SUCCESSOR OF SINGULARS: COMBINATORICS
AND NOT COLLAPSING CARDINALS
\leq \kappa \text{ IN } (<\kappa)\text{-SUPPORT ITERATIONS}

SH667

SAHARON SHELAH

The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem 91904, Israel

Department of Mathematics
Hill Center - Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019 USA

Department of Mathematics
University of Wisconsin
Madison, WI USA

ABSTRACT. On the one hand we deal with (<\kappa)-supported iterated forcing notions which are (\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)-complete, have in mind problems on Whitehead groups, uniformizations and the general problem. We deal mainly with the case of a successor of the singular cardinal. This continues [Sh 587]. On the other hand we deal with complimentary ZFC combinatorial results.

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§1  GCH implies for successor of singular no stationary $S$ has uniformization

[For $\lambda$ strong limit singular, for stationary $S \subseteq S^\lambda_{\text{cf}(\lambda)}$ we prove strong negation of uniformization for some $S$-ladder system and even weak versions of diamond. E.g. if $\lambda$ is singular strong limit and $2^\lambda = \lambda^+$, then there are $\gamma_i^\delta < \delta$ increasing in $i < \text{cf}(\lambda)$ with limit $\delta$ for each $\delta \in S$ such that for every $f : \lambda^+ \to \alpha^* < \lambda$ for stationarily many $\delta \in S$, for every $i$ we have $f(\gamma_{2i}^\delta) = f(\gamma_{2i+1}^\delta).$]

§2  Forcing for successor of singulars

[Let $\lambda$ be strong limit singular $\kappa = \lambda^+ = 2^\lambda, S \subseteq S_{\text{cf}(\lambda)}^\kappa$ stationary not reflecting. We present the consistency of a forcing axiom implying e.g.: if $h_\delta$ is a function from $A_\delta$ to $\theta, A_\delta \subseteq \delta = \text{sup}(A_\delta), \text{otp}(A_\delta) = \text{cf}(\lambda), \theta < \lambda$ then for some $h : \kappa \to \theta$ for every $\delta \in S$ we have $h_\delta \subseteq h.$]

§3  $\kappa^+$-c.c. and $\kappa^+$-pic

[In the forcing axioms we would like to allow forcing notions of cardinality $> \kappa$; for this we use a suitable chain condition (allowed here and in [Sh 587]). This sheds more light on the strongly inaccessible case and we comment on this (and forcing against cases of diamonds).]

§4  Existence of non-free Whitehead groups (and Ext$(G,\mathbb{Z}) = 0$) abelian groups in successor of singulars

[We use the information on the existence of weak version of the diamond for $S \subseteq S_{\text{cf}(\lambda)}^\lambda, \lambda$ strong limit singular with $2^\lambda = \lambda^+$, to prove that there are some abelian groups with special properties (from reasonable assumptions). We also get more combinatorial principles on $\lambda = \mu^+, \mu > \text{cf}(\mu)$ (even if just $\lambda = \lambda^{2^\omega}$).]
§1 GCH implies for successor of singular
no stationary $S$ has uniformization

We show that a major improvement in [Sh 587] over [Sh 186] for inaccessible (every ladder on $S$ has uniformization rather than some ladder on $S$) cannot be done for successor of singulars. This is continued in §4.

1.1 Fact: Assume

(a) $\lambda$ is strong limit singular with $2^\lambda = \lambda^+$, let $\text{cf}(\lambda) = \sigma$

(b) $S \subseteq \{ \delta < \lambda^+ : \text{cf}(\delta) = \sigma \}$ is stationary.

Then we can find $\langle \gamma^\delta_i : i < \sigma : \delta \in S \rangle$ such that

(α) $\gamma^\delta_i$ is increasing (with $i$) with limit $\delta$

(β) if $\mu < \lambda$ and $f : \lambda^+ \to \mu$ then the following set is stationary:

$$\{ \delta \in S : f(\gamma^\delta_i) = f(\gamma^\delta_{i+1}) \text{ for every } i < \sigma \}.$$ Moreover

(β) if $f_i : \lambda^+ \to \mu_i, \mu_i < \lambda$ for $i < \sigma$ then the following set is stationary:

$$\{ \delta \in S : f_i(\gamma^\delta_i) = f_i(\gamma^\delta_{i+1}) \text{ for every } i < \sigma \}.$$

Proof. This will prove 1.2, too. We first concentrate on (α) + (β) only.

Let $\lambda = \sum_{i < \sigma} \lambda_i, \lambda_i$ a cardinal increasing continuous with $i$, $\lambda_{i+1} > 2^{\lambda_i}, \lambda_0 > 2^\sigma$. For $\alpha < \lambda^+$, let $\alpha = \bigcup_{i < \sigma} a_{\alpha,i}$ such that $|a_{\alpha,i}| \leq \lambda_i$. Without loss of generality $\delta \in S \Rightarrow \delta$ divisible by $\lambda^\alpha$ (ordinal exponentiation). For $\delta \in S$ let $\langle \beta^\delta_i : i < \sigma \rangle$ be increasing continuous with limit $\delta, \beta^\delta_i$ divisible by $\lambda$ and $> 0$. For $\delta \in S$ let $\langle b^\delta_i : i < \sigma \rangle$ be such that: $b^\delta_i \subseteq \beta^\delta_i, |b^\delta_i| \leq \lambda_i, b^\delta_i$ is increasing continuous with $i$ and $\delta = \bigcup_{i < \sigma} b^\delta_i$ (e.g. we can let $b^\delta_i = \bigcup_{j_1, j_2 < i} a_{j_1,j_2} \cup \lambda_i$). We further demand $\lambda_i \subseteq b^\delta_i \cap \lambda_i$. Let $\langle f^{\alpha}_i : \alpha < \lambda^+ \rangle$ list the two-place functions with domain an ordinal $< \lambda^+$ and range $\subseteq \lambda^+$. Let $S = \bigcup_{\mu < \lambda} S_\mu$, with each $S_\mu$ stationary and $\langle S_\mu : \mu < \lambda \rangle$ pairwise disjoint. We now fix $\mu < \lambda$ and will choose $\bar{\gamma}^\delta = \langle \gamma^\delta_i : i < \sigma \rangle$ for $\delta \in S_\mu$ such that clause (α) holds and clause (β) holds (that is for every $f : \lambda^+ \to \mu$ for stationary many $\delta \in S_\mu$ the conclusion of clause (β) holds), this clearly suffices.

Now for $\delta \in S_\mu$ and $i < j < \sigma$ we can choose $\zeta^\delta_{i,j,\epsilon}$ (for $\epsilon < \lambda_j$) (really here we use just $\epsilon = 0, 1$) such that:
(A) \[ \langle \zeta^\delta_{i,j,\varepsilon} : \varepsilon < \lambda_j \rangle \] is a strictly increasing sequence of ordinals.

(B) \[ \beta^\delta_i < \zeta^\delta_{i,j,\varepsilon} < \beta^\delta_{i+1} \] (can even demand \( \zeta^\delta_{i,j,\varepsilon} < \beta^\delta_i + \lambda \))

(C) \[ \zeta^\delta_{i,j,\varepsilon} \notin \{ \zeta^\delta_{i_1,j_1,\varepsilon_1} : j_1 < j, \varepsilon_1 < \lambda_{j_1} \} \] (and \( i_1 < \sigma \), really only \( i_1 = i \) matters)

(D) for every \( \alpha_1, \alpha_2 \in b^\delta_j \), the sequence \( \langle \min \{ \lambda_j, f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,\varepsilon}) \} : \varepsilon < \lambda_j \rangle \) is constant i.e.: one of the following occurs:

\[ \begin{align*}
(\alpha) & \quad \varepsilon < \lambda_j \Rightarrow (\alpha_2, \zeta^\delta_{i,j,\varepsilon}) \notin \text{Dom}(f^*_{\alpha_1}) \\
(\beta) & \quad \varepsilon < \lambda_j \Rightarrow f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,\varepsilon}) = f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,0}), \text{ well defined} \\
(\gamma) & \quad \varepsilon < \lambda_j \Rightarrow f^*_{\alpha_1}(\alpha_2, \zeta^\delta_{i,j,\varepsilon}) \geq \lambda_j, \text{ well defined.}
\end{align*} \]

For each \( i < j < \sigma \) we use “\( \lambda \) is strong limit \( \lambda_j \geq \sum_{j_1 < j} \lambda_{j_1} + \sigma^\varepsilon \).

Let \( G = \{ g : g \text{ a function from } \sigma \text{ to } \sigma \text{ such that } (\forall i < \sigma)( i < g(i) \} \). For each function \( g \in G \) we try \( \bar{g}^\delta = \langle \zeta^\delta_{i,g(i),0}, \zeta^\delta_{i,g(i),1} : i < \sigma \rangle \) i.e. \( \langle \zeta^\delta_{2i}, \zeta^\delta_{2i+1} \rangle = \langle \gamma^\delta_{i,g(i),0}, \gamma^\delta_{i,g(i),1} \rangle \).

Now we ask for each \( g \in G \):

**Question^\mu:** Does \( \langle \bar{g}^\delta : \delta \in S_\mu \rangle \) satisfy

\[ (\forall f \in \lambda^\ast \mu)(\exists \text{stat } \delta \in S_\mu)(\bigwedge_{i < \sigma} f(\bar{g}^\delta_{2i}) = f(\bar{g}^\delta_{2i+1}))? \]

If for some \( g \in G \) the answer is yes, we are done. Assume not, so for each \( g \in G \) we can find \( f_g : \lambda^+ \to \mu \) and a club \( E_g \) of \( \lambda^+ \) such that:

\[ \delta \in S_\mu \cap E_g \Rightarrow (\exists i < \sigma)(f_g(\bar{g}^\delta_{2i}) \neq f_g(\bar{g}^\delta_{2i+1})) \]

which means

\[ \delta \in S_\mu \cap E_g \Rightarrow (\exists i < \sigma)[f_g(\zeta^\delta_{i,g(i),0}) \neq f_g(\zeta^\delta_{i,g(i),1})]. \]

Let \( G = \{ g_\varepsilon : \varepsilon < 2^\sigma \} \), so we can find a 2-place function \( f^* \) from \( \lambda^+ \) to \( \mu \) satisfying

\[ f^*(\varepsilon, \alpha) = f_{g_\varepsilon}(\alpha) \text{ when } \varepsilon < 2^\sigma, \alpha < \lambda^+. \]

Hence for each \( \alpha < \lambda^+ \) there is \( \gamma[\alpha] < \lambda^+ \) such that \( f^* \upharpoonright \alpha = f^*_{\gamma[\alpha]} \).

Let \( E^* = \bigcap_{\varepsilon < 2^\sigma} E_{g_\varepsilon} \cap \{ \delta < \lambda^+ : \text{ for every } \alpha < \delta \text{ we have } \gamma[\alpha] < \delta \} \). Clearly it is a club of \( \lambda^+ \), hence we can find \( \delta \in S_\mu \cap E^* \). Now \( \beta^\delta_{i+1} < \delta \) hence \( \gamma[\beta^\delta_{i+1}] < \delta \) (as \( \delta \in E^* \)) but \( \delta = \bigcup_{i < \sigma} b^\delta_i \) hence for some \( j < \sigma, \gamma[\beta^\delta_{i+1}] \in b^\delta_j \); as \( b^\delta_j \) increases with
we can define a function $h : \sigma \to \sigma$ by $h(i) = \text{Min}\{j : j > i + 1 \text{ and } \mu < \lambda_j \text{ and } \gamma[\beta_{i+1}^\delta] \subseteq b_j^\delta\}$. So $h \in H$ hence for some $\varepsilon(*) < 2^\sigma$ we have $h = g_{\varepsilon(*)}$. Now looking at the choice of $\zeta_{i,h(i),0}, \zeta_{i,h(i),1}$ we know (remember $2^\sigma < \lambda_0 \subseteq b_j^\delta$ and $\mu < \lambda_{h(i)}$)

$$(\forall \varepsilon < 2^\sigma)(\forall \alpha \in b_{h(i)}^\delta)[\text{Rang}(f_{\alpha}^*) \subseteq \mu \& \text{ Dom}(f_{\alpha}^*) \supseteq \beta_{i+1}^\delta \rightarrow f_{\alpha}^*(\varepsilon, \zeta_{i,h(i),0}) = f_{\alpha}^*(\varepsilon, \zeta_{i,h(i),1})].$$

In particular this holds for $\varepsilon = \varepsilon(*)$, $\alpha = \gamma[\beta_{i+1}^\delta]$, so we get

$$f_{\gamma[\beta_{i+1}^\delta]}^*(\varepsilon(*), \zeta_{i,h(i),0}) = f_{\gamma[\beta_{i+1}^\delta]}^*(\varepsilon(*), \zeta_{i,h(i),1}).$$

By the choice of $f^*$ and of $\gamma[\beta_{i+1}^\delta]$ this means

$$f_{g_{\varepsilon(*)}}(\zeta_{i,h(i),0}) = f_{g_{\varepsilon(*)}}(\zeta_{i,h(i),1})$$

but $h = g_{\varepsilon(*)}$ and the above equality means $f_{g_{\varepsilon(*)}}(\gamma_{2i}^{g_{\varepsilon(*)}}) = f_{g_{\varepsilon(*)}}(\gamma_{2i+1}^{g_{\varepsilon(*)}})$, and this holds for every $i < \sigma$, and $\delta \in E^* \Rightarrow \delta \in E_{g_{\varepsilon(*)}}$ so we get a contradiction to the choice of $(f_{g_{\varepsilon(*)}}, E_{\varepsilon(*)})$.

So we have finished proving $(\alpha) + (\beta)$.

How do we get $(\beta)^+$ of 1.1, too?

The first difference is in phrasing the question, now it is, for $g \in G$:

**Question** $\mu$: Does $\langle \gamma^{g,\delta} : \delta \in S^g \rangle$ satisfy:

$$
\left((\forall f_0 \in \lambda^+ \mu_0)(\forall f_1 \in \lambda^+ \mu_1)\ldots(\forall f_i \in \lambda^+ \mu_i)\ldots\right.)_{i < \sigma}(\exists \text{stat } \delta \in S^g)(\bigwedge_{i < \sigma} f_i(\gamma_{2i}^{g,\delta}) = f_i(\gamma_{2i+1}^{g,\delta})).
$$

If for some $g$ the answer is yes, we are done, so assume not so we have $f_{g,i} \in \lambda^+(\mu_i)$ for $g \in G, i < \sigma$ and club $E_g$ of $\lambda^+$ such that

$$\delta \in S^g \cap E_g \Rightarrow (\exists i < \sigma)(f_{g,i}(\gamma_{2i}^{g,\delta}) \neq f_{g,i}(\gamma_{2i+1}^{g,\delta})).$$

A second difference is the choice of $f^*$ as $f^*(\sigma \varepsilon + i, \alpha) = f_{g_{\varepsilon,i}}(\alpha)$ for $\varepsilon < 2^\sigma,$ $i < \sigma, \alpha < \lambda^+.$

Lastly, the equations later change slightly.

$\Box_{1.1}$
1.2 Fact: 1) Under the assumptions (a) + (b) of 1.1 letting $\bar{\lambda} = \langle \lambda_i : i < \sigma \rangle$ be increasingly continuous with limit $\lambda$ such that $2^\sigma < \lambda_0, 2^{\lambda_i} < \lambda_{i+1}$ we have $(*)_1 + (*)_2$ where

$(*)_1$ we can find $\langle \gamma^\delta_\zeta : \zeta < \lambda : \delta \in S \rangle$ such that

$(\alpha)$ $\gamma^\delta_\zeta$ is increasing in $\zeta$ with limit $\delta$

$(\beta)^+$ if $f_i : \lambda^+ \to \lambda_{i+1}$, for $i < \sigma$, then the following set is stationary

$\{ \delta \in S : f_i(\gamma^\delta_\zeta) = f_i(\gamma^\delta_{\xi}) \text{ when } \zeta, \xi \in [\lambda_i, \lambda_{i+1}) \text{ for every } i < \sigma \}$

$(*)_2$ moreover if $F_i : [\lambda^+] < \lambda \to [\lambda^+] < \lambda$ for $i < \sigma$ (or just $F_i : [\lambda^+] < \lambda \to [\lambda^+] < \lambda$) and $\sup(w) < \min(F_i(w))$ for $w \in [\lambda^+] < \lambda$, for each $i < \sigma$, then in addition we can demand

$(i)$ $\{ \gamma^\delta_\zeta : \zeta \in [\lambda_i, \lambda_{i+1}] \} \subseteq F_i(\{ \gamma^\delta_\zeta : \zeta < \lambda_i \})$

$(ii)$ $|\{ \gamma^\delta_\zeta : \zeta < \lambda^* : \gamma^\delta_\zeta = \gamma \}| \leq \lambda$ for each $\gamma < \lambda^*$ and $\lambda^* < \sigma$

2) Assume $\lambda, \langle \lambda_i : i < \sigma \rangle$ are as in part (1) and $\langle C_\delta : \delta \in S \rangle$ is given, it guess clubs (for $\lambda^+$, which mean that for every club $E$ of $\lambda^+$ the set $\{ \delta \in S : C_\delta \subseteq E \}$ is a stationary subset of $\lambda^+$) and $C_\delta = \{ \alpha[\delta, i] : i < \sigma, \alpha[\delta, i] \text{ divisible by } \lambda^\omega \}$ increasing in $i$ with limit $\delta, \langle \text{cf}(\alpha[\delta, i+1]) : i < \sigma \rangle$ is increasing with limit $\lambda$ and let $\beta(\delta, i) = \sum_{j<i} \lambda_j \times \text{cf}(\alpha[\delta, j])$. Then

$(*)$ we can find $\langle \gamma^\delta_\zeta : \zeta < \lambda^* : \delta \in S \rangle$ such that

$(\alpha)$ $\langle \gamma^\delta_\zeta : \zeta < \lambda \rangle$ is increasing with limit $\delta$, (for $\delta \in S$)

$(\beta)$ $\sup\{ \gamma^\delta_\zeta : \gamma^\delta_\zeta < \beta[\delta, j+1] \} = \alpha[\delta, j]$

$(\gamma)$ for every $f_i \in (\lambda^+)(\mu_i)$ for $i < \sigma$ where $\mu_i < \lambda$ and club $E$ of $\lambda^+$, for stationarily many $\delta \in S$ we have $\{ \gamma^\delta_\zeta : i < \lambda \} \subseteq E$ and $f_i(\gamma^\delta_\zeta) = f_i(\gamma^\delta_{\xi})$, when $\zeta, \xi \in [\beta[\delta, i] + \lambda_i \xi, \beta[\delta, i] + \lambda_i \xi + \lambda_i)$ and $\xi < \text{cf}(\alpha[\delta, i])$.

Proof. 1) The same proof as in 1.1 for $(*)_1$, but see a proof after the proof of 4.2. 2) Should be clear, too.
§2 Case C: Forcing for successor of singular

We continue [Sh 587].

2.1 Hypothesis. 1) $\lambda$ strong limit singular $\sigma = \text{cf}(\lambda) < \lambda, \kappa = \lambda^+, \mu^* \geq \kappa, 2^\lambda = \lambda^+$.

2.2 Definition. 1) Let $\mathcal{C}_{<\kappa}(\mu^*)$ be the family of $\hat{\mathcal{E}}_0 \subseteq \{ \bar{a} : \bar{a} = \langle a_i : i \leq \alpha \rangle \}$ where $\alpha < \kappa, a_i \in [\mu^*]^{<\kappa}$ increasing continuous, and $a_i \subseteq \kappa \in \kappa$ such that: for every $\theta = \text{cf}(\theta) < \lambda, \chi$ large enough and $x \in \mathcal{H}(\chi)$ we can find $\langle N_i : i \leq \theta \rangle$ obeying $\bar{a} \in \hat{\mathcal{E}}_0$ (with error some $n$ see [Sh 587, B.5.1(2)]) and such that $x \in N_0$; this repeats [Sh 587, B.5.1(2)]; formally we should say that $\bar{N}$ obeys $\bar{a}$ for $\mu^*$.
2) $\mathcal{C}_{<\kappa}^1(\mu^*)$ is the family of $\hat{\mathcal{E}}_1 \subseteq \{ \bar{a} : \bar{a} = \langle a_i : i \leq \sigma \rangle, a_i$ increasing continuous, $i < \sigma \Rightarrow |a_i| < \lambda$ and $\lambda + 1 \subseteq \bigcup_{i<\sigma} a_i \}$.

2.3 Definition. 1) We say $\tilde{M} = \langle M_i : i \leq \sigma \rangle$ is ruled by $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ if, for some $\chi > \mu^*$:
   
   (a) $\hat{\mathcal{E}}_0 \subseteq \mathcal{C}_{<\kappa}(\mu^*), \hat{\mathcal{E}}_1 \subseteq \mathcal{C}_{<\kappa}^1(\mu^*)$
   
   (b) for some $\langle M^i : -1 \leq i < \sigma \rangle$ and $\langle \tilde{N}^i : -1 \leq i < \sigma \rangle$ we have:
      
      (\alpha) $M_i < (\mathcal{H}(\chi), \in, <^\chi)$
      
      (\beta) $\tilde{M}$ obeys some $\bar{a} \in \hat{\mathcal{E}}_1$ for some finite error (so for some $n$, for every $i, a_i \subseteq M_i \cap \mu^* \subseteq a_{i+n}$) and $\tilde{M} \upharpoonright (i+1) \in M_{i+1}$ and $j < i \Rightarrow M_j < M_j$ and $M_i$ is increasing continuous
      
      (\gamma) $[M_{i+1}]^{2\|M_i\|} \subseteq M_{i+1}$ for $i$ a limit ordinal $\sigma$
      
      (\delta) $\tilde{M}^i = \langle \tilde{M}^i_\alpha : \alpha \leq \delta_i \rangle, \tilde{N}^i = \langle N^i_\alpha : \alpha \leq \delta_i \rangle$ and $M^i_\alpha \subseteq N^i_\alpha < (\mathcal{H}(\chi), \in, <^\chi)$ and $\lambda + 1 \subseteq N^i_\alpha$ and $\|M^i_\alpha\| = \|M^i_\alpha\|^{\|M_i\|}$ for $\alpha < \delta_i$ non limit, $\|M^i_\alpha\|^{\|M_i\|} \subseteq M^i_{\beta+1}, \beta < \delta_i$
      
      (\epsilon) $\langle N^i_\alpha : \alpha \leq \delta_i \rangle = \tilde{N}^i$ obeys some $\bar{b}_i \in \hat{\mathcal{E}}_0$ for some finite error and $M^i, \tilde{N}^i$ are increasing continuous
      
      (\zeta) $M_{i+1} = M^i_\delta \subseteq N^i_\delta$ and $\langle (\tilde{M}^i, \tilde{N}^i) : j < i \rangle \in M^i_0$
      
      (\eta) $\delta_i \subseteq M_{i+1}$ (hence $\delta_i < \lambda$) and $\lambda \subseteq N^i_{\alpha_i}$
      
      (\theta) $\text{cf}(\delta_i) > 2\|M_i\|$ for $i$ limit,
      
      (\iota) $\tilde{N}^i \upharpoonright (\alpha + 1), \tilde{M}^i \upharpoonright (\alpha + 1) \in M^i_{\alpha+1}$ for $\alpha < \delta_i, i < \sigma$ hence $N^i_\beta = \text{Sk}_{(\mathcal{H}(\chi), \in, <^\chi)(M^i_\beta \cup \lambda)}$ when $i < \omega \sigma$ and $\beta \leq \delta_i$ is a limit ordinal

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1We may later ignore the $i = -1$ in our notation.
(κ) \( N^i_{\delta_i} < N^j_0 \) for \( i < j \)
(λ) \( M_i < M^0_i, M_i \in M_0 \).

2) We say above that \( \langle \tilde{M}^i : i < \sigma \rangle, \langle \tilde{N}^i : i < \sigma \rangle \) is an \( (\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \)-approximation to \( \bar{M} \).

3) Let \( \mathcal{C}_{<\kappa}(\mu^*) \) be the family of \( (\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \) such that:

(a) \( \tilde{\mathcal{E}}_0 \in \mathcal{C}_{<\kappa}(\mu^*) \) and \( \tilde{\mathcal{E}}_1 \in \mathcal{C}^1_{<\kappa}(\mu^*) \)
(b) for \( \chi \) large enough and \( x \in \mathcal{H}(\chi) \) we can find \( \tilde{M} \) which is ruled by \( (\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \) and \( x \in M_0 \)
(c) \( \tilde{\mathcal{E}}_0 \) is closed (see below).

4) \( \tilde{\mathcal{E}}_0 \) is closed if \( \langle a_i : i \leq \alpha \rangle \in \tilde{\mathcal{E}}_0, \gamma \leq \beta \leq \alpha \) implies \( \langle a_i : i \in [\beta, \gamma] \rangle \in \tilde{\mathcal{E}}_0 \).

Remark. 1) In Definition 2.3(1), letting \( \bar{N} = \bar{N}^0 \cup \bar{N}^1 \ldots \) i.e. \( \bar{N} = \langle N_i : i < \lambda \rangle, N_\varepsilon =: N^\varepsilon_i \) if \( \varepsilon = \sum_{j<i} \delta_j + \alpha \); so \( \ell g(\bar{N}) = \lambda \) and \( \bar{N} \upharpoonright (i_0 + 1) \in N_{i_0 + 1} \) so \( \bar{N} \) is \( \prec \)-increasingly continuous, and \( \gamma < \lambda \Rightarrow \bar{N} \upharpoonright \gamma \in N_{\gamma + 1} \).

2.4 Claim. 1) Assume \( \tilde{\mathcal{E}}_0 \in \mathcal{C}_{<\kappa}(\mu^*) \) and \( \bar{Q} = \langle P_\alpha, Q_i : i < \gamma \rangle \) is a \( (\kappa) \)-support iteration such that \( \Vdash_{P_i} \text{“} Q_i \text{ is strongly } \tilde{\mathcal{E}}_0 \text{-complete” for each } i < \gamma \), see [Sh 587, B.5.3(3)].
Then \( P_\gamma \) is strongly \( \tilde{\mathcal{E}}_0 \)-complete (hence \( P_\gamma / P_\beta \)).
2) If \( Q \) is \( \tilde{\mathcal{E}}_0 \)-complete, then \( \Vdash Q \models \tilde{\mathcal{E}}_0 \text{ non-trivial.} \)

Proof. By [Sh 587, B.5.6] (here the choice “for any regular cardinal \( \theta < \kappa \)” rather than “for any cardinal \( \theta < \kappa \)” in [Sh 587, B.5.1(2)] is important). \( \square \)

2.5 Definition. Let \( (\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \in \mathcal{C}_{<\kappa}(\mu^*) \) and let \( Q \) be a forcing notion.
1) For a sequence \( \tilde{M} = \langle M_i : i \leq \sigma \rangle \) ruled by \( (\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \) with an \( (\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \)-approximation \( \langle \tilde{M}^i : i < \sigma \rangle, \langle \tilde{N}^i : i < \sigma \rangle \) and a condition \( r \in Q \) we define a game \( \mathcal{G}_{\tilde{M}, \langle \tilde{M}^i : i < \sigma \rangle, \langle \tilde{N}^i : i < \sigma \rangle}(Q, r) \) between two players COM and INC.
The play lasts \( \sigma \) moves during which the players construct a sequence \( \langle i_0, p, \langle p_i, q_i : i_0 - 1 \leq i < \sigma \rangle \rangle \) such that \( i_0 < \sigma \) is non-limit, \( p \in M_{i_0} \cap Q, p_i \in M_{i+1} \cap Q, q_i = \langle q_i, \varepsilon : \varepsilon < \delta_i \rangle \subseteq Q \) (where \( \delta_i + 1 = \ell g(\tilde{N}^i) \)).
The player INC first decides what is $i_0 < \delta$ and then it chooses a condition $p \in \mathbb{Q} \cap M_{i_0}$ stronger than $r$. Next, at the stage $i \in [i_0 - 1, \delta)$ of the game, COM chooses $p_i \in \mathbb{Q} \cap M_{i+1}$ such that:

(i) $p \leq \mathbb{Q} p_i$

(ii) $(\forall j < i)(\forall \varepsilon < \delta_j)(q_{j, \varepsilon} \leq \mathbb{Q} p_i)$

(iii) if $i$ is a non-limit ordinal, then $p_i \in \hat{\mathbb{Q}}$ is minimal satisfying (i) + (ii)

(iv) if $i$ is a limit ordinal, then $p_i \in \mathbb{Q}$.

Now the player INC answers choosing an increasing sequence $\bar{q}_i = \langle q_{i, \varepsilon} : \varepsilon < \delta_i \rangle$ such that $p_i \leq \mathbb{Q} q_i$ and $\bar{q}_i$ is $\langle N^i \upharpoonright [\alpha, \delta], \mathbb{Q} \rangle^*$-generic for some $\alpha < \delta_i$ (see [Sh 587, B.5.3.1]) and $\beta < \delta_i \Rightarrow \bar{q}_i \upharpoonright (\beta + 1) \in M_{i, \beta + 1}$.

The player COM wins if it has always legal moves and the sequence $\langle p_i : i < \omega \sigma \rangle$ has an upper bound in $\mathbb{Q}$.

2) We say that the forcing notion $\mathbb{Q}$ is complete for $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ or $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$-complete if

(a) $\mathbb{Q}$ is strongly complete for $\hat{\mathcal{E}}_0$ and

(b) for a large enough regular $\chi$, for some $x \in \mathcal{H}(\chi)$, for every sequence $\bar{M}$ ruled by $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$ with an $\hat{\mathcal{E}}_0$-approximation $\langle M^i : i < \sigma \rangle, \langle N^i : i < \sigma \rangle$ and such that $x \in M_0$ and for any condition $r \in \mathbb{Q} \cap M_0$, the player INC does not have a winning strategy in the game $\mathcal{G}_{M, \langle M^i : i < \sigma \rangle, \langle N^i : i < \sigma \rangle}(\mathbb{Q}, r)$.

2.6 Proposition. Assume

(a) $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1) \in \mathcal{C}_{< \kappa}(\mu^*)$,

(b) $\mathbb{Q}$ is a forcing notion for $(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1)$.

Then $\Vdash \mathbb{Q} "(\hat{\mathcal{E}}_0, \hat{\mathcal{E}}_1) \in \mathcal{C}_{< \kappa}(\mu^*)"$.

Proof. Straightforward (and not used in this paper).

2.7 Proposition. Assume that $\hat{\mathcal{E}} \in \mathcal{C}_{< \kappa}(\mu^*)$ is closed and $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \gamma \rangle$ is a $(< \kappa)$-support iteration of forcing notions which are strongly complete for $\hat{\mathcal{E}}$. Let $\mathcal{I} = (T, <^+ M, \text{rk})$ be a standard $(w, \alpha_0)^\gamma$-tree (see [Sh 587, A.3.3]), $\|T\| < \lambda, w \subseteq \gamma, \alpha_0$ an ordinal, and let $\bar{p} = \langle p_t : t \in T \rangle \in FTr'(\bar{\mathbb{Q}})$, see [Sh 587, A.3.2]. Suppose that $\mathcal{I}$ is an open dense subset of $\mathbb{P}_\gamma$. Then there is $\bar{q} = \langle q_t : t \in T \rangle \in FTr'(\bar{\mathbb{Q}})$ such that $\bar{p} \leq \bar{q}$ and for each $t \in T$
2.8 Proposition. Assume that \( \hat{A} \). The difference with \([\text{Sh} 587, \text{B}.7.2]\) is the appearance of the \( \bar{\alpha} \).

Our next proposition corresponds to \([\text{Sh} 587, \text{B}.7.2]\) which corresponds to \([\text{Sh} 587, \text{A}.3.6]\). The difference with \([\text{Sh} 587, \text{B}.7.2]\) is the appearance of the \( M, M^i \).

**Proof.** Just like the proof of \([\text{Sh} 587, \text{B}.7.1]\).

Our next proposition corresponds to \([\text{Sh} 587, \text{B}.7.2]\) which corresponds to \([\text{Sh} 587, \text{A}.3.6]\). The difference with \([\text{Sh} 587, \text{B}.7.2]\) is the appearance of the \( M, M^i \).

2.8 Proposition. Assume that \( \hat{E} \in \mathcal{C}_{<\kappa}(\mu^+) \) is closed and \( \hat{Q} = (\mathbb{P}_\alpha, Q_\alpha : \alpha < \gamma) \) is a \(<\kappa\)-support iteration and \( x = \langle x_\alpha : \alpha < \gamma \rangle \) is such that

\[ \models_{\mathbb{P}_\alpha} " Q_\alpha \text{ is strongly complete for } \hat{E} \text{ with witness } x_\alpha " \]

(for \( \alpha < \gamma \)). Further suppose that

(a) \( (N, a) \) is an \( \hat{E} \)-complementary pair (see \([\text{Sh} 587, \text{B}.5.1]\)), \( \tilde{N} = \langle N_i : i \leq \delta \rangle \) and \( x, \hat{E}, \tilde{Q} \in N_0 \),

(b) \( \mathcal{F} = (T, \lt^+, \text{rk}) \in N_0 \) is a standard \( (w, \alpha_0)^\gamma \)-tree, \( w \subseteq \gamma \cap N_0, \|w\| < \text{cf}(\delta), \alpha_0 \) is an ordinal, \( \alpha_1 = \alpha_0 + 1 \) and \( 0 \in w \),

(c) \( \bar{p} = \langle p_t : t \in T \rangle \in \text{FT}r'(\tilde{Q}) \cap N_0, w \in N_0 \), (of course \( \alpha_0 \in N_0 \), on \( \text{FT}r' \) see \([\text{Sh} 587, \text{A}.3.2]\)),

(d) \( \hat{M} = \langle M_i : i \leq \delta \rangle, M_i \prec (\mathcal{H}(\chi), \varepsilon, <^*_{\varepsilon}) \), \( M_i \) is increasing continuous, \( [M_i]\|w\| + \|\mathcal{F}\| \subseteq M_{i+1} \) and the pair \( (\hat{M} \upharpoonright (i+1), \hat{N} \upharpoonright (i+1)) \) belongs to \( M_{i+1}, M_i < N_i \) and \( w \cup \{x, \bar{\delta}_0, \bar{Q}\} \in M_0 \)

(e) for \( i \leq \delta \), \( \mathcal{F}_i = (T_i, \lt_i, \text{rk}_i) \) is such that \( T_i \) consists of all sequences \( t = \langle t_\zeta : \zeta \in \text{dom}(t) \rangle \) such that \( \text{dom}(t) \) is an initial segment of \( w \), and

(i) each \( t_\zeta \) is a sequence of length \( \alpha_1 \)

(ii) \( \langle t_\zeta \upharpoonright \alpha_0 : \zeta \in \text{dom}(t) \rangle \in T \)

(iii) for each \( \zeta \in \text{dom}(t) \), either \( t_\zeta(\alpha_0) = * \) or \( t_\zeta(\alpha_0) \in M_i \) is a \( \mathbb{P}_\zeta \)-name for an element of \( Q_\zeta \) and

if \( t_\zeta(\alpha) \neq * \) for some \( \alpha < \alpha_0 \), then \( t_\zeta(\alpha_0) \neq * \),
(iv) \( \text{rk}_i(t) = \min(w \cup \{\zeta\} \setminus \text{dom}(t)) \) and \(<_i\) is the extension relation.

Then

(a) each \( \mathcal{T}_i \) is a standard \((w, \alpha_1)^\gamma\)-tree, \( \|T_i\| \leq \|T\| \cdot \|M_i\|^{\|w\|} \) and if \( i < \delta \) then \( T_i \in N_{i+1} \)

(b) \( \mathcal{T} \) is the projection of each \( \mathcal{T}_i \) onto \((w, \alpha_0)\) and \( \mathcal{T}_i \) is increasing with \( i \)

(c) there is \( \bar{q} = \langle q_t : t \in T_\delta \rangle \in FT_{r'}(\tilde{Q}) \) such that

(i) \( \tilde{p} \leq_{\text{proj}_{\mathcal{T}_t}^{\mathcal{T}_t}} \bar{q} \)

(ii) if \( t \in T_\delta \{<>\} \) then the condition \( q_t \in \mathcal{P}_{\text{rk}_3(t)} \) is an upper bound of an \((\tilde{N} \upharpoonright [i_0, \delta], \mathcal{P}_{\text{rk}_3(t)})^*\)-generic sequence (where \( i_0 < \delta \) is such that \( t \in T_{i_0} \)) and for every \( \beta \in \text{dom}(q_t) = N_\delta \cap \text{rk}(t) \), \( q_t(\beta) \) is a name for the least upper bound in \( \tilde{Q}_{\beta} \) of an \((\tilde{N}[G_\beta] \upharpoonright [\xi, \delta], \tilde{a} \upharpoonright [\xi, \delta]) \) is an \( \mathcal{T}^* \)-complementary pair.

[Note that by [Sh 587, B.5.5], the first part of the demand on \( q_t \) implies that if \( i_0 \leq \xi \) then \( q_t \upharpoonright [i, \delta_0] \) forces that \((\tilde{N}[G_\beta] \upharpoonright [\xi, \delta], \tilde{a} \upharpoonright [\xi, \delta]) \) is an \( \mathcal{S}^* \)-complementary pair.]

(iii) if \( t \in T_\delta, t' = \text{proj}_{\mathcal{T}_t}^{\mathcal{T}_t}(t) \in T, \zeta \in \text{dom}(t) \) and \( t_\zeta(\alpha_0) \neq * \), then \( q_t \upharpoonright [\zeta] \models \text{"} p_{t'}(\zeta) \leq_{\tilde{Q}_{\zeta}} t_\zeta(\alpha_0) \Rightarrow t_\zeta(\alpha_0) \leq_{\tilde{Q}_{\zeta}} q_t(\zeta) \” \),

(iv) \( q_{<>} = p_{<>} \).

Proof. Clauses (a) and (b) should be clear. Clause (c) is proved as in [Sh 587, B.7.2]. \( \square \)

Remark. In 2.9 below is proved as in the inaccessible case i.e. the proofs of ([Sh 587, B.7.3]) with \( M, \langle N_i : i < \sigma \rangle \) as in Definition 2.5. We define the trees point: in stage \( i \) using trees \( \mathcal{T}_i \) with set of levels \( w_i = M_i \cap \gamma \) and looking at all possible moves of COM, i.e. \( p_i \in M_{i+1} \cap \mathcal{P}_\gamma \), so constructing this tree of conditions in \( \delta_i \) stages, in stage \( \varepsilon < \delta_i \), has \( |N^i \cap M_{i+1}|^{2^{|M_{i+1}|}} \) nodes.

Now

\[
\begin{align*}
p \in \mathcal{P}_\gamma \cap M_{i+1} \Rightarrow & \quad \text{Dom}(p) \subseteq M_{i+1} \quad \text{but} \\
p \in \mathcal{P}_\gamma \cap M_{i+1} \Rightarrow & \quad \text{Dom}(p) \subseteq M_{\sigma} = \bigcup_{i < \omega_\sigma} N_i^i \\
p \in \mathcal{P}_\gamma \cap N^i \Rightarrow & \quad \text{Dom}(p) \subseteq N^i.
\end{align*}
\]
So in limit cases \( i < \sigma \): the existence of limit is by the clause \((\mu)\) of Definition 2.3. In the end we use the winning of the play and then need to find a branch in the tree of conditions of level \( \sigma \): like Case A using \( \hat{\gamma} \).

\[ \square_{2.9} \]

2.9 Theorem. Suppose that \((\hat{\gamma}_0, \hat{\gamma}_1) \in C_{<\kappa}(\mu^*)\) (so \( \hat{\gamma}_0 \in C_{<\kappa}(\mu^*) \)) and \( \hat{\gamma} = \langle p_{\alpha}, q_{\alpha} : \alpha < \gamma \rangle \) is a \((< \kappa)\)-support iteration such that for each \( \alpha < \kappa \)

\[ \models p_{\alpha} \text{ "} Q_{\alpha} \text{ is complete for } (\hat{\gamma}_0, \hat{\gamma}_1) \text{".} \]

Then

(a) \( \models p_{\gamma} (\hat{\gamma}_0, \hat{\gamma}_1) \in C_{<\kappa}(\mu^*) \), moreover

(b) \( P_{\gamma} \) is complete for \((\hat{\gamma}_0, \hat{\gamma}_1)\).

Proof. We need only part (a) of the conclusion, so we concentrate on it. Let \( \chi \) be a regular large enough regular cardinal, \( x \) be a name for an element of \( \mathcal{H}(\chi) \) and \( p \in P_{\gamma} \). Let \( x_{\alpha} \in \mathcal{H}(\chi) \) be a \( P_{\alpha} \)-name for the witness that \( Q_{\alpha} \) is (forced to be) complete for \( \hat{\gamma}_0, \hat{\gamma}_1 \) and let \( \bar{x} = \langle x_{\alpha} : \alpha < \gamma \rangle \). Since \( (\hat{\gamma}_0, \hat{\gamma}_1) \in C_{<\kappa}(\mu^*) \), we find \( \bar{M} = \langle M_i : i \leq \sigma \rangle \) which is ruled by \( (\hat{\gamma}_0, \hat{\gamma}_1) \) with an \( \hat{\gamma}_0 \)-approximation \( \langle \bar{M}_i, \bar{N}_i : -1 \leq i < \sigma \rangle \) and such that \( p, \bar{Q}, \bar{x}, \bar{\hat{\gamma}}_0, \bar{\hat{\gamma}}_1 \in M_0 \) (see 2.3). Let \( \bar{N}_i = \langle N_i^i : i \leq \delta_i \rangle \) and let \( \bar{a}_i \in \hat{\gamma}_0 \) be such that \( (\bar{N}_i, \bar{a}_i) \) is an \( \hat{\gamma}_0 \)-complementary pair and let \( \bar{M}_i = \langle M^i_j : i \leq \delta_i \rangle \). Let \( w_i = \{0\} \cup \bigcup \{\gamma \cap M^i_\omega \} \) (for \( i \leq \delta \)).

By induction on \( i \leq \sigma \) we define standard \((w_i, i)^{-}\)-trees \( T_{\gamma} \in M_{i+1} \) and \( \bar{p}^i = \langle p_i^i : t \in T_i \rangle \in FTr'(\bar{Q}) \cap M_{i+1} \) such that \( |T_i| \leq |M_i||w_i| \leq |M_{i+1}| \) if \( i \) is limit or \( 0, w_{i+1} = w_i \) hence \( T_{i+1} = T_i \), and if \( j < i \leq \delta \) then \( T_j = \text{proj}_{(w_j, j+1)}(T_i) \) and \( \bar{p}^j = \text{proj}_{\gamma_j} \bar{p}^i \).

**CASE 1:** \( i = 0 \).

Let \( T_0^* \) consist of all sequences \( \langle t_\zeta : \zeta \in \text{dom}(t) \rangle \) such that \( \text{dom}(t) \) is an initial segment of \( w_0 \) and \( t_\zeta =<> \) for \( \zeta \in \text{dom}(t) \). Thus \( T_0^* \) is a standard \((w_0, 0)^{-}\)-tree, \( |T_0^*| = |w_0| + 1 \). For \( t \in T_0^* \) let \( p_{t_0}^0 = p \upharpoonright \text{rk}_0(t) \). Clearly the sequence \( \bar{p}^0 = \langle p_{t_0}^0 : t \in T_0^* \rangle \) is in \( FTr'(\bar{Q}) \cap N_0^{-1} \). Apply 2.8 to \( \hat{\gamma}_0, \bar{Q}, N^{-1}, T_0^*, w_0 \) and \( \bar{p}^0 \) (note that \( |M_0^{-1}| |w_0| \leq |M_1^{-1}| \) for \( \epsilon < \delta_0 \)). As a result we get a \((w_0, 1)^{-}\)-tree \( T_0 \) (the one called \( \mathcal{T}_{\delta_0} \) there) and \( \bar{p}^0 = \langle p_t^0 : t \in T_0 \rangle \in FTr'(\bar{Q}) \cap M_1 \) (the one called \( \bar{q} \) there) satisfying
 clauses \((\varepsilon), (c)(i)-(iv)\) of 2.8 and such that \(\|T_0\| \leq \|N_{\delta_0}^{-1}\|\bigwedge_{\omega_0} = \|M_0\|\bigwedge_{\omega_0} = \|M_0\|\) (remember cf\((\delta_0) > 2\|M_0\|\)). So, in particular, if \(t \in T_0\), \(\zeta \in \text{dom}(t)\) then \(t_\zeta(0) \in M_1\) is either \(*\) of a \(\mathbb{P}_\zeta\)-name for an element of \(\mathcal{Q}_\zeta\).

Moreover, we additionally require that \((\mathcal{I}_0, p^0)\) is the \(\prec^*\)-first with all these properties, so \(\mathcal{I}_0, p^0 \in M_1\).

**CASE 2:** \(i = i_0 + 1\).

We proceed similarly to the previous case. Suppose we have defined \(\mathcal{I}_{i_0}\) and \(p^{i_0}\) such that \(\mathcal{I}_{i_0}, p^{i_0} \in M_{i_0+1}, \|T_{i_0}\| \leq \|M_{i_0+1}\|\). Let \(\mathcal{I}^*_i\) be a standard \((w_i, i_0)^\gamma\)-tree such that

\[T^*_i \text{ consists of all sequences } \langle t_\zeta : \zeta \in \text{dom}(t) \rangle \text{ such that } \text{dom}(t) \text{ is an initial segment of } w_i \text{ and} \]

\[\langle t_\zeta : \zeta \in \text{dom}(t) \cap w_{i_0} \rangle \in T_{i_0} \text{ and } (\forall \zeta \in \text{dom}(t) \setminus w_{i_0})(\forall j < i_0)(t_\zeta(j) = *)\].

Thus, \(\mathcal{I}_{i_0} = \text{proj}_{(w_{i_0}, i_0)}^{(w_i, i)}(\mathcal{I}^*_i)\) and \(\|T^*_i\| \leq \|M_i\|\). Let \(p^*_i = p^{i_0}_i \upharpoonright \text{rk}^*_i(t)\) for \(t \in T^*_i, t' = \text{proj}_{\mathcal{I}_{i_0}}(t)\). Now apply 2.8 to \(\mathcal{I}_{i_0}, \mathcal{I}^*_i, \mathcal{I}^*_j, w_i, p^*_i\) (check that the assumptions are satisfied). So we get a standard \((w_i, i_0 + 1)^\gamma\)-tree \(\mathcal{I}_i\) and a sequence \(\bar{p}^i\) satisfying \((\varepsilon), (c)(i)-(iv)\) of 2.8, and we take the \(\prec^*\)-pair \((\mathcal{I}_i, \bar{p}^i)\) with these properties. In particular, we will have \(\|T_i\| \leq \|M_{i_0}\| \cdot \|N_{\delta_0}^{-1}\|\bigwedge_{\omega_0} = \|M_{i_0+1}\|\) and \(\bar{p}^i, \mathcal{I}_i \in M_{i+1}\).

**CASE 3:** \(i\) is a limit ordinal.

Suppose we have defined \(\mathcal{I}_j, \bar{p}^j\) for \(j < i\) and we know that \(\langle (\mathcal{I}_j, \bar{p}^j) : j < i \rangle \in M_{i+1}\) (this is the consequence of taking “the \(\prec^*\)-first such that ...”). let \(\mathcal{I}^*_i = \text{lim}\langle (\mathcal{I}_j : j < i) \rangle\). Now, for \(t \in T^*_i\), we would like to define \(p^*_i\) as the limit of \(\text{proj}_{\mathcal{I}^*_i}^j(t)\). However, our problem is that we do not know if the limit exists.

Therefore, we restrict ourselves to these \(t\) for which the respective sequence has an upper bound. To be more precise, for \(t \in \mathcal{I}^*_i\) we apply the following procedure.

\(\otimes\) Let \(t^i = \text{proj}_{\mathcal{I}^*_i}^j(t)\) for \(j < i\). Try to define inductively a condition \(p^*_i \in \mathbb{P}_{\text{rk}^*_i(t)}\) such that dom\((p^*_i) = \bigcup\{\text{dom}(p^j_i) \cap \text{rk}^*_i(t) : j < i\}\). Suppose we have successfully defined \(p^*_i \upharpoonright \alpha \upharpoonright \alpha = p^j \upharpoonright \alpha\) for \(\alpha \in \text{dom}(p^*_i)\), in such a way that \(p^*_i \upharpoonright \alpha \upharpoonright \alpha \leq p^j \upharpoonright \alpha\) for all \(j < i\). We know that

\[p^*_i \upharpoonright \alpha \upharpoonright \alpha \vdash_{\mathbb{P}_\alpha} \text{ "the sequence } \langle p^j(\alpha) : j < i \rangle \text{ is } \leq_{\mathbb{Q}_\alpha} \text{-increasing} \text{".} \]
So now, if there is a \( P_\alpha \)-name \( \tau \) for an element of \( Q_\alpha \) such that
\[
p^*_i \restr\alpha \Vdash \forall j < i (p^j_i(\alpha) \leq_{\hat{Q}_\alpha} \tau),
\]
then we take the \( P_\alpha \)-name of the lub of \( \langle p^j_i(\alpha) : j < i, p^j_i(\alpha) \neq * \rangle \) in \( \hat{Q}_\alpha \), and we continue. If there is no such \( \tau \) then we decide that \( t \notin \mathcal{I}_i^+ \) and we stop the procedure\(^2\).

Now, let \( \mathcal{I}_i^+ \) consist of those \( t \in T_i^* \) for which the above procedure resulted in a successful definition of \( p^*_i \in P_{\text{rk}_i^+(t)} \). It might not be clear at the moment if \( \mathcal{I}_i^+ \) containss anything more than \( <> \), but we will see that this is the case. Note that
\[
\|T_i^+\| \leq \|T_i^*\| \leq \prod_{j < i} \|T_j\| \leq \prod_{j < i} \|M_j\| \leq 2^\|M_i\| \leq \|M_i^t\|.
\]
Moreover, for nonlimit \( \varepsilon > 2 \) we have \( \|M_i^t\|^{\omega_\varepsilon} + \|T_i^+\| \leq \|M_i^t\|^{\|M_i\|} \subseteq M_{\varepsilon+1}^t \) and \( \mathcal{I}_i^+ = \mathcal{I}_i^*, \bar{p}^i = \bar{p}^*i \) (this time there is no need to take the \( \langle \chi \rangle \)-first pair as the process leaves no freedom). So we have finished Case 3.

After the construction is carried out we continue in a similar manner as in \([\text{Sh} 587, \text{A.3.7}]\) (but note slightly different meaning of the *'s here).

So we let \( \mathcal{I}_\sigma = \lim(\langle \mathcal{I}_i : i < \sigma \rangle) \). It is a standard \( (\sigma, \sigma)\gamma \)-tree. By induction on \( \alpha \in w_\sigma \cup \{\gamma\} \) we choose \( q_\alpha \in P_\alpha \) and a \( P_\alpha \)-name \( t_\alpha \) such that:

\( a \) \( \forces_{P_\alpha} "t_\alpha \in T_{\omega_\sigma} \& \text{rk}_\delta(t_\alpha) = \alpha" \) and let \( i_0^\sigma = \min\{i < \delta : \alpha \in M_i\} < \sigma \),

\( b \) \( \forces_{P_\alpha} "t_\beta = t_\alpha \restr \beta" \) for \( \beta < \alpha \),

\( c \) \( \text{dom}(q_\alpha) = w_\delta \cap \alpha \),

\( d \) if \( \beta < \alpha \) then \( q_\beta = q_\alpha \restr \beta \),

\( e \) \( p^i_{\text{proj}_{\mathcal{I}_i^+}(t_\alpha)} \) is well defined and \( p^i_{\text{proj}_{\mathcal{I}_i^+}(t_\alpha)} \restr\alpha \leq q_\alpha \) for each \( i < \omega_\sigma \),

\( f \) for each \( \beta < \alpha \)

\(^2\)Generally in such situation we can act as in 2.7 to get a real decision, i.e. if \( p^*_i \restr(\alpha + 1) \) is not well defined while \( p^*_i \restr\alpha \) is well defined then \( p^*_i \restr\alpha \forces "\text{the sequence } \langle p^j_i(\alpha) : j < i \rangle \text{ has no } \leq_{\hat{Q}_\alpha} \text{-upper bound}. \) But the need has not arisen here.
q_\alpha \Vdash_{P_\alpha} "(\forall i < \delta)((t_{\beta+1})_\beta(i) = * \iff i < i_0^\beta)" and the sequence
\langle i_0^\beta, b^\beta_{\text{proj}_{\beta}^\delta}(t_{\beta+1})(\beta), \langle (t_{\beta+1})_\beta(i), p^i_{\text{proj}_{\beta}^\delta}(t_{\beta+1})(\beta) : i_0^\beta \leq i < \delta) \rangle

is a result of a play of the game $\mathcal{G}^{\downarrow}_{M[G_\beta], \langle \tilde{N}_i[G_\beta] : i < \delta \rangle}(Q_\beta, 0_{Q_\beta})$, won by player COM".

(Remember: $\hat{e}_1$ is closed under end segments). This is done completely parallel to the last part of the proof of [Sh 587, A.3.7]

Finally, look at the condition $q_\gamma$ and the clause (g) above.

$\square_{2.9}$

2.10 Generalization 1) $\hat{e}_1$ is a set of triples $\langle \tilde{a}, \langle \tilde{b}^i, \tilde{a}^i : i < \sigma \rangle, \tilde{\lambda} \rangle, \tilde{a} = \langle a_i : i \leq \sigma \rangle, \tilde{a}^i = \langle a^i_\alpha : \alpha \leq \delta_i \rangle, \tilde{b}^i = \langle b^i_\alpha : \alpha \leq \delta_i \rangle \in \hat{e}_0, a^i_{\delta_i} = a_{i+1}, a_i \subseteq b^i_0, \lambda = \langle \lambda_i : i < \sigma \rangle$ an increasing sequence of cardinals $< \lambda, \sum \lambda_i = \lambda$.

2) We say $(M, \langle M^i : i < \sigma \rangle, \langle \tilde{N}^i : i < \sigma \rangle)$ obeys $\langle \tilde{a}, \langle \tilde{b}^i : i < \tilde{\lambda} \rangle \rangle : M_i \cap \mu^* = a_i, \tilde{N}^i$ obeys $\tilde{b}^i$ all things in 2.3 but $\lambda_i \geq \|M_i\|, \lambda_i \geq \prod_{j \leq i} \|M_j\|, [M^i_\alpha]^{\lambda_i} \subseteq M^i_{\alpha+1}$ for $\alpha < \delta_i$ (so earlier $\lambda_i = 2\|M_i\|$).

2.11 Conclusion 1) Assume

(a) $S \subseteq \{\delta < \kappa : \text{cf}(\delta) = \sigma\}$ is stationary not reflecting

(b) $\tilde{a} = \langle \tilde{a}_\delta : \delta \in S \rangle, \tilde{a}_\delta = \langle a_{\delta,i} : i \leq \sigma \rangle, \delta = a_{\delta,\sigma}$ and $a_{\delta,i}$ increasing with $i$ and $i < \sigma \Rightarrow |a_{\delta,i}| < \lambda$ and $\sup(a_{\delta,i}) < \delta$

[variant: $\tilde{\lambda}^\delta = \langle \lambda^\delta_i : i < \sigma \rangle$ increasing with limit $\lambda$]

(c) we let $\mu^* = \kappa, \hat{e}_0 = \hat{e}_0[S] = \{\tilde{a} : \tilde{a} = \langle a_i : i \leq \alpha \rangle, \alpha < \kappa, a_i \in \kappa \setminus S$ increasing continuous}$

(d) $\hat{e}_1 = \{\tilde{a}_\delta : \delta \in S\}$

(or $\{\langle \tilde{a}_\delta, \langle \tilde{a}^i_\delta, \tilde{b}^i_\delta : i < \sigma \rangle, \tilde{\lambda}_\delta \rangle : \delta \in S\}$ appropriate for (2.10))

(e) we assume the pair $(\hat{e}_0, \hat{e}_1) \in \mathcal{C}^{\downarrow}_{\kappa}(\mu^*)$

(f) $\mu = \mu^*, \kappa < \tau = \text{cf}(\tau) < \mu$. 

(667)
Then for some $(\mathcal{E}_0, \mathcal{E}_1)$-complete forcing notion $P$ of cardinality $\mu$ we have

$$\Vdash_P \text{"forcing axiom for } (\mathcal{E}_0, \mathcal{E}_1)\text{-complete forcing notion of cardinality } \leq \kappa \text{ and } < \tau \text{ of open dense sets"}$$

and in $V^P$ the set $S$ is still stationary (by preservation of $(\mathcal{E}_0, \mathcal{E}_1)$-nontrivial).

2) If clauses (a),(c) holds and $\Diamond_S$, then for some $\bar{a}$, if we define $\mathcal{E}_1$ as in clause (d) then clause (b),(d),(e) holds.

**Proof.** 1) See more in the end of §3.

2) Easy. $\square_{2.11}$

2.12 Application: In $V^P$ of 2.11:

(a) if

(i) $\theta < \lambda, A_\delta \subseteq \delta = \sup(A_\delta)$ for $\delta \in S$,

(ii) $|A_\delta| < \theta$

(iii) $\bar{h} = (h_\delta : \delta \in S), h_\delta : A \rightarrow \theta$

(iv) $A_\delta \subseteq \bigcup\{a_{\delta,i+1} \setminus a_{\delta,i} : i < \sigma\}$,

then for some $h : \kappa \rightarrow \theta$ and club $E$ of $\kappa$ we have $(\forall \delta \in S \cap E)[h_\delta \subseteq^* h]$ where $h' \subseteq^* h''$ means that $\sup(\text{Dom}(h')) > \sup\{\alpha : \alpha \in \text{Dom}(h') \text{ and } \alpha \notin \text{Dom}(h'') \text{ or } \alpha \in \text{Dom}(h'') \& \ h'(\alpha) \neq h''(\alpha)\}$

(b) if we add: “$h_\delta$ constant”, then we can omit the assumption (iii)

(c) we can weaken $|A_\delta| < \theta$ to $|A_\delta \cap a_{\delta,i+1}| \leq |a_{\delta,i}|$

(d) in (c) we can weaken $|A_\delta| \leq \theta \lor |A_\delta \cap a_{\delta,i+1}| \leq |a_{\delta,i}|$ to $h_\delta \upharpoonright a_{\delta,i+1}$ belongs to $M_{i+1} \cap N_\alpha^i$ for some $\alpha < \delta_i$

(remember $\text{cf}(\sup a_{\delta,i+1}) > \lambda_i^\delta$).

2.13 Remark. 1) Compared to [Sh 186] the new point in the application is (b).

2) You may complain why not having the best of (a) + (b), i.e. combine their good points. The reason is that this is impossible by §1, §4; the situation is different in the inaccessible case.

**Proof.** Should be clear. Still we say something in case $h_\delta$ constant, that is (b). Let
\[ Q = \{ (h, C) : h \text{ is a function with domain an ordinal} \]
\[ \alpha < \kappa = \lambda^+, \]
\[ C \text{ a closed subset of } \alpha + 1, \alpha \in C \]
\[ \text{and } (\forall \delta \in C \cap S \cap (\alpha + 1))(h_\delta \subseteq^* h) \} \]

with the partial order being inclusion.

For \( p \in Q \) let \( p = (h^p, C^p) \).

So clearly if \((h, C) \in Q\) and \( \alpha = \text{Dom}(h) < \beta \in \kappa \) then for some \( h_1 \) we have \( h \subseteq h_1 \in \mathbb{Q}_1 \), \( \text{Dom}(h_1) = \beta \); moreover, if \( \gamma < \theta \) \& \( \beta \notin S \) then \((h, C) \leq (h \cup \gamma_{[\alpha, \beta]}, C \cup \{\beta\}) \in Q\).

The main point is proving \( Q \) is complete for \((\mathcal{E}_0, \mathcal{E}_1)\). Now “\( Q \) is strongly complete for \( \mathcal{E}_0\)” is proved as in [Sh 587, B.6.5.1,B.6.5.2] (or 3.14 below which is somewhat less similar). The main point is clause (b) of 2.5(2); that is, let \( \bar{M}, \langle \bar{M}^i : i < \omega \sigma \rangle, \langle \bar{N}^i : i < \omega \sigma \rangle \) be as there. In the game \( G_{\bar{M}, \langle \bar{N}^i : i < \omega \sigma \rangle}(r, \mathbb{Q}) \) from 2.5(1), we can even prove that the player COM has a winning strategy: in stage \( i \) (non-trivial): if \( h_\delta \) is constantly \( \gamma < \theta \) or just \( h_\delta \upharpoonright (\bar{A}_\delta \cap \bar{a}_{\delta,i+1} \setminus \bar{a}_{\delta,i}) \) is constantly \( \gamma < \theta \) then we let

\[
p_i = \left( \bigcup \{ h^{q^i_j} : j < i \text{ and } \zeta < \delta_i \} \cup \gamma_{[N_i, \cap \kappa, \beta_i]} \right)
\]

\[
\text{closure}(\bigcup \{ C^{q^i_j} : j < i \text{ and } \zeta < \delta_i \} \cup \{ \beta_i \} \}
\]

for some \( \beta_i \in \mathbb{M}_{i+1} \cap \kappa \setminus M_i \) large enough such that \( A_\delta \cap M_{i+1} \cap \kappa \subseteq \beta_i \). \( \square \)

\[ \rightarrow \text{ scite} \{2.10\} \text{ undefined} \]

\[ \text{Remark. In the example of uniformizing (see [Sh 587]) if we use this forcing, the density is less problematic.} \]

\[ \textbf{2.14 Claim.} \]

1) In \(?\)’s conclusion we can omit the club \( E \) that is let \( E = \kappa \) and

\[ \rightarrow \text{ scite} \{2.10\} \text{ undefined} \]

\[
\text{demand } (\forall \delta \in S)(h_\delta \subseteq^* h) \text{ provided that we add in } ?, \text{ recalling } S \subseteq \kappa \text{ does not}
\]

\[ \rightarrow \text{ scite} \{2.10\} \text{ undefined} \]

\[ \text{reflect is a set of limit ordinals and} \]

\[ \bar{A} = \langle A_\delta : \delta \in S \rangle, A_\delta \subseteq \delta = \sup(A_\delta) \]
satisfies

\[(*)\] \(\delta_1 \neq \delta_2 \text{ in } S \Rightarrow \sup(A_{\delta_1} \cap A_{\delta_2}) < \delta_1 \cap \delta_2.\)

2) If \((\forall \delta \in S)(\otp(A_\delta) = \theta)\) this always holds.

Proof. We define \(Q = \{ h : \Dom(h) \text{ is an ordinal } < \kappa \text{ and } h(\beta) \neq 0 \wedge \beta \in \Dom(h) \rightarrow (\exists \delta \in S)[h_\delta(\beta) = h(\beta)] \text{ and } \delta \in (\Dom(h) + 1) \cap S \text{ implies } h_\delta \subseteq^* h\}\) ordered by \(\subseteq\). Now we should prove the parallel of the fact:

\[\exists' \text{ if } p \in Q, \alpha = \Dom(p) < \beta < \kappa \text{ then there is } q \text{ such that } p \leq q \in Q \text{ and } \Dom(q) = \beta.\]

Why this holds? We can find \(\langle A'_\delta : \delta \in S \cap (\beta + 1) \rangle\) such that \(A'_\delta \subseteq A_\delta, \sup(A_\delta \setminus A'_\delta) < \delta\) and \(\bar{A}' = \langle A'_\delta : \delta \in S \cap (\beta + 1) \rangle\) is pairwise disjoint.

Now choose \(q\) as follows

\[ q(j) = \begin{cases} p(j) & \text{if } j < \alpha \\ h_\delta(j) & \text{if } j \in A'_\delta \setminus \alpha \text{ and } \delta \in S \cap (\beta + 1) \setminus (\alpha + 1) \\ 0 & \text{if otherwise.} \end{cases} \]

Why does \(\bar{A}'\) exist? Prove by induction on \(\beta\) that for any \(\bar{A}^1, \langle A'_\delta : \delta \in S \cap (\alpha + 1) \rangle\) as above and \(\beta\) satisfying \(\alpha < \beta < \kappa\), we can end extend \(\bar{A}^1\) to \(\langle A'_\delta : \delta \in S \cap (\beta + 1) \rangle\) which is as above. \(\Box\)

2.15 Remark. Note: concerning \(\kappa\) inaccessible we could imitate what is here: having \(M_{i+1} \preceq N_i^i, \bigcup_{i<\delta} M_i = \bigcup_{i<\delta} N_i^i\).

As long as we are looking for a proof that no sequence of length \(< \kappa\) are added, the gain is meagre (restricting the \(\tilde{q}\)'s by \(\tilde{q} \upharpoonright \alpha \in N''_{\alpha+1}\)). Still if you want to make the uniformization and some diamond we may consider this.

2.16 Comment: We can weaken further the demand, by letting \(\text{COM}\) have more influence. E.g. we have (in 2.3) \(\delta_i = \lambda_i = \cf(\lambda_i) = ||M_{i+1}||, D_i\) a \(|a_i|^+\)-complete filter on \(\lambda_i\), the choice of \(\tilde{q}^i\) in the result of a game in which \(\text{INC}\) should have chose a set of player \(\in D_i\) and \(\Diamond D_i\) holds (as in the treatment of case \(E^*\) here).

The changes are obvious, but I do not see an application at the moment.
We intend to generalize pic of [Sh:f, Ch.VIII, §1]. The intended use is for iteration with each forcing \( \kappa \) - see use in [Sh:f]. In [Sh 587, B.7.4] we assume each \( Q_i \) of cardinality \( \leq \kappa \). Usually \( \mu = \kappa^+ \).

Note: \( \hat{E}_0 \) is as in the accessible case, in [Sh 587] but this part works in the other cases. In particular, in Cases A,B (in [Sh 587]'s context) if the length of \( \bar{a} \in \hat{E}_0 \) is \( < \lambda \) (remember \( \kappa = \lambda^+ \)), then we have \( (< \lambda) \)-completeness implies \( \hat{E}_0 \)-completeness AND in 3.7 even \( \bar{a} \in \hat{E}_0 \Rightarrow \ell g(\bar{a}) = \omega \) is O.K.

In Case A on the \( S_0 \subseteq S^\lambda_\kappa \) if \( \ell g(\bar{a}) = \lambda, a_\lambda \in S_0 \) is O.K., too. STILL can start with other variants of completeness which is preserved.

3.1 Context: We continue [Sh 587, B.5.1-B.5.7(1)] (except the remark [Sh 587, B.5.2(3)]) under the weaker assumption \( \kappa = \kappa^{<\kappa} > \aleph_0 \), so \( \kappa \) is not necessarily strongly inaccessible; also in our \( \hat{E}_0 \)'s we allow \( \bar{a} \) such that \( |a_\delta| = |\delta| \) is strongly inaccessible.

3.2 Definition. Assume:

\[ (a) \quad \mu = \text{cf}(\mu) > |\alpha|^{<\kappa} \text{ for } \alpha < \mu \]

(b) the triple \((\kappa, \mu^*, \hat{E}_0)\) satisfies: \( \kappa = \text{cf}(\kappa) > \aleph_0, \mu^* \geq \kappa, \hat{E}_0 \subseteq \{ \bar{a} : \bar{a} \text{ an increasing continuous sequence of members of } [\mu^*]^{<\kappa} \text{ of limit length } < \kappa \) with \( a_i \cap \kappa \in \kappa \) \]

(c) \( S^{\square} \subseteq \{ \delta < \mu : \text{cf}(\delta) \geq \kappa \} \) stationary.

For \( \ell = 1, 2 \) we say \( Q \) satisfies \((\mu, S^{\square}, \hat{E}_0)\)-pic\( \ell \) if: for some \( x \in \mathcal{K}(\chi) \) (can be omitted, essentially, i.e. replaced by \( Q \)) we have

\[ (\ast) \text{ if } \]

\[ (\alpha) \quad S \subseteq S^{\square} \text{ is stationary and } \langle \mu, S, \hat{E}_0, x \rangle \in N_0^{\alpha} \]

\[ (\beta) \text{ for } \alpha \in S, \delta_\alpha < \kappa, \text{ and } \]

\[ (i) \text{ if } \ell = 1, \tilde{N}^\alpha = \langle N_i^\alpha : i \leq \delta_\alpha \rangle \text{ and } c_\alpha = \delta_\alpha \text{ and } \tilde{N}^{\alpha\star} = \tilde{N}^\alpha \]

\[ (ii) \text{ if } \ell = 2 \text{ then } \tilde{N}^{\alpha\star} = \langle N_i^\alpha : i \leq \delta_\alpha \rangle, \tilde{N}^\alpha = \langle N_i^\alpha : i \in c_\alpha^+ \rangle \]

where \( c_\alpha \subseteq \delta_\alpha = \sup(c_\alpha), c_\alpha^+ = c_\alpha \cup \{ \delta_\alpha \}, c_\alpha \text{ is closed, } \gamma < \beta \in c_\alpha \Rightarrow c_\alpha \cap \gamma \in N_0^\gamma \)

\[ (\gamma) \quad (\tilde{N}^\alpha, \bar{a}^\alpha) \text{ is } \hat{E}_0\text{-complementary (see [Sh 587, B.5.3]); so } \tilde{N}^\alpha \text{ obeys } \bar{a}^\alpha \in \hat{E}_0 \text{ (with some error } n_\alpha) \text{ (so here we have } ||N_0^{\alpha_\delta}|| < \kappa, \delta_\alpha < \kappa) \]

\[ (\delta) \quad \bar{p}^\alpha \text{ is } (\tilde{N}^\alpha, Q)^1\text{-generic (see [Sh 587, Definition B.5.3.1])} \]
\( (ε) \) \( α ∈ N_0^α \) and

\( (i) \) if \( ℓ = 1 \), then for some club \( C \) of \( μ \) for every \( α ∈ S \) we have

\[ ⟨(\bar{N}_β, \bar{p}_β) : β ∈ S ∩ C ∩ α⟩ \] belongs to \( N_0^α \)

\( (ii) \) if \( ℓ = 2 \), then for some club \( C \) of \( μ \) for every \( α ∈ S ∩ C \) and

\[ i < δ_α \] we have \( ⟨(\bar{N}_β, | (i + 1), \bar{p}_β | (i + 1)) : β ∈ S ∩ C⟩ \) belongs to \( N_0^α \)

\( (ε) \) we define a function \( g \) with domain \( S \) as follows: \( g(α) = (g_0(α), g_1(α)) \)

where \( g_0(α) = N_0^α \cap (∪_{β < α} N_0^β) \) and \( g_1(α) = (N_0^α, N_0^α, c)_{i < δ_1, c ∈ g_0(α)} / ∼_h \),

then we can find a club \( C \) of \( μ \) such that:

if \( α < β \) \& \( g(α) = g(β) \) \& \( α ∈ C ∩ S \) \& \( β ∈ C ∩ S \) then \( δ_α = δ_β, g(α) = g(β) \), for some \( h, N_0^α ≅ N_0^β \) (really unique), and for each \( i < δ_α \)

the function \( h \) maps \( N_0^i, p_i^α \) to \( p_i^β \) and \( \{p_i^β : i < δ_α\} ∪ \{p_i^β : i < δ_β\} \)

has an upper bound.

3.3 Claim. Assume \( \boxdot \), i.e. \( (a), (b), (c) \) of 3.2 and

\( (d) \) \( \hat{E}_0 \) is non-trivial, which means:

for every \( χ \) large enough and \( x ∈ H(χ) \) there is \( \bar{N} = (N_i : i ≤ δ) \) incrementally continuous, \( N_i < (H(χ), ε), x ∈ N_i, ||N_i|| < k, \bar{N} | (i + 1) ∈ N_{i+1} \)

and \( \bar{N} \) obeys some \( \bar{a} ∈ \hat{E}_0 \) with some finite error \( n \)

\( (e) \) \( Q \) is a strongly \( cl(\hat{E}_0) \)-complete forcing notion (hence adding no new bounded subsets of \( κ \)) where \( cl(\hat{E}_0) =: \{\bar{a} | [α, β] : \bar{a} ∈ \hat{E}_0 \) and \( α ≤ β ≤ ℓg(\bar{a})\} \)

\( (f) \) \( Q \) satisfies \( (μ, S^{□}, \hat{E}_0)-\text{pic}_ℓ \) where \( ℓ ∈ \{1, 2\} \).

Then \( Q \) satisfies the \( μ \)-c.c. provided that

\( (*) \) \( ℓ = 1 \) or \( ℓ = 2 \) and \( \hat{E}_0 \) is fat, see below.

3.4 Definition. We say \( \hat{E}_0 ∈ C_≤κ(μ^α) \) is fat, if in the following game \( Đ_{κ, μ^∗}(\hat{E}_0) \)

between fat and lean, the fat player has a winning strategy.

A play last \( κ \) moves; in the \( α \)-th move:

Case 1: \( α \) nonlimit.

The player lean chooses a club \( Y_α ⊆ [μ^∗]|κ \), the fat player chooses \( a_α ∈ Y_α \) and \( P_α ⊆ \{c : c ⊆ α \text{ is closed}\} \) of cardinality \( < κ \).

Case 2: \( α \) limit.
We let $Y_\alpha = [\mu_0]^{<\kappa}_\kappa$ and $a_\alpha = \cup \{a_\beta : \beta < \alpha\}$ and the player fat chooses $\mathcal{P}_\alpha \subseteq \{C : C \subseteq \alpha \text{ is closed}\}$ of cardinality $< \kappa$.

In a play, fat wins if for some limit ordinal $\alpha$ and $c \in \mathcal{P}_\alpha$ we have:

\begin{enumerate}[(i)]
    \item $\beta \in c \Rightarrow c \cap \beta \in \mathcal{P}_\beta$
    \item $\alpha = \sup(c)$
    \item $\{a_\beta : \beta \in c \cup \{\alpha\}\} \in \hat{\mathcal{E}}_0$.
\end{enumerate}

3.5 Remark. 0) With more care in the game Definition 3.10 we incorporate choosing the $\bar{p}^\alpha$’s. In 3.7(*)(ii) we can add $\langle N^{1}_{\beta + 1} : \beta \in \alpha \cap c \rangle$ belongs to $N_{\alpha}^0$.

1) In the Definition 3.4, without loss of generality $c \in \mathcal{P}_\alpha$ & $\beta \in c \Rightarrow c \cap \beta \in \mathcal{P}_\beta$.

2) If $\kappa$ is strongly inaccessible without loss of generality we have $\mathcal{P}_\alpha = \mathcal{P}(\alpha)$, so fat has a winning strategy.

3) In general being fat is a weak demand, e.g. if $\hat{\mathcal{E}}_0 \supseteq \{a : \bar{a} = \langle a_i : i \leq \omega\rangle, a_\omega = \bigcup_n a_n, a_i \in [\mu]^*_{<\kappa}\}$ is increasing.


Assume $p_\alpha \in \mathbb{Q}$ for $\alpha < \mu$ and let $\chi$ be large enough and $x$ as in Definition 3.2. We choose $\langle \bar{N}^\alpha, \bar{p}^\alpha \rangle$ by induction on $\alpha < \mu$ as follows. If $\langle \langle \bar{N}^\beta, \bar{p}^\beta \rangle : \beta < \alpha\rangle$ is already defined, as $\hat{\mathcal{E}}_0$ is non-trivial there is a pair $(\bar{N}^\alpha, \bar{a}^\beta)$ which is $\hat{\mathcal{E}}_0$-complementary and $\langle \langle \bar{N}^\beta, \bar{p}^\beta \rangle : \beta < \alpha\rangle, \mathbb{Q}, \langle p_\beta : \beta < \mu\rangle, p_\alpha, x$ belong to $N_0^\alpha$ and let $N^\alpha = \langle N^\alpha_i : i \leq \delta_i\rangle$. So $p_\alpha \in N_0^\alpha$ and we can choose $p_{\alpha,i} \in N_{i+1}^\alpha$ such that $p_\alpha = p_{\alpha,0}$ and $\langle p_{\alpha,i} : i < \delta_\alpha\rangle$ is $(\bar{N}^\alpha, \mathbb{Q})$-generic.

[Why? By the proof of [Sh 587, B.5.6.4].] Now by “$\mathbb{Q}$ is $(\mu, S^\alpha, \hat{\mathcal{E}}_0)$-pic”, for some $\alpha < \beta$ in $S^\alpha$, $\{p^\alpha_i : i < \delta_\alpha\} \cup \{p^\beta_i : i < \delta_\beta\}$ has a common upper bound hence in particular, $p_\alpha, p_\beta$ are compatible.

Case 2: $\ell = 2$.

Assume $p_\alpha \in \mathbb{Q}$ for $\alpha < \mu$ and let $\chi$ be large enough. Let $\textbf{St}$ be a winning strategy for the player fat in the game $\mathcal{E}_{\kappa, \mu^*}(\hat{\mathcal{E}}_0)$. Now we choose by induction on $i < \kappa$. The tuple $(N^\alpha_i, \mathcal{P}^\alpha_i, Y^\alpha_i, \bar{p}^\alpha_i)$ where $\bar{p}^\alpha_i = \langle p^\alpha_{i,c} : c \in \mathcal{P}^\alpha_i\rangle$ for $\alpha < \mu$ such that:

\begin{enumerate}[(a)]
    \item $M^\alpha_i < (\mathcal{H}(\chi), c, <^\chi)$
    \item $M^\alpha_i$ increasing continuous in $i$
    \item $\|M^\alpha_i\| < \kappa$ and $\langle M^\alpha_j : j \leq i\rangle \in M^\alpha_{i+1}$ and $M^\alpha_i \cap \kappa = \kappa$ and $p_\alpha \in M^\alpha_i$,
    \item $\langle Y^\alpha_j, M^\alpha_j \cap \mu^*, \mathcal{P}^\alpha_j : j \leq i\rangle$ is an initial segment of a play of $\mathcal{E}_{\kappa, \mu^*}(\hat{\mathcal{E}}_0)$ in which the player fat uses his winning strategy $\textbf{St}$
\end{enumerate}
\[ (M_2, \P_2, Y_2, p_2) : j \leq i, \beta \in S \) belongs to \( N_{i+1}^\alpha \) (hence \( \P_2 \subseteq M_{j+1}^\alpha \), etc.)

\( p_{i,c}^\alpha \in Q \cap N_{i+1}^\alpha \)

\( \text{if } c \in \P_i^\alpha \) and \( \langle p_{i,c}, j \in c \rangle \) has an upper bound then \( p_{i,c}^\alpha \) is such a bound.

Can we carry the induction?

For \( i \) limit let \( M_i^\alpha = \{ M_j^\alpha : j < i \} \) and choose \( Y_i^\alpha, \P_i^\alpha \) by clause (d) i.e. by the rules of the game \( \mathcal{D}_{\kappa, \mu^*}(\widehat{\alpha}_0) \) and \( p_i^\alpha \) by clause (g) + (h) (possible as forcing by \( Q \) adds no new sequences of length \( < \kappa \) of members of \( V \)). For \( i \) non-limit, let \( x_i = \langle (M_j^\beta, \P_j^\beta, Y_j^\beta, p_j^\beta) : j \leq i, \beta \in S \rangle \) let \( Y_i^\alpha = \{ a : a \in [\mu^*]^{<\kappa} \text{ and } \alpha \in a \text{ and } a = \mu^* \cap \text{Sk}^{\kappa}_{<\kappa}(\langle x_i \times Q, \text{St}, \alpha \rangle) \rangle \} \). Let \( a_i^\alpha, \P_i^\alpha \) be the move which the strategy \( \text{St} \) dictate to the player fat if the \( i \)-th move of lean is \( Y_i^\alpha \), and the play so far is \( \langle (Y_j^\alpha, M_j^\alpha \cap \mu^*, \P_{\alpha,j}) : j < i \rangle \rangle \). Now we choose \( M_i^\alpha = \text{Sk}^{\kappa}_{<\kappa}(\langle x_i \times Q, \text{St}, \alpha \rangle) \rangle \) and \( \P_i^\alpha \) has already been chosen and \( \bar{p}_i^\alpha = \langle p_{i,c}^\alpha : c \in \P_i^\alpha \rangle \) as in the limit case.

Having carried the induction, for each \( \alpha \in S \) in the play \( \langle (Y_i^\alpha, M_i^\alpha \cap \mu^*, \P_i^\alpha) : i < \kappa \rangle \), the player fat wins the game having used the strategy \( \text{St} \), hence there are a limit ordinal \( i_\alpha < \kappa \) and closed \( c_\alpha \in \P_\alpha \) and \( i_\alpha = \sup\{c_\alpha \} \) and \( \langle M_j^\alpha : j \in c_\alpha \cup \{i_\alpha \} \rangle \) obeys some member \( \bar{a}_\alpha \) of \( \widehat{\alpha}_0 \). As \( Q \) is \( c\ell(\widehat{\alpha}_0) \)-complete, we can prove by induction on \( j \in c_\alpha \cup \{i_\alpha \} \) that \( \varepsilon < j \text{ and } \varepsilon \in C_\alpha \Rightarrow Q \models p_{i_\alpha}^\alpha, c_\alpha \cap \varepsilon \leq p_{i_\alpha}^\alpha, c_\alpha \cap j \).

Let \( \delta_\alpha = i_\alpha, N_\alpha^\alpha = M_\alpha^\alpha \) for \( i \leq \delta_\alpha \) and \( \bar{p}_\alpha = \langle p_i^\alpha : i \in c_\alpha \rangle \). Now continue as in Case 1. \( \square \)

3.6 Claim. If \( (\ast) \) of Definition 3.2, we can allow \( \text{Dom}(g) \) to be a subset of \( \text{ScapC}, \langle A_i : i < \mu \rangle \rangle \) be an increasingly continuous sequence of sets, \( |A_i| < \mu \), \( N_{\delta_\alpha}^\alpha \subseteq A_{\alpha+1} \), replacing the definition of \( g, g_0 \) and by \( g_0(\alpha) = N_{\delta_\alpha}^\alpha \cap A_\alpha \) and \( g_1 \) by \( g_1(\alpha) = (N_{\delta_\alpha}^\alpha, N_{\alpha}^\alpha, c)\langle i < \delta_\alpha, c \in g_0(\alpha) \rangle \approx (\mu, S^{\square}, \widehat{\alpha}_0)\text{-pic}_\epsilon \) (and get equivalent definition).

Remark. If \( \text{Dom}(g) \cap S^{\square} \) is not stationary, the definition says nothing.

Proof. Straight.

3.7 Claim. Assume clauses \( \exists \), i.e. (a), (b), (c) of 3.2 and (d) of 3.3.

For \( (\kappa) \)-support iteration \( Q = \langle P_i, Q_i : i < \alpha \rangle \), if we have \( \models_{P_i} \) “\( Q_i \) is \( (\mu, S^{\square}, \widehat{\alpha}_0)\text{-pic}_\epsilon \)” for each \( i < \alpha \) and forcing with \( \text{Lim}(Q) \) add no bounded subsets of \( \kappa \), then \( P_\alpha \) and \( P_\beta/P_\alpha \), for \( \beta \leq \gamma \leq \ell g(Q) \) are \( \widehat{\alpha}_0 \)-complete \( (\mu, S^{\square}, \widehat{\alpha}_0)\text{-pic}_\epsilon \).
3.8 Remark. We can omit the assumption “\(\text{Lim} (\bar{Q})\) add no bounded subsets of \(\kappa\)” if we add the assumption \(cl (\hat{D}_0) \in \mathcal{C}_{< \kappa} (\mu^*)\), see [Sh 587, Def.B.5.1(2)], because with the later assumption the former follows by [Sh 587, B.5.6].

Proof. Similar to [Sh:f, Ch.VIII]. We first concentrate on

Case 1: \(\ell = 1\).

It is enough to prove for \(P_\alpha\).
We prove this by induction on \(\alpha\). Let \(\models_{P_i} \text{“} Q_i \text{ is} (\mu, S^\square, \hat{D}_0)\)-pic \(\ell\) as witnessed by \(x_i\) and let \(\chi_i = \text{Min}\{ \chi : x_i \in \mathcal{H} (\chi) \}\).

Let \(x = (\mu^*, \kappa, \mu, S^\square, \hat{D}_0, (\chi_i, x_i) : i < \ell g (\bar{Q}))\) and assume \(\chi\) is large enough such that \(x \in \mathcal{H} (\chi)\) and let \(\langle (\bar{N}_\alpha, \bar{P}_\alpha) : \alpha \in S \rangle\) be as in Definition 3.2, so \(S \subseteq S^\square\) is stationary and \(\bar{N}_\alpha = \langle N^\alpha_i : i \leq \delta_\alpha \rangle\). We define a \(g\) by

\[
\begin{align*}
\exists_1 \ g & \text{ is a function with domain } S \\
\exists_2 \ g (\alpha) = (g_\ell (\alpha) : \ell < 2) \text{ where } \\
g_0 (\alpha) = (N^\alpha_0) \cap ( \bigcup_{\beta < \alpha} N^\beta_0 ) \\
g_1 (\alpha) = \text{the isomorphic type of } (N^\alpha_0, N^\alpha_1, P^\alpha, c)_{c \in g_0 (\alpha)}.
\end{align*}
\]

Let \(C\) be a club of \(\mu\) such that \(\alpha \in S \cap C \Rightarrow \langle (\bar{N}_\beta, \bar{P}_\beta) : \beta < \alpha \rangle \in N^\alpha_0\), (recall \(\ell = 1\)).

Fix \(y\) such that \(S_y = \{ \alpha \in S : g (\alpha) = y \text{ and } \alpha \in C \}\) is stationary.

Let \(w_\alpha = \bigcup_{i < \delta_\alpha} \text{Dom} (p^\alpha_i), w^*_y = w_\alpha \cap g_0 (\alpha)\) for \(\alpha \in S_y\) (as \(\alpha \in S_y\), clearly the set does not depend on the \(\alpha\)). For each \(\zeta \in w^*_y\) we define a \(P_\zeta\)-name, \(S_{y, \zeta}\) as follows:

\[S_{y, \zeta} = \{ \alpha \in S_y : (\forall i < \delta_\alpha) (p^\alpha_i \upharpoonright \zeta \in G_\zeta) \}.\]

Now we try to apply Definition 3.2 in \(V^{P_\zeta}\) to

\[\langle (N^\alpha_i [G_\zeta] : i \leq \delta_\alpha), (p^\alpha_i (\zeta) [G_\zeta] : i < \delta_\alpha) \rangle : \alpha \in S_{y, \zeta} [G_\zeta] \rangle\].

Clearly, if \(S_{y, \zeta} [G_\zeta]\) is a stationary subset of \(\mu\), we can apply it and \(g_{y, \zeta}\) be the \(P_\zeta\)-name of a function with domain \(S_{y, \zeta}\) defined like \(g\) in (*) of Definition 3.2. Now \(g_{y, \zeta}\) is well defined.

and actually can be computed if we use \(A_\beta = \bigcup \{ N^\alpha_{\delta_\alpha} : \alpha < \beta \}\). So by an induction hypothesis on \(\alpha\) there is a suitable \(P_\zeta\)-name \(C_\zeta\) of a club of \(\mu\) such that in addition, if \(S_{y, \zeta} [G_\zeta]\) is not a stationary subset of \(\mu\), let \(C_\zeta [G_\zeta]\) be a club of \(\mu\).
disjoint to it. But as \( P_\zeta \) satisfies the \( \mu \)-c.c. without loss of generality \( C_\zeta = C_\zeta \) so \( C' = C \cap \bigcap_{\zeta \in w'} C_\zeta \) is a club of \( \mu \). Now choose \( \alpha_1 < \alpha_2 \) from \( S_y \cap C' \) and we choose by induction on \( \varepsilon \in w' = w_y^* \cup \{0, \ell g(Q)\} \) a condition \( q_\varepsilon \in P_\varepsilon \) such that:

\[
\exists \exists (i) \varepsilon_1 < \varepsilon \Rightarrow q_{\varepsilon_1} = q_\varepsilon \upharpoonright \varepsilon_1
\]

\[
(ii) \ q_\varepsilon \text{ is a bound to } \{p_\alpha^{\varepsilon_1} \upharpoonright \varepsilon : \delta < \delta_\alpha \} \cup \{p_\beta^{\varepsilon_2} \upharpoonright \varepsilon : i < \delta_\alpha \}.
\]

For \( \varepsilon = 0 \) let \( q_0 = \emptyset \). We have nothing to do really if \( \varepsilon \) is with no immediate predecessor in \( w \), we let \( q_\varepsilon \) be \( \bigcup \{q_{\varepsilon_1} : \varepsilon_1 < \varepsilon, \varepsilon_1 \in w'\} \). So let \( \varepsilon = \varepsilon_1 + 1, \varepsilon_1 \in w' \); now if \( q_\varepsilon \in G \subseteq P_{\varepsilon_1,2}, G \) generic over \( V \), then \( \alpha_1, \alpha_2 \in S_{y,\varepsilon_1}[G] \), hence \( S_{y,\zeta}[G] \cap C_{\varepsilon_1} \) is non-empty, hence is stationary, and we use Definition 3.2.

**Case 2:** \( p = 2 \).

Similar proof. \( \square \)

**3.9 Claim.** Assume \( \mu = \text{cf}(\mu) > \kappa, (\forall \alpha < \mu)(|\alpha|^{< \kappa} < \mu), S \subseteq \{\delta < \mu : \text{cf}(\delta) \geq \kappa\} \) is stationary. If \( |Q| \leq \kappa \) or just \( < \mu, \mathcal{E}_0 \in \mathcal{C}_{<\kappa}^<(\mu^*) \), that is \( \subseteq \{a : a \text{ increasing continuous of length } < \kappa, a_i \in [\mu^*]^{< \kappa} \text{ and } a_i \in \kappa \} \) non-trivial, possibly just for one cofinality say \( \kappa_0 \), then \( Q \) satisfies \( \kappa^+\)-pic.

**Proof.** Trivial, we get same sequence of condition or just see the proof of [Sh 587, B.7.4]. \( \square \)

**3.10 Discussion:** 1) What is the use of pic?

In the forcing axioms instead \( \text{"}|Q| \leq \kappa \" \) we can write \( \text{"Q satisfies the } (\mu, S^\Box, \mathcal{E}_0)-\text{pic"} \). This strengthens the axioms.

In [Sh:f] in some cases the length of the forcing is bounded (there \( \omega_2 \)) but here no need (as in [Sh:f, Ch.VII, \S 1]).

This section applies to all cases in [Sh 587] and its branches.

2) Note that we can demand that the \( p_\alpha^\varepsilon \) satisfies some additional requirements (in Definition 3.2) say \( p_\alpha^{\varepsilon_i} = F_\varepsilon(\bar{N} \upharpoonright (2i + 1), \bar{p}_\alpha \upharpoonright (2i + 1)) \).

Let us see how this gives some improvement of the results of [Sh 576, B.8] on \( \mathcal{C}_{<\kappa}^<(\mu^*) \), see [Sh 587, B.5.7.3].

**3.11 Definition.** Assume

\( \oplus \kappa > \kappa_0 \) is strongly inaccessible and \( (\mathcal{E}_0, \mathcal{E}_1) \in \mathcal{C}_{<\kappa}^<(\mu^*) \) and \( \theta_0, \theta_1 \) are regular cardinals \( > \kappa, \theta_2 \) a cardinal \( > \kappa \) (let \( \theta = (\theta_0, \theta_1, \theta_2) \), the usual case is \( \theta_0 = \kappa^+ \) and \( \mathcal{E} \subseteq \mathcal{E}_1 \) is nontrivial (see in Definition 3.3, clause (d)) and \( \ell \in \{1, 2\} \).
Let $Ax_{\theta_1,\theta_2}^\kappa(\hat{E}_0,\hat{E}_1,\bar{s})$, the forcing axiom for $(\hat{E}_0,\hat{E}_1,\bar{s})$, and $\bar{\theta} = (\theta_0,\theta_1,\theta_2)$ be the following statement:

\[ \text{\textbullet} \text{ if} \]

(i) $\mathbb{Q}$ is a focing notion of cardinality $< \theta_1$
(ii) $\mathbb{Q}$ is complete for $(\hat{E}_0,\hat{E}_1)$, see Definition [Sh 587, B.5.9(3)]
(iii) $\mathbb{Q}$ satisfies $(\theta_0,S^{\square,\hat{e}})$-pic $\ell$
(iv) $\mathcal{I}_i$ is a dense subset of $\mathbb{Q}$ for $i < i^* < \theta_2$,

then there is a directed $H \subseteq \mathbb{Q}$ such that $(\forall i < i^*)(H \cap \mathcal{I}_i \neq \emptyset)$.

3.12 Theorem. Assume $\oplus$ of Definition 3.11 and $\mu = \mu^{\theta_1} < \theta_0 \geq \theta_0 + \theta_2$.
Then there is a forcing notion $\mathbb{P}$ such that:

$(\alpha)$ $\mathbb{P}$ is complete for $\hat{E}_0$
$(\beta)$ $\mathbb{P}$ has cardinality $\mu$
$(\gamma)$ $\mathbb{P}$ satisfies the $\theta_0$-c.c. and even the $(\kappa,\theta_0,\hat{s})$-pic $\ell$
$(\delta)$ $\mathbb{P}$ is complete for $(\hat{E}_0,\hat{E}_1)$, hence $\Vdash \mathbb{P}$ “$(\hat{E}_0,\hat{E}_1) \in C^{\bullet}_{<\kappa}(\mu^*)”$ and more
$(\epsilon)$ $\Vdash \mathbb{P}$ “$Ax_{\bar{\theta}}^\kappa(\hat{E}_0,\hat{E}_1,\bar{s})$.

Proof. Like the proof of [Sh 587, B.8.2], using 3.7 instead of [Sh 587, B.7.4]. $\square_{3.12}$

We may wonder how large can a stationary $S \subseteq \kappa$ be?

3.13 Claim. 1) Assume

$\oplus(a)$ $\kappa$ is strongly inaccessible $> \aleph_0$
$\oplus(b)$ $S \subseteq \kappa$ is stationary
$\oplus(c)$ for letting $\mu^* = \kappa$ and $\hat{E}_0 = \hat{E}_0[S] = \{\bar{a} \in C_{<\kappa}(\mu^*)$: for every $i \leq \ell g(\bar{a})$ we have $a_i \notin S \}$ we have $\hat{E}_0 \in C_{<\kappa}(\mu^*)$
$\oplus(d)$ we let $\hat{E}_1 = \hat{E}_1[S] = \{\bar{a} \in C_{<\kappa}(\mu^*)$: for every nonlimit $i \leq \ell g(\bar{a})$ we have $a_i \notin S \}.$

Then

$(\alpha)$ $(\hat{E}_0,\hat{E}_1) \in C^{\bullet}_{<\kappa}(\mu^*)$, see [Sh 587, B.5.7(3)].

2) The parallel of 2.11.

We now deal with forcing the failure of diamond on the set of inaccessibles.
3.14 Claim. Assume

(a) \( \kappa, S, \hat{E}_0, \hat{E}_1 \) are as in 3.13
(b) if \( S_{bd} = \{ \theta < \kappa : \theta \) strongly inaccessible, \( S \cap \theta \) is stationary in \( \theta \) and \( \diamond S \cap \theta \} \) is not a stationary subset of \( \kappa \)
(c) \( \bar{A} = \langle A_\alpha : \alpha \in S \rangle, A_\alpha \subseteq \alpha \)
(d) \( Q = Q_{\bar{A}} \) is as in Definition 3.15 below
(e) \( \hat{E} \subseteq \hat{E}_0 \) is nontrivial.

Then

\((\alpha)\) \( Q \) is complete for \( (\hat{E}_0, \hat{E}_1) \)

\((\beta)\) \( Q \) satisfies the \((\kappa, \kappa^+, \hat{E}^\ell)\)-pic

\((\gamma)\) \( Q \) satisfies the \( \kappa^+\)-c.c.

3.15 Definition. For \( \kappa = \text{cf}(\kappa), S \subseteq \kappa = \text{sup}(S), \bar{A} = \langle A_\alpha : \alpha \in S \rangle, \) with \( A_\alpha \subseteq \alpha \) we define the forcing notions \( Q = Q^d_{\bar{A}} \) as follows:

(a) \( p \in Q \) iff

\((i)\) \( p = (c, A) = (c^p, A^p) \)

\((ii)\) \( c \) is \( \emptyset \) or a closed bounded subset of \( \kappa \) hence has a last element

\((iii)\) \( A \subseteq \text{sup}(c) \) such that

\((iv)\) if \( \alpha \in C \cap S \) then \( A \cap \alpha \neq A_\alpha \)

(b) \( p \leq q \) iff

\((i)\) \( c^p \) is an initial segment of \( c^q \)

\((ii)\) \( A^p = A^q \cap \text{sup}(c^p) \).

Proof of 3.14. We concentrate on part (1), part (2)'s proof is similar. Now

\((\ast)_1\) for every \( \alpha < \kappa \), \( \mathcal{I}_\alpha = \{ p \in Q : \alpha < \text{sup}(c^p) \} \) is dense open.

[Why? If \( p \in Q \), let \( \beta = \text{sup}(c^p) + 1 + \alpha \) and \( q = (c^p \cup \{ \beta \}, A^p) \), so \( p \leq q \in \mathcal{I}_\alpha \).]

\((\ast)_2\) If \( \delta < \kappa \) is a limit ordinal, \( \langle p_i : i < \delta \rangle \) is \( \leq Q \)-increasing and \( \text{sup}(c^{p_i}) \leq \alpha_{i+1} < \text{sup}(c^{p_{i+1}}) \) for \( i < \delta \), and for limit \( i, \alpha_i = \cup \{ \alpha_j : j < i \} \) and \( \{ \alpha_{1+i} : i < \delta \} \)
is disjoint to $S$, then $p = \left( \bigcup_{i < \delta} c_i^p, \bigcup_{i < \delta} A_i^p \right)$ is a $\leq \kappa$-lub of $\langle p_i : i < \delta \rangle$.

[Why? Just think.]

$(*)_3$ forcing with $\mathcal{Q}$ add no new sequences of length $< \kappa$ of ordinals (or members of $\mathbf{V}$).

[Why? By $(*)_2$+ the assumption @, clause (c) of Claim 3.13 as in [Sh 587, B.6].]

$(*)_4$ $\mathcal{Q}$ is complete for $\hat{\delta}_0$

[Why? Just think.]

$(*)_5$ $\mathcal{Q}$ is complete for $(\hat{\delta}_0, \hat{\delta}_1)$, see [Sh 587, Def.B.5.9(3)].

[Why? Let $\chi$ be large enough and let $\langle M_i : i < \delta \rangle$ be ruled by $\langle \hat{\delta}_0, \hat{\delta}_1 \rangle$, with $\hat{\delta}_0$-approximation $\langle (\hat{N}_i, \hat{a}_i) : i < \delta \rangle$, see [Sh 587, Def.B.5.9(1)] and $r \in \mathcal{Q} \cap M_0$ and $S, \kappa, \bar{A} \subseteq M_0$ and we have to prove that the player COM has a winning strategy in the game $\mathcal{G}_{\bar{M}, (\hat{N}_i, i < \delta)}(\mathcal{Q}, r)$.

For this we proved by induction on $0 < \kappa$ (a limit ordinal) the statement $\exists_\delta$ if $\langle M_i : i \leq \delta \rangle, \langle \hat{N}_i : i < \delta \rangle, r$ are as above (but $\alpha$ may be a nonlimit ordinal) $\bar{b} = \langle b_i : i < \delta \rangle, b_i \subseteq M_{i+1} \cap \kappa \setminus M_i \leq \|M_i\|$ and $B \subseteq M_\delta \cap \kappa$ (or just $B \subseteq \cup\{b_i : i < \delta \}$, then we can find $p$ such that $r \leq p \in \mathcal{Q}$ and $A^p \cap b_i = B \cap b_i$ for every $i < \delta$ and $\sup(c^p) = M_\delta \cap \kappa$.

Case 1: $\alpha$ nonlimit. Trivial.

Case 2: $\alpha$ limit and for some $\beta < \alpha$ we have $\text{cf}(\delta) \leq \|M_i\|$. Let $\theta = \text{cf}(\theta)$ and let $\langle \delta_\varepsilon : \varepsilon \leq \theta \rangle$ be increasing continuous, $\delta_0 = 0, \|M_\delta\| > \theta$ and $\delta_\theta = \delta$.

Choose $b \subseteq M_{\delta_\theta + 1} \cap \kappa \setminus M_{\delta_1} \setminus b_{\delta_1}$ of cardinality $\theta$ and choose $b' \subseteq b$ such that $\zeta \in (\varepsilon, \delta] \Rightarrow A_{M_{\delta_\theta} \cap \kappa \cap b} = b'$. By the induction hypothesis, we can find $r_{\delta_1} \in M_{\delta_1 + 1}$ such that $\text{sup}(c^{r_1}) = M_{\delta_1} \cap \kappa, r \leq r_{\delta_1}, \beta < \delta_1 \Rightarrow A^{r_1} \cap b_{\beta} = B \cap b_{\beta}$ and $r_1$ is $(M_\beta, \mathcal{Q})$-generic for every $\beta \leq \delta_1$. Let $r_{\delta_1}^+$ be such that $r_{\delta_1} \leq r_{\delta_1}^+ \in \mathcal{Q} \cap M_{\delta_1 + 1}$ and $\sup(b_{\delta_1} \cup b) < \sup(r_{\delta_1}^+)$ and $A_{r_{\delta_1}^+}^{r_{\delta_1}^+} \cap b_{\delta_1} = B \cap b_{\delta_1}$ and $A_{r_{\delta_1}^+}^{r_{\delta_1}^+} \cap b = b'$. Now we choose by induction on $\varepsilon \in [2, \delta]$, a condition $r_\varepsilon$ such that $r_\varepsilon \in M_{\delta_\varepsilon + 1}$, $\text{sup}(c^{r_\varepsilon}) = M_{\delta_\varepsilon} \cap \kappa, r_\varepsilon^+ \leq r_{\varepsilon}, [\zeta \in [2, \varepsilon] \Rightarrow r_\zeta \leq r_\varepsilon]$ and $\beta < \delta_\varepsilon \Rightarrow A^{r_\varepsilon} \cap b_{\beta} = B \cap b_{\beta}$ and $r_{\varepsilon}$ is $(M_\beta, \mathcal{Q})$-generic for every $\gamma \leq \delta_\varepsilon$. For limit $\varepsilon, r_\varepsilon$ is uniquely determined and it $\in \mathcal{Q}$ by the choice of $r_1^+$. For $\varepsilon$ nonlimit use the induction hypothesis for $\langle M_\beta : \beta \in [\delta_\varepsilon + 1, \delta_{\varepsilon + 1}] \rangle$.

Case 3: Neither Case 1 nor Case 2.

So $\alpha$ is strongly inaccessible, call it $\theta$ and $\theta = M_\theta \cap \kappa$; so as $\{\kappa, S\} \subseteq M_\theta \prec (\mathcal{H}(\chi), \in, <^*_\chi)$, necessarily $\delta = \sup(S), \delta \in S_{bd}$ and $\neg \Diamond_{\theta \cap S}$ (e.g. $\theta \cap S$ is not
stationary in $S$). Choose for each $\beta < \theta$, an ordinal $\gamma_\beta \in M_{\beta+1} \cap \kappa \setminus M_{\beta} \setminus b_\beta$ and let $A'_i = \{ j < i : \gamma_j \in A_{M_\beta \cap \kappa} \}$ for $i \in S \cap \theta$.

Now $\langle A'_i : i \in S \cap \theta \rangle$ cannot be a diamond sequence for $\theta$ hence we can find $X \subseteq \theta$ and club $C^-$ of $\theta$ such that $\delta \in X \cap S \Rightarrow A^-_\delta \neq X \cap \delta$. Let $C = \{ i < \theta : i \text{ limit, } (\forall j < i)(\gamma_j < i) \text{ and } i \in C^- \text{ and } M_i \cap \kappa = i \}$, clearly $C$ is a club of $\theta$. Let $b^+_\beta = a_\beta \cup \{ \gamma_\beta \}, B^+ = B \cup \{ \gamma_\beta : \beta \in X \}$, and proceed naturally. \hfill \Box_{3.14}

3.16 Remark. So we can iterate and get that (G.C.H. and) diamond fail for “most” stationary subsets of any strongly inaccessibles. We shall return to this elsewhere.
§4 Existence of non-free Whitehead (and $\text{Ext}(G, \mathbb{Z}) = \{0\}$) abelian groups in successor of singulars

In [Sh 587], the consistency with GCH of the following is proved for some regular uncountable $\kappa$: there is a $\kappa$-free nonfree abelian group of cardinality $\kappa$, and all such groups are Whitehead. We use $\kappa$ inaccessible, here we ask: is this assumption necessary for the first such $\kappa$?

The following claim seems to support the hope for a positive answer.

4.1 Claim. Assume

(a) $\lambda$ is strong limit singular, $\sigma = \text{cf}(\lambda) < \lambda, \kappa = \lambda^+ = 2^\lambda$

(b) $S \subseteq \{\delta < \kappa : \text{cf}(\delta) = \sigma\}$ is stationary

(c) $S$ does not reflect or at least

(c) $\tilde{A} = \langle A_\delta : \delta \in S \rangle$, $\text{otp}(A_\delta) = \sigma$, $\text{sup}(A_\delta) = \delta$ and

$\tilde{A}$ is $\lambda$-free, i.e., for every $\alpha < \kappa$ we can find $\langle \alpha_\delta : \delta \in \alpha^* \cap S \rangle, \alpha_\delta < \delta$ such that $\langle A_\delta \setminus \alpha_\delta : \delta \in S \cap \alpha^* \rangle$ is a sequence of pairwise disjoint sets

(d) $\langle G_i : i \leq \sigma \rangle$ is a sequence of abelian groups such that:

(α) $\delta < \sigma$ limit $\Rightarrow G_\delta = \bigcup_{i < \delta} G_i$

(β) $i < j \leq \sigma \Rightarrow G_j/G_i$ free and $G_i \subseteq G_j$

(γ) $G_\sigma/\bigcup_{i < \sigma} G_i$ is not Whitehead

(δ) $|G_\sigma| < \lambda$

(ε) $G_0 = \{0\}$.

Then

1) There is a strongly $\kappa$-free abelian group of cardinality $\kappa$ which is not Whitehead, in fact $\text{Γ}(G) \subseteq S$.

2) There is a strongly $\kappa$-free abelian group $G^*$ of cardinality $\kappa$ satisfying $\text{HOM}(G^*, \mathbb{Z}) = \{0\}$, in fact $\text{Γ}(G^*) \subseteq S$ (in fact the same abelian group can serve).

3) We can rephrase clause (d)(γ) of the assumption, i.e. “$G_\sigma/\bigcup_{i < \sigma} G_i$ is not Whitehead” by:

(d)(γ) some $f^* \in \text{HOM}(\bigcup_{i < \sigma} G_i, \mathbb{Z})$ cannot be extended to $f' \in \text{HOM}(G_\sigma, \mathbb{Z})$.

We first note:
4.2 Claim. Assume

(a) $\lambda$ strong limit singular, $\sigma = cf(\lambda) < \lambda, \kappa = 2^\lambda = \lambda^+$
(b) $S \subseteq \{ \delta < \kappa : cf(\delta) = \sigma$ and $\lambda^\delta$ divides $\delta$ for simplicity $\}$ is stationary
(c) $A_\delta \subseteq \delta = sup(A_\delta), \text{otp}(A_\delta) = \sigma, A_\delta = \{ \alpha_{\delta, \zeta} : \zeta < \sigma \}$ increasing with $\zeta$
(d) let $h_0 : \kappa \rightarrow \kappa$ and $h_1 : \kappa \rightarrow \sigma$ be such that
\begin{align*}
&(\forall \alpha < \kappa)((\forall \zeta < \sigma)(\forall \gamma \in (\alpha, \kappa))(\exists \lambda^\beta \in [\gamma, \gamma + \lambda])(h_0(\beta) = \alpha$ and $h_1(\beta) = \zeta), \\
&\text{and } (\forall \alpha < \kappa)h_0(\alpha) \leq \alpha
\end{align*}
(e) Let $\lambda = \langle \lambda_{\zeta} : \zeta < \sigma \rangle$ be increasing continuous with limit $\lambda$ such that $\lambda_0 = 0$ and $\zeta < \sigma \Rightarrow \lambda_{\zeta+1} = cf(\lambda_{\zeta+1}) > \sigma$.

Then we can choose $\langle (g_\delta, \gamma^\delta_{\zeta} : \zeta < \lambda) : \delta \in S \rangle$ such that
\begin{align*}
\odot_1(i) \quad &\langle \gamma^\delta_{\zeta} : \zeta < \lambda \rangle \text{ is strictly increasing with limit } \delta \\
(ii) \quad &\text{if } \lambda_{\zeta} \leq \xi < \lambda_{\zeta+1} \text{ then } h_0(\gamma^\delta_{\xi}) = h_0(\gamma^\delta_{\lambda_{\zeta}}) = \alpha_{\delta, \zeta} \text{ and } h_1(\gamma^\delta_{\xi}) = h_1(\gamma^\delta_{\lambda_{\zeta}}) = \zeta \\
(iii) \quad &h^*_\delta \text{ a partial function from } \kappa \text{ to } \kappa, sup(Dom(h^*_\delta)) < \gamma^\delta_{\zeta} \text{ for } \delta \in S
\end{align*}
\begin{align*}
\odot_2 \quad &\text{for every } f : \kappa \rightarrow \kappa, B \in [\kappa]^{<\lambda} \text{ and } g^2_\zeta : \kappa \rightarrow \lambda_{\zeta+1} \text{ for } \zeta < \sigma \text{ there are stationarily many } \delta \in S \text{ such that: }
\begin{align*}
(i) \quad &h^*_\delta = f \upharpoonright B \\
(ii) \quad &\text{if } \lambda_{\zeta} \leq \xi < \lambda_{\zeta+1} \text{ then } g^2_\zeta(\gamma^\delta_{\xi}) = g^2_\zeta(\gamma^\delta_{\lambda_{\zeta}}).
\end{align*}
\end{align*}

Remark. Note that when subtraction or division$^3$ is meaningful, $\odot_2$ is quite strong.

Proof. By the proofs of 1.1, 1.2 (can use guessing clubs by $\alpha_{\delta, \zeta}$'s, can demand that $\beta^\delta_{2\zeta}, \beta^\delta_{2\zeta+1} \in [\alpha_{\delta, \zeta}, \alpha_{\delta, \zeta} + \lambda]$).

But to help the reader we give a proof.

Let $\lambda = \sum \lambda_i, \lambda_i$ increasing continuous, $\lambda_{i+1} > 2^{\lambda_i}, \lambda_0 = 0, \lambda_1 > 2^\sigma$. Let $M_i < (\mathcal{H}(2), \in, <^*)$ be increasing continuous, $||M_i|| = \lambda, \langle M_j : j \leq i \rangle \in M_{i+1}, \lambda + 1 \subseteq M_i$ and $\{ A, h_0, h_1, \bar{\lambda} \} \in M_0$. For $\alpha < \lambda^+$, let $\alpha = \bigcup_{i < \sigma} a_{\alpha, i}$ such that $|a_{\alpha, i}| \leq \lambda_i$ and $a_{\alpha, i} \in M_{\alpha+1}$ and even $\langle a_{\beta, i} : i < \sigma : \beta \leq \alpha \rangle \in M_{\alpha+1}$. Without loss of generality $\delta \in S \Rightarrow \delta$ divisible by $\lambda^\omega$ (ordinal exponentiation). For $\delta \in S$

$^3$ i.e. $x_\beta$ belongs to some additive group $G^*$ for $\beta < \kappa, \gamma \in \text{Hom}(G^*, H^*), g(\beta) = \gamma(x_\beta)$ then for some $\delta$ as in $\odot_2$, we have $g(x^0_{\beta, \delta} - x_{\beta, \delta}) = 0_{H^*}$; similarly for multiplicative groups.
let \( \bar{\beta}_i^\delta = (\beta_i^\delta : i < \sigma) \) be increasing continuous with limit \( \delta, \beta_\delta^\delta \) divisible by \( \lambda \) and \( > 0 \). For \( \delta \in S \) let \( (b_i^\delta : i < \sigma) \) be such that: \( b_i^\delta \subseteq \beta_i^\delta, |b_i^\delta| \leq \lambda, b_i^\delta \) is increasingly continuous in \( i \) and \( \delta = \bigcup_{i<\sigma} b_i^\delta \) (e.g. \( b_i^\delta = \bigcup_{j_1,j_2<i} a_{j_1,j_2}^\delta \cup \lambda_i \)). We further demand \( \lambda_i \subseteq b_i^\delta \cap \lambda \). Let \( (f_{\alpha}^*: \alpha < \lambda^+) \) list the two-place functions with domain an ordinal \( < \lambda^+ \) and range \( \subseteq \lambda^+ \). Let \( H \) be the set of functions \( h, \text{Dom}(h) \in [\kappa]^{<\lambda}, \text{Rang}(h) \subseteq \kappa, \text{so } |H| = \kappa \). Let \( S = \bigcup \{ S_h : h \in H \} \), with each \( S_h \) stationary and \( \langle S_h : h \in H \rangle \) pairwise disjoint. Without loss of generality \( \delta \in S_h \Rightarrow \sup(\text{Dom}(h)) < \beta_\delta^\delta \). Let \( h_\delta^* \) be \( h \) when \( \delta \in S_h \). We now fixed \( h \in H \) and will choose \( \bar{\gamma}^\delta = (\gamma_i^\delta : i < \lambda) \) for \( \delta \in S_h \) such that clauses \( (1) + (2) \) for our fixed \( h \) (and \( \delta \in S_h \) ignoring \( h \) in \( (2) \)) hold, clearly suffices.

Now for \( \delta \in S_h \) and \( i < \sigma \) and \( g \in \sigma \sigma \) we can choose \( \zeta_i^\delta \) (for \( \varepsilon < \lambda_{i+1} \)) such that:

\begin{align*}
(A) & \quad (\zeta_i^\delta : \varepsilon < \lambda_{i+1}) \text{ is a strictly increasing sequence of ordinals} \\
(B) & \quad \beta_i^\delta < \zeta_i^\delta < \beta_{i+1}^\delta, \text{ (can even demand } \zeta_i^\delta < \beta_i^\delta + \lambda) \\
(C) & \quad h_0(\zeta_i^\delta, \varepsilon) = \alpha_{\delta,i} \text{ and } h_1(\zeta_i^\delta, \varepsilon) = i \\
(D) & \quad \text{for every } \alpha_1, \alpha_2 \in B_{g(i)}, \text{ the sequence } (\text{Min}\{\lambda_{g(i)}, f_{\alpha_1}^*(\alpha_2, \zeta_i^\delta) : \varepsilon < \lambda_{i+1}\}) \text{ is constant i.e. one of the following occurs: } \\
& \quad (\alpha) \quad \varepsilon < \lambda_{i+1} \Rightarrow (\alpha_2, \zeta_i^\delta) \notin \text{Dom}(f_{\alpha_1}^*) \\
& \quad (\beta) \quad \varepsilon < \lambda_{i+1} \Rightarrow f_{\alpha_1}^*(\alpha_2, \zeta_i^\delta) = f_{\alpha_1}^*(\alpha_2, \zeta_{i,0}^\delta) \text{ well defined} \\
& \quad (\gamma) \quad \varepsilon < \lambda_{j}, f_{\alpha_1}^*(\alpha_2, \zeta_i^\delta) \geq \lambda_j, \text{ well defined. We can add } (f_{\alpha_1}^*(\alpha_2, \zeta_i^\delta) : \\
& \quad \varepsilon < \lambda_i) \text{ is constant or strictly increasing.} \\
\end{align*}

\( (E) \) for some \( j < \sigma \), we have \( (\forall \varepsilon < \lambda_{i+1})[\zeta_i^\delta : \varepsilon < \lambda_{i+1}] \) where \( \alpha = \sup\{\zeta_i^\delta : \varepsilon < \lambda_{i+1}\} \), (remember \( \sigma \neq \lambda_{i+1} \) are regular).

For each function \( g \in \sigma \sigma \) we try \( \bar{\gamma}^\delta \) if \( \lambda_i \leq \varepsilon < \lambda_{i+1} \) then \( \gamma_i^\delta \varepsilon = \zeta_i^\delta \).

Now for some \( g \) it works. \( \square \)

\begin{proof}[Proof of 1.2(1)] Let \( M = \bigcup\{ M_\alpha : \alpha < \kappa \} \), \( M_\alpha \prec (\mathcal{H}(2^\kappa)^+, \varepsilon) \) has cardinality \( \lambda, M_\alpha \) is increasing continuous, \( (M_\beta : \beta \leq \alpha) \in M_\alpha \) and \( (F_i : i < \sigma) \) belongs to \( M_0 \).

Let \( E_0 = \{ \delta < \kappa : M_\delta \cap \kappa = \delta \} \) and \( E = \text{acc}(E_0) \). The proof is like the proof of 4.2 with the following changes:

\begin{enumerate}
\item[\( i \)] \( \beta_i^\delta \in E_0 \) for \( \delta \in S \cap E \)
\end{enumerate}

\end{proof}
(ii) in clause (A) we demand $\langle \zeta_{i,g}^\delta : g \in G, \varepsilon < \lambda_{i+1} \rangle$ belongs to $M_{\beta_{i+1}^\delta}$ (hence also $\langle \zeta_{j,g}^\delta : g \in G, \varepsilon < \lambda_{j+1} : j \leq i \rangle$ belongs to $M_{\beta_{i+1}^\delta}$)

(iii) clause (c) is replaced by: $\zeta_{i,g}^\delta, \varepsilon \in F_i(\{\zeta_{j,g_{(j+1)},\varepsilon}^\delta : \varepsilon < \lambda_{j+1} \text{ and } j < i\})$.

$\square_{1,2}$

Proof of 4.1. 1) We apply 4.2 to the $\langle A_\delta : \delta \in S \rangle$ from 4.1, and any $h_0, h_1$ as in clause (d) of 4.2.

Let $\{t_{i,j}^i + G_i : \gamma < \theta^{i,j} \}$ be a free basis of $G^i/G_i$ for $i < j \leq \sigma$. If $i = 0, j = \sigma$ we may omit the $i, j$, i.e. $t_{\zeta} = t_{\zeta}^{0,\sigma}$ and $\theta = \theta^{0,\sigma}$. Let $\theta + \aleph_0 = |G_\sigma| < \lambda$; actually $\theta^{\zeta,\zeta+1} < \lambda_{\zeta}$ is enough; without loss of generality $\theta < \lambda_1$ in 4.2. Let $\beta_{\zeta,i}^\delta = \gamma_{\zeta(\zeta,i)}^\delta$

where $\zeta(\zeta,i) = \bigcup \lambda_\varepsilon + 1 + i$ for $\delta \in S, \zeta < \sigma, i < \theta$.

Let $\beta_\delta(\ast) = \min \{\beta : \beta \in \text{Dom}(h_0^\delta), h_0^\delta(\beta) \neq 0\}$, if well defined where $h_0^\delta$ is from 4.2.

Clearly (see $\bigcirc_1(iii)$ of 4.2) we have $\beta_\delta(\ast) \notin \{\beta_{\zeta,i}^\delta : \zeta < \sigma, i < \theta\}$ (or omit $\lambda_\zeta, \beta_{\zeta,i}^\delta$ for $\zeta$ too small). We define an abelian group $G^*_{\delta}$: it is generated by $\{x_\alpha : \alpha < \kappa\} \cup \{y_{\gamma}^\delta : \gamma < \theta \text{ and } \delta \in S\}$ freely except for the relations:

$$(\ast)_1 \sum_{\gamma < \theta} a_\gamma y_{\gamma}^\delta = \sum \{b_{\zeta,\gamma}(x_{\beta_{\zeta,\gamma}^\delta} - x_{\gamma_{\zeta}^\delta}) : \zeta < \sigma \text{ and } \gamma < \theta^{\zeta,\zeta+1}\}$$

when $G_\sigma \models \sum_{\gamma < \theta^{0,\sigma}} a_\gamma t_{\gamma} = \sum \{b_{\zeta,\gamma}t_{\gamma}\zeta^{\zeta+1} : \zeta < \sigma \text{ and } \gamma < \theta^{\zeta,\zeta+1}\}$ where $a_\gamma, b_{\zeta,\gamma} \in \mathbb{Z}$ but all except finitely many are zero.

There is a (unique) homomorphism $g_\delta$ from $G_\sigma$ into $G^*$ induced by $g_\delta(t_{\gamma}) = y_{\gamma}^\delta$. As usual it is an embedding. Let $\text{Rang}(g_\delta) = G^{\langle \delta \rangle}$. For $\beta < \kappa$ let $G^*_{\beta,\delta}$ be the subgroup of $G^*$ generated by $\{x_\alpha : \alpha < \beta\} \cup \{y_{\gamma}^\delta : \gamma < \theta^{0,\sigma} \text{ and } \delta \in \beta \cap S\}$. It can be described similarly to $G^*$.

Fact A: $G^*$ is strongly $\lambda$-free.

Proof. For $\alpha^* < \beta^* < \kappa$, we can find $\langle \alpha_\delta : \delta \in S \cap (\alpha^*, \beta^*) \rangle$ such that $\langle A_\delta \setminus \alpha_\delta : \delta \in S \cap (\alpha^*, \beta^*) \rangle$ are pairwise disjoint and disjoint to $\alpha^*$ hence the sequence $\langle \beta_{\zeta,i}^\delta : i < \theta, \zeta \in (\min\{\zeta < \sigma : \beta_{\zeta,0}^\delta > \alpha_\delta\}, \sigma)\rangle : \delta \in S \cap (\alpha^*, \beta^*) \rangle$ is a sequence of pairwise disjoint sets.

For $\delta \in S \cap (\alpha^*, \beta^*)$, let $\zeta_\delta = \min\{\zeta : \beta_{\zeta,0}^\delta > \alpha_\delta\} < \sigma$. Now easily $G^*_{\beta^*+1,\delta}$ is generated as an extension of $G^*_{\alpha^*+1,\delta}$ by $\{g_{\delta}(t_{\gamma_{\zeta}^\delta}) : \gamma < \theta^{\zeta_\delta,\sigma} \text{ and } \delta \in S \cap (\alpha^*, \beta^*)\} \cup \{x_\alpha : \alpha < \beta^*\}$. 


\( \alpha \in (\alpha^*, \beta^*) \) and for no \( \delta \in S \cap (\alpha^*, \beta^*) \) do we have \( \alpha \in \{ \beta^*_\gamma : i < \theta \gamma, \zeta < \zeta \delta \} \); moreover \( G^*_{\alpha^*+1} \) is freely generated (as an extension of \( G^*_\alpha +1 \)). So \( G^*_{\beta^*+1}/G^*_\alpha^*+1 \) is free, as also \( G^*_1 \) is free we have shown Fact A.

**Fact B:** \( G^* \) is not Whitehead.

**Proof.** We choose by induction on \( \alpha \leq \kappa \), an abelian group \( H_\alpha \) and a homomorphism \( h_\alpha : H_\alpha \to G^*_{\alpha} = \langle \{ x_\beta : \beta < \alpha \} \cup \{ y_\gamma : \gamma < \theta, \delta \in S \cap \alpha \} \rangle \) increasing continuous in \( \alpha \), with kernel \( Z \), \( h_0 = \text{zero} \) and \( k_\alpha : G^*_{\alpha} \to H_\alpha \) is a not necessarily linear mapping such that \( h_\alpha \circ k_\alpha = \text{id}_{G^*_{\alpha}} \). We identify the set of members of \( H_\alpha \), \( G^*_{\alpha} \), \( Z \) with subsets of \( \lambda \times (1 + \alpha) \) such that \( H^\delta_\alpha = Z \).

Usually we have no freedom or no interesting freedom. But we have for \( \alpha = \delta + 1, \delta \in S \). What we demand is (\( G^*(\delta) - \) see before Fact A):

\[
\begin{align*}
(\alpha) & \quad f_\delta(x_{\beta^*_\gamma,i} - x_{\beta^*_\lambda,i}) \in Z \text{ is the same for all } i \in (\bigcup_{\varepsilon < \zeta} \lambda_\varepsilon, \lambda_\zeta) \\
(\beta) & \quad h_{\delta+1} \circ f_\delta = \text{id}_{G^*(\delta)}. 
\end{align*}
\]

[Why is this possible? By non-Whiteheadness of \( G^*/ \bigcup_{i<\sigma} G^i \) that is see (d)(\( \gamma \))\(^{-}\) in 4.1.]

The rest should be clear.

**Proof of 4.1(2).** Of course, similar to that of 4.1(1) but with some changes.

**Step A:** Without loss of generality there is a homomorphism \( f^* \) from \( \bigcup_{i<\sigma} G^i \) to \( Z \), which cannot be extended to a homomorphism from \( G_{\sigma} \) to \( Z \).

[Why? Standard, see [Fu].]

**Step B:** During the construction of \( G^* \), we choose \( G^*_\alpha \) by induction on \( \alpha \leq \kappa \), but if \( h^*_\alpha(0) \) from 4.2 is a member of \( G^*_\delta \) in (\( \ast \))\( _1 \) we replace \( (x_{\beta^*_\gamma} - x_{\beta^*_\lambda}) \) by \( (x_{\beta^*_\gamma} - x_{\beta^*_\lambda} + f^*(t_{\gamma}^{\zeta+1})g_\delta(0)) \), note that \( f^*(t_{\gamma}^{\zeta+1}) \in Z \) and \( h^*_\delta(0) \in G^*_\delta \).

So if in the end \( f : G^* \to Z \) is a non-zero homomorphism, let \( x^* \in G^* \) be such that
We also note the following consequence of a conclusion of an instance of GCH.

4.3 Claim. Assume

(a) \( \lambda = \mu^+ \) and \( \mu > \sigma = \text{cf}(\mu) \)
(b) \( \lambda = \lambda^\theta \) where \( \theta = 2^\sigma \)
   (equivalently \( \mu^0 = \mu^+ > 2^0 \))
(c) \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \sigma \} \) is stationary
(d) \( \bar{\eta} = \{ \eta_\delta : \delta \in S \} \) with \( \eta_\delta \) an increasing sequence of length \( \sigma \) with limit \( \delta \).

Then we can find \( \bar{A}^\delta : \delta \in S \) such that:

(\( \alpha \)) \( \bar{A}^\delta = \langle A^\delta_i : i < \sigma \rangle \)
(\( \beta \)) \( A^\delta_i \in [\delta]^\leq \mu \) and \( \sup(A^\delta_i) < \delta \)
(\( \beta^+ \)) for some \( \langle \lambda^*_i : i < \sigma \rangle \) increasing with limit \( \lambda \), \( |A^\delta_i| < \lambda^*_i \),
(\( \gamma \)) for every \( h : \lambda \to \lambda \), for stationarily many \( \delta \in S \) we have \( (\forall i < \sigma)[h(\eta_\delta(i))] \in A^\delta_i \).

4.4 Remark. 1) We can restrict ourselves to \( h : \lambda \to \mu \) in clause (\( \gamma \)), and then, of course, can use \( \langle A^\delta_i : i < \sigma ; : \delta \in S \rangle \) with \( A^\delta_i \subseteq \mu \).
2) We can add to the conclusion “\( A^\delta_i \subseteq \eta_\delta(i+1) \)” if \( \eta \) guess clubs.

Proof. Let \( \langle \lambda_i : i < \sigma \rangle \) be increasing continuous with limit \( \mu \). Let \( \langle \bar{\alpha}_\gamma : \gamma < \lambda \rangle \) list \( \theta \lambda \), so \( \bar{\alpha}_\gamma = \langle \alpha_{\gamma, \varepsilon} : \varepsilon < \theta \rangle \) and without loss of generality \( \alpha_{\gamma, \varepsilon} \leq \gamma \). For each \( \delta \in S \) let \( \langle b^\delta_i : i < \sigma \rangle \) be an increasing continuous sequence of subsets of \( \delta \) with union \( \delta \) such that \( |b^\delta_i| < \mu \) and \( \sup(b^\delta_i) < \delta \); for (\( \beta^+ \)), moreover \( |b^\delta_i| \leq \lambda^*_i \);

\(^5\)What does this mean? \( f^*(x^*) \) is an integer so its absolute value is well defined.
this is possible as \( \text{cf}(\delta) = \sigma = \text{cf}(\mu) < \mu \). Let \( \langle g_\varepsilon : \varepsilon < \theta \rangle \) list \( \sigma \sigma \) and define
\[
A_i^{\varepsilon,\delta} =: \{ \alpha_{\gamma,\varepsilon} : \gamma \in b_\varepsilon^\delta (i) \}. 
\]
Now \( A_i^{\varepsilon,\delta} \) is a set of cardinality \( \leq |b_\varepsilon^\delta (i)| < \mu \) and
\[
\sup(A_i^{\varepsilon,\delta}) \leq \sup(b_\varepsilon^\delta (i)) \) (as we have demanded that \( \alpha_{\gamma,\varepsilon} \leq \gamma \) but \( \sup(b_\varepsilon^\delta (i)) < \delta \)
by the choice of the \( b_\varepsilon^\delta \)'s hence \( \sup(A_i^{\varepsilon,\delta}) < \delta \). So for each \( \varepsilon < \theta \) the sequence
\[
\bar{A}_i^{\varepsilon} =: \langle \bar{A}_i^{\varepsilon,\delta} : \delta \in S \rangle, 
\]
where \( \bar{A}_i^{\varepsilon,\delta} = \langle A_i^{\varepsilon,\delta} : i < \sigma \rangle \) satisfies clauses \((\alpha) + (\beta)\) and \((\beta)^+ \) when relevant. Hence it suffices to prove that for some \( \varepsilon < \theta \) the sequence \( \bar{A}_i^{\varepsilon} \) satisfy clause \((\gamma)\), too. Assume toward contradiction that for every \( \varepsilon < \theta \) the sequence \( \bar{A}_i^{\varepsilon} \) fails clause \((\gamma)\) hence there is \( h_\varepsilon : \lambda \to \lambda \) which exemplifies this, that is for some club \( \bar{E}_\varepsilon \) of \( \lambda, \delta \in E_\varepsilon \cap S \Rightarrow (\exists i < \sigma)[h_\varepsilon(\eta_\delta(i)) \notin A_i^{\varepsilon,\delta}] \). So for every \( \beta < \lambda \) the sequence \( \langle h_\varepsilon(\beta) : \varepsilon < \theta \rangle \) belongs to \( \theta \lambda \), hence is equal to \( \bar{\alpha}_h(\beta) \) for some \( h(\beta) < \lambda \). Clearly \( E = \{ \delta \in \lambda : \delta \) is well defined as, for \( i < \sigma \) the ordinal \( h(\eta_\delta(i)) \) is \( < \delta(*) \) (as \( \delta(*) \in E \) and \( \eta_\delta(i) < \delta(*) \)) and \( \delta = \bigcup_{j<n} b_j^\delta \). As \( g^* \in \sigma \sigma \) clearly for some \( \varepsilon(*) \) \( < \theta \) we have \( g_\varepsilon(*) = g^* \).

So, for any \( i < \sigma \) let \( \gamma_i = h(\eta_\delta(i)) \), now \( h(\eta_\delta(i)) \in b_\varepsilon^\delta (i) \) (by the choice of \( g^* \)) and \( g^*(i) = g_\varepsilon(i) \) by the choice of \( \varepsilon(*) \), together \( \gamma_i \in b_\varepsilon^\delta (i) \). But \( A_i^{\varepsilon(*)} \) hence \( \alpha_{\gamma_i,\varepsilon(*)} \in A_i^{\varepsilon(*)} \), but as
\[
\gamma_i = h(\eta_\delta(i)), 
\]
by the choice of \( h \) we have \( h_{\varepsilon(*)}(\eta_\delta(i)) = \alpha_{\gamma_i,\varepsilon(*)} \in A_i^{\varepsilon(*)} \).

So \((\forall i < \sigma)(h_{\varepsilon(*)}(\eta_\delta(i)) \in A_i^{\varepsilon(*)}) \), which by the choice of \( h \) implies \( \delta(*) \notin E_{\varepsilon(*)} \) but \( \delta(*) \in E \subseteq \bigcap_{\varepsilon < \sigma} E_{\varepsilon} \), contradiction. \( \square_{4.3} \)
REFERENCES.


