order into which none of them embeds is necessarily a Specker order (= take an Aronszajn tree, order it lexicographically).

You may ask: Can there be a "minimal" order among those? But there cannot. It is known ([134], answering a question of Countryman) that there is a Specker order L such that the product $L \times L$ (with the product order) is the union of countably many chains (comes from a very special Aronszajn tree). Hence L and L^* (its inverse) embed no common uncountable chain. Now, consistently any two candidates for L are isomorphic or anti-isomorphic. So, if we can also put the additional forcing together, then we will have a candidate for a basis which seems extremely likely. We just need to start with a Specker order, i.e., an Aronszajn tree with a lexicographic order, under the circumstances (okay to force a little), without loss of generality with any node having two immediate successors. Look at it as an Aronszajn tree, and remember Abraham and Shelah [3]. So without loss of generality, on a club the tree is isomorphic to the one for L, and let $\mathrm{Dom}(c) = T$,

$$c(\eta) = \begin{cases} \text{green} & \text{if the two linear orders make the same} \\ & \quad\text{decision about the two immediate successors of } \eta, \\ \text{red} & \text{otherwise,} \end{cases}$$

and so the problem 5.1 arises.

* * *

A common property of some of the problems discussed below (5.2, 5.3, 5.4) is a difference between asking about $S^\lambda_\theta = \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ and asking about a stationary $S \subseteq S^\lambda_\theta$ such that $S^\lambda_\theta \setminus S$ is stationary too; a difference of which I became aware in [136] (e.g. $\diamondsuit_{S_1}$ & $\neg\diamondsuit_{S_2}$ is possible for disjoint stationary subsets of ω_1) after much agony.

Question 5.2 (GCH)**.** If μ is singular, do we have $\diamondsuit_{S^{\mu^+}_{\mathrm{cf}(\mu)}}$? (Those are the only cases left.)

Similarly for inaccessibles, see [118].

If we try to force consistency of the negation, note that (for μ strong limit singular)

$$2^\mu = \mu^+ + \Box_{\mu^+} \quad \Rightarrow \quad \diamondsuit_{S^{\mu^+}_{\mathrm{cf}(\mu)}} \qquad\qquad \text{when } \mathrm{cf}(\mu) > \aleph_0$$

(see [147, 3.2,p.1030]). So we need large cardinals (hardly surprising for successors of singular cardinals). See more in Džamonja and Shelah [31]).

Probably it is wiser to try to force this for "large" μ. Changing the cofinality of a supercompact cardinal μ to $\aleph_0$, where "μ is prepared" is not helpful as after the forcing some old $S^{\mu^+}_{\aleph_0}$ is added to the old $S^{\mu^+}_{\mu}$ to make the new $S^{\mu^+}_{\aleph_0}$.

If $\mathbf{V} \models \neg\diamondsuit_{S^{\mu^+}_{\mu}}$ (which may be forced but you need enough indestructibility of measure), you get $\neg\diamondsuit_S$ for some non-reflecting stationary $S \subseteq S^{\mu^+}_{\aleph_0}$, but we have it "cheaply" by forcing (say, starting with $\mathbf{L}$; see [147], better [119]). So maybe it is wiser to start with μ of cofinality $\aleph_0$.

Let μ be a limit of large cardinals and try to add enough subsets to μ to "kill" $\diamondsuit^{\mu^+}_{\mathrm{cf}(\mu)}$. Our knowledge of such forcing for such cases is limited at the present. But ZFC + GCH still give an approximation (see more [119]):

(∗) if $S \subseteq S^{\mu^+}_{\mathrm{cf}(\mu)}$ is stationary then for some $\langle\langle\alpha_{\delta,i} : i < \mathrm{cf}(\mu)\rangle : \delta \in S\rangle$ we have:
 (a) $\alpha_{\delta,i}$ is increasing with limit δ,
 (b) if $\theta < \mu$, $f : \mu^+ \longrightarrow \theta$, *then*

$$(\exists \text{ stationarily many } \delta \in S)(\forall i < \mathrm{cf}(\mu))(f(\alpha_{\delta,2i}) = f(\alpha_{\delta,2i+1})).$$

Note: Having two equal values inside a group calls for division (or subtraction) so that we get a known value. So, if we are trying to guess homomorphisms from a group G with $|G| = \mu^+$ to H, $|H| = \theta < \mu$, $G = \bigcup\limits_{i<\mu^+} G_i$, G_i strictly increasing continuous, $|G_i| < \mu^+$, and $S \subseteq S^{\mu^+}_{\mathrm{cf}(\mu)}$ is stationary, then we can find $\bar\eta = \langle\langle g_{\delta,i} : i < \mathrm{cf}(\mu)\rangle : \delta \in S\rangle$ such that $g_{\delta,i} \in G_{\alpha_{\delta,2i+2}} \setminus G_{\alpha_{\delta,2i}}$ and for every homomorphism $h : G \longrightarrow H$ there are stationarily many δ such that $(\bigwedge\limits_i h(g_{\delta,i}) = e_H)$ (relevant to Whitehead groups). Without loss of generality, $|G_{i+1} \setminus G_i| = \mu$, $|G_0| = \mu$, the universe of G_i is $\mu \times (1 + i)$ (or $g_i \in G_{i+1} \setminus G_i$ uses the question on $f : f(i) = f(g_i)$). See [119].

Question: Can we have something similar for any sequence $\bar\eta$?

Answer: No. We have quite a bit of freedom (e.g., demand $\alpha_{\gamma,i} \in A^*$, $A^* \in [\mu^+]^{\mu^+}$ fixed) but certainly not for any.

In fact, for any stationary non-reflecting set $S \subseteq S^{\mu^+}_{\mathrm{cf}(\mu)}$ and any sequence $\bar\eta = \langle\langle\alpha_{\delta,i} : i < \mathrm{cf}(\mu)\rangle : \delta \in S\rangle$ with $\alpha_{\delta,i}$ increasing with limit δ we can force:

for every $\langle h_\delta \in {}^{\mathrm{cf}(\mu)}\theta : \delta \in S\rangle$, $\theta < \mu$, there is $h \in {}^{(\mu^+)}\theta$ such that

$$(\forall \delta \in S)(\forall^\infty i)(h(\alpha_{\delta,i}) = h_\delta(i)).$$

This is a strong negation of the earlier statements (see [147], [119]). A related ZFC result is that for a singular cardinal μ, the restriction of the club filter $\mathcal{D}_{\mu^+} \restriction S^{\mu^+}_{\mathrm{cf}(\mu)}$ is not μ^{++}–saturated (see Gitik and Shelah [47]).

A well known problem is

Question 5.3. (1) For a regular cardinal $\lambda > \aleph_2$, can $\mathcal{D}_{\lambda^+} \restriction S^{\lambda^+}_\lambda$ be λ^{++}–saturated?

(2) Similarly just adding the assumption GCH.

* * *

Many "club guessings" are true (see [98]), but I have looked in vain several times on:

Question 5.4. Let λ be a regular uncountable cardinal. Can we find a sequence $\langle\langle\alpha_{\delta,i} : i < \lambda\rangle : \delta \in S^{\lambda^+}_\lambda\rangle$ such that $\alpha_{\delta,i}$ are increasing continuous in i with limit δ, and for every club E of λ^+

$$(\exists \text{ stationarily many } \delta)(\exists \text{ stationarily many } i)(\alpha_{\delta,i+1}, \alpha_{\delta,i+2} \in E),$$

or other variants (just $\alpha_{\delta,i+1} \in E$ is provable, $\alpha_{\delta,i} \in E$ is trivial under the circumstances).

This is interesting even under GCH, particularly as by Kojman and Shelah [70] (essentially) we get from it that there is a λ^+-Souslin tree.

We may think instead of trying to prove for $S, S_1 \subseteq S^{\lambda^+}_\lambda$ being stationary disjoint, that we can force the failure for S (with GCH). This works (see [118]) but $S = S^{\lambda^+}_\lambda$ is harder. The present forcing proofs fail, but also using "first counterexample" fails. We may consider proving: GCH $\Rightarrow$ $\neg$GSH (where GSH is Generalized Souslin Hypothesis: for no uncountable regular λ there is a λ–Souslin tree). Let us look at two successor cases λ^+, λ^{++} (λ regular). How can this help? Assume that there is no λ^{++}–Souslin tree and GCH holds. It follows that every stationary $S \subseteq S^{\lambda^{++}}_{\leq\lambda}$ reflects in $S^{\lambda^{++}}_{\lambda^+}$ (see Gregory [50]), moreover it is enough to assume just that there is no $(<\lambda^+)$–complete Souslin tree, by Kojman and Shelah [70]. Hence

($\otimes$) there is $S^* \subseteq \lambda^+$ such that:

(a) (square on S^*)
$\bar{C} = \langle C_\alpha : \alpha \in S^*\rangle$, $C_\alpha \subseteq \alpha$ closed, $\mathrm{otp}(C_\alpha) \leq \lambda$,
if α is limit then $\sup(C_\alpha) = \alpha$, and
$\beta \in C_\alpha \ \Rightarrow\ C_\beta = \beta \cap C_\alpha$,

(b) $S^* \cap S^{\lambda^+}_\lambda$ is stationary.

(Why? This follows by [159, §4] which says that, e.g., $S^{\lambda^{++}}_{<\lambda^+}$ is the union of λ^+ sets with squares.) This looks as if it should help, but I did not find yet how.

Džamonja and Shelah [31] introduced

Definition 5.5. We say that λ strongly reflects at θ if $\theta < \lambda$ are regular uncountable cardinals and for some $F : \lambda \longrightarrow \theta$ for every $\delta \in S^\lambda_\theta$ for some club C of δ, $F \restriction \delta$ is strictly increasing (equivalently, is one–to–one).

This helps to prove variants of $\clubsuit$ on the critical stationary subset of μ^+ when μ is singular, i.e., on $S^{\mu^+}_{\mathrm{cf}(\mu)}$, see Džamonja and Shelah [31], [30], and on independence results Cummings, Džamonja and Shelah [22] and Džamonja and Shelah [29].

Question 5.6. Can we get something parallel when $\mathrm{cf}(\mu) = \aleph_0$?

Question 5.7. Can we prove that for some strong limit singular cardinal μ and a regular cardinal $\theta < \mu$ we have $\clubsuit_S$, where $S = S^{\mu^+}_\theta$?

* * *

On $I[\lambda]$ see [138], [152], [163]. We know that e.g.

$$\{\delta < \aleph_{\omega+1} : \mathrm{cf}(\delta) = \aleph_1\} \notin I[\aleph_{\omega+1}]$$

is consistent with GCH, but

Problem 5.8. (1) Can $\{\delta < \aleph_{\omega+1} : \mathrm{cf}(\delta) = \aleph_2\} \notin I[\aleph_{\omega+1}]$?
(2) Can $\{\delta < (2^{\aleph_0})^{+\omega+1} : \mathrm{cf}(\delta) = (2^{\aleph_0})^+\} \notin I[(2^{\aleph_0})^{+\omega+1}]$?

Now [159, §4], dealing with successors of regulars, raises the question

Question 5.9. (1) Let λ be inaccessible $> \aleph_0$. Is I^{sq}_λ (see Definition 5.10 below) non-trivial, i.e., does it include stationary sets of cofinality $\sigma \in (\aleph_0, \lambda) \cap \mathrm{Reg}$? Does it include such S which is large in some sense (e.g., for every such σ)?
(2) Similarly for successor of singular.

Definition 5.10. For a regular cardinal $\lambda > \aleph_0$ let

$$I^{\mathrm{sq}}_\lambda = \Big\{A \subseteq \lambda : \begin{array}{l}\text{for some partial square } \bar{C} = \langle C_\delta : \delta \in S^1\rangle,\\ S^1 \subseteq \lambda \text{ and the set } A \setminus S^1 \text{ is not stationary in } \lambda\end{array}\Big\}.$$

* * *

Definition 5.11. A linear order I is μ–entangled if: for any pairwise distinct $t_{i,\ell} \in I$, $i < \mu$, $\ell < n$, for any $w \subseteq \{0, \dots, n-1\}$ there are $i < j$ such that

$$t_{i,\ell} < t_{j,\ell} \quad \Leftrightarrow \quad \ell \in w.$$

If $|I| = \mu$ then we omit μ.

Question 5.12. Is there an entangled linear order of cardinality λ^+, where $\lambda = \lambda^{\aleph_0}$?

A "yes" answer will solve a problem of Monk [84] on the spread of ultraproducts of Boolean Algebras; see [174].

With the help of pcf we can build entangled linear orders in λ^+ for many λ which means: provably for a proper class of λ's; see [166].

The interesting phenomenon is: from instances of GCH, we can give a positive answer, but also from strong negations of GCH we may get a positive answer. On the one hand, if $\mu^{\aleph_0} = \mu$, $2^\mu > \aleph_{\mu^{+4}}$ we have: for many $\delta < \mu^{+4}$ we get a positive answer to 5.12 with $\lambda = \aleph_\delta$. On the other hand, from $\mu = \mu^{\aleph_0}$, $2^\mu < \mu^{+\omega}$ we can also prove a positive answer.

In fact, in the remaining case there are quite heavy restrictions on pcf. A typical universe with negative answer to 5.12 (we think that it) will satisfy: for a strong limit cardinal μ, $2^\mu = \mu^{+\omega+1}$, and for $\mathfrak{a} \subseteq \mathrm{Reg} \cap \mu$, $\mu = \sup(\mathfrak{a})$, $\mathrm{pcf}(\mathfrak{a}) \setminus \mu$ essentially concentrates on $\mu^{+\omega+1}$ (say $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle$ increasing continuous, if $\mathfrak{a}$ is disjoint from $\{\mu_i^{+n} : i < \mathrm{cf}(\mu),\ 0 < n < \omega\}$ then $\emptyset = \mathrm{pcf}(\mathfrak{a}) \cap (\mu, \mu^{+\omega})$.) See [174] and §9 here.

Maybe our knowledge of forcing will advance. Note that we need not only to have pcf structure as indicated, but also to take care of the non-pcf phenomena as well for constructing entangled linear order as in 5.12.

Considering a ZFC proof of existence, it seems most reasonable to assume toward a contradiction that the answer is no and consider strong limit singular μ of uncountable cofinality. So we know $2^\mu < \aleph_{\mu^{+4}}$ and being more careful even $2^\mu < \aleph_{\mu^+}$. Let $\gamma(*) = \min\{\gamma : 2^{(\mu^{+\gamma})} > 2^\mu\}$, so necessarily $\gamma(*)$ is a successor ordinal, say $\gamma(*) = \beta(*)+1$. Let $\lambda =: \mu^{+\beta(*)}$. We may consider trying to construct an entangled linear order of cardinality $(2^\mu)^+$, using the weak diamond on $\lambda^+ = \mu^{+\beta(*)+1}$. Moreover, we know that there are trees T with λ^+ levels and $\leq \lambda^+$ nodes and at least $(2^\mu)^+$ many λ^+–branches (even ${}^{\lambda^+}2 = \bigcup\limits_{\zeta<2^\mu} \lim_{\lambda^+}(T_\zeta)$ for some subtrees T_ζ of ${}^{\lambda^+>}2$, $|T_\zeta| = \lambda^+$ above).

Moreover, a relative of $\diamondsuit^*_{\lambda^+}$ holds. All this seems reasonably promising, but has failed so far to solve the problem.

I have also considered to repeat the proof of the weak diamond for λ^+ to try to show that a tree with infinite splitting in the above representation is necessary.

Problem 5.13. Can we prove that a stronger version of the weak diamond holds for some λ? E.g., a version with more than two colours and/or fixing the cofinality. We shall be glad to get even just the definable weak diamond. See [150], [102, §3], [113] and [107].

* * *

Our ignorance about such problems may well come from our gaps in forcing theory.

A major problem (more exactly a series of problems) is

Problem 5.14. (1) Can we have a reasonable theory of iterations (and/or forcing axioms) for $(<\lambda)$-complete forcing notions ($\lambda = \lambda^{<\lambda}$)?

(2) Similarly for forcing notions not changing cofinalities of cardinals $< \lambda$?
(3) Similarly for forcing notions preserving μ^+ and not adding bounded subsets to μ, μ a strong limit singular cardinal?

See some recent information on the first in [118], [110], [119], and even much less on the second [176, Ch.XIV], and on the third Mekler and Shelah [80], Džamonja and Shelah [28].

Though much was done on forcing for the function 2^λ and some specific problems, our flexibility is not as good as for $2^{\aleph_0}$ in forcing theory.

Particularly intriguing are solutions where we know some λ exists but do not know which. The dual problem is iterated forcing of length Ord (class forcing); now for such iteration it is particularly hard to control in the neighborhood of singulars.

Problem 5.15. Prove the consistency of: for every λ (or regular λ) a suitable forcing axiom holds.

Relevant is "GCH fails everywhere" (see Foreman and Woodin [36]). Now Cummings and Shelah [24], [25] is a modest try and 1.22 is relevant.

Specific well known targets are

Problem 5.16. Is GSH consistent? (GSH is the generalized Souslin hypothesis: for every regular uncountable λ there is no λ-Souslin tree.)

Problem 5.17. Is it consistent that for no regular $\lambda > \aleph_1$ do we have a λ-Aronszajn tree (see Abraham [1], Cummings and Foreman [23]).

A relevant problem is 6.4.

* * *

I have found partition theorems on trees with ω levels very useful and interesting (see Rubin and Shelah [93], and [144], [142], [176, X, XI, XV,2.6]). In [156, 13,p.1453] and [137, Ch.VIII,§1] trying to prove a theorem on the number of non-isomorphic models of a pseudo elementary class we arrived at the following problem [without loss of generality, try with $2^\lambda = \lambda^+$ and see [127], by absoluteness]:

Question 5.18. Assume $m(*) < \omega$, $2^{\lambda_n} < \lambda_{n+1}$, M is a model with vocabulary of cardinality θ, $\theta + \mu < \lambda_0$, $a^i_\eta \in M$ for $i < \mu$, $\eta \in T = \bigcup_n \prod_{\ell<n} \lambda_\ell$. Can we find a strictly increasing function $h : \omega \longrightarrow \omega$ and one-to-one functions $f^i_n : \prod_{k<n} \lambda_k \longrightarrow \prod_{k<h(n)} \lambda_k$ such that

(a) for $n < m$, $\eta \in \prod_{\ell<m} \lambda_\ell$ we have $f^i_n(\eta \restriction \ell) = (f^i_m(\eta)) \restriction h(\ell)$,

(b) for $n < \omega$, $m(*) < \omega$, $i_0 < \ldots < i_{m(*)-1} < \mu$ and $\eta_\ell, \nu_\ell \in \prod_{k<n} \lambda_k$ for $\ell < m(*)$, the tuples $\langle a^{i_0}_{\eta_0}, a^{i_1}_{\eta_1}, \ldots, a^{i_{m(*)-1}}_{\eta_{m(*)-1}} \rangle$, $\langle a^{i_0}_{\nu_0}, a^{i_1}_{\nu_1}, \ldots, a^{i_{m(*)-1}}_{\nu_{m(*)-1}} \rangle$ realize the same type in M ?

(If on λ_n there is a "large ideal" (see [171]) life is easier, see [156].)

6. Boolean Algebras and iterated forcing

We turn to Boolean Algebras. Monk has made extensive lists of problems about Boolean Algebras (which inspired more than few works of mine). His problems mostly go systematically over all possible relations; our perspective is somewhat different.

Among my results on Boolean Algebras I like 6.1 stated below (see [151]), but the result did not draw much attention though the paper was noticed (see Bonnet and Monk [17], Juhász [63]).

Theorem 6.1. *If B is a Boolean Algebra of cardinality $\geq \beth_\omega$ and $\lambda = \mathrm{Id}(B)$ (the number of ideals of the Boolean Algebra)* then $\lambda = \lambda^{\beth_\omega}$.
(We can instead of ideals of Boolean Algebras speak about open subsets of a compact Hausdorff topology, and we can replace $\beth_\omega$ by any singular strong limit).

So we are left with

Question 6.2. Is it true that for any large enough Boolean Algebra B we have $\mathrm{Id}(B) = \mathrm{Id}(B)^{<\theta}$ when, e.g., $\theta = \log_2(|B|)$, or at least for some constant n, $\theta = \min\{\mu : \beth_n(\mu) \geq |B|\}$?
(Similarly for compact spaces).

By [151], for every B there is such n. Of course, in non-specially constructed universes the answer is yes. If you like to try consistency, you have to use the phenomena proved consistent in Gitik and Shelah [45].

On the other hand, a ZFC proof may go in a different way than [151]. Related (see Juhász [62]) is

Question 6.3. What can be the number of open sets of a T_2 topology? T_3 topology? One with clopen basis?

It seems interesting to consider the following

Problem 6.4. Is there a class of cardinals λ (or just two) such that there is a (λ^+, λ)–thin tall superatomic Boolean Algebra B (i.e., $|B| = \lambda^+$, B is superatomic and for every $\alpha < \lambda^+$, B has $\leq \lambda$ atoms of order α), provably in ZFC?

It is well known that if $\lambda = \lambda^{<\lambda}$ then there is a (λ^+, λ)–thin tall superatomic Boolean Algebra, so for $\lambda = \aleph_0$ there is one, so for negative consistency we need "GCH fails everywhere or at least for every large enough λ". Also note that trivially if there is a λ^+-tree (i.e., one with λ^+ levels each of cardinality $\leq \lambda$), then there is such (λ^+, λ)–thin tall superatomic Boolean Algebra.

The point is that for several problems in Monk [84]: Problems 72, 74, 75 and ZFC versions of Problems 73, 77, 78, 79 (all solved in Roslanowski and Shelah [87] in the original version, i.e. showing consistency) there is no point to try to get positive answers as long as we do not know it for 6.4.

Also for several problems of [84] (Problems 49, 57, 58, 61, 63, 87) there is no point to try to get consistency of non-existence as long as we have not proved the consistency of the GSH (generalized Souslin hypothesis) which says there are no λ^+-Souslin trees or there is no λ-Souslin tree for $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. For some others this is not provable, but it still seems very advisable to wait for the resolution of GSH.

Problem 6.5. Usually the question on cardinal invariants $\mathbf{inv}_1$, $\mathbf{inv}_2$ is "do we always have $\mathbf{inv}_1(B) \leq \mathbf{inv}_2(B)$?", or "do we always have $2^{\mathbf{inv}_1(B)} \geq \mathbf{inv}_2(B)$?" But maybe there are relations like $\mathbf{inv}_1(B) \leq (\mathbf{inv}_2(B))^{+n}$ for some fixed $n < \omega$ (or for ω). A particularly suspicious case is $|B| \leq (\mathrm{irr}_n(B))^+$, where $n \in [2, \omega]$ and

$$\mathrm{irr}_n(B) = \sup\big\{|X| : \begin{array}{l} X \subseteq B \text{ is } n\text{-irredundant which means:} \\ \text{if } m < 1+n \text{ and } a_0, \ldots, a_m \in X \text{ are pairwise distinct} \\ \text{then } a_0 \text{ is not in the subalgebra generated by} \\ \{a_1, \ldots, a_m\}\big\}. \end{array}$$

(If $n = \omega$ we may omit it.)

I think that for $n = \omega$ the case $|B| = \aleph_2$ had appeared in an old list of Monk, but not in [84]. We may also ask $|B| \leq (\mathrm{irr}_n(B))^{+m}$ for $n, m \in [2, \omega]$.

Of course, the open case is when, say, $|B| = \lambda^{++}$, $2^\lambda \geq \lambda^{++}$. Thinking about this problem, I was sure the answer is consistently no (consistently yes is easy, even in the Easton model). Moreover, I feel I know how to do it: let $\lambda = \lambda^{<\lambda}$, $\lambda^+ < \theta < \mu$, $\mathfrak{B}$ be a suitable algebra on μ with $\leq \lambda$ functions. A member p of the forcing notion $\mathbb{P}$ consists of $w^p \in [\mu]^{<\lambda}$ and a Boolean algebra B^p generated by $\{x_i : i \in w^p\}$ but such that

> if $B^p \models$" $x_{i_0} = \sigma(x_{i_1}, \ldots, x_{i_n})$ " for ordinals $i_0, \ldots, i_n \in w^p$
> and a Boolean term σ
> then $i_0 \in c\ell_{\mathfrak{B}}(\{i_1, \ldots, i_n\})$ and possibly more

(the order of $\mathbb{P}$ is the natural one). This is to reconcile the demand "the Boolean algebra has cardinality μ, so without loss of generality we have to ask $i < j \Rightarrow x_i \neq x_j$" and the λ^+-c.c. Of course, $\underset{\sim}{B} = \bigcup\{B^p : p \in \underset{\sim}{G}_{\mathbb{P}}\}$.

Note: if $\lambda = \aleph_0$ we have more freedom. (The expected proof goes: if

$$p \Vdash \text{`` } \underset{\sim}{X} \subseteq \underset{\sim}{B},\ \underset{\sim}{X} = \{\underset{\sim}{y}_i : i < \theta\} \text{ with no repetition exemplifies } \mathrm{irr}(\underset{\sim}{B})^+ \geq |B| \text{ ''}$$

then we can find p_i, $p \leq p_i \in \mathbb{P}$, $p_i \Vdash$“ $y_i = \sigma_i(x_{\alpha(i,0)}, \ldots, x_{\alpha(i,n_i)})$ ”. Hence, if $(\forall \alpha < \theta)(|\alpha|^{<\lambda} < \theta = \mathrm{cf}(\theta))$, without loss of generality, $\sigma_i = \sigma$, $n_i = n^*$ and $\langle (p_i, \langle \alpha(i,\ell) : \ell < n^* \rangle) : i < \theta \rangle$ forms a Δ-system (moreover indiscernible as a sequence of sequences of ordinals of length $< \lambda$). Let $\{\alpha(i,\ell) : \ell < n^*\} \subseteq w^p = w^* \cup \{\gamma_{i,\zeta} : \zeta < \zeta^*\}$, $\zeta^* < \lambda$, etc, and we have to find $n < \omega_1$, $i_0 < \ldots < i_n < \theta$ and q above $p_{i_0}, \ldots, p_{i_n}$ such that $B^q \models$“ $y_{i_0} = \sigma^*(y_{i_1}, \ldots, y_{i_n})$ ”. So it is natural to demand $\zeta < \zeta^* \ \Rightarrow\ \gamma_{i_0,\zeta} \in c\ell_{\mathfrak{B}}\{\gamma_{i_1,\zeta}, \ldots, \gamma_{i_n,\zeta}\}$. So if $\lambda = \aleph_0$ we may use $n > |w^{p_i}|$. But this approach has not converged to a proof.)

So (see Monk [84, Problem 28])

Question 6.6. (1) Is there a class of (or just one) λ such that for some Boolean Algebra B of cardinality λ^+ we have $\mathrm{irr}(B) = \lambda$?
(2) Similarly for $\mathrm{irr}_n(B)$.

Colouring theorems (e.g. [172]) are not enough for a construction.

Question 6.7. (1) For which pairs (λ, θ) of cardinals $\lambda \geq \theta$ is there a superatomic Boolean Algebra with $> \lambda$ elements, λ atoms and every $f \in \mathrm{Aut}(B)$ moves $< \theta$ atoms?
(That is $|\{x : B \models$“ x an atom and $f(x) \neq x$ ”$\}| < \theta$).
(2) In particular, is it true that for some θ, for a proper class of λ's there is such Boolean Algebra?
(3) Replace “automorphism” by “one–to-one endomorphism”.

See some results in [104, §1,§2] for θ strong limit singular. (It may be interesting to try: with n depending on the arity of the term as in [104].)

Concerning attainment in ZFC:

Question 6.8. (1) Can we show the distinction made between the attainments of variants of hL (and hd), in a semi-ZFC way? That is, in Rosłanowski and Shelah [86] such examples are forced. “Semi ZFC” means can we prove such examples exist after adding to ZFC only restrictions on cardinal arithmetic?
(2) Similarly for other consistency results. (Well, preferably of low consistency strength).

In view of [103], [127] it is reasonable to consider

Problem 6.9. In 6.5, 6.6 replace $\mathrm{irr}_n(B)$ by $\mathrm{irr}_n(\bar{a}, B)$ (this is true for other cardinal invariants as well), see Definition 6.10 below.

Definition 6.10. Let B be a Boolean Algebra and $\bar{a} = \langle a_i : i < \lambda \rangle$ be a sequence of elements of B.

(1) $\mathrm{irr}_n(\bar{a}, B) = \sup\{|X| : X \subseteq \lambda$ and $\langle a_i : i \in X \rangle$ is n–irredundant$\}$.
(2) Similarly for other invariants of the "universal family" from Rosłanowski and Shelah [91] (see Definition 1.1 there).

Question 6.11. Is there (at least consistently) a Boolean Algebra B^*, such that if B is a Boolean Algebra extending B^* *then* for some ultrafilters D_1, D_2 on B we have: $(B, D_1), (B, D_2)$ are not isomorphic, i.e., no automorphism of B maps D_1 onto D_2.
A close topological relative is: "is there a homogeneous compact Hausdorff space of cellularity $> 2^{\aleph_0}$" (van Douwen, see Kunen [74]).

* * *

There are some lemmas in [151] which help to prove 6.1, and I would like to know whether the bounds used there are the best possible. Those lemmas also show that for some cardinal invariants (for Boolean Algebras or topologies), defined by supremum, if the supremum is not attained, then the value is "almost" regular (the classical result of Erdös and Tarski on the cellularity on Boolean Algebra (or topology) says it is regular, whereas we get in [151] that, e.g., the spread satisfies $2^{\mathrm{cf}(\mathrm{s}^+(B))} > \mathrm{s}^+(B)$, for a Boolean Algebra B, $2^{2^{\mathrm{cf}(\mathrm{s}^+(X))}} > \mathrm{s}^+(X)$ for a Hausdorff space, $2^{\mathrm{cf}(\mathrm{s}^+(X))} > \mathrm{s}^+(X)$ for a T_3 space X).

Question 6.12. Can we find more applications of the theorems (and proofs) in [151] implying (or saying) that if the supremum in some cardinal invariants for a space $\mathcal{U}$ (or a Boolean Algebra B, or whatever) is not attained, then it has large cofinality?

A recent application is in Rosłanowski and Shelah [87, §6]. This is a converse to 6.8.

For the spread (and the hereditarily Lindelof degree and the hereditarily density) the results are best possible (see Juhász and Shelah [64]), and for regular spaces we have better results ([151, 5.1] also best possible) but are the bounds in the claims below best possible?

Definition 6.13. (1) φ is nice for X if φ is a function from the family of subsets of the topological space X to cardinals satisfying
$$\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B) + \aleph_0$$
(i.e., monotonicity and subadditivity).
(2) We say φ is (χ, μ)–complete provided that if $A_i \subseteq X, \varphi(A_i) < \chi$ for $i < \mu$ then $\varphi(\bigcup_{i<\mu} A_i) < \chi$.
Let $C(\varphi, \mu) = \{\chi : \varphi$ is (χ, μ)–complete$\}$.

(3) We say φ is $(<\lambda,\mu)$–complete, if for arbitrarily large $\chi<\lambda$, φ is (χ,μ)–complete.

(4) Let Ch_φ be the following function from X to cardinals:

$$\mathrm{Ch}_\varphi(y)=\min\{\varphi(u): y\in u\in\tau(X)\},$$

where $\tau(X)$ is the topology of X, that is the family of open sets.

Remark 6.14. (1) We can replace μ by $<\mu$ and $i<\mu$ by $i<\alpha<\mu$, and make suitable changes later.

(2) In our applications we can restrict the domain of φ to the Boolean Algebra generated by $\tau(X)$ and even more, e.g., in 6.15, 6.17 below to simple combinations of the $u_{i,\xi,\zeta}$.

(3) We can change the definition of $(<\lambda,\mu)$–complete to

if $A_i\subseteq X$ (for $i<\mu$) and $\sup\limits_{i<\mu}\varphi(A_i)<\lambda$ then $\varphi(\bigcup\limits_{i<\mu}A_i)<\lambda$,

without changing our subsequent use. [We then will use: if $\varphi(A_\alpha)<\chi_i$ for $\alpha<\mu$ then $\varphi(\bigcup\limits_{\alpha<\mu}A_\alpha)<\chi_{i+1}$].

Lemma 6.15. *Suppose λ is a singular cardinal of cofinality θ, $\lambda=\sum\limits_{i<\theta}\chi_i$, $\chi_i<\lambda$, $\theta<\lambda$ and $\mu=\beth_5(\theta)^+$, or even just $\mu=\beth_2(\beth_2(\theta)^+)^+$. Assume that*

(i) *φ is nice for X,*

(ii) *$X_{\chi_i}=\{y\in X:\mathrm{Ch}_\varphi(y)\geq\chi_i\}$ has cardinality $\geq\mu$ for $i<\theta$,*

(iii) *φ is $(<\lambda,\mu)$–complete.*

Then *there are open sets $u_i\subseteq X$ (for $i<\theta$) such that*

$$\varphi(u_i\setminus\bigcup_{j\neq i}u_j)\geq\chi_i.$$

Remark 6.16. If $|\{y\in X:\mathrm{Ch}_\varphi(y)\geq\chi_i\}|<\mu$, it essentially follows from (χ_i,μ)–completeness that $\varphi(X_{\chi_i})\geq\lambda$, where $X_\chi=\bigcup\{u\in\tau(X):\varphi(u)<\chi\}$. Otherwise, $\varphi(X\setminus X_{\chi_i})\geq\lambda$ by subadditivity, but $\varphi(X\setminus X_{\chi_i})\leq\prod\{\varphi(\{y\}):y\in X\setminus X_{\chi_i}\}$, so by (χ_i,μ)–completeness for some $y\in X$, $\varphi(\{y\})\geq\chi_i$, which is impossible for the instances which interest us.

Lemma 6.17. *Suppose that X is a Hausdorff space, λ is a singular cardinal, $\theta=\mathrm{cf}(\lambda)$, $\lambda=\sum\limits_{i<\theta}\chi_i$, $\chi_i<\lambda$, $\mu<\lambda$ and clauses (i), (ii), (iii) of 6.15 hold (for φ).*

(1) *If $\mu=\beth_2(\theta)^+$ (or even $\sum\limits_{\sigma<\theta}\beth_2(\sigma)^+$),* then *there are open sets u_i (for $i<\theta$) such that $\varphi(u_i\setminus\bigcup\limits_{j>i}u_j)\geq\chi_i$.*

(2) *If* $X = \bigcup\{u : \varphi(u) < \lambda\}$*, and* μ *is as in part (1),* then *there are open sets* u_i *(for* $i < \theta$*) such that* $\varphi(u_i \setminus \bigcup_{j \neq i} u_j) \geq \chi_i$.

(3) *If* $\mu \geq \beth_3(2^{<\theta})^+$*,* φ *is* $(< \chi_0, \mu)$*–complete,* then *there are* u_i *(for* $i < \theta$*) such that* $\varphi(u_i \setminus \bigcup_{j \neq i} u_j) \geq \chi_0$ *(so* λ*,* χ_i *(*$0 < i < \theta$*) are irrelevant).*

Remark 6.18. Part (1) of the lemma is suitable to deal with Boolean Algebras, part (2) with the existence of $\{x_\alpha : \alpha < \lambda\}$ such that for every $\alpha < \lambda$ for some u, $x_\alpha \in u \cap \{x_\beta : \beta < \lambda\} \subseteq \{x_\beta : \beta \leq \alpha\}$.

Now,

Question 6.19. Are the cardinal bounds in 6.14 — 6.17 best possible?

7. A taste of Algebra

I have much interest in Abelian groups, but better see Eklof and Mekler [32].

Thomas prefers to deal just with short elegant proofs of short elegant problems (for me the second demand suffices). So he was rightly happy when proving that for any infinite group G with no center, $\gamma(G) < (2^{|G|})^+$, where $G^{[0]} = G$, $G^{[1]}$ is the automorphism group of G considered as an extension of G, $G^{[i+1]} = (G^{[i]})^{[1]}$, $G^{[\delta]} = \bigcup_{i<\delta} G^{[i]}$, so $G^{[i]}$ is an increasing sequence of groups with no center, and $\gamma(G) = \min\{\gamma : G^{[\gamma]} = G^{[\gamma+1]}\}$.

But is there a better cardinal bound? No, for $|G|$ regular $> \aleph_0$, see Just, Shelah and Thomas [66], but we are left with:

Question 7.1. If G is a countable group with a trivial center, then do we have $\gamma(G) < \omega_1$? What about singular $|G|$?

I heard about the following problem (see Hamkins [56]).

Problem 7.2. If G is a group possibly with center, $G^{[i]}$ is defined as above but we have just a homomorphism $h_{i+1,i} : G^{[i]} \longrightarrow G^{[i+1]}$ with the center of $G^{[i]}$ being the kernel (and in limit stages take the direct limit), is there a bound to $\gamma(G)$ really better than the first strongly inaccessible $> |G|$ (gotten by Hamkins [56])?

Thomas also had started investigating cofinalities of some natural groups, (see Sharp and Thomas [95], [96], Thomas [192]). He drew me to it and I was particularly glad to see that pcf pops in naturally; e.g., (see Shelah and Thomas [184]) if $\lambda_n \in \mathrm{CF}(\mathrm{Sym}(\omega))$ and $\lambda \in \mathrm{pcf}\{\lambda_n : n < \omega\}$ then $\lambda \in \mathrm{CF}(\mathrm{Sym}(\omega))$, where

Definition 7.3. (1) $\mathrm{CF}(G) = \{\theta : \theta = \mathrm{cf}(\theta)$ and there is an increasing sequence of proper subgroups of G of length θ with union $G\}$.
(2) $\mathrm{cf}(G) = \min[\mathrm{CF}(G) \setminus \{\aleph_0\}]$.

Though we found some information about $\mathrm{cf}(\prod_n \mathrm{alt}(n))$ (see Saxl, Shelah and Thomas [179], where $\mathrm{alt}(n)$ is the group of even permutations of $\{0, \dots, n-1\}$), we remained baffled by

Question 7.4. Is it consistent that $\aleph_2 \leq \mathrm{cf}(\prod\limits_{n<\omega} \mathrm{alt}(n))$?

It is natural to try to use iterations of length ω_2, where each iterand consists of trees with norms (see Rosłanowski and Shelah [92]). Naturally, a norm on $\mathcal{P}(\mathrm{alt}(n))$ will be such that if $\mathbf{nor}(A) \geq m+1$ and σ is a group term, then we can have for "many" $f_1, \dots, f_k, g \in \mathrm{alt}(n)$ that $A' = \{h \in A : \sigma(h, f_1, \dots, f_{k^*}) = g\}$ has $\mathbf{nor}(A') \geq m$ toward destroying a guess on an approximation to a lower subgroup exemplifying $\omega_1 \in \mathrm{CF}(\prod\limits_{n<\omega} \mathrm{alt}(n))$. This helps for $\aleph_0 \notin \mathrm{CF}(\prod\limits_{n<\omega} \mathrm{alt}(n))$, but fails for the purpose of 7.4.

* * *

My interest in lifting for the measure algebra started when Talagrand promised me "flowers on your grave from every measure theorist" (a little hard to check), and Fremlin said essentially the same, conventionally (see [145], [176, Ch.IV]). But this does not solve some other problems from Fremlin's list, from which I particularly like

Question 7.5. Assume CH (or even GCH or just prove consistency).
Do we have lifting for every measure algebra?
Which means: let $\mathcal{B}(I)$ be the algebra of subsets of ${}^I 2$ generated from clopen ones by countable unions and intersections, $\mu_{\mathcal{B}}$ the Lebesgue measure on $\mathcal{B}$ (so we get the so-called Maharam algebra), $\mathbf{I} = \{A \in \mathcal{B} : \mu_{\mathcal{B}}(A) = 0\}$ (so $\mathbf{I}$ is the ideal of null sets). A lifting is a homomorphism from $\mathcal{B}/\mathbf{I}$ into $\mathcal{B}$ such that

$$X \in \mathcal{B} \quad \Rightarrow \quad f(X/\mathbf{I}) = X \mod \mathbf{I}.$$

Naturally, we think $I_0 \subseteq I_1 \Rightarrow \mathcal{B}(I_0) \subseteq \mathcal{B}(I_1)$ (by identifying) and for an increasing sequence $\bar{I} = \langle I_\alpha : \alpha < \alpha^* \rangle$ we let $\mathcal{B}(\bar{I}) = \bigcup\limits_{\alpha<\alpha^*} \mathcal{B}(I_\alpha)$. In the positive direction we may try to prove by induction on λ; then we will be naturally drawn to proving: for any $\mathcal{P}^-(n)$-diagram $\langle \mathcal{B}(\bar{I}_s) : s \in \mathcal{P}^-(n) \rangle$, where $\mathcal{P}^-(A) = \{u : u \subseteq A,\ u \neq A\}$, and a sequence of liftings $\bar{f} = \langle f_s : s \in \mathcal{P}^-(n) \rangle$ satisfying a reasonable induction hypothesis, $\bigcup\limits_s f_s$ can be extended to a lifting of $\bigcup\limits_s I_{s,\alpha}$ (as e.g. in [143], Sageev and Shelah [94]). For the negative direction we may think of using a partition theorem.

* * *

For a long time I have been interested in compactness in singular cardinals; i.e., whether if something occurs for "many" subsets of a singular λ of cardinality $< \lambda$, it occurs for λ. For the positive side (on the filters see 7.7 below)

Theorem 7.6. *Let λ be a singular cardinal, $\chi^* < \lambda$. Assume that $\mathbf{F}$ is a set of pairs (A, B) (written usually as B/A; $\mathbf{F}$ stands for free) $A, B \subseteq \mathcal{U}$ satisfying the axioms II, III, IV, VI, VII below. Let $A^*, B^* \subseteq \mathcal{U}$, $|A^*| = \lambda$,* then *$B^*/A^* \in \mathbf{F}$ if B^*/A^* is λ-free in a weak sense which means (see Definition 7.7 below):*

- $(*)_0$ *for the $\mathcal{D}_{\chi^*}(B^*)$–majority of $B \in [B^*]^{<\lambda}$ we have $B/A^* \in \mathbf{F}$,*
  or just
- $(*)_1$ *the set $\{\mu < \lambda : \{B \in [B^*]^\mu : B/A^* \in \mathbf{F}\} \in \mathcal{E}_\mu^{\mu^+}(B^*)\}$ contains a club of λ,*
  or at least
- $(*)_2$ *for some set C of cardinals $< \lambda$, unbounded in λ and closed (meaningful only if* $\mathrm{cf}(\lambda) > \aleph_0$*), for every $\mu \in C$, for an $\mathcal{E}_\mu^{\mu^+}(B^*)$–positive set of $B \in [B^*]^\mu$ we have $B/A^* \in \mathbf{F}$.*

The axioms are

Ax II: $B/A \in \mathbf{F} \quad \Leftrightarrow \quad A \cup B/A \in \mathbf{F}$,

Ax III: if $A \subseteq B \subseteq C$, $B/A \in \mathbf{F}$ and $C/B \in \mathbf{F}$ *then* $C/A \in \mathbf{F}$,

Ax IV: if $\langle A_i : i \leq \theta\rangle$ is increasing continuous, $\theta = \mathrm{cf}(\theta)$, $A_{i+1}/A_i \in \mathbf{F}$ then $A_\theta/A_0 \in \mathbf{F}$,

Ax VI: if $A/B \in \mathbf{F}$ then for the $\mathcal{D}_{\chi^*}$–majority of $A' \subseteq A$, $A'/B \in \mathbf{F}$ (see below),

Ax VII: if $A/B \in \mathbf{F}$ then for the $\mathcal{D}_{\chi^*}$–majority of $A' \subseteq A$, $A/B \cup A' \in \mathbf{F}$.

(Of course we can get variants by putting more or less into the statement.)

Definition 7.7. (1) Let $\mathcal{D}$ be a function giving for any set B^* a filter $\mathcal{D}(B^*)$ on $\mathcal{P}(B^*)$ (or on $[B^*]^\mu$). Then to say "for the $\mathcal{D}$–majority of $B \subseteq B^*$ (or $B \in [B^*]^\mu$) we have $\varphi(B)$" means $\{B \subseteq B^* : \varphi(B)\} \in \mathcal{D}(B^*)$ (or $\{B \in [B^*]^\mu : \neg\varphi(B)\} = \emptyset \mod \mathcal{D}$).

(2) Let $\mathcal{D}_\mu(B^*)$ be the family of $Y \subseteq \mathcal{P}(B^*)$ such that for some algebra M with universe B^* and $\leq \mu$ functions,

$$Y \supseteq \{B \subseteq B^* : B \neq \emptyset \text{ is closed under the functions of } M\}.$$

(3) $\mathcal{E}_\mu^{\mu^+}(B^*)$ is the collection of all $Y \subseteq [B^*]^\mu$ such that: for some χ, x such that $\{B^*, x\} \in \mathcal{H}(\chi)$, *if* $\bar{M} = \langle M_i : i < \mu^+\rangle$ is an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\chi), \in)$ such that $x \in M_0$ and $\bar{M} \restriction (i+$

$1) \in M_{i+1}$,
then for some club C of μ^+, $i \in C \;\Rightarrow\; M_i \cap B^* \in Y$.

On the filters see Kueker [72], and [133, §3]. The theorem was proved in [133] but with two extra axioms, however it included the full case of varieties (i.e., including the non-Schreier ones). Later, the author eliminated those two extra axioms: Ax V in Ben David [14], and Ax I in [135] (answering a question of Fleissner on providing a combinatorial proof of the compactness). Hodges [57] contains also presentation of variants of this result.

There are some cases of incompactness (see Fleissner and Shelah [35], and [157]).

Problem 7.8. (1) Are there general theorems covering the incompactness phenomena?
(2) Are there significantly better compactness theorems (for uncountable cofinality, of course)?

Related is

Question 7.9. What can be

$$\{\lambda : \text{there is a } \lambda\text{–free for } \mathcal{V} \text{ algebra } M \text{ of cardinality } \lambda \text{ which is not free}\},$$

for a variety $\mathcal{V}$ (at least with countable vocabulary) ?
(See Eklof and Mekler [32], Mekler and Shelah [81], Mekler, Shelah and Spinas [82].)

* * *

There are cases of strong dichotomy: if $\geq \lambda$ then $\geq 2^\lambda$, related to groups (see [153], Grossberg and Shelah [51], [52], and [129]; on Abelian groups see Fuchs [39]).

Question 7.10. [$\mathbf{V} = \mathbf{L}$] If $\lambda > \mathrm{cf}(\lambda) > \aleph_0$, G is a torsion free Abelian group of cardinality λ, can $\lambda = \nu_p(\mathrm{Ext}(G, \mathbb{Z}))$?

The cardinal $\nu_p(\mathrm{Ext}(G, \mathbb{Z}))$ is the dimension of $\{x \in \mathrm{Ext}(G, \mathbb{Z}) : px = 0\}$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. To avoid Ext note that (see Fuchs [39]) this group can be represented as $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\{h/p : h \in \mathrm{Hom}(G, \mathbb{Z})\}$. If G is torsion free, then the group $\mathrm{Ext}(G, \mathbb{Z})$ is divisible and hence the ranks $\nu_p(\mathrm{Ext}(G, \mathbb{Z}))$ for p prime, and $\nu_0(\mathrm{Ext}(G, \mathbb{Z}))$, the rank of $\{x \in \mathrm{Ext}(G, \mathbb{Z}) : x \text{ torsion}\}$, determine $\mathrm{Ext}(G, \mathbb{Z})$ up to an isomorphism. If we assume ($\mathbf{V} = \mathbf{L}$ and) there is no weakly compact cardinal, this question is the only piece left for characterizing the possible such $\mathrm{Ext}(G, \mathbb{Z})$, see Mekler, Rosłanowski and Shelah [79].

Question 7.11. What is the first cardinal $\lambda = \lambda_\kappa$ such that: for every ring R of cardinality $\leq \kappa$, if there is endorigid (or rigid, or 1-to-1 rigid) R–module of size $\geq \lambda$, *then* there are such R-modules in arbitrarily large cardinals? (I.e. Hanf numbers).

8. Partitions and colourings

Remember (see Erdős, Hajnal, Maté and Rado [34])

Definition 8.1.
(1) $\lambda \to (\alpha)^n_\kappa$ means: for every colouring $c : [\lambda]^n \longrightarrow \kappa$ there is a set $X \subseteq \lambda$ of order type α such that $c \restriction [X]^n$ is constant.
(2) $\lambda \to [\alpha]^n_\kappa$ means: for every colouring $c : [\lambda]^n \longrightarrow \kappa$ there is a set $X \subseteq \lambda$ of order type α such that $\mathrm{Rang}(c \restriction [X]^n) \neq \kappa$.
(3) $\lambda \to [\alpha]^n_{\kappa,\sigma}$ means: for every colouring $c : [\lambda]^n \longrightarrow \kappa$ there is a set $X \subseteq \lambda$ of order type α such that $\mathrm{Rang}(c \restriction [X]^n)$ has cardinality $< \sigma$.

Definition 8.2. M is a Jonsson algebra if it is an algebra with countably many functions with no proper subalgebra of the same cardinality.
(See [166], [98].)

Definition 8.3. $\mathrm{Pr}_1(\lambda, \mu, \theta, \sigma)$ means: there is $c : [\lambda]^2 \longrightarrow \theta$ such that if $u_i \in [\lambda]^{<\sigma}$ (for $i < \mu$) are pairwise disjoint and $\gamma < \theta$ *then* for some $i < j < \mu$ we have $c \restriction (u_i \times u_j)$ is constantly γ.
(See [166], [98].)

There are many more variants.

It irritates me that after many approximations, I still do not know (better for me "consistently no", better for set theory "yes") the answer to the following.

Question 8.4. If μ is singular, is there a Jonsson algebra on μ^+? (and even better $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$?)

Also the requirements on an inaccessible to get colouring theorems may well be an artifact of our inability, so let us state minimal open cases.

Question 8.5.
(1) Let λ be the first ω–Mahlo cardinal. Does $\lambda \nrightarrow [\lambda]^2_\lambda$ or at least $\lambda \nrightarrow [\lambda]^2_\theta$ for $\theta < \lambda$?
(2) Let λ be the first inaccessible cardinal which is $(\lambda \cdot \omega)$–Mahlo. Is there a Jonsson algebra on λ? (Even better $\lambda \nrightarrow [\lambda]^2_\lambda$?)

In both parts it is better to have $\mathrm{Pr}_1(\lambda, \lambda, \lambda, \aleph_0)$, etc; it is interesting even assuming GCH.

Whereas under GCH the relation $\to$ for cardinals is essentially understood (see Erdős, Hajnal, Maté and Rado [34]), the case of ordinals is not. As by [132] (GCH for simplicity) $\alpha < \lambda = \mathrm{cf}(\lambda) \;\Rightarrow\; \lambda^{++} \to (\lambda^+ + \alpha)^2_2$ (on the problem see [34]), and by Baumgartner, Hajnal and Todorčević [10] for $k < \omega$,

$$\alpha < \lambda = \mathrm{cf}(\lambda) \quad \Rightarrow \quad \lambda^{++} \rightarrow (\lambda^+ + \alpha)^2_k,$$

it remains open:

Question 8.6 (GCH)**.** When does $\lambda^{++} \rightarrow (\lambda^+ + \alpha)^2_{\aleph_0}$, where $2 \leq \alpha < \lambda$? (The GCH assumption is for simplicity only.)

By [165] we know that if $\lambda = \lambda^{<\lambda}$ (e.g., $\lambda = \aleph_1 = 2^{\aleph_0}$) then possibly 2^λ is very large (in particular $> \lambda^{+\omega}$), but $2^\lambda \nrightarrow (\lambda \times \omega)^2_2$. However, $\lambda^{+2k} \rightarrow (\lambda \times n)^2_k$ (by [165, bottom of p.288]), so

Question 8.7. For $\lambda = \lambda^{<\lambda}$, $\lambda > \aleph_0$, and $k < \omega$ and $n < \omega$, what is the minimal m such that $\lambda^{+m} \rightarrow (\lambda \times n)^2_k$?
(Baumgartner, Hajnal and Todorčević [10, p.2, end of §0] prefer to ask whether $\lambda^{++} \rightarrow (\lambda + \omega)^2_3$ for $\lambda = \aleph_1$, so $\lambda = \lambda^{<\lambda}$ means CH as they choose another extreme case of the unknown).

Now, for me a try at consistency of negative answers for 8.6 calls for using historic forcing (see Shelah and Stanley [182], Rosłanowski and Shelah [86, §3]; it is explained below). On the other hand, large cardinals may make some positive results easier.

Question 8.8. Assume that $\lambda > \kappa > |\zeta| + \sigma$, and κ is a compact cardinal, and $\lambda = \lambda^{<\lambda}$. Does it follow that $\lambda^+ \rightarrow (\lambda + \zeta)^2_\sigma$?

I tend to think the answer is yes. If so, then we cannot expect, when $\lambda = \lambda^{<\lambda}$, that a λ–complete λ^+-c.c. forcing notion will not add a counterexample c to $\lambda^+ \rightarrow (\lambda + \alpha^*)^2_{\aleph_0}$. Let for simplicity $\alpha^* = 2$. Also we should expect that for such c, for every $u \in [\lambda^+]^{<\lambda}$ we can find $f_n : u \times u \longrightarrow \omega$ for $n < \omega$ such that $f_n(\alpha, \alpha) = n$, and for no distinct $\alpha, \beta, \gamma \in u$ do we have:

$$c(\{\alpha, \beta\}) = n = f_n(\alpha, \beta) \quad \text{or} \quad f_n(\alpha, \beta) = f_n(\alpha, \gamma) = c(\{\beta, \gamma\}).$$

The point of historic forcing is that we know what kind of object our forcing notion has to add. In our case, we assume $\lambda = \lambda^{<\lambda}$ and a condition p has to give $u^p \in [\lambda^+]^{<\lambda}$ and $c^p : [u^p]^2 \longrightarrow \omega$ and by the above considerations also $f_n : u^p \times u^p \longrightarrow \omega$ (for $n < \omega$), and for having some leeway we let $f_n : u^p \times u^p \longrightarrow [\omega]^{\aleph_0}$ such that $n \in f_n(\alpha, \alpha)$, and we demand

($*$) for no distinct $\alpha, \beta \in u^p$ and m do we have

$$m = c(\{\alpha, \beta\}) \in f_n(\alpha, \beta) \cap f_n(\alpha, \alpha),$$

($**$) for no distinct α, β, γ from u^p do we have

$$c(\{\beta, \gamma\}) \in f_n(\alpha, \beta) \cap f_n(\alpha, \gamma).$$

Moreover, we choose $\langle A_\eta : \eta \in {}^{\omega>}\omega\rangle$, $A_\eta \in [\omega]^{\aleph_0}$, $\langle A_{\eta\frown\langle\ell\rangle} : \ell < \omega\rangle$ are pairwise disjoint subsets of A_η and demand $f_n(\alpha,\beta) \in \{A_\eta : \eta \in {}^{\omega>}\omega\}$.

Considering that our forcing will be strategically $(<\lambda)$–complete (as λ–complete seems too much both if the answer to 8.8 is yes and because of the properties of historic forcing in general), in order that there will be no $A \subseteq \lambda^+$, $\mathrm{otp}(A) = \lambda + \alpha^*$ such that $\underset{\sim}{c} \restriction [A]^2$ is constant, we have a set $\mathcal{T}$ of cardinality $< \lambda$ such that (for simplicity $\lambda = \mu^+$, maybe also μ regular):

(a) each $x \in \mathcal{T}^p$ has the form $x = (a, \delta, b) = (a^x, \delta^x, b^x)$, where
  (i) $\delta < \lambda^+$, $\mathrm{cf}(\delta) = \lambda$,
  (ii) $a \subseteq u^p \setminus \delta$, $\mathrm{otp}(a) = \zeta$,
  (iii) $b \subseteq u^p \cap \delta$, $\mathrm{otp}(b) = \mu$,
(b) if $(a', \delta', b'), (a'', \delta'', b'') \in \mathcal{T}^p$ then $|b' \cap b''| < \mu$ (probably not necessary),
(c) $c^p \restriction [a \cup b]^2$ is constantly $n(x)$,
(d) for no $\alpha \in u^p \cap \delta \setminus \sup(b)$ do we have

$$(\forall \beta \in a \cup b)[c^p(\{\alpha, \beta\}) = n(x)].$$

A p like above will be a precondition. Now the "history" enters and the proof should be clear just as the definition of the set of the conditions should roll itself.

So atomic conditions will have u^p a singleton, each condition will have a history telling how it was created: each step in the history corresponds to one of the reasons for creating a condition in the proof. Naturally, a major reason is the proof of the λ^+-c.c. by the Δ–system lemma. So assume for $\ell = 1, 2$ that $p_\ell \in \mathbb{P}$, $\alpha_1 < \alpha_2$, $u^{p_1} \cap \alpha_1 = u^{p_2} \cap \alpha_2$, $u^{p_1} \subseteq \alpha_2$, $\mathrm{otp}(u^{p_1}) = \mathrm{otp}(u^{p_2})$ and the order preserving mapping $\mathrm{OP}_{u^{p_2}, u^{p_1}}$ from u^{p_1} onto u^{p_2} maps p_1 to p_2. We have to amalgamate p_1 and p_2 getting q, so we have to determine f^q_n on $(u^{p_1} \setminus u^{p_2}) \times (u^{p_2} \setminus u^{p_1})$, and $c^q(\{\alpha, \beta\})$ for $\alpha \in u^{p_1} \setminus u^{p_2}$, $\beta \in u^{p_2} \setminus u^{p_1}$.

But our vision is that there is one line of history, so should the history of q continue the history of p_1 with p_2 joining or should the history of q continue the history of p_2 with p_1 joining? Both are O.K., but we get two distinct conditions q', q'' which, however, are equivalent; i.e. $q' \le q'' \le q'$.

Generally going back and changing the history as above we get an equivalent condition if this is done finitely many times (this also explains why we get strategical $(<\lambda)$–completeness and not λ–completeness). That is, we define $p \le_{\mathrm{pr}} q$ iff p appears in the history of q, $p \sim q$ if p is gotten from q by finitely many changes as above in the history, and lastly $p \le q$ iff $(\exists p')(p' \sim p \ \&\ p' \le_{\mathrm{pr}} q)$.

What is left? "Only" carrying out the amalgamation (using and guaranteeing the conditions, and probably changing them so that they will fit).

Question 8.9. What are the best cardinals needed for the canonization theorems in [140]?

An old well-known problem is

Question 8.10. Is $\aleph_1 \to [\aleph_1; \aleph_1]^2_2$ consistent? (And variants, connected to the L-space problem; this seems related to 5.1).

Another problem of Erdös is (the answer is consistently yes, see [155], even colouring also no edges, but provability in ZFC is not clear):

Question 8.11. Is there a graph G with no K_4 (complete graph on 4 vertices) such that $G \to (K_3)^2_{\aleph_0}$, that is for any colouring of the edges by $\aleph_0$ colours there is a monochromatic triangle?

We can ask it for K_k, K_{k+1} instead of $k = 3$ and colouring r–tuples instead of pairs; the answer still is consistently yes (see [155]), so the problem is in ZFC. See [114, Ch.III,§1] on a connection to model theory.

* * *

A very nice theorem of Hajnal [53] says that, e.g., for any finite graph G and κ for some graph H, $H \to (G)^2_\kappa$, but leaves as a mystery:

Question 8.12. Let G be a countable graph, is there a graph H such that

$$H \to (G)^2_{\aleph_0} \ ?$$

Starting from a problem of Erdös and Hajnal [33], I have been very interested in consistency results, e.g., of the form $\lambda \to [\mu]^2_3$, $\kappa < \mu < \lambda \leq 2^\kappa$ (see [131]). Usually those are really canonization theorems, for a fixed natural coloring, any other is, on a large set, computable from it. Those results help sometimes in consistency results (just as, e.g., the Erdös-Rado theorem helps in ZFC results). Still it seems to me worthwhile to know.

Question 8.13.
(1) Can we put together the results of, e.g., [131], Shelah and Stanley [181] and [160]? Assume that $\kappa = \kappa^{<\kappa} < \lambda$, λ is, e.g., strongly inaccessible large enough. Can we find a $(<\kappa)$–complete, κ^+–c.c. forcing notion $\mathbb{P}$ such that in $\mathbf{V}^{\mathbb{P}}$:
   (a) for $\sigma < \kappa$ and $\mu < \lambda$ we can find μ', λ' such that $\mu < \mu' < \lambda' < \lambda$ and $\lambda' \to [\mu']^2_{\sigma,2}$;
   (b) if κ is a measurable indestructibly by adding many Cohens, then also the parallel results for colouring n-tuples (see [160]);
   (c) if $\kappa = \aleph_0$, we also have results on colouring n–tuples simultaneously for all n ?
(2) Add the hopeful consistency answer for 8.14, 8.15.

Probably easier are, e.g.

Question 8.14. Is it consistent that for some n, $2^{\aleph_0} = \aleph_n = \lambda \to [\aleph_1]^2_3$? (The exact n is less exciting for me, the main division line seems to me $\aleph_\omega$, of course best to know the exact λ.)

Question 8.15. Is it consistent that $2^{\aleph_0} > \lambda \to [\aleph_1]^2_3$?

* * *

On Finite Combinatorics. Spencer, Szemeredi and Alon told me that finding $\lim(\log(r^2_2(n)/n))$ is a major problem (see Definition 8.16 below), but the difference between lower and upper bounds seems to me negligible.

Definition 8.16. $r^m_k(n)$ is the minimal r such that $r \to (n)^m_k$.

Erdös and Hajnal ask, and I find more convincing, the following.

Question 8.17. What is the order of magnitude of $r^3_2(n)$?

We expect it should be 2^{2^n}, or e.g. $2^{2^{(n^\varepsilon)}}$ for some $\varepsilon > 0$. But we cannot rule out its being 2^n or e.g. $2^{n^{1/\varepsilon}}$ for some $\varepsilon > 0$.
Here the difference is large.

Note that for four colours the problem (what is $r^3_4(n)$) is settled; but I think the true question is:

Question 8.18. Determine (order of magnitude is OK) $f_k(n,r), f^+_k(n,c)$ where:

(a) $f_3(n,c)$ is the minimal m such that
for every $\mathbf{d} : [m]^3 \longrightarrow \{0,\dots,c-1\}$, there are $A \in [m]^n$ and a strictly increasing function $h : A \longrightarrow \{0,\dots,2^n-1\}$ such that for $\ell_0 < \ell_1 < \ell_2$ in A, the value of $\mathbf{d}(\{\ell_0,\ell_1,\ell_2\})$ is determined by the quantifier–free type $\langle h(\ell_0), h(\ell_1), h(\ell_2)\rangle$ in B^3_n,
where B^3_n has the universe $\{0,\dots,2^n-1\}$ and two relations: (viewed as ${}^n 2$) the lexicographic order and

$$\{(\eta_0,\eta_1,\eta_2) : \eta_0 <_{\ell x} \eta_1 <_{\ell x} \eta_2 \text{ and } \ell g(\eta_0 \cap \eta_1) < \ell g(\eta_1 \cap \eta_2)\}.$$

(b) $f^+_3(n,c)$ is defined similarly but for every pregiven $\mathrm{Rang}(h)$ we can find such h.

(c) B^k_n is defined below by induction of k, and then $f_k(n,c), f^+_k(n,c)$ are defined analogously to f_3, f^+_3.

(d) Define canonization numbers $g_k(n), g^+_k(n)$, which is the first m such that: if $d : [n]^k \to C$, with no restriction on the cardinality of C, then we can find A, h as above, and a quantifier free formula ϕ in the vocabulary of B^k_n such that for any $u_1, u_2 \in [A]^k$ we have

$$d(u_1) = d(u_2) \quad \text{iff} \quad \phi(\dots, h(\ell_1), \dots; \dots, h(\ell_2), \dots)_{\substack{\ell_1 \in u_1, \\ \ell_2 \in u_2}} \text{ is satisfied in } B^k_n.$$

The explicit way to describe B^k_n, by induction on k is: it has a linear order $<_k$:
B^2_n is the structure $(n,<)$,
B^{k+1}_n has universe ${}^{B^k_n}\{0,1\}$ and the relation:
$\eta <_{k+1} \nu$ if and only if
for some $y = y(\eta,\nu) \in B^k_n$ we have $\eta(y) = 0$, $\nu(y) = 1$ and

$$\eta \restriction \{x \in B^k_n : x <_k y\} = \nu \restriction \{x \in B^k_n : x <_k y\}.$$

For an m-place relation $R^{B^k_n}$ of B^k_n, $R^{B^{k+1}_n}$ is a $2m$-place relation on B^{k+1}_n, namely

$$\{\langle \eta_0,\dots,\eta_{2m-1}\rangle : \quad \eta_\ell \in B^{k+1}_n,\ \eta_\ell <_{k+1} \eta_{\ell+1} \text{ and } \langle y(\eta_0,\eta_1), y(\eta_2,\eta_3),\dots\rangle \in R^{B^k_n}\}.$$

Now it is not clear how fast the number in 8.18 grows, e.g., we cannot exclude $2^{2^{n+c+k}}$. The main question is whether it grows like $h(k)$–iterated exponentiation in n (say c fixed) with h going to infinity, or with h constant. Of course, enriching somewhat the structure is not a great loss to me.

Question 8.19. Let $f^*(n,c)$ be the first m such that if $\langle A_\ell : \ell < m\rangle$ are pairwise disjoint, $|A_\ell| = m$ for $\ell < n$ and

$$F : \{w \subseteq \bigcup_\ell A_\ell : |w \cap A_\ell| \in \{1,2\},\ (\exists!\ell)(|w| = 1)\} \longrightarrow C,$$

where $|C| = c$ *then* for some $x_\ell \neq y_\ell$ from A_ℓ for $\ell < n$ we have

$$\ell^* < n \ \Rightarrow\ F(\{x_\ell, y_\ell : \ell \neq \ell^*\} \cup \{x_{\ell^*}\}) = F(\{x_\ell, y_\ell : \ell \neq \ell^*\} \cup \{y_{\ell^*}\}).$$

Again the main question for me is: Does $f^*(n,c)$ grow as a fixed iterated exponentiation?
(This is connected to the van der Waerden theorem, see [154]).

On the Ramsey Theory see Graham, Rothschild and Spencer [49].

Definition 8.20. (1) For a group G and a subset A of G, and a group H let $H \to (G)^A_\sigma$ mean:
if $\mathbf{d}$ is a function with domain H and range of cardinality $\le \sigma$,
then for some embedding h of G into H the function $\mathbf{d}$ restricted to $h(A)$ is constant.

(2) For a group G and a subset A of G, and group H let $H \to [G]^A_{\sigma,\tau}$ mean:
if $\mathbf{d}$ is a function with domain H and range of cardinality $\le \sigma$,
then for some embedding h of G into H the range of the function $\mathbf{d}$ restricted to $h(A)$ has cardinality $< \tau$.

If we omit τ, we mean just that the range is not equal to σ.

(3) For a group G and an equivalence relation E on G, and group H let $H \to (G)^E_\sigma$ mean:
if $\mathbf{d}$ is a function with domain H and range of cardinality $\leq \sigma$,
then for some embedding h of G into H, for any $x, y \in G$ which are E-equivalent we have $\mathbf{d}(h(x)) = \mathbf{d}(h(y))$.

(4) In part (1) (or (2), or (3)) we can replace A (or the domain of E) by the family of subgroups of G isomorphic to a fixed group K, and then $\mathbf{d}$ is a function with domain being the set of subgroups of G isomorphic to K.

(5) Like part (4), but we replace "subgroups isomorphic to K" by "embedding of K", and then replace "$\to$" by "$\to^*$".

Discussion 8.21. There is a connection between the two last definitions: the first implies a special case of the second one, when we restrict ourselves to permutation groups of some finite set, and A is the set of conjugates of the permutation just interchanging two elements.

Problem 8.22. (1) Investigate the arrows from Definition 8.20.

(2) In particular, consider the case when A is a set of pairwise conjugate members of G each of order two.

9. Except Forcing

Problem 9.1. Are there methods to prove independence except forcing?

As mathematicians do not report their failures, not much is said in the literature. I do not mean results that simply follow from the consistency strength, e.g., by [146] in some forcing extensions of $\mathbf{L}$ by a forcing notion not collapsing cardinals (in fact satisfying the c.c.c.), PB $\equiv$" every projective set of reals has the Baire property ", whereas the consistency strength of PM $\equiv$" every projective set of reals is Lebesgue measurable" is larger. Hence we have the consistency with ZFC of PB + $\neg$PM, but we cannot use it to prove the consistency of ZFC + $\neg$PB + PM (on the other hand see hopefully [125]). The point is that the issue of the consistency strength is elsewhere, so it gives such results as a byproduct, and we do not have strong control, i.e., we would like to have:

Desire 9.2. A method to get independence of statements starting with the problem and then having a natural direction.

A major issue of set theory is:

Dream 9.3. Find a parallel of forcing for ZFC + $\mathbf{V} = \mathbf{L}$ (or even for ZFC with no statements on $\mathbf{L}$) and even more so:

Dream 9.4. Find a parallel of forcing for **PA** (or even ZFC with statements on number theory).

A possible direction is trying to construct non-well-founded models (note that wellfoundedness is inherent in forcing as it preserves the ordinals). It would be interesting for set theory:

Speculation 9.5. In ZFC+ "there is a supercompact cardinal " we can prove that the theory

$$\text{ZFC} + \text{“ there is a compact cardinal ”}$$

has a model, but not a wellfounded model.

This would explain well why it remains a mystery if supercompact and compact have the same consistency strength (Magidor proved the first compact may be the first measurable or first supercompact; but having no inner model, the usual method does not work now).

But back to 9.3, a traditional way to attack such problems is to chose a good test question, for independence from ZFC, CH has served excellently. Unfortunately, for independence from ZFC + $\mathbf{V} = \mathbf{L}$, there is no obvious candidate. Now, trying to shoot the enemy without seeing him may end, as many times, in first shooting the arrow and then marking the target, an old practice.

It is not totally unreasonable that forcing might be unique in some sense. I mean the syntactical statements. Note that the theories should have the same ordinals in some sense, and we should exclude forcing with a proper class, so we should in some sense restrict ourselves to bounded sentences, i.e.,

$$(\exists \chi)((\mathcal{H}(\chi), \in) \models \psi).$$

Definition 9.6. For bounded sentences ψ_1, ψ_2 let $\psi_1 \leq_{\text{cs}} \psi_2$ mean that for every finite subset Φ of ZFC + ψ_1, in ZFC + ψ_2, we can prove it has a well-founded model.

Still we have here a problem: by playing with statements on the existence of models we can find "erratic" behaviour which it is widely felt does not occur in " NMM (= non-metamathematical) statements ". (I do not call them mathematical, as the sentences like CON(ZFC+ there exists an inaccessible cardinal) are excellent mathematical statements in my eyes). See [75] and [190] on finding NMM statements closely related to such statements.

This appears in several directions.

Problem 9.7. Can we find a large family Φ_ℓ of bounded sentences in set theory which formalize "being NMM" at least to some extent (and so at least) is reasonably wide and natural, for which we answer at least some of the following:

$(A)_0$ for Φ_0, $\mathrm{ZFC}+\mathbf{V}=\mathbf{L}$ is complete, i.e.,

$$\psi \in \Phi_0 \quad \Rightarrow \quad [\mathrm{ZFC}+\mathbf{V}=\mathbf{L} \vdash \psi \text{ or } \mathrm{ZFC}+\mathbf{V}=\mathbf{L} \vdash \neg\psi].$$

[This is the other side of the lack of good candidates, for in other words, the successes in solving the Souslin hypothesis, the Kurepa hypothesis, n–cardinal transfer theorems in $\mathbf{L}$ deprives us of good candidates.] As large cardinals may enter, we may consider:

$(A)_1$ for $\psi \in \Phi_1$, ZFC+"there exists a supercompact cardinal" decides $\psi^{\mathbf{L}}$.

$(A)_2$ for all $\psi_1, \psi_2 \in \Phi_2$, the consistency strengths of $\mathrm{ZFC}+\psi_1$ and $\mathrm{ZFC}+\psi_2$ are comparable.

$(A)_3$ if $\mathrm{ZFC}+\psi$ is consistent, $\psi \in \Phi_3$, then we can get it by forcing.

We may consider weakenings. Of course, judging the success may be disputable, in fact, I am sure that 9.3, 9.4 will be eventually solved, whereas 9.7 probably will remain with answers like "completeness for reasonable logics in $\mathbf{L}$" (which is still very interesting).

There are works on independence for fragments of $\mathbf{PA}$ (with bounded induction, without $(\forall x)(2^x$ exists) (possibly with $(\forall x)(x^{[\log(x)]}$ exists)), they speak about pigeonhole principles when we add a new set as a predicate (see Ajtai [5]), this seems to me parallel to the Frankel–Mostowski method. I have thought about it lately because of [124].

Concerning 9.3, let us consider the following thought.

In $\mathrm{ZFC}+\mathbf{V}=\mathbf{L}$, every element has the form

$$\tau_{\varphi(x,\bar{y})}(\alpha,\bar{\beta}) = \{x \in \mathbf{L}_\alpha : (\mathbf{L}_\alpha, E) \models \varphi(x,\bar{\beta})\},$$

where $\bar{\beta} = \langle \beta_0, \dots, \beta_{n-1}\rangle$, $\beta_\ell < \alpha$ (wlog, $\beta_0 < \beta_1 < \dots$) and φ is first order. So let $\mathbf{I}$ be a linear order (which later can be non-standard) and let $\bar{\Phi} = \langle \Phi_\ell : \ell < n^*\rangle$ be an increasing sequence of sets of first order formulas (in set theory vocabulary $\{\in\}$). We may consider trying to define $\langle M_{\mathbf{I},\Phi_\ell} : \ell < n^*\rangle$ with $M_{\mathbf{I},\Phi_\ell} = \{\tau_\varphi(a,\bar{b}) : a \in \mathbf{I},\ b_\ell <_{\mathbf{I}} a,\ \varphi \in \Phi_\ell\}$ as in [149, §1].

10. Recent advances/comments

Section 1:

During the winter of 1999, Gitik told me that he can start with

$$\mathbf{V} \models \text{“}\ \kappa_n \text{ hypermeasurable of order } \lambda_n, \quad \lambda_n \text{ first (strongly) inaccessible } > \kappa_n,\ \lambda_n < \kappa_{n+1},\ \lambda > \kappa = \sum_n \kappa_n \text{”},$$

and find a forcing notion $\mathbb{P}$, not adding bounded subsets of $\kappa = \sum_{n<\omega} \kappa_n$, satisfying the κ^{++}–c.c., and making $2^\kappa \geq \lambda$. I have conjectured that combining this proof with earlier proofs, you can demand that forcing with $\mathbb{P}$ makes λ_n be the n–th inaccessible cardinal, $\kappa = \sum_n \lambda_n$, GCH holds below κ and $2^\kappa \geq \lambda$, so $\mathrm{pp}(\kappa) \geq \lambda$. Gitik has confirmed this conjecture with λ_n the n^{th} Mahlo cardinal. This proves that though 1.20 is open, other theorems which hold for λ of cofinality $\aleph_1$ cannot be generalized to cofinality $\aleph_0$.

On what the pcf theory tells you what you cannot do toward proving the consistency of the negation of the weak hypothesis 1.12(A) see Gitik and Shelah [43].

Section 2:
Concerning 2.2, 3.7, i.e., 2.13 and more generally on making the continuum large, see related work in progress [101].

Section 3:
Question 3.2 seems to be solved (see [111]), but certainly not along the lines described here. We show CON($\mathfrak{a} > \mathfrak{d}$) in the following way: we make the continuum large, and use ultrapower of the forcing notion with model theoretic point of view. This works also for $\mathfrak{u}$. However, we do not know

Question 10.1. (1) Is ZFC+$2^{\aleph_0} = \aleph_2$ consistent with $\mathfrak{a} > \mathfrak{d}$?
(2) What is the consistency strength of ZFC $+\mathfrak{a} > \mathfrak{u}$? (For $\mathfrak{a} > \mathfrak{d}$, ZFC suffices.)

Section 4:
Some examples of forcing notions as required in Problem 4.24 are given in Rosłanowski and Shelah [89, §1].

Section 5:
Regarding ($\otimes$) (formulated after 5.4), let us state explicitly:

Theorem 10.2. *If κ is a regular cardinal then one of the following occurs*
(a) *there is a subset S of κ^+ with a square on it (see ($\otimes$)(a) after 5.4) and such that stationarily many $\delta \in S$ has cofinality κ,*
(b) *there is a subset S of κ^{++} with square on it and such that stationarily many $\delta \in S$ has cofinality κ and $S \cap S^{\kappa^{++}}_\kappa$ does not reflect in any ordinal of cofinality κ^+.*

I think that 10.2 is nice and useful, e.g., for constructing objects in $\aleph_n$ for infinitely many n. In fact, I mention it as lately Eklof told me he has built sufficiently separable Abelian groups giving negative answers to Kaplanski's test problems in every $\aleph_n$ by induction on n, using, for $n < \omega$, the statement $I[\aleph_{n+2}] \restriction S^{\aleph_{n+2}}_{\aleph_{n+1}}$ is not trivial (that is, there is a stationary subset S of $S^{\aleph_{n+2}}_{\aleph_{n+1}}$ which belongs to $I[\aleph_{n+2}]$). I have advise him to use 10.2 (at the price of advancing from n to $n+1$ or $n+2$).

Proof. We prove slightly more. Let λ be regular, $\theta = \mathrm{cf}(\theta) < \lambda$ (in the theorem as stated $\lambda = \kappa^+$, $\theta = \kappa$). By [159, §4], there are $S \subseteq \lambda^+$ and a square $\langle C_\delta : \delta \in S\rangle$ (as defined in $\otimes$(a) mentioned above), such that

- $\delta \in S \quad \Rightarrow \quad \mathrm{otp}(C_\delta) < \lambda$, and
- $S_1 =: \{\delta \in S : \mathrm{cf}(\delta) = \theta\}$ is a stationary subset of λ^+.

Let $\gamma(*) < \lambda$ be the minimal γ such that the set

$$S^1_\gamma =: \{\delta \in S_1 : \mathrm{otp}(C_\delta) = \gamma\}$$

is stationary (so necessarily $\mathrm{cf}(\gamma(*)) = \theta$). Let e be a closed unbounded subset of $\gamma(*)$ of order type θ such that

$$\varepsilon \in \mathrm{nacc}(e) \quad \Rightarrow \quad \varepsilon \text{ is a successor ordinal.}$$

Let

$$S_2 =: \{\delta \in S : \mathrm{otp}(C_\delta) \in e \cup \{\gamma(*)\}\},$$

and for $\alpha \in S$ let $C^2_\alpha =: C_\alpha \cap S_2$. Clearly $S^1_{\gamma(*)} \subseteq S_2$, and $\alpha \in S_2 \Rightarrow \mathrm{otp}(C^2_\alpha) \leq \theta$.

Case 1: For some $\delta^* < \lambda^+$ of cofinality λ, the intersection $S^1_{\gamma(*)} \cap \delta^*$ is a stationary subset of δ^*.

Let $\langle \alpha_\varepsilon : \varepsilon < \lambda\rangle$ be an increasing continuous sequence of ordinals $< \delta^*$ with limit δ^* such that for any non limit $\varepsilon < \lambda$, the ordinal α_ε is non limit. Let

$$S^* =: \{\varepsilon < \lambda : \alpha_\varepsilon \in S_2\}$$

and for $\varepsilon \in S^*$ let $C^*_\varepsilon =: \{\zeta < \varepsilon : \alpha_\zeta \in C^2_{\alpha_\varepsilon}\}$. Now easily S^* is a stationary subset of λ, and

$$S^*_1 =: \{\varepsilon \in S^* : \mathrm{cf}(\varepsilon) = \theta\}$$

is a stationary subset of λ, and is equal to $\{\varepsilon < \lambda : \alpha_\varepsilon \in S^1_{\gamma(*)}\}$.

If $\theta = \aleph_0$, by [159, §4], we are done.

So we can assume that $\theta > \aleph_0$; hence $\varepsilon \in S^*_1 \Rightarrow \varepsilon = \sup(C^*_\varepsilon)$. Also, $\mathrm{otp}(C^*_\varepsilon) \leq \theta$ and C^*_ε is a closed subset of ε for $\varepsilon \in S^*$, and

$$\zeta \in C^*_\varepsilon \ \&\ \varepsilon \in S^* \quad \Rightarrow \quad C^*_\zeta = C^*_\varepsilon \cap \zeta.$$

So $\langle C^*_\varepsilon : \varepsilon \in S^*\rangle$ is as required in clause (a) of the theorem, except the requirement

($*$) if ε is a limit ordinal in S^*, then C^*_ε is unbounded in ε.

But this is a minor point — we can replace such ε's by their successors.

Case 1: Not case 1.

Then clause (b) of our theorem holds as exemplified by $\langle C^2_\alpha : \alpha \in S_2\rangle$. $\square$

Concerning the discussion on 5.12 ("is there an entangled linear order of cardinality λ^+ such that $\lambda = \lambda^{\aleph_0}$?"), we mentioned that if μ is strong limit singular of uncountable cofinality and $2^\mu > \aleph_{\mu^+}$, then it holds. Let me justify this:

Proposition 10.3. (1) *Assume:*

(a) *μ is a singular cardinal of cofinality κ,*

(b) $\lambda = \aleph_{\mu^+}$,

(c) *μ is strong limit, $\lambda \le 2^\mu$*
or at least

(c)$^-$ *$\mu = \mu^{<\kappa}$ is a limit cardinal, $\lambda \le \mu^\kappa$.*

Then *there is an entangled linear order of cardinality λ^+ and $\lambda = \lambda^{<\kappa}$.*

(2) *Assume*

(a) $\kappa = \mathrm{cf}(\kappa) < \mu \le \mathrm{cf}(\lambda) < \lambda$,

(b) $\lambda \le \mu^\kappa$, $\mu = \mu^{<\kappa}$,
or at least there is a tree with κ levels, $\le \mu$ nodes and at least λ κ–branches,

(c) *if $\mathfrak{a}$ is a set of $< \kappa$ regular cardinals from the interval (μ, λ), then* $\max \mathrm{pcf}(\mathfrak{a}) < \lambda$,

(d) *if $\theta \in (\mu, \lambda)$ and $\chi \in (\theta, \lambda)$ is the minimal singular cardinal χ such that* $\mathrm{pp}(\chi) \ge \lambda^+$, *then* $\mathrm{cf}(\chi) \le \kappa$.

Then *there is an entangled linear order of cardinality λ^+.*

Proof. (1) First note that

$(*)_1$ if $\theta < \lambda$ then $\theta^{<\kappa} < \lambda$.

[Why? By [166, Ch IX, 4.2 p.394] as $\mu^{<\kappa} = \mu < \lambda = \aleph_{\mu^+}$, or see [164].]

$(*)_2$ $\lambda^{<\kappa} = \lambda$.

[Why? By $(*)_1$ as $\mathrm{cf}(\lambda) > \kappa$.]

If for some $\alpha < \mu^+$, for every $\beta \in [\alpha, \mu^+)$ we have

$$\max \mathrm{pcf}(\{\aleph_{\gamma+1} : \gamma \in [\alpha, \beta)\}) < \lambda,$$

then we can get the desired conclusion as in the proof of [164, 4.2 p. 416, case I]. So assume

$(*)_3$ for every $\alpha < \mu^+$, for some $\beta \in [\alpha, \mu^+)$ we have

$$\lambda \le \max \mathrm{pcf}(\{\aleph_{\gamma+1} : \gamma \in [\alpha, \beta)\});$$

note that the equality never holds.

For every $\alpha < \mu^+$ which is above μ let $\beta(\alpha) \in (\alpha, \mu^+)$ be the first ordinal β such that $\max \mathrm{pcf}(\{\aleph_{\gamma+1} : \gamma \in [\alpha, \beta)\}) > \lambda$. So $\aleph_{\beta(\alpha)}$ is a singular cardinal $> \aleph_\alpha$ of cofinality $\le \mu$ such that $\lambda \le \mathrm{pp}(\aleph_{\beta(\alpha)})$. Choose by induction of $i < \mu^+$ an ordinal $\alpha(i) < \mu^+$ such that $j < i \Rightarrow \beta(\alpha(j)) < \alpha(i)$. Let $\alpha(*) = \sup\{\alpha(i) : i < \mu\}$ and let

$$\mathfrak{a}_i = \{\theta : \theta \text{ is a regular cardinal } \in [\aleph_{\alpha(i)}, \aleph_{\alpha(i+1)})\}$$

and

$$\mathfrak{b}_i = \{\theta \in \mathfrak{a}_i : \quad \text{for no set } \mathfrak{c} \subseteq \mathfrak{a}_i \cap \theta \text{ of cardinality } \leq \mu \text{ do we have } \theta \in \mathrm{pcf}(\mathfrak{c})\}.$$

Now, for every $i < \mu$ and regular $\theta \in [\aleph_{\alpha(*)}), \lambda)$ there is a subset $\mathfrak{c}_{\theta,i}$ of $\mathfrak{a}_i$ of cardinality $\leq \mu$ such that $\theta = \max\mathrm{pcf}(\mathfrak{c}_{\theta,i})$. In fact, wlog $\mathfrak{c}_{\theta,i} \subseteq \mathfrak{b}_i$. By [166, Ch II, 4.11 p.81], this implies that

$(*)_4$ we have $\mathrm{Ens}(\theta, \mu)$ for any regular $\theta \in [\aleph_{\alpha(*)}, \aleph_{\mu^+})$.

Let δ^* be the first limit ordinal $\delta > \alpha(*)$ such that $\lambda \leq \mathrm{pp}(\aleph_\delta)$. By a previous assumption, $\delta^* < \mu^+$, hence $\mathrm{cf}(\delta^*) < \mu$, and also $\kappa \leq \mathrm{cf}(\delta^*)$. Can $\mathrm{cf}(\delta^*) > \kappa$? No, by [164, 3.4 p.412].

So necessarily $\mathrm{cf}(\delta^*) = \kappa$ and we can find a set $\mathfrak{a}$ of $\leq \kappa$ regular cardinals in the interval $(\aleph_{\alpha(*)}, \aleph_{\delta^*})$ such that:

(i) $\max\mathrm{pcf}(\mathfrak{a}) = \lambda^+$,
(ii) if $\theta \in \mathfrak{a}$ then $\max\mathrm{pcf}(\mathfrak{a} \cap \theta) < \theta$,
(iii) $\mathfrak{a}$ is unbounded in $\aleph_{\delta^*}$.

Recall that $\mathrm{Ens}(\theta, \mu)$ for every $\theta \in \mathfrak{a}$, so by the proposition below we are done.

(2) Similarly, using sets of at most $\mathrm{cf}(\lambda)$ regular cardinals in (μ, λ). $\square$

Proposition 10.4. (1) *Assume*

(a) $\mathfrak{a}$ *is a set of regular cardinals* $> |\mathfrak{a}|$,
(b) *for every* $\theta \in \mathfrak{a}$ *we have* $\max\mathrm{pcf}(\mathfrak{a} \cap \theta) < \theta$,
(c) J *is an ideal on* $\mathfrak{a}$ *extending* $J^{\mathrm{bd}}_{\mathfrak{a}}$,
(d) $\prod \mathfrak{a}/J$ *has true cofinality* λ,
(e) $\mathrm{cf}(\sup(\mathfrak{a})) = \kappa$,
(f) $\langle \mu_i : i < \kappa\rangle$ *is a non-decreasing sequence of cardinals such that* $\prod\limits_{j<i} \mu_j \leq \mu_i$,
(g) $\sup(\mathfrak{a}) \leq \prod\limits_{j<\kappa} \mu_j$,
(h) $\langle i_\theta : \theta \in \mathfrak{a}\rangle$ *is non-decreasing with limit* κ *and such that* $\mathrm{Ens}(\theta, \mu_{i_\theta})$ *for* $\theta \in \mathfrak{a}$.

Then *there is an entangled linear order of cardinality* λ^+.

(2) *In part (1), we can omit assumption (f) and do not require* $\langle i_\theta : \theta \in \mathfrak{a}\rangle$ *to be non-decreasing, but then we demand*

(i) *for each* $\theta^* \in \mathfrak{a}$, $\{\theta \in \mathfrak{a} : i_\theta = i_{\theta^*}\} \notin J_{<\lambda}[\mathfrak{a}]$, $\lambda = \max\mathrm{pcf}(\mathfrak{a})$,
(j) $\langle E_i : i < \kappa\rangle$ *is a sequence of equivalence relations on* λ *such that for every finite* $w \subseteq \lambda$, *for some* i, $E_i \restriction w$ *is the equality.*

Proof. Let $\langle f_\alpha : \alpha < \lambda\rangle$ be a sequence of pairwise distinct members of $\prod \mathfrak{a}$ such that for each $\theta \in \mathfrak{a}$ the set $T_\theta = \{f_\alpha \restriction \theta : \alpha < \lambda\}$ has cardinality $< \theta$ —

exists by the assumptions (the only place we use $\prod \mathfrak{a}/J$ has true cofinality λ), see [166, II 3.1].

Let $\langle \theta(\zeta) : \zeta < \kappa \rangle$ be a strictly increasing sequence of cardinals $\in \mathfrak{a}$, unbounded in it. Let $\langle E_{i,j} : j < \kappa \rangle$ be a sequence of equivalence relations on $T_{\theta(i)}$. For $\theta \in \mathfrak{a}$ let $\zeta(\theta) = \min\{\zeta : \theta \leq \theta(\zeta)\}$ and let E_θ be the following equivalence relation on T_θ:

$f\ E_\theta\ g$ if and only if

$f, g \in T_\theta$ and for every $i, j < \zeta$ we have $(f \restriction i)\ E_{i,j}\ (g \restriction i)$.

Clearly

$(*)_5$ E_θ is an equivalence relation with $\leq \mu_{\zeta(\theta)}$ equivalence classes,

$(*)_6$ if $\theta < \chi$ are from $\mathfrak{a}$ and $f, g \in T_\chi$ and $f\ E_\chi\ g$ then $(f \restriction \theta)\ E_\theta\ (g \restriction \theta)$.

So for each $\theta \in \mathfrak{a}$, $f \in T_\theta$ we can choose a linear order $<_f$ on θ such that

$(*)_7$ if $f\ E_\theta\ g$ then $<_f = <_g$,

$(*)_8$ if $n < \omega$, and $f_1, \ldots, f_n \in T_\theta$ are pairwise non-E_θ-equivalent then $\langle \theta, <_{f_m} : m = 1, \ldots, n \rangle$ is an entangled sequence of linear orders.

Now we linearly order $\{f_\alpha : \alpha < \lambda\}$ by:

$(*)_9$ if $f_\alpha \restriction \theta = f_\beta \restriction \theta$, call it g, and $f_\alpha(\theta) < f_\beta(\theta)$ then:

$f_\alpha <^* f_\beta$ if and only if

$f_\alpha(\theta) <_g f_\beta(\theta)$.

The rest as in [166, Ch II section 4]. $\square$

Section 6:

Concerning questions from Monk [84] of the form "can the invariant of the ultraproduct of the Boolean Algebras be smaller then the ultraproducts of the invariants of the B_i", Magidor and Shelah [77] deal with consistency results. It is continued in Shelah and Spinas [180], and see more in [121].

Concerning [84, Problems 10, 11] on depth see [112].

About 6.7(2): By [106], the answer is no, i.e., if B is a superatomic Boolean Algebra with $> \beth_4(\theta)$ elements, then it has an automorphism moving $> \theta$ atoms. It is essentially reduced to problems on $Pr(\lambda, \lambda, \theta, \aleph_0)$ where

Definition 10.5. $Pr(\mu, \lambda, \theta, \kappa)$ means that there is a family $\mathcal{A}$ of μ subsets of λ each of cardinality κ, the intersection of any two of cardinality $< \kappa$, such that any subset of λ of cardinality θ contains one of them.

(Mild pcf conjectures guarantee existence in many cases.)

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