ANTI-HOMOGENEOUS PARTITIONS
OF A TOPOLOGICAL SPACE

SH668

SAHARON SHELAH

The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem 91904, Israel

Department of Mathematics
Hill Center-Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019 USA

Abstract. We prove the consistency (modulo supercompact) of a negative answer to the Cantor discontinuum partition problem (i.e., some Hausdorff compact space cannot be partitioned to two sets not containing a closed copy of Cantor discontinuum). In this model we have CH. Without CH we get consistency results using a pcf assumption, close relatives of which are necessary for such results; so we try to deal with equiconsistency.

Saharon: §5 + glossary?
§1 General spaces: consistency from strong assumptions

[We define $X^* \rightarrow (Y^*)^1_\theta$ for topological spaces $X^*, Y^*$. Then starting with a Hausdorff space $Y^*$ with $\theta$ points such that any set of $< \sigma$ members is discrete and $\kappa = \kappa^{<\kappa} \in (\theta, \lambda)$ and appropriate $\mathcal{A} \subseteq [\lambda]^{\theta}$ such that any two members has intersection $< \sigma < \theta$, we force appropriate $X^*$. We then show that the assumption holds under appropriate pcf assumption and finish with some improvements, varying the topological and set theoretical assumptions.]

§2 Consistency from supercompact, with clopen basis

[We deal here with the set theoretic assumption. We show that the assumptions can be gotten from supercompact for the case we agree to have CH, relying on earlier consistency results. We also investigate the order of consistency between relatives of “$S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ is stationary with no stationary subset in $I[\lambda]”$, and existence of non trivial (in an appropriate sense) of $\mathcal{A} \subseteq [\lambda]^\kappa$, $[A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma]$.]

§3 Equi-consistency

[We show that some versions of the topological question and suitable combinatorial questions are equi-consistents. See [Sh 108], [HJSh 249], [Sh 460], [HJSh 697]. We then indicate the changes needed for the not necessarily closed subspace case colouring by more colours and other spaces. For discussion see [Sh 666],§1.]

§4 Decomposing families of almost disjoint functions
§1 General spaces: consistency from strong assumptions

In our main theorem, 1.2, we give set theoretic sufficient conditions for being able to force counterexamples to the Cantor discontinuum partition problem, possibly replacing the Cantor discontinuum by any other space. It has a version for spaces with clopen basis. Then (in claim 1.4) we connect this to pcf theory: after easy forcing the assumptions of Theorem 1.2 can be proved, if we start with a suitable (strong) pcf assumption (whose status is not known). Then in claim 1.7 we deal with variants of the theorem, weakening the topological and/or set theoretic assumptions. Further variants are discussed in the end of the section (\(T_3\) spaces without clopen basis and variants of 1.4). This continues Juhasz Hajnal Shelah \[\text{JHSh:249}\]. By \[\text{JHSh:249}\] if \(\aleph_0 > \aleph_\omega\) then it is doubtful if \((\exists X)(X \rightarrow (\text{Cantor discontinuum})^n_{\aleph_0})\) is consistent; e.g. if \(|a| \leq \aleph_\alpha \Rightarrow |\text{pcf}(a)| \leq \aleph_{\alpha+732}\) or if \(V = V_1^Q\) where \(Q\) is a c.c.c. forcing making the continuum \(\geq \beth_\omega\), then there is no such space. On the case with \(\geq \text{cf}(\theta)\) colours see 4.17. Bill Weiss proved the existence of such problems \((X \nrightarrow (2^\omega)^1_2)\) under \(V = L\).

Recall

1.1 Definition. Let \(n \in [1, \omega)\) (though we concentrate on \(n = 1\)).
1) We say \(X^* \rightarrow (Y^*)^n_\theta\), if \(X^*, Y^*\) are topological spaces and for every \(h : [X^*]^n \rightarrow \theta\) there is a closed subspace \(Y\) of \(X^*\), homeomorphic to \(Y^*\) such that \(h \upharpoonright [Y]^n\) is constant (if \(n = 1\) we may write \(h : X^* \rightarrow \theta\) and \(h \upharpoonright Y\)).
2) If we omit the “closed”, we shall write \(\rightarrow^w\) instead of \(\rightarrow\). We write \((Y^*)^n_\sigma < \theta\) meaning: for every \(h : [X^*]^n \rightarrow \gamma < \theta\) there is a \(A \subseteq [\lambda]^\theta\) and \(A_1 \neq A_2 \in \mathcal{A}\) such that for every \(\varepsilon < \zeta\) we have \(|A_\varepsilon \cap \bigcup_{\xi < \varepsilon} A_\xi| < \sigma\).

1.2 Theorem. Assume

- \((A)\)
  - \((i)\) \(\lambda > \kappa > \theta > \sigma \geq \aleph_0\) and \(\kappa = \kappa^{<\kappa}\)
  - \((ii)\) \((\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)\) and \(\kappa > \theta^* \geq \theta\)

- \((B)\)
  - \(\mathcal{A} \subseteq [\lambda]^\theta\) and \(A_1 \neq A_2 \in \mathcal{A}\) such that \(|A_1 \cap A_2| < \sigma\)

- \((B)\)
  - \(\mathcal{A}\) is \((<\kappa)\)-free (or \(\kappa\)-free or \((\kappa, \sigma)\)-free) which means: if \(\mathcal{A}' \subseteq \mathcal{A}\), \(|\mathcal{A}'| < \kappa\)
  - then for some list \(\{A_\varepsilon : \varepsilon < \zeta\}\) of \(\mathcal{A}'\), for every \(\varepsilon < \zeta\) we have \(|A_\varepsilon \cap \bigcup_{\xi < \varepsilon} A_\xi| < \sigma\)

- \((C)\)
  - \(F : \lambda \rightarrow [\lambda]^\leq \kappa\), then some \(A \in \mathcal{A}\) (or just some \(A\) such that \((\exists A')(A \subseteq A' \text{ and } |A| = \theta \text{ and } A' \in \mathcal{A}'\)) is \(F\)-free which means \((\ast)\)
  - \(\text{for } \alpha \neq \beta \text{ from } A \text{ we have } \alpha \notin F(\beta)\)
(D) $Y^*$ is a Hausdorff space with set of points $\theta$ and a basis $\mathcal{B} = \{b_i : i < \theta^*\}$
(E) if $Y$ is a subset of $Y^*$ with $< \sigma$ points, then $Y$ is a discrete subset, i.e. there is a sequence of open (for $Y^*$) pairwise disjoint sets $\langle \mathcal{U}_y : y \in Y \rangle$ such that $y \in \mathcal{U}_y$
(if $\sigma = \aleph_0$ this follows from being Hausdorff).

Then
1) for some $\kappa$-complete $\kappa^+$-c.c. forcing notion $\mathbb{P}$, in $\mathbb{V}^\mathbb{P}$ there is $X^*$ such that:
(a) $X^*$ is a Hausdorff topological space with $\lambda$ points and basis of size $|\mathcal{A}| + \theta^*$
(b) $X^* \to (Y^*)_{< \text{cf}(\theta)}^1$ (that is, if $X^* = \bigcup_{i < (\ast)} X_i$ where $i(\ast) < \text{cf}(\theta)$) then some closed subspace $Y$ of $X^*$ homeomorphic to $Y^*$ is included in some single $X_i$ (i.e. $\exists i(Y \subseteq X_i)$).

2) If in addition $Y^*$ has a clopen basis $\mathcal{B}$ of cardinality $\leq \theta^*$ such that the union of any $< \sigma$ members of $\mathcal{B}$ is clopen, then we can require that $X^*$ has a clopen basis.

1.3 Remark. We may define the conditions historically (see [ShSt 258], [RoSh 599], so put only the required conditions). Then we can allow $\theta^* = \kappa$, but see 1.7.

Proof. We write the proof for part (1) and indicate the changes for part (2). Without loss of generality

\[ \forall \alpha < \beta < \lambda)(\forall B \in [\lambda]^{< \lambda})(\exists \geq \kappa^+ A \in \mathcal{A})[\{\alpha, \beta\} \subseteq A \& A \cap B \subseteq \{\alpha, \beta\}] \]

[Why? As we can use $\{\{2 \alpha : \alpha \in A\} : A \in \mathcal{A}\}$, without loss of generality $\bigcup\{A : A \in \mathcal{A}\} = \{2 \alpha : \alpha < \lambda\}$ and choose $A_{\alpha, \beta, \gamma} \in [\lambda]^\theta$ for $\alpha < \beta < \gamma < \lambda$ such that $\{\alpha, \beta\} \subseteq A_{\alpha, \beta, \gamma}$ and $\langle A_{\alpha, \beta, \gamma}\setminus\{\alpha, \beta\} : \alpha < \beta < \gamma < \lambda\rangle$ are pairwise disjoint subsets of $\{\alpha + 1 : \alpha < \lambda\}$, each of cardinality $\theta$ and we replace $\mathcal{A}$ by $\mathcal{A}^* =: \mathcal{A} \cup \{A_{\alpha, \beta, \gamma} : \alpha < \beta < \gamma < \lambda\}$. Now clauses (A), (D), (E) are not affected. Clearly clause (B)_1 holds (i.e. $\mathcal{A}^* \subseteq [\lambda]^\theta$ and $A \neq B \in \mathcal{A}^* \Rightarrow |A \cap B| < \sigma$). Also clause (C) is inherited by any extension of the original $\mathcal{A}$. Lastly for clause (B)_2, if $\mathcal{A}' \subseteq \mathcal{A}^*, |\mathcal{A}'| < \kappa$, let $\langle A_\zeta : \zeta < \zeta^*\rangle$ be a list of $\mathcal{A} \cap \mathcal{A}$ as guaranteed by (B)_2 and let $\langle A_\zeta : \zeta < \zeta^* + |\mathcal{A}'\setminus\mathcal{A}|\rangle$ list with no repetitions $\mathcal{A}'\setminus\mathcal{A}$, now check.]

\[ \exists_2 \mathcal{B} \text{ is a basis of } Y^* \text{ of cardinality } \theta^*, \text{ and for part (2), } \mathcal{B} \text{ is as there.} \]

[Why? Straight.]
Let $\mathcal{A} = \{ A_\zeta : \zeta < \lambda^* \}$ and $\mathcal{B} = \{ b_i : i < \theta^* \}$.
We define a forcing notion $\mathbb{P}$:

$p \in \mathbb{P}$ has the form $p = (u, u_*, v, v_*, \bar{w}) = (u^p, u_*^p, v^p, v_*^p, \bar{w}^p)$ such that:

\begin{align*}
(\alpha) & \quad u_* \subseteq u \subseteq [\lambda]^{< \kappa} \\
(\beta) & \quad v_* \subseteq v \subseteq [\lambda^*]^{< \kappa} \\
(\gamma) & \quad \bar{w} = \bar{w}^p = \langle w_{\zeta,i} : \zeta \in v_* \text{ and } i < \theta^* \rangle = \langle w_{\zeta,i}^p : \zeta \in v_*, i < \theta^* \rangle \\
(\delta) & \quad w_{\zeta,i} \subseteq u_* \text{ and } b_i \cap b_j = \emptyset \Rightarrow w_{\zeta,i} \cap w_{\zeta,j} = \emptyset; \text{ this is toward being Hausdorff} \\
(\epsilon) & \quad \zeta \in v_* \Rightarrow A_\zeta \subseteq u \\
(\zeta) & \quad \text{letting } A^p_\zeta =: \bigcup \{ w_{\zeta,i}^p : p \in G \}_\zeta \text{ for } \zeta \in v_*^p, \text{ it has cardinality } \theta \text{ and for simplicity even order type } \theta, \text{ of course } A^p_\zeta \subseteq u^p_\zeta \text{ and for some sequence } \\
& \quad \langle \gamma^p_{\zeta,j} : j < \theta \rangle \text{ listing its members with no repetitions we have } \\
& \quad w_{\zeta,i}^p \cap A^p_\zeta = \{ \gamma^p_{\zeta,j} : j < \theta \text{ and } j \in b_i \} \\
(\eta) & \quad \text{if } \zeta \in v_*^p, i < \theta^* \text{ and } \xi \in v_*^p \text{ then the set } \mathcal{U}_{\zeta,i,\xi}^p \text{ is an open subset (for part (2), clopen subset) of the space } Y^* \text{ where } \mathcal{U}_{\zeta,i,\xi}^p =: \{ j < \theta : \gamma^p_{\xi,j} \in w_{\zeta,i}^p \}. \\
\end{align*}

\[ \bigoplus \text{ convention} \] if $\zeta \in \lambda^* \setminus v_*^p$ we stipulate $w_{\zeta,i}^p = \emptyset$.

The order is: $p \leq q$ iff $u^p \subseteq u^q, u_*^p = u_*^q \cap u^p, v_*^p \subseteq v_*^q, v_*^p \subseteq v_*^q \cap v^p$ and $\zeta \in v_*^p \Rightarrow w_{\zeta,i}^p = w_{\zeta,i}^q \cap u^p$.

Clearly

\[ (*)_0 \] $\mathbb{P}$ is a partial order.

What is the desired space in $V^\mathbb{P}$? We define a $\mathbb{P}$-name $X^*$ of a topological space as follows:

\[ \bigotimes \text{ set of points } \bigcup \{ u_*^p : p \in G \}_\zeta \]

The topology is defined by the following basis:

\[ \{ \bigcap_{\ell < n} \mathcal{U}_{\zeta, i_\ell} \cap \omega < \zeta, i_\ell < \theta^* \text{ for } \ell < n \} \]

\[ \bigotimes \mathcal{U}_{\zeta,i}(G_\mathbb{P}) = \bigcup \{ w_{\zeta,i}^p : p \in G_\mathbb{P} \text{ satisfies } \zeta \in v_*^p \}; \text{ so } \]

\[ \vdash \text{ "if } \zeta \in \lambda^* \setminus \bigcup \{ v_*^p : p \in G_\mathbb{P} \} \text{ and } i < \theta^* \text{ then } \mathcal{U}_{\zeta,i}(G_\mathbb{P}) = \emptyset" \]

(for part (2), also their compliments and hence their Boolean combinations).

Now we shall prove
\((*)_1\) for \(\alpha < \lambda, \zeta < \lambda^*\) and \(p \in \mathbb{P}\) we have

\((i)\) \(p \Vdash \text{"}\alpha \in X^*\text{"}\) iff \(\alpha \in u^*_p\) and

\((ii)\) \(p \Vdash \text{"}\alpha \not\in X^*\text{"}\) iff \(\alpha \in u^*_n \setminus u^*_p\) and

\((iii)\) \(\Vdash \mathbb{P} \text{ "}\lambda^* = \bigcup \{v^p : p \in G_{\mathbb{P}}\}\)

\((iv)\) if \(\zeta \in v^*_p\) and \(i < \theta\) then

\(p \Vdash \text{"}\forall \zeta,i \in w^*_p = w_{\zeta,i}^*\text{"}\)

\((v)\) \(\{p \in \mathbb{P} : \alpha \in u^p\}\) is a dense open subset of \(\mathbb{P}\)

\((vi)\) \(\{p \in \mathbb{P} : \zeta \in v^p\}\) is a dense open subset of \(\mathbb{P}\).

[Why? Easy, e.g. let \(p \in \mathbb{P}, \xi \in \lambda^* \setminus v^p\) and \(\alpha \neq \beta\) are from \(\lambda \setminus u^p\), we define \(q \in \mathbb{P}\) by: \(u^q = u^p \cup \{\alpha, \beta\}, u^*_q = u^*_p \cup \{\alpha\}, v^q = v^p \cup \{\xi\}, v^*_q = v^*_p\) and \(w^*_q = w^*_p\) for \(i < \theta^*\). Easily \(\mathbb{P} \models \text{"}p \leq q, \beta \in u^q \setminus u^*_p, \alpha \in u^*_q\) and \(\zeta \in v^q \setminus v^*_q\).

\((*)_2\) \(\mathbb{P}\) is \(\kappa\)-complete, in fact if \(\langle p_\varepsilon : \varepsilon < \delta\rangle\) is increasing in \(\mathbb{P}\) and \(\delta < \kappa\) then \(p = \bigcup_{\varepsilon < \delta} p_\varepsilon\) is an upper bound where \(u^p = \bigcup_{\varepsilon < \delta} u^{p_\varepsilon}, u^*_p = \bigcup_{\varepsilon < \delta} u^{*_p}, v^p = \bigcup_{\varepsilon < \delta} v^{p_\varepsilon}, v^*_p = \bigcup_{\varepsilon < \delta} v^{*_p}\) and \(w^p = \bigcup_{\varepsilon < \delta} w^{p_\varepsilon}\).

[Why? Straight.]

\((*)_3\) \(\mathbb{P}' = \{p \in \mathbb{P} : \text{if } \zeta < \lambda^* \text{ and } |A_\zeta \cap u^p| \geq \sigma \text{ then } \zeta \in v^p\}\) is a dense subset of \(\mathbb{P}\)

[why? for any \(p \in \mathbb{P}\) we define by induction\(^1\) on \(\varepsilon \leq \sigma^+ : p_\varepsilon \in \mathbb{P}\) is increasing continuous with \(\varepsilon\). Let \(p_0 = p\), if \(p_\varepsilon\) is defined, we define \(p_{\varepsilon+1}\) by

\[v^{p_{\varepsilon+1}} = \{\zeta < \lambda^* : \zeta \in v^{p_\varepsilon} \text{ or } |A_\zeta \cap u^{p_\varepsilon}| \geq \sigma\}\]

\[v^{p_{\varepsilon+1}} = v^{p_{\varepsilon}} (= v^{p_\varepsilon})\]

\[u^{p_{\varepsilon+1}} = u^{p_\varepsilon} \cup \{A_\zeta : \zeta \in v^{p_{\varepsilon+1}}\}\]

\[u^{p_{\varepsilon+1}} = u^{p_\varepsilon} (= u^{p_\varepsilon})\]

\[w^{p_{\varepsilon+1}} = w_{\zeta,i}^{p_{\varepsilon+1}} = w_{\zeta,i}^{p_{\varepsilon}} = w_{\zeta,i}^{p_\varepsilon}\] if \(\zeta \in v^{p_\varepsilon}, i < \theta^*\)

\(^1\)of course, if \(\delta^* < \kappa\) is a limit ordinal such that \(\text{cf}(\delta^*) \neq \text{cf}(\sigma)\) then we may use \(\langle p_\varepsilon : \varepsilon \leq \delta^*\rangle\) and \(p_{\delta^*}\) is as required
(and there are no other cases).

By assumption (A)(ii), the set \( v^{p_{e+1}} \) has cardinality < \( \kappa \), so \( p_{e+1} \) belongs to \( \mathbb{P} \).

Clearly \( p_e \leq p_{e+1} \in \mathbb{P} \). Now for \( e \) limit let \( p_e = \bigcup_{\zeta < \xi} p_\zeta \). Clearly we can carry the definition. Now \( p_{\sigma^+} = \bigcup_{\xi < \sigma} p_\xi \) is as required because if \( A_\zeta \in \mathcal{A} \setminus A_\zeta \cap v^{p_{\sigma^+}} \geq \sigma \) then for some \( \varepsilon < \sigma^+ \) we have \( |A_\zeta \cap v^{p_{\varepsilon+1}}| \geq \sigma \) hence \( \zeta \in v^{p_{\varepsilon+1}} \) hence \( A_\zeta \subseteq v^{p_{\varepsilon+1}} \subseteq v^{p_{\varepsilon^+}} \).

Note that we use here \( \sigma \), \( \varepsilon < \sigma^+ \) which follows from \( \sigma < \theta < \kappa \).

\[
\mathbb{P} \text{ satisfies the } \kappa^+-c.c.
\]

[Why? Let \( p_j \in \mathbb{P} \) for \( j < \kappa^+ \), without loss of generality \( p_j \in \mathbb{P}' \) for \( j < \kappa^+ \). Now by the \( \Delta \)-system lemma for some unbounded \( S \subseteq \kappa^+ \) and \( v^\otimes \in [\lambda]^<\kappa \), \( u^\otimes \in [\lambda]^<\kappa \) we have:

\[
j \in S \Rightarrow v^\otimes \subseteq v^{p_j} \quad \& \quad u^\otimes \subseteq u^{p_j} \text{ and } \langle v^{p_j} \setminus v^\otimes : j \in S \rangle \text{ are pairwise disjoint and } \langle u^{p_j} \setminus u^\otimes : j \in S \rangle \text{ are pairwise disjoint. Without loss of generality otp}(p^{p_j}) \text{, otp}(u^{p_j}) \text{ are constant for } j \in S \text{ and any two } p_i,p_j \text{ are isomorphic over } v^\otimes,u^\otimes \text{ (if not clear see 1.7).}
\]

Now for \( j_1,j_2 \in S \) the condition \( p_{j_1},p_{j_2} \) are compatible because of the following \((*)_5\)]

\[
(*)_5 \text{ assume } p^1,p^2 \in \mathbb{P} \text{ satisfies}
\]

\[
(i) \quad v^\uparrow_i \cap (v^\uparrow_j \setminus v^\uparrow_i) = \emptyset \quad \text{and} \quad u^\uparrow_i \cap (u^\uparrow_j \setminus u^\uparrow_i) = \emptyset
\]

\[
(ii) \quad v^\uparrow_j \cap (v^\uparrow_i \setminus v^\uparrow_j) = \emptyset \quad \text{and} \quad u^\uparrow_j \cap (u^\uparrow_i \setminus u^\uparrow_j) = \emptyset
\]

\[
(iii) \quad \text{if } \zeta \in v^\uparrow_i \cap u^\uparrow_j \text{ then } A^\uparrow_\zeta = A^\uparrow_j \text{ and } i < \theta^* \Rightarrow u^\uparrow_i \cap (u^\uparrow_j \cap v^\uparrow) = u^\uparrow_i \cap (u^\uparrow_j \cap u^\uparrow)
\]

\[
(iv)_1 \quad \text{if } \zeta \in v^\uparrow_i \setminus v^\uparrow_j \text{ then } |A_\zeta \cap u^\uparrow| < \sigma \text{ or just } |A^\uparrow_\zeta \cap u^\uparrow| < \sigma
\]

\[
(iv)_2 \quad \text{similarly}\ 2 \text{ for } \zeta \in v^\uparrow_j \setminus v^\uparrow_i \text{ we have } |A^\uparrow_\zeta \cap u^\uparrow| < \sigma.
\]

Then there is \( q \in \mathbb{P} \) such that:

\[
(a) \quad v^q = v^{p^1} \cup v^{p^2}
\]

\[
(b) \quad v^q_i = v^q_j \cup v^q_k
\]

\[
(c) \quad v^q = u^{p^1} \cup u^{p^2}
\]

\(^2\text{Note that if } p^1,p^2 \in \mathbb{P}' \text{, then clauses } (iv)_1,(iv)_2 \text{ holds automatically, but the proof of 1.7 which is very similar to the proof of 1.2, uses this version.}
(d) $u^q_i = u^p_i \cup u^p_j$

(e) $p^1 \leq q$ and $p^2 \leq q$.

[Why? To define the condition $q$ by clauses (a)-(d) above we just have to define $w^q_{\zeta,i}$ (for $\zeta \in v^q_i = v^p_i \cup v^p_j$ and $i < \theta^*$. If $\zeta \in v^p_i \cap v^p_j$ we let $w^q_{\zeta,i} = w^p_{\zeta,i} \cup w^p_{\zeta,j}$ for $i < \theta^*$ (clearly $\ell \in \{1, 2\}$) $\Rightarrow w^q_{\zeta,i} \cap w^p_j = w^p_{\zeta,j}$); this will be enough to guarantee $\mathbb{P} \models \ "p^1 \leq q \ & \ p^2 \leq q^*"$ provided that we have $q \in \mathbb{P}$ and that for $\ell = 1, 2$ we shall define $w^q_{\zeta,i}$ for $\zeta \in v^p_{3-\ell} \setminus v^p_\ell$ such that $w^q_{\zeta,i} \cap u^p_{3-\ell} = w^p_{\zeta,i}$ and $w^q_{\zeta,i} \subseteq u^q_i$; so only clauses (\delta) + (\eta) in the definition of membership in $\mathbb{P}$ are problematic.

Now for $\ell = 1, 2$, let $v^p_\ell \setminus v^p_{3-\ell}$ be listed as $\langle \mathbb{Y}(\varepsilon, \ell) : \varepsilon < \varepsilon^*_\ell \rangle$ with no repetitions such that $B^\ell_\varepsilon = A^{\ell,\varepsilon}_\mathbb{Y}(\varepsilon, \ell) \cap (\bigcup_{\xi < \varepsilon} A^{\ell,\xi,\varepsilon}_\mathbb{Y}(\varepsilon, \ell) \cup u^q_{3-\ell})$ is of cardinality $< \sigma$.

[Why possible? By the assumption (B)2 and clause (iv)$\ell$ above.] Now for each $\zeta \in v^p_{3-\ell} \setminus v^p_\ell$ we choose by induction on $\varepsilon \leq \varepsilon^*_\ell$ the sequence $\langle w^x_{\zeta,i} : i < \theta^* \rangle$ such that

1) $w^x_{\zeta,i} \subseteq u^p_{3-\ell} \cup \bigcup_{\xi < \varepsilon} A^{\ell,\xi,\varepsilon}_\mathbb{Y}(\varepsilon, \ell) (\subseteq u^p_\ell \cup u^p_{3-\ell})$.

2) $w^x_{\zeta,i}$ is increasing continuous with $\varepsilon$.

3) $w^x_{\zeta,0} = w^p_{3-\ell}$.

4) $\varepsilon' < \varepsilon \Rightarrow w^x_{\zeta,i} \cap (u^p_{3-\ell} \cup \bigcup_{\xi < \varepsilon'} A^{\ell,\xi,\varepsilon'}_\mathbb{Y}(\varepsilon, \ell)) = w^x_{\zeta,i}$.

5) If $i < j < \theta^*$ and $b_i \cap b_j = \emptyset$ (hence $w^3_{\zeta,i} \cap w^3_{\zeta,j} = \emptyset$) then $w^x_{\zeta,i} \cap w^x_{\zeta,j} = \emptyset$.

6) For each $i < \theta^*$ the set $\{j < \theta : \gamma^{p^*,\ell}_\mathbb{Y}(\varepsilon, \ell, j) \in w^x_{\zeta,i+1} \}$ is an open set in $Y^\ast$ (for part (2) of 1.2: clopen].

If we succeed then we let $w^q_{\zeta,i}$ be $w^x_{\zeta,i}$ for $\ell \in \{1, 2\}, \zeta \in v^p_{\ell}$; clearly by clauses (3) + (4) in the construction for $\varepsilon' = 0$ we have $w^q_{\zeta,i} \cap u^p_{3-\ell} = w^p_{\zeta,i}$ and by clause (1) in the construction we have $w^q_{\zeta,i} \subseteq u^q_i$ and clause (\delta) in the definition of $q \in \mathbb{P}$ holds by (5), and clause (\eta) by (6) in the construction. So let us carry the induction.

For $\varepsilon = 0$ use clause (3) and for limit $\varepsilon$ take unions (see clause (2)). Suppose we have defined for $\varepsilon$ and let us define for $\varepsilon + 1$. By an assumption above $B^\varepsilon_\ell = A^{\ell,\varepsilon}_\mathbb{Y}(\varepsilon, \ell) \cap (\bigcup_{\xi < \varepsilon} A^{\ell,\xi,\varepsilon}_\mathbb{Y}(\varepsilon, \ell) \cup u^q_{3-\ell})$ has cardinality $< \sigma$ and so $Z^\varepsilon_\ell : = \{j < \theta : \gamma^{p^*,\ell}_\mathbb{Y}(\varepsilon, \ell, j) \in B^\varepsilon_\ell \}$ is a subset of $\theta$ of cardinality $< \sigma$. Hence, by assumption (E) of the theorem 1.2, we can find a sequence $\langle t_j(\varepsilon, \ell) : j \in Z^\varepsilon_\ell \rangle$ such that: $t_j(\varepsilon, \ell) < \theta^*$ and $j \in b_{t_j(\varepsilon, \ell)}$ for
Lastly, we let

\[ w_{\xi,i}^{\ell+1} = w_{\xi,i}^{\ell} \cup \{ \gamma_{\psi(\xi,i)},s : \text{for some } j \in Z_\xi^\ell \text{ we have} \]
\[ \gamma_{\psi(\xi,i),j} \in w_{\xi,i}^{\ell} \text{ and} \]
\[ s \in b_{t_j(\xi,i)} \} . \]

Clearly this is O.K. and we are done. Remember that the union of \(< \sigma \) set from \( \mathcal{B} \) is clopen for part (2) of 1.2.

So we have proved \((*)_5\) hence also \((*)_4\). We sometime need a stronger version of \((*)_5\)

\[(*)_6\text{ in }(*)_5 \text{ if in addition for } \ell = 1,2 \text{ we are given } Z_\xi \subseteq u^{\psi,\ell}_\ast \backslash u^{\Psi,\ell} \text{ such that}\]
\[ (\forall \xi \in u^{\ast}_\ast)([A_{\xi}^{\psi,\ell} \cap Z_\xi] < \sigma) \text{ then we may add to the conclusion:} \]

\[(f) \ell \in \{1,2\}, \xi \in u^{\psi,\ell}_\ast \backslash u^{\rho,\ell}_\ast, i < \theta^* \Rightarrow w_{\xi,i}^{\ell} \cap Z_\xi = \emptyset. \]

More generally

\[(f)^+ \text{ if } g_\ell : (u^{\psi,\ell}_\ast \backslash u^{\rho,\ell}_\ast) \times \theta^* \times Z_\xi \rightarrow \{0,1\} \text{ satisfies} g(\xi,j_1,1) = 1 = g(\xi,j_2,\gamma) \Rightarrow b_{j_1} \cap b_{j_2} \neq \emptyset \text{ for } \ell = 1,2, \text{ then we can add} \]
\[ \ell \in \{1,2\}, \xi \in u^{\psi,\ell}_\ast \backslash u^{\rho,\ell}_\ast, i < \theta^*, \gamma \in Z_\xi \Rightarrow [\gamma \in w_{\xi,i}^{\ell} \leftrightarrow g_\ell(\xi,i,\gamma) = 1]. \]

[Why? During the proof of \((*)_5\) when for each \( \xi \in u^{\psi,\ell}_\ast \backslash u^{\rho,\ell}_\ast \), we define \( \langle w_{\xi,i}^{\ell} : i < \theta^* \rangle \) by induction on \( \varepsilon \) we add

\[ (7) i < \theta^*, \gamma \in Z_\xi \cap (u^{\Psi,\ell} \cup \bigcup_{\xi < \varepsilon} A_{\xi}^{\psi,\ell}(\xi,i)) \implies \gamma \in w_{\xi,i}^{\ell} \leftrightarrow g_\ell(\xi,i,\gamma) = 1. \]

In the proof when we use assumption (E), instead of using \( B_\xi^\ell = A_{\xi}^{\psi,\ell}(\xi,i) \cap (u^{\Psi,\ell} \cup \bigcup_{\xi < \varepsilon} A_{\xi}^{\psi,\ell}(\xi,i)) \cup u^{\Psi,\ell} \) we use \( B_\xi^\ell = A_{\xi}^{\psi,\ell}(\xi,i) \cap (\bigcup_{\xi < \varepsilon} A_{\xi}^{\psi,\ell}(\xi,i) \cup u^{\Psi,\ell} \cup Z_\xi) \) which still has cardinality \(< \sigma \). In the end if \( Z_\xi \not\subseteq \bigcup_{\xi < \varepsilon} A_{\xi}^{\psi,\ell} \cup u^{\Psi,\ell} \) we let \( w_{\xi,i}^{\ell} =: w_{\xi,i}^{\ell+1} =: w_{\xi,i}^{\ell\ast} \cup \{ \gamma \in Z_\xi : g(\xi,i,\gamma) = 1 \}. \]

Now we come to the main point

\[ (*)_7 \text{ in } \mathbf{V}^p, \text{ if } i(*) < \text{cf}(\theta) \text{ and } X^* = \bigcup_{i<i(*)} X_i \text{ then some closed } Y \subseteq X^* \text{ is homeomorphic to } Y^*. \]
Why? Toward contradiction assume $p^* \in \mathbb{P}$ and $p^* \Vdash \langle X_i : i < i(\ast) \rangle$ is a counterexample to $\ast \lambda$. So in particular $p^* \Vdash \langle X_i : i < i(\ast) \rangle$ is a partition of $X^*$, i.e., of $\bigcup \{ u^p : p \in G_{\mathbb{P}} \}$ and let $X_{i(\ast)} = \lambda \setminus \{ X_i : i < i(\ast) \}$.

For each $\alpha < \lambda$ let $\langle (p_{\alpha,j}, i_{\alpha,j}) : j < \kappa \rangle$ be such that:

- (i) $\langle p_{\alpha,j} : j < \kappa \rangle$ is a maximal antichain$^3$ of $\mathbb{P}'$ above $p^*$
- (ii) $p_{\alpha,j} \Vdash \alpha \in X_{i_{\alpha,j}}$ and $\alpha \in u^{p_{\alpha,j}}$, so $i_{\alpha,j} \leq i(\ast)$ and $\alpha \in u^{p_{\alpha,j}} \iff i_{\alpha,j} < i(\ast)$
- (iii) $p^* \leq p_{\alpha,j}$
- (iv) $p_{\alpha,j} \in \mathbb{P}'$.

Why can we demand $\alpha \in u^{p_{\alpha,j}}$? By clause (v) of $\ast_1$. Why not $j < j_\alpha \leq \kappa$?

For notational simplicity and as above any member $p$ of $\mathbb{P}$ there are $\kappa$ pairwise contradictory members (by the information on the set of points of $X^*$, see $\ast_1$ above).

Now we define a function $F$, Dom($F$) = $\lambda$ as follows:

$$F(\alpha) = \bigcup \{ u^{p_{\alpha,j}} : j < \kappa \} \subseteq [\lambda]^{\leq \kappa}.$$  

So by clause (c) of the assumption of 1.2 we can find $\zeta(\ast) < \lambda^*$ and $A \subseteq A_{\zeta(\ast)}$ of order type $\theta$ such that: if $\alpha \neq \beta$ are from $A$ then $\alpha \notin F(\beta)$. Let $A = \{ \beta_\varepsilon : \varepsilon < \theta \}$ with no repetitions. We can find $\zeta(\ast\ast) \in \lambda^* \setminus \bigcup \{ u^{p_{\alpha,j}} : j < \kappa, \alpha \in A_{\zeta(\ast)} \} \setminus \{ \zeta(\ast) \}$ the set $A_{\zeta(\ast\ast)}$ is disjoint to $\bigcup \{ F(\alpha) : \alpha \in A_{\zeta(\ast)} \}$, let $\langle \gamma_j : j < \theta \rangle$ be an increasing sequence of members of $A_{\zeta(\ast\ast)}$ and let $p^+ \in \mathbb{P}$ be defined by: $u^{p^+} = u^{p^*} \cup A_{\zeta(\ast\ast)}, u^{p^+}_* = u^{p^*}_* \cup \{ \gamma_j : j < \theta \}, v^{p^*} = v^{p^*} \cup \{ \zeta(\ast\ast) \}, v^{p^+} = v^{p^+}_* \cup \{ \zeta(\ast\ast) \}$, $w^{p^+}_{\zeta,i} = w^{p^*}_{\zeta,i} \iff w^{p^*}_{\zeta,i}$ is well defined and $w^{p^+}_{\zeta(i),i} = \{ \gamma_j : j \in b_i \}$. Clearly $p \leq p^+$. Now we shall choose by induction on $\varepsilon \leq \theta, p_\varepsilon, g_\varepsilon$ and if $\varepsilon < \theta$ also $j_\varepsilon < \kappa$ such that:

- $p_\varepsilon \in \mathbb{P}, p^+_\varepsilon \leq p_\varepsilon$
- $u^{p_\varepsilon} = u^{p^+_\varepsilon} \cup \bigcup_{\varepsilon(1) < \varepsilon} u^{p_{\varepsilon(1),j_{\varepsilon(1)}}}$
- $u^{p^+_\varepsilon} = u^{p^+_\varepsilon} \cup \bigcup_{\varepsilon(1) < \varepsilon} u^{p_{\varepsilon(1),j_{\varepsilon(1)}}}$
- $v^{p_\varepsilon} = v^{p^+_\varepsilon} \cup \bigcup_{\varepsilon(1) < \varepsilon} v^{p_{\varepsilon(1),j_{\varepsilon(1)}}}$

$^3$if we demand only $\in \mathbb{P}$ then we should increase $F(\alpha)$ accordingly
\[ v_\varepsilon^p \subseteq v_\varepsilon^p \cup \bigcup_{\varepsilon(1)<\varepsilon} v_\varepsilon^{p_{\beta(1),j(1)}} \]

(of course \( p_0 = p^+ \))

(b) \( p_{\varepsilon+1} \geq p_{\beta, j_\varepsilon} \) and \( j_\varepsilon = \min\{ j < \kappa : p_{\beta, j} \) is compatible with \( p_\varepsilon \) and \( i_{\beta, j} \leq i(\varepsilon) \} \)

(c) \( g_\varepsilon \) is a function, increasing with \( \varepsilon \), from \( v_\varepsilon^p \times \theta^* \times i(\varepsilon) \) into the family of open subsets of \( Y^* \) (for part (2), clopen)

(d) if \( b_{\varepsilon1} \cap b_{\varepsilon2} = \emptyset \) then \( g_\varepsilon(\zeta, j_\varepsilon, i_\varepsilon) \cap g_\varepsilon(\zeta, j_\varepsilon, i_\varepsilon) = \emptyset \) when both are defined

(e) letting \( \Upsilon_\varepsilon = \text{otp}\{ \xi < \varepsilon : i_{\beta, j} = i_{\beta, j_\varepsilon} \} \), for every \( \zeta \in v_\varepsilon^p \) and \( i < \theta^* \) and \( \xi < \varepsilon \) we have:

\[ \beta_\xi \in w_{\varepsilon, j}^p \iff \Upsilon_\xi \in g_\varepsilon(\zeta, j, i_{\beta, j_\varepsilon}) \]

(f) \( p_\varepsilon \) is increasing continuous with \( \varepsilon \) (in \( \mathbb{P} \))

(g) if \( \xi < \varepsilon \) and \( j < \theta^* \) then \( \beta_\xi \in w_{\varepsilon, (\varepsilon, j)}^p \iff \Upsilon_\xi \in b_\varepsilon \).

It is easy to carry the definition. For \( \varepsilon = 0, p_\varepsilon = p^* \). If they are defined for \( \varepsilon \) let us define for \( \varepsilon + 1 \) so \( p_\varepsilon, j_\varepsilon, g_\varepsilon \) are well defined, hence \( p_\varepsilon, p_{\beta, j_\varepsilon} \) are two compatible members of \( \mathbb{P} \) hence the assumptions (i) - (iv), (iv)2 in (5) holds with \( p_\varepsilon, p_{\beta, j_\varepsilon} \) here standing for \( p^1, p^2 \) there.

First we define \( g_{\varepsilon+1} \) with domain \((v_\varepsilon^p \cup v_\varepsilon^{p_{\beta, j_\varepsilon}}) \times \theta^* \) extending \( g_\varepsilon \), so we have to define \( \langle g_{\varepsilon+1}(\zeta, j, i) : \zeta \in v_\varepsilon^{p_{\beta, j_\varepsilon}} \setminus v_\varepsilon^p \) and \( j < \theta^* \) and \( i < i(\varepsilon) \) \) and the restriction are for each \( (\zeta, i) \) separately. For each \( \zeta \in v_\varepsilon^{p_{\beta, j_\varepsilon}} \setminus v_\varepsilon^p \) set \( Z_{\zeta, i}^\varepsilon = \{ \xi < \varepsilon : i_{\beta, j_\varepsilon} = i \) and \( \Upsilon_\xi \in w_\varepsilon^p \} \subseteq \theta \) has cardinality < \( \sigma \) hence we can find a sequence \( \langle \mathcal{U}_\varepsilon^{i, j} : \xi \in Z_{\zeta, i}^\varepsilon \rangle \) of pairwise disjoint open sets of \( Y^* \) such that \( \xi \in Z_{\zeta, i}^\varepsilon \Rightarrow \Upsilon_\xi \in \mathcal{U}_\varepsilon^{i, j} \).

Now we define

\[ g_{\varepsilon+1}(\zeta, j, i) = \bigcup\{ \mathcal{U}_\zeta^{i, j} : \xi \in Z_{\zeta, i}^\varepsilon \} \]

It is easy to check that \( g_{\varepsilon+1} \) is as required in clauses (c) + (d).

We intend to use (5)5 toward this, let

\[ Z_\varepsilon^2 = \{ \beta_\varepsilon \} \]

\[ g_\varepsilon^1 : (v_\varepsilon^p \setminus v_\varepsilon^{p_{\beta, j_\varepsilon}}) \times \theta^* \rightarrow \{0, 1\} \]

be defined by \( g_\varepsilon^1(\zeta, j, \beta_\varepsilon) = 1 \iff \Upsilon_\varepsilon \in g_\varepsilon(\zeta, j, i_{\beta, j_\varepsilon}) \)

\[ Z_\varepsilon^1 = \{ \beta_\varepsilon : \xi < \varepsilon \} \]
$g_2^\varepsilon : (v_\varepsilon^{p_{\beta_\varepsilon,j_\varepsilon}} \setminus v_\varepsilon^{\ast}) \times \theta^\varepsilon \times Z_\varepsilon^1 \to \{0, 1\}$ be defined by $g_2^\varepsilon(\zeta, j, \beta_\varepsilon) = 1 \Leftrightarrow \Upsilon_\xi \in g_{\varepsilon+1}(\zeta, j, \beta_\varepsilon, j_\varepsilon)$.

In limit $\varepsilon$ take union. In all cases $j_\varepsilon$ is well defined by clause $(i)$ above noting then $\beta_\varepsilon \notin u^{p_\varepsilon}$ so by $(\ast)_1$ we know that $p_\varepsilon \Vdash \beta_\varepsilon \notin X^\ast$, i.e., $\beta_\varepsilon \in X_{i(\ast)}$.

Having carried the induction let $i^* < i(\ast)$ be minimal such that the set $Z = \{\varepsilon < \theta : i_{\beta_\varepsilon,j_\varepsilon} = i^*\}$ has cardinality $\theta$; it exists as $i(\ast) < \text{cf}(\theta)$. Note: $\zeta(\ast) \notin v^{p_{\beta_\varepsilon,j_\varepsilon}}$ for $\varepsilon < \theta$, $j < \kappa$ as $A \cap F(\beta_\varepsilon)$ is the singleton $\{\beta_\varepsilon\}$ so $|A \cap u^{p_{\beta_\varepsilon,j_\varepsilon}}| \leq 1$. Now we define $p$:

\[ u^\varepsilon = u^{p_\varepsilon} \]
\[ u_*^\varepsilon = u_*^{p_\varepsilon} \]
\[ v^\varepsilon = v^{p_\varepsilon} \cup \{\zeta(\ast)\} \]
\[ v_*^\varepsilon = v_*^{p_\varepsilon} \cup \{\zeta(\ast)\} \]

$A_{\zeta(\ast)}^\varepsilon = \{\beta_\varepsilon : \varepsilon \in Z\}$ and $\gamma_{\zeta(\ast),\varepsilon}^\varepsilon$ is the $\varepsilon$-th member of $A_{\zeta(\ast)}^\varepsilon$; equivalently the unique $\beta_\xi$ such that $i_{\beta_\xi,j_\xi} = i^* \& \Upsilon_\xi = \varepsilon$ and $u_{\xi,i}^\varepsilon$ is

\[ (\alpha) \ u_{\xi,i}^{p_\varepsilon} \text{ if } \zeta \in u^{p_\varepsilon} \]
\[ (\beta) \ u_{\zeta(\ast),i}^{p_{\zeta(\ast),\xi}} \text{ if } \zeta = \zeta(\ast). \]

We can easily check that $p \in \mathbb{P}$ and $p^* \leq p_{\beta_\varepsilon,j_\varepsilon}, p^+ \leq p \in \mathbb{P}$ (but we do not ask $p_\varepsilon \leq p$). Clearly $p$ forces that $\{\beta_\varepsilon : \varepsilon \in Z\}$ is included in one $X_i$, that is $X_{i^*}$.

Let $g : \theta \to \lambda$ be $g(\varepsilon) = \beta_\varepsilon$ when $\xi < \theta, \varepsilon \in Z$, $\text{otp}(Z \cap \varepsilon) = \xi$. Now $p \geq p^*$ and we are done by $(\ast)_8$ below.

$(\ast)_8$ if $p \in \mathbb{P}$ and $\zeta \in v_\varepsilon^\ast$ then $p \Vdash \text{"the mapping } j \mapsto \gamma_{\zeta,j}^\varepsilon \text{ for } j < \theta \text{ is a homeomorphism from } Y^\ast \text{ onto the closed subspace } X \setminus \{\gamma_{\zeta,j}^\varepsilon : j < \theta\} \text{ of } X"$.

[Why? Let $p \in G, G \subseteq \mathbb{P}$ be generic over $\mathbb{V}$.

\[ (\alpha) \ \text{If } b \in \mathcal{B} \text{ (recall that } \mathcal{B} \text{ is a basis of } Y^\ast) \text{, then for some open set } \mathcal{U} \text{ of } X[G] \text{ (clopen for part (2)) we have} \]

\[ \mathcal{U} \cap \{\gamma_{\zeta,j}^\varepsilon : j < \theta\} = \{\gamma_{\zeta,j}^\varepsilon : j \in b\} \]
(\beta) If \( b \) is an open set for \( Y^* \), then for some open subset \( \mathcal{U} \) of \( X \) we have

\[
\mathcal{U} \cap \{ \gamma^p_{\xi,j} : j < \theta \} = \{ \gamma^p_{\xi,j} : j \in b \}
\]

[Why? As \( b = b_i \) for some \( i < \theta^* \) and \( p \) forces by clause (\( \varepsilon \)) + (\( \zeta \)) of the definition of \( p \in \mathbb{P} \) and clause (iv) of (\( \ast \)) above that
\( \mathcal{U}_{\xi,i} \cap \{ \gamma^p_{\xi,j} : j < \theta \} = \{ \gamma^p_{\xi,j} : j \in b_i \} \), see \( \mathbb{S}_1 \) above.]

\[
\gamma^p_{\xi,j(*)} \in w^p_{\xi,i(*)} \cap \{ \gamma^p_{\xi,j} : j < \theta \} \subseteq \mathcal{U}_{\xi,i(*)}[G] \cap \{ \gamma^p_{\xi,j} : j < \theta \} \subseteq \mathcal{U}.
\]

[Why? By the definition of the topology \( X^*[G] \) we can find \( n < \omega, \xi_\ell < \lambda^* \) and \( j_\ell < \theta^* \) for \( \ell < n \) such that \( \gamma^p_{\xi,j(*)} \in \bigcap_{\ell < n} \mathcal{U}_{\xi_\ell,j_\ell}[G] \subseteq \mathcal{U} \). [Why?
By \( \mathbb{S} \). We can find \( q \in G \subseteq \mathbb{P} \) such that \( p \leq q \) and \( \xi_\ell \in v_\ell^q \) for \( \ell < n \).
[Why? Recall (\( \ast \)) and \( \mathbb{S}_1 \).] For each \( \ell < n \), by clause (\( \eta \)) in the definition of \( \mathbb{P} \) we know that \( \mathcal{U}_{\xi_\ell,j_\ell}^q =: \{ j < \theta : \gamma^p_{\xi,j} \in \mathcal{U}_{\xi_\ell,j_\ell}^q \} \) is an open set for \( Y^* \), and necessarily \( j(*) \in \mathcal{U}_{\xi_\ell,j_\ell}^q \). Let \( i(*) < \theta \) be such that \( j(*) \in b_i(*) \subseteq \bigcap_{\ell < n} \mathcal{U}_{\xi_\ell,j_\ell}^q \) hence \( \gamma^p_{\xi,j(*)} \in \mathcal{U}_{\xi,i(*)}[G] \cap \{ \gamma^p_{\xi,j} : j < \theta \} \subseteq \bigcap_{\ell < n} \mathcal{U}_{\xi_\ell,j_\ell}[G] \subseteq \mathcal{U} \) as required. So \( i(*) \) is as required.]

(\delta) \( \{ \gamma^p_{\xi,j} : j < \theta \} \) is a closed subset of \( X^* \)

[Why? Let \( \beta \in \lambda \setminus \{ \gamma^p_{\xi,j} : j < \theta \} \) and let \( p \leq q \in \mathbb{P} \); it suffices to find \( q^+, \eta \leq q^+ \in \mathbb{P} \) and \( \xi \in v_\ell^q \) and \( i < \theta^* \) such that \( \beta \in u^q \cup u^\eta \) or \( \beta \in w^q_{\xi,i} \) and \( w^q_{\xi,i} \cap \{ \gamma^p_{\xi,j} : j < \theta \} = \emptyset \). If \( \beta \notin u^q \) define \( q^+ \) like \( q \) except that \( u^{q^+} = u^q \cup \{ \beta \} \) (but \( u^q \cup \{ \beta \} \) as in clause (i) of (\( \ast \))). So without loss of generality \( \beta \in u^q \).

We can find a set \( u \subseteq u^q \) such that \( \beta \in u, A^q_\xi \cap u = \emptyset \) and \( \zeta' \in v^{q^+}_\ell \Rightarrow \{ j < \theta : \gamma^p_{\xi,j} \in u \} \) is an open subset of \( Y^* \), (why? just as in the proof of (\( \ast \))_5; that is let \( \langle \xi_{\varepsilon} : \varepsilon < \varepsilon^* \rangle \) be a list \( v^{q^+}_\ell \) such that \( B_{\varepsilon} = A_{\xi_{\varepsilon}} \cup \{ A_{\xi_{\varepsilon}(1)} : \varepsilon(1) < \varepsilon \} \) has cardinality < \( \sigma \), and as any two members of \( \mathcal{A} \) has intersection of cardinality < \( \sigma \) without loss of generality \( \xi_0 = \zeta \), and choose \( u_{\varepsilon} \subseteq \bigcup \{ A_{\xi_{\varepsilon}(1)} : \varepsilon(1) < \varepsilon \} \) by induction on \( \varepsilon \leq \varepsilon^* \) such that]
\[
\varepsilon' < \varepsilon \Rightarrow \mu_{\varepsilon'} = \mu_{\varepsilon} \cup \{ A_{\xi_{\varepsilon'(i)}} : \varepsilon(1) < \varepsilon'(i) \} \text{ and } \beta \in \bigcup \{ A_{\xi_{\varepsilon'(i)}} : \varepsilon(1) < \varepsilon'(i) \} \Rightarrow \beta \in \mu_{\varepsilon} \text{ and } \varepsilon(1) < \varepsilon \Rightarrow \{ j : \gamma_{\xi_{\varepsilon'(i)},j} \in \mu_{\varepsilon} \} \text{ is open in } Y^* \] and \( u_1 = u_0 = \emptyset \). For part (2) we ask “clopen subset of \( Y^* \).” In the end let \( u = u_{\varepsilon'} \cup \{ \beta \} \).

By \( \otimes_1 \) in the beginning of the proof we can find \( \xi \in \lambda \setminus \nu^q \) such that \( \emptyset = A_{\xi} \cap u^q \) (why? apply \( \otimes_1 \) with \( \alpha' < \beta' \in \lambda \setminus u^q \) and then \((\ast)_1\)) and let \( \gamma_{\xi,i} \in A_{\xi} \) for \( i < \theta \) be increasing. We define \( q^+ \) as follows.

\[
v^q_+ = v^q \cup \{ \xi \}
\]

\[
v^q_+ = v^q \cup \{ \xi \}
\]

\[
u^q_+ = u^q \cup A_{\xi}
\]

\[
u^q_+ = u^q \cup \{ \gamma_{\xi,j} : j < \theta \}
\]

\( u^q_+ \) is \( v^q_+ \) if \( \zeta \in v^q_+ \) and is \( \{ \gamma_{\xi,j} : j \in b_i \} \cup u \) if \( \zeta = \xi \text{ and } 0 \in b_i \) and is \( \{ \gamma_{\xi,j} : j \in b_i \} \) if \( \zeta = \xi \text{ and } 0 \notin b_i \). So \( q^+ \) is as required above and this suffices.

Lastly, we would like to know that \( X^* \) is a Hausdorff space. We prove more

\[(\ast)_9 \text{ In } V^P \text{ if } u_1 \subseteq u_2 \in [\lambda]^\sigma \text{ then for some } \zeta, i \text{ we have} \]

\[
w_{\zeta,i} \cap u_2 \cap X^* = u_1 \cap X^*
\]

[Why? Let \( p_0 \in P \) force that \( u_1 \subseteq u_2 \) form a counterexample, as \( P \) is \( \kappa \)-complete some \( p_1 \geq p_0 \) forces \( u_1 = u_1, u_2 = u_2 \) and without loss of generality \( p_1 \in P' \). By \( (\ast)_1 \) and \( \kappa \)-completeness without loss of generality \( u_2 \subseteq u^{p_1} \) and as by \( (\ast)_1 \) we have \( p_1 \vdash "u_2 \cap X^* = u_2 \cap u^{p_1}_*" \) we can ignore the elements of \( u_2 \setminus u^{p_1}_* \) so without loss of generality \( u_2 \subseteq u^{p_1}_* \).

Let \( \zeta(\ast) \in \lambda \setminus v^{p_1} \) be such that \( A_{\zeta(\ast)} \cap u^{p_1} = \emptyset \) (as in the proof of \( (\ast)(\delta) \)). Let \( \gamma_{\zeta(\ast),j} \in A_{\zeta(\ast)} \), for \( j < \theta \) be increasing. Let \( u \subseteq u^{p_1}_* \) be such that \( u \cap u_2 = u_1 \) and \( \zeta'' \in v^{p_1} \Rightarrow \{ j < \theta : \gamma_{\zeta''} \in u \} \) is open (for part (1)) or is clopen (for part (2)) in \( Y^* \) (exists as in the proof of \( (\ast)_5 \) and of \( (\ast)_8(\delta) \)) and define \( q \in P \):]
\[ u^q = u^{p_1} \cup A_\xi \]

\[ u^q_* = u^{p_1} \cup \{ \gamma_{\xi(j)} : j < \theta \} \]

\[ v^q = v^p \cup \{ \xi(*) \} \]

\[ v^q_* = v^{p_1} \cup \{ \xi(*) \} \]

\[ w^q_{\xi,i} \text{ is: } w^{p_1}_{\xi,i} \text{ if } \xi \in v^{p_1}_i \text{, is } \{ \gamma_{\xi(j)} : j \in b_i \} \cup u \text{ if } \xi = \xi(*) \& 0 \in b_i \text{ and is } \{ \gamma_{\xi(j)} : j \in b_i \} \text{ if } \xi = \xi(*) \& 0 \notin b_i. \text{ It is easy to see that } q \text{ is required.} \]

Together all is done. □

Now when do the assumptions of 1.2 hold?

1.4 Claim. 1) Assume

(a) \( a \in [\text{Reg} \cap \lambda \gamma \kappa]^{\theta} \) and \( J = [a]^{<\sigma} \)

(b) \( \Pi a/J \) is \( (\lambda^*)^+ \)-directed,

(c) \( \sigma \) is regular, \( \lambda > \kappa^{++}, \kappa > \theta > \sigma \)

(d) \( \lambda^* > \lambda > \kappa^{<\kappa} = \kappa > \theta \);

(e) \( \lambda^* < 2^\lambda \) is regular.

Then

(f) In \( V_1 = V^{\text{Levy}(\lambda^*,2^\lambda)} \) we have (a),(c),(d) and (e) and \( 2^\lambda = \lambda^* \) and

(g) the assumptions \((A)(i),(B_1),(B_2),(C)\) of Theorem 1.2 hold (recall \((A)(i)\)

means we omit \( \theta^* \) and the demand \((\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)\).

2) We can in (g) above strengthen \((C)\)

\((C)^*\) if \( x_\beta = \langle x_{\beta,i} : i < i_\beta \rangle \) for \( \beta < \lambda \) and \( i_\beta < \kappa \) then we can find \( i(*) \) and \( A \in [\lambda]^\theta \) such that

(i) \( \beta \in A \Rightarrow i_\beta = i(*) \)

(ii) if \( \beta_1 < \beta_2 \) are from \( A \) and \( i_1, i_2 < i(*) \) then \( x_{\beta_1,i_2} = x_{\beta_2,i_2} \Rightarrow x_{\beta_1,i_1} = x_{\beta_2,i_1} \) & \( x_{\beta_1,i_1} = x_{\beta_2,i_2} \)

(iii) if \( i < i(*) \) and \( \beta_1, \beta_2, \beta_3 \in A \) are distinct then \( x_{\beta_1,i} = x_{\beta_2,i} \Rightarrow x_{\beta_1,i} = x_{\beta_3,i} \).

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Proof. Let $a = \{\lambda_\varepsilon : \varepsilon < \theta\}$ without repetitions; without loss of generality $\lambda_\varepsilon > \kappa^+$ for $\varepsilon < \theta$. Let $J' = [\theta]^{<\sigma}$. In $\mathbf{V}$ we can find $\langle f_\alpha : \alpha < \lambda^* \rangle$ such that $\mathbb{E}$ below holds for this proof $1.5$ below.

1.5 Observation. Assume $a = \{\lambda_\varepsilon : \varepsilon < \theta\}$ is a set of regular cardinals, $J'$ is an ideal on $\theta$ such that $\Pi \lambda_\varepsilon/J'$ is $(\lambda^*)^+$-directed, $\lambda^* = \text{cf}(\lambda^*) > \sup(a)$, and $\sigma$ regular and $\theta < \kappa \leq \text{Min}(a)$ (or at least for every regular $\kappa' < \kappa$ the set $\{\varepsilon < \theta : \lambda_\varepsilon \leq \kappa'\}$ belongs to $J'$ and, of course, $a$ is a set of regular cardinals $> \theta$). Then

$\mathbb{E}$ we can find $\langle f_\alpha : \alpha < \lambda^* \rangle$ satisfying $f_\alpha \in \prod_{\varepsilon < \theta} \lambda_\varepsilon$ such that

(a) $\bar{f} = \langle f_\alpha : \alpha < \lambda^* \rangle$ is $<_{J'}$-increasing

(b) $\bar{f}$ has a $<_{J'}$-lub $f^*$

(c) if $a \in (J')^+$ then $\sup\{\text{cf}(f^*(i)) : i \in a\} \geq \sup(\{\lambda_\varepsilon : \varepsilon \in a\})$

(d) if $\alpha_1 < \lambda^* \& \alpha_2 < \lambda^* \& \varepsilon_1 < \theta \& \varepsilon_2 < \theta \& f_\alpha_1(\varepsilon_1) = f_\alpha_2(\varepsilon_2)$ then $\varepsilon_1 = \varepsilon_2$

(e) for every $Z \in [\lambda^*]^{\leq \kappa}$ we can find $\bar{a} = \langle a_\alpha : \alpha \in Z \rangle$ such that $a_\alpha \in J'$ and $\alpha < \beta \& \alpha \in Z \& \beta \in Z \& \varepsilon \in \theta \setminus a_\alpha \setminus a_\beta \Rightarrow f_\alpha(\varepsilon) < f_\beta(\varepsilon)$

(f) if $\sigma < \theta$ and $J' = [\theta]^{<\sigma}$, then for every $Z \in [\lambda^*]^{<\kappa}$ for some sequence $\bar{a} = \langle a_\alpha : \alpha \in Z \rangle$ satisfying $a_\alpha \in J'$ for $\alpha \in Z$ and some well ordering $<^*$ of $Z$ we have $\alpha \in Z \& \beta \in Z \& \alpha <^* \beta \& \varepsilon \in \theta \setminus a_\alpha \setminus a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon)$.

Proof. By the proof of [Sh:g, II.1.4, pg.50] or see [Sh 506] we get: for some $\langle f_\alpha : \alpha < \lambda^* \rangle$ with $f_\alpha \in II_a$ we have (a) + (b) + (c) + (e).

Clause (d) is easy, just replace $f_\alpha$ by $f'_\alpha \in \prod_{\varepsilon < \theta} \lambda_\varepsilon$ which is defined by $f'_\alpha(\varepsilon) =:

\theta \times f_\alpha(\varepsilon) + \varepsilon$ and replace $f^*$ by $f^{**}$, $\text{Dom}(f^{**}) = \theta$, $f^{**}(\varepsilon) = \theta \times f^*(\varepsilon)$ recalling $\theta < \text{Min}(a)$. We shall prove that $\langle f'_\alpha : \alpha < \lambda^* \rangle$ is also as required in clause (f). So let $Z \in [\lambda^*]^{\leq \kappa}$ be given; let $Z = \{\gamma_\varepsilon : \varepsilon < |Z|\}$ be with no repetitions. Let $\langle a_\beta : \beta \in Z \rangle$ be as guaranteed by clause (e). We can choose by induction on $\zeta < |Z|$, $Z_\zeta \subseteq Z$ increasing continuous in $\zeta$ and $Z^\zeta$ such that $Z_0 = \emptyset$, $|Z_{\zeta+1} \setminus Z_{\zeta}| \leq \sigma$, $Z_\zeta \neq Z \Rightarrow Z_{\zeta} \neq Z_{\zeta+1}$ and $Z_{\zeta+1} \setminus Z_{\zeta} \subseteq Z^\zeta \subseteq Z_{\zeta+1} \setminus Z^\zeta \subseteq \sigma$, $\gamma_\varepsilon \in Z_{\varepsilon+1}$ and $\alpha \in Z^\zeta \& \beta \in Z^\zeta \& \varepsilon \notin a_\beta \& \varepsilon \in a_\alpha \& f_\alpha(\varepsilon) = f_\beta(\varepsilon) \Rightarrow \beta \in Z^\zeta$ (actually the
Why does such a sequence exist? The only problem is, given \(Z_\xi\) to choose \(Z^\xi\). Now as \(|a_\alpha| < \sigma\) (because \(a_\alpha \in J' \subseteq [a]^{<\sigma}\)) and by the choice of \(\langle a_\alpha : \alpha \in Z \rangle\) we have

\[
(\forall \varepsilon < \theta)(\forall \gamma < \lambda_\varepsilon)(\exists^{\leq 1}\beta \in Z)(\varepsilon \notin a_\beta \& f_\beta(\varepsilon) = \gamma)
\]

hence there are no problems, (in fact if \(\sigma\) is uncountable we can ask \(< \sigma\)). With more details, we define \(Z^\alpha\) by induction on \(n < \omega\) as follows; \(Z^\alpha_0 = \{\gamma_\varepsilon\}, Z^\alpha_{n+1} = Z^\alpha_n \cup \{\beta : \text{for some } \alpha \in Z^\alpha_n \text{ and } \varepsilon \in a_\alpha \text{ we have } \varepsilon \notin a_\beta \& f_\beta(\varepsilon) = f_\alpha(\varepsilon)\} \).

As \(\sigma\) is a regular and \(\alpha \in Z \Rightarrow |a_\alpha| < \sigma\), by (*) we can prove by induction on \(n\) that \(|Z^\alpha_n| < \sigma\), hence \(Z^\alpha = \cup\{Z^\alpha_n : n < \omega\}\) is as required. Now list \(Z^\xi\) as \(\langle a^\xi_\alpha : \xi < \xi^*_\alpha\rangle\) such that \(\xi^*_\alpha = |Z^\xi| \leq \sigma\).

Define a well ordering \(<^*\) of \(Z\) as follows

\[
\alpha < \beta \Leftrightarrow (\exists \zeta)[\alpha \in Z_\zeta \land \beta \notin Z_\zeta] \lor (\exists \zeta)(\alpha \in Z^\zeta \land \beta \in Z^\zeta \land (\exists \xi_1, \xi_2)_{\alpha = a^\zeta_{\xi_1} \land \beta = a^\zeta_{\xi_2}} < \xi_1 < \xi_2).
\]

Now we define \(a'_\alpha \in J'\) for \(\alpha \in Z\) as follows: if \(\alpha = a^\zeta_\xi \in Z^\zeta_{\xi+1} \setminus Z_\zeta\) then \(a'_\alpha = \cup\{a^\zeta_\varepsilon : \varepsilon \leq \xi\}\); as \(\xi < |Z^\zeta| \leq \sigma\) and \(\sigma\) is regular and \(\beta \in Z \Rightarrow |a_\beta| < \sigma\) clearly \(|a'_\alpha| < \sigma\). Now suppose \(\langle a'_\alpha : \alpha \in Z\rangle\) fails clause (f) for \(<^*\) which is a well ordering of \(Z\). So there are \(\varepsilon < \theta\) and \(\alpha <^* \beta\) from \(Z\) which exemplifies this and let \(\zeta\) be such that \(\alpha \in Z^\zeta \setminus Z_\zeta\) and let \(\xi\) be such that \(\alpha = a^\zeta_{\xi};\) so by the definition of \(<^*\) we have \(\beta \notin Z^\zeta\) and \(\beta \notin \{a^\zeta_{\xi'} : \xi' \leq \xi\}\). As \(a_\alpha \subseteq a'_\alpha\) and the choice of \(\langle a_\alpha : \alpha \in Z\rangle\) and as \(\varepsilon \notin a_\beta\) (by the choice of \(\alpha, \beta, \varepsilon\)) necessarily \(\varepsilon \in a_\alpha\) hence \(\beta \in Z^\zeta\) (so \(\beta \in Z^\zeta \setminus Z_\zeta\)) but as said above \(\beta \notin \{a^\zeta_{\xi'} : \xi' \leq \xi\},\) so by the choice of \(a'_\beta\) we get easy contradiction.

\(\square\) \(1.5\)

Continuation of the proof of 1.4. Clearly in \(V_1\) we have (a),(c),(d) of 1.4 and \(\Xi\) of 1.5 above.

As in \(V\), \(\lambda < \lambda^* = \text{cf}(\lambda^*) < 2^\lambda\), clearly in \(V_1\) we have \(2^\lambda = \lambda^*\) (and we can forget \(V\) and the assumption (b), recall (b) says “\(\Pi\alpha\) is \((\lambda^*)^+\)-directed”. More on the existence of \(\bar{f}\) as in \(\Xi\), see [Sh:g, Ch.VIII,§5]).

So we can in \(V_1\) let \(\langle h_\alpha : \alpha < \lambda^*\rangle\) list the functions \(h : \lambda \rightarrow [\lambda]^\alpha\). Now for each \(\zeta < \lambda^*\) we define a function \(g_\zeta : \kappa^{++} \rightarrow [\kappa^{++}]^{\leq \alpha}\) by\(^4\)

\[
g_\zeta(\gamma) = \{\beta < \kappa^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \gamma \text{ we have}
\]

\[
\theta \times \kappa^{++} \times f_\zeta(\varepsilon_1) + \theta \times \beta + \varepsilon_1 \in
\]

\[
h_\zeta[\theta \times \kappa^{++} \times f_\zeta(\varepsilon_2) + \theta \times \gamma + \varepsilon_2]
\]

\(^4\)actually \(\theta \times \kappa^{++} \times \gamma = \kappa^{++} \times \gamma\)
So we can ([Ha61]) for each $\zeta < \lambda^*$ find $Z_\zeta \in [\kappa^+]^{\kappa^+}$ such that

$$\beta_1 \neq \beta_2 \in Z_\zeta \Rightarrow \beta_1 \notin g_\zeta(\beta_2).$$

For $\zeta < \lambda^*$ let

$$A_\zeta = \{ \theta \times \kappa^+ \times f_\zeta(\varepsilon) + \theta \times \beta + \varepsilon : \varepsilon < \theta \text{ and } \beta < \kappa^+ \text{ is the } \varepsilon\text{-th member of } Z_\zeta \}.$$ 

Now we shall check.

Let $\mathcal{A} = \{ A_\zeta : \zeta < \lambda^* \}$. Clearly

1. $A_\zeta \in [\lambda]^\theta$ (hence $\mathcal{A} \subseteq [\lambda]^\theta$)
2. $\zeta_1 \neq \zeta_2 \Rightarrow |A_{\zeta_1} \cap A_{\zeta_2}| < \sigma$
3. $|\mathcal{A}| = \lambda^*$
4. if $F : \lambda \rightarrow [\lambda]^{<\kappa}$, then some $A \in \mathcal{A}$ is $F$-free
5. if $\mathcal{A}' \in [\mathcal{A}]^{<\kappa}$, then we can list $\mathcal{A}'$ as $\{ A_\zeta : i < i(*) \}$ such that

$$|A_{\zeta_i} \cap \bigcup_{j<i} A_{\zeta_j}| < \sigma \text{ for each } i < i(*)$$

which has cardinality $< \sigma$ where $\beta_{\zeta_i, \varepsilon}$ is the $\varepsilon$-th member of $Z_{\zeta_i}$.
So clause (A)(i) holds by clause (d) of our assumption (note, $\theta^*$ does not appear here), clause (B) holds by $(\ast)_1 + (\ast)_2$ and (B) holds by $(\ast)_5$ and lastly (C) holds by $(\ast)_4$.

2) Similar, just in order to get more in the proof of $(\ast)_4$ we let $\langle h_\alpha : \alpha < \lambda^* \rangle$ list the relevant $h$'s and choose $Z_\zeta$ accordingly. 

\[ \square \]

1.6 Remark. We can get $(C)^*$ from Claim 1.4(2) for any $\kappa$ such that $\{ \theta \in \mathfrak{a} : (2^{<\kappa})^+ \leq \theta \} \in J$.

1.7 Claim. 1) We can change the assumptions of Theorem 1.2 by omitting (A)(ii) and by replacing (E) by $(E)^-$ and (C) by $(C)'$, i.e., having:

$(A)(i)$ \[ \lambda > \kappa > \theta \geq \sigma \geq \mathfrak{A} \] and $\kappa = \kappa^{<\kappa} \neq \kappa$ and $\theta^* \leq \kappa$

(i.e., this is $(A)(i)$ without (A)(ii), i.e., omitting “$(\forall \alpha < \kappa)(|\alpha|^{\sigma} < \kappa), \kappa > \theta^* \geq \theta^*$”)

$(C)'$ if $F : \lambda \rightarrow [\lambda]^{\leq\kappa}$ then we can find $A' \in \mathfrak{A}$ and $A \subseteq A'$ of order type $\theta$ such that:

if $\beta \in A$ then $\beta \notin \bigcup \{ F(\alpha) : \alpha \in A \cap \beta \}$

$(E)^-$ if $\mathcal{Y}_0, \mathcal{Y}_1$ are disjoint subsets of $\mathcal{Y}^*$ each with $< \sigma$ points, then there are open disjoint sets $\mathcal{U}_0, \mathcal{U}_1$ of $\mathcal{Y}^*$ such that $\mathcal{Y}_0 \subseteq \mathcal{U}_0, \mathcal{Y}_0 \subseteq \mathcal{Y}_1$.

2) We can similarly weaken the assumptions in 1.2(2), omitting “the union of $< \sigma$ members of $\mathcal{B}$ belong to $\mathcal{B}$”, but in $(E)^-$ demand $\mathcal{U}_0$ to be clopen.

Proof. We indicate the changes in the proof.

We can further demand from $\langle b_i : i < \theta^* \rangle$ that

$\otimes_3 b_{2i} \cap b_{2i+1} = \emptyset$ and if $b_{ti} \cap b_{tj} = \emptyset$ then for some $j$ we have $(b_{2j}, b_{2j+1}) = (b_{ti}, b_{tj})$ and if $Y_0, Y_1 \in [\mathcal{Y}^*]^{<\sigma}$ are disjoint then for some $i$ we have $Y_0 \subseteq b_{2i}, Y_1 \subseteq b_{2i+1}$.

Note that $\theta^{<\sigma} \leq \kappa$ so no problem arises with the number of $b_i$'s.

In the definition of $\mathbb{P}$ we replace clause $(\delta)$ by

$(\delta)^- w_{\zeta, i} \subseteq u_s$ and $w_{\zeta, 2i} \cap w_{\zeta, 2i+1} = \emptyset$

and in the definition of the order when $\mathbb{P} \models p \leq q$ we add $\zeta \in v_s^p \setminus v_s^q \Rightarrow |A_\zeta \cap u_p| < \sigma$.

However, as we have weakened assumption $(A)$, the $\kappa^+$-c.c. may fail. So we define: the pair $(f, g)$ is an isomorphism from $p \in \mathbb{P}$ onto $q \in \mathbb{P}$ if:

$(i)$ $f$ is a one-to-one order preserving mapping from $u_p$ onto $u_q$
We say \( p, q \) are isomorphic if such \( (f, g) \) exists. Clearly being isomorphic is an equivalent relation. Let \( \chi \) be large enough and \( \mathcal{C} \) be an elementary submodel of \((\mathcal{H}(\chi), \in, <^*)\) of cardinality \( \kappa \) such that \( \lambda, \kappa, \theta^*, \theta, \sigma, Y^*, \langle b_i : i < \theta^* \rangle, \mathcal{A}, \mathbb{P} \) belong to \( \mathcal{C} \) and \( \kappa > \mathcal{C} \subseteq \mathcal{C} \). Let

\[
\mathbb{Q} = \{ p \in \mathbb{P} : \text{for some } q \in \mathbb{P} \cap \mathcal{C} \text{ the conditions } p, q \text{ are isomorphic} \}.
\]

In the rest of the proof \( \mathbb{P} \) is replaced by \( \mathbb{Q} \), each time we construct a condition we have to check if it belongs to \( \mathbb{Q} \).

The only place we use \((\forall \alpha < \kappa)(|\alpha|^\sigma < \kappa)\) is in the proof of \((*)_3\). So omit \((*)_3\), and this requires us first to improve the proof of \((*)_4\) (and second \((*)_7\), see later). Let \( p_j \in \mathbb{Q} \) for \( j < \kappa^+ \) and let \( v_j = \{ \zeta < \lambda^* : A_{\zeta} \cap u^{p_j} \text{ has cardinality } \geq \sigma \} \cup v^{p_j} \), so clearly \( |v_j| \leq \kappa \) and \( v^{p_j} \subseteq v_j \).

For some stationary \( S \subseteq \{ \delta < \kappa^+ : \text{cf}(\delta) = \kappa \} \), the conditions \( p_j \) for \( j \in S \) are pairwise isomorphic and \( j \in S \) implies \( v^{p_j} \cap (\bigcup_{i<j} v_i) = v^\otimes \) and \( u^{p_j} \cap (\bigcup_{i<j} (u^{p_i} \cup \bigcup_{\zeta \in v_i} A_{\zeta})) = u^\otimes \). Also without loss of generality for \( j_1, j_2 \in S \) the isomorphism \((f, g)\) from \( p_{j_1} \) to \( p_{j_2} \) satisfies \( f \upharpoonright u^\otimes = \text{id}_{u^\otimes}, g \upharpoonright v^\otimes = \text{id}_{v^\otimes} \). We would like to apply \((*)_5\) from the proof of 1.2 to \( p_i, p_j \) for any \( j > i \) from \( S \), so we have to verify clauses \((i), (ii), (iii), (iv)_{1, 2} \) there. Now only clause \((iv)_{2} \) is problematic. Now if \( i \neq j \) are from \( S \) and \( \zeta \in v^{p_i} \setminus v^{p_j} \) and \( |A_{\zeta}^{p_i} \cap u^{p_i}| \leq \sigma \), note that \( A_{\zeta}^{p_i} \subseteq u^{p_i} \), hence \( A_{\zeta}^{p_i} \cap u^{p_j} \subseteq u^{p_i} \cap u^{p_j} = u^\otimes \subseteq v_{\text{min}(S)+1} \). So for \( i, j \in S \setminus \{ \text{min}(S), p_i, p_j \} \) are compatible by \((*)_5\) in the proof of 1.2.

In the proof of \((*)_5\) and \((*)_6\) (hence \((*)_7\)), clause \((E)^-\) gives us less but the change in the definition of \( \mathbb{P} \) (weakening \((\delta)\) to \((\delta)^-\)) demands less and they fit, e.g., during the proof of \((*)_5\) we can deal with each pair \((w_{\zeta,2i}, w_{\zeta,2i+1})\) separately.

The second place in which we use \((*)_3\) is during the proof of \((*)_7\). The proof is similar but:
\( F(\alpha) \) now is a subset of \( \lambda \) which includes \( \cup \{ u^{p_{i,j}} : j < \kappa \} \) and satisfies 
\[ A \in \mathcal{A} \quad \& \quad |A \cap F(\alpha)| \geq \sigma \Rightarrow A \subseteq F(\alpha). \]

[Why such \( F(\alpha) \) exists? As in the proof of (\( \ast \))_3. Then, first we apply clause \( (C)' \)
from 1.7 to find \( \zeta(\ast) < \lambda^* \) and \( A \subseteq A_{\zeta(\ast)} \) of order type \( \theta \) such that \( A \) satisfy the
demands of \( (C)^\ast \). Second, choosing \( p_\varepsilon \) by induction on \( \varepsilon \), choosing \( p_{\varepsilon+1} \), verifying
the conditions in \( (\ast)_5 \) they hold because of the change in the definition of the order
of \( \mathbb{P} \).

Lastly, for proving “\( X \) is Hausdorff”, clause \( (\delta)^- \) is weaker but as \( Y^* \) is Hausdorff
(and the choice of \( (b_i : i < \theta^*) \)) there is no problem. \( \square_{1.7} \)

1.8 Concluding Remarks. 1) We could make in 1.2 only some of the changes from
1.7, i.e.,

\( (\alpha) \) in 1.2 we replace \( (A),(C) \) by \( (A)(i),(C)^* \) of 1.7
\( (\beta) \) in 1.2 replace \( (E) \) by \( (E)^- \).

2) In 1.2(1) can we make the space regular \( (T_3) \)?

In view of 1.2(2) this may be not so interesting, still needed for a regular \( X^* \to \)(\( \mathbb{R} \))\^{1}_{\mathcal{A}_0} . \)
Note that for \( X \) to be a \( T_3 \)-space it is enough that there is a family \( \mathcal{B} \) of
open subsets such that their finite intersections forms a basis, \( (\forall x \neq y \in X)(\exists U \in \mathcal{B})(x \in U \& y \notin U) \) and \( x \in \mathcal{U}_0 \in \mathcal{B} \Rightarrow (\exists U_1, U_2 \in \mathcal{B})(x \in U_1 \subseteq U_0 \& \mathcal{U}_0 \cup U_2 = X \& U_1 \cap U_2 = \emptyset) \). Let \( R_0 \subseteq \{(i,j) : b_i \cap b_j = \emptyset \} \) (so to include
generalization, as in 1.7 we chose \( R_0 \subseteq \{(2i,2i+1) : i < \theta^* \} \) and \( R_1 \subseteq \{(i,j) : b_i \cup b_j = Y^* \} \), \( R_2 \subseteq \{(i,j) \) : \( b_i \subseteq \emptyset \} \).

We need: for \( i_0 < \theta^*, j < \theta \) such that \( j \in b_{i_0} \) there are \( i_1, i_2 < \theta^* \) such that
\( j \in b_{i_1}, b_{i_1} \subseteq b_{i_0}, b_{i_0} \cup b_{i_2} = Y^*, b_{i_1} \cap b_{i_2} = \emptyset \) and moreover \( (i_1,i_0) \in R_2, (i_0,i_2) \in R_1, (i_1,i_2) \in R_0 \). If \( Y^* \) is a \( T_3 \)-space with a basis of cardinality \( \leq \theta^* \) then there is
no problem to find such \( b \).

Then we should change the definition of \( \mathcal{P} \), clause \( (\delta) \) to

\( (\delta)^- \)

\( (a) \quad w_{\zeta,i}^p \subseteq u_x^p \)
\( (b) \quad (i,j) \in R_0 \Rightarrow w_{\zeta,i}^p \cap w_{\zeta,j}^p = \emptyset; \)
\( (c) \quad (i,j) \in R_1 \Rightarrow w_{\zeta,i}^p \cup w_{\zeta,j}^p = u_x^p \)
\( (d) \quad (i,j) \in R_2 \Rightarrow w_{\zeta,i}^p \subseteq w_{\zeta,j}^p \)

\( (\theta) \) if \( \alpha \in w_{\zeta,i}^p \) then for some \( i_1,i_2 \) we have \( \alpha \in w_{\zeta,i_1}^p, (i_1,i_0) \in R_2, (i_0,i_2) \in R_1, (i_1,i_2) \in R_0 \)
(or use a three place relation \( R \)).
So there is no problem to generalize the proof of 1.2.

3) In 1.4, as indicated in the proof, we can replace in the assumption (b) + (e), i.e. “\( \Pi a/J \) is \((\lambda^*)^+\)-directed, \( \lambda^* < 2^\lambda \) is regular” by: \( \lambda^* = 2^\lambda \) and

\[
(*) \text{ there is } \vec{f} = \langle f_\alpha : \alpha < \lambda^* \rangle, f_\alpha \in \Pi a \text{ such that for every } Z \in [\lambda^*]^{<\kappa} \text{ we can find } \langle a_\alpha : \alpha \in Z \rangle, a_\alpha \in J \text{ such that } \alpha \neq \beta \in Z \land \varepsilon \in \emptyset \setminus a_\alpha \setminus a_\beta \Rightarrow f_\alpha(\varepsilon) \neq f_\beta(\varepsilon).
\]

4) By the proof of 1.5, if \( a, J = [a]^{<\sigma} \) and \( \vec{f} \) are as in \((*)\) of 1.8(3) then

\[
(*)' \text{ there is } \vec{f}' = \langle f'_\alpha : \alpha < \lambda^* \rangle, f'_\alpha \in \Pi a \text{ such that for every } Z \in [\lambda^*]^{<\kappa} \text{ we can find } \langle a_\alpha : \alpha \in Z \rangle, a_\alpha \in J \text{ and well ordering } <^* \text{ of } Z \text{ such that } \alpha <^* \beta \in Z \land \varepsilon \in \emptyset \setminus a_\beta \Rightarrow f'_\alpha(\varepsilon) \neq f'_\beta(\varepsilon) \text{ (in fact } \vec{f}' = \vec{f}).
\]

5) See more 4.17: for more colours.

6) If \( \text{CON(ZFC} + \exists \text{ supercompact)} \) then \( \text{CON(CH} + \text{ there is a } T_3\text{-topological space with } \aleph_{\omega+1} \text{ needs such that } X \to (\mathbb{R})_{\aleph_0}^1) \).

[Why? Similar to the proof of 2.6 below, using 1.8(2).]
§2 Consistency from supercompact

In the first section we got consistency results concerning Cantor discontinuum partition problem but using pcf statement of unclear consistency status (they come from 1.4); this is very helpful toward finding the consistency strength, and unavoidable if e.g. we like CH to fail (see §3), but it does not give a well grounded consistency result. Here relying on Theorem 1.2 of the first section, we get consistency results using "only" supercompact cardinals. First we give a sufficient condition for clause (C) of Theorem 1.2 which is reasonable under instances of G.C.H. We then (2.2) quote Hajnal Juhasz Shelah [HJSh 249], [HJSh 697] (for $\sigma = \aleph_0, \sigma > \aleph_0$, respectively) and from it (in claim 2.3), in the natural cases, prove that the assumptions of 1.2 hold deducing (2.6) the consistency of CH + there is a $T_3$-space $X$ with clopen basis with $\aleph_\omega + 1$ point such that $X \to (\text{Cantor set})^1_{\aleph_0}$ starting with a supercompact cardinal. This gives a (consistent) negative answer to the Cantor discontinuum partition problem. We can even make it compact. We also try to clarify the relations between such properties of, e.g., $\aleph_\omega + 1$.

2.1 Observation: If clauses\(^5\) (A)(i) + (B)\(1\) of Theorem 1.2 holds, then clause (C) there follows from

$$(C)^+ \text{ if } (Y_i : i < \kappa^+) \text{ is a partition of } \lambda \text{ then for some } A \in \mathcal{A} \text{ and } i < \kappa^+$$

we have $A \subseteq Y_i$.

Proof. Let $F : \lambda \to [\lambda]^{\leq \kappa}$ be given. Choose by induction on $\zeta \leq \lambda$ a set $U_\zeta \subseteq \lambda$ and $g_\zeta : U_\zeta \to \kappa^+$, both increasing continuous with $\zeta$ such that:

\((*)\)

(i) if $\alpha \in U_\zeta$ then $F(\alpha) \subseteq U_\zeta$ and

(ii) if $\alpha \in U_\zeta$ then $F(\alpha) \setminus \{\alpha\} \subseteq \{\beta \in U_\zeta : g_\zeta(\beta) \neq g_\zeta(\alpha)\}$.

For $\zeta = 0$ let $U_0 = \emptyset = g_0$, for $\zeta$ limit take unions. If $U_\zeta = \lambda$, let $U_{\zeta+1} = U_\zeta$ and $g_{\zeta+1} = g_\zeta$, otherwise let $\alpha_\zeta = \text{Min}\{\lambda \setminus U_\zeta\}$ and let $W_\zeta \in [\lambda]^{< \kappa}$ be such that $\alpha_\zeta \in W_\zeta$ and $(\forall \alpha \in W_\zeta)[F(\alpha) \subseteq W_\zeta]$. Let $\epsilon_\zeta = \sup\{g_\zeta(\beta) : \beta \in U_\zeta \cap W_\zeta\}$ so $\epsilon_\zeta < \kappa^+$ and let $U_{\zeta+1} = U_\zeta \cup W_\zeta$ and let $g_{\zeta+1}$ extend $g_\zeta$ such that $g_{\zeta+1} \upharpoonright (W_\zeta \setminus U_\zeta)$ is one to one with range $\subseteq [\epsilon_\zeta, \epsilon_\zeta + \kappa)$. Clearly $\zeta \subseteq U_\zeta = \text{Dom}(g_\zeta)$ so $g = \cup\{g_\zeta : \zeta < \lambda\}$ is a function with domain $\lambda$.

Now applying $(C)^+$ to the partition which $g$ defines, we get some $A \in \mathcal{A}$ on which $g$ is constant so by $(*)$ we are done. $\square_{2.1}$

By [HJSh 249], [HJSh 697], or see more below in 2.7 - 2.9, (toward equiconsistency) we have:

\(^5\)actually from (B)\(1\), only "$(B)^+_{\mathcal{A}} \subseteq [\lambda]^\theta$" is used; as we do not change $\mathcal{A}$ and the cardinals this is O.K.
2.2 Claim. Assume $V \models GCH$ (for simplicity) and $\sigma < \chi < \chi_0^\omega \leq \kappa < \mu < \mu^+ = \lambda$ and $\sigma, \chi, \chi_0, \kappa, \lambda$ are regular, $\text{cf}(\mu) = \sigma$ and $\chi$ is a supercompact cardinal (or just $\lambda$-supercompact), e.g. $\mu = \chi_0^\sigma$.

Then for some forcing notion $\mathbb{P}$, which is $\sigma$-complete of cardinality $\chi_0$, in $V^{\mathbb{P}}, 2^\sigma = \sigma^+, 2^{\sigma^+} = \chi_0 = \sigma^{++}$ (and $GCH$ holds) and some $\mathbb{B} = \langle \mathcal{B}_\delta : \delta \in S \rangle$ satisfies for any regular $\kappa \in (\chi_0, \mu)$:

\[(*) (i) \quad S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \sigma^+ \} \text{ is stationary,} \]
\[\mathcal{B}_\delta \subseteq \delta, \text{otp}(\mathcal{B}_\delta) = \sigma^+ \text{ and } \delta_1 \neq \delta_2 \in S \Rightarrow |\mathcal{B}_{\delta_1} \cap \mathcal{B}_{\delta_2}| < \sigma\]
\[(ii) \quad \lambda = \mu^+ = 2^\mu, \mu \text{ strong limit and letting } \theta = \sigma^+ \text{ we have } \sigma < \theta < \kappa = \kappa^{<\kappa} < \mu; \text{ note that if } \mu = \chi_0^\omega, \text{ then } \lambda = \kappa_{\omega+1}\]
\[(iii) \quad \text{cf}(\mu) = \sigma \text{ (this actually follows by (i) and (ii))}\]

What we need is getting in such model, condition $(C)^+\text{ of 2.1}$ which also is from [HJSh 697] but for completeness we shall prove what we use.

2.3 Claim. Assume

(a) $\mathcal{B} = \langle \mathcal{B}_\delta : \delta \in S \rangle, \sigma, \kappa, \mu, \lambda$ are as in the conclusion $(\ast)$ of the previous claim 2.2 and

(b) $S$ reflects in no ordinal of cofinality $\leq \kappa$ (holds automatically if $\kappa < \sigma^{++}$, see [Sh 108], [Sh 88a]), but see 2.7, 2.8.

Then without loss of generality $\sigma, \theta =: \sigma^+, \lambda, \mathcal{A} = \{ \mathcal{B}_\delta : \delta \in S \}$ (and $\theta^* = \theta$) satisfies the set theoretic requirements $(A), (B)_1, (B)_2, (C)$ in Theorem 1.2 and even $(C)^*$ of 1.4.

Proof. Without loss of generality “$\delta \in S \Rightarrow \mu^\omega$ divides $\delta^\omega$”, and as we are assuming $\mu$ is strong limit of cofinality $\sigma$ and $\lambda = \mu^+ = 2^\mu$ and $\delta \in S \Rightarrow \text{cf}(\delta) = \sigma^+ \neq \sigma = \text{cf}(\mu)$ we have $\diamond_S$ ([Sh 108]). So let $\langle f_\delta : \delta \in S \rangle$ be such that $f_\delta : \delta \rightarrow [\delta]^{\kappa}$ satisfy $(\forall f : \lambda \rightarrow [\lambda]^{\kappa})(\exists \text{stat } \delta \in S)(f_\delta = f \upharpoonright \delta)$. For each $\delta \in S$, let $B_\delta = \{ \alpha_{\delta, \varepsilon} : \varepsilon < \sigma^+ \}$ be increasing with $\varepsilon$ and let $g_\delta : \kappa^{++} \rightarrow [\kappa^{++}]^{\leq \kappa}$ be defined by

\[g_\delta(\beta) = \{ \gamma < \kappa^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \sigma^+ \text{ we have} \]
\[\kappa^{++} \times \alpha_{\delta, \varepsilon_1} + \gamma \in f_\delta(\kappa^{++} \times \alpha_{\delta, \varepsilon_2} + \beta) \}.

So by the free subset lemma (Hajnal [Ha61]) there is $Z_\delta \in [\kappa^{++}]^{\kappa^{++}}$ such that $\gamma_1 \neq \gamma_2 \in Z_\delta \Rightarrow \gamma_1 \notin g_\delta(\gamma_2)$. Let $\gamma_{\delta, \varepsilon} \in Z_\delta$ be strictly increasing with $\varepsilon < \sigma^+$ and
let \( B'_\delta = \{ \kappa^{++} \times \alpha_{\delta, \varepsilon} + \gamma_{\delta, \varepsilon} : \varepsilon < \sigma^+ \} \). So clauses \((A), (B)_1\) are immediate. Now clearly \((C)\) of 1.2 holds and lastly \((B)_2\) of 1.2 follow from the assumption \((b)\) on \( S \) (see [Sh 108]). In order to get \((C)^*\) of 1.7 we should shrink \( Z_\delta \) further.

Now \( \mathcal{A} = \{ B'_\delta : \delta \in S \} \) are as required in Theorem 1.2.

In similar spirit, we do further analysis.

**2.4 Claim.** Assume

\[(a)\] \( \lambda \) is regular and \( \theta^\sigma < \lambda \)

\[(b)\] \( \bar{B} = \langle B_\delta : \delta \in S \rangle \), where \( S \subseteq \lambda \) is stationary

\[(c)\] \( B_\delta \subseteq \delta \) and \( B_\delta \) has cardinality \( \theta \)

\[(d)\] if \( \delta_1 \neq \delta_2 \) are from \( S \) then \( \sigma > |B_{\delta_1} \cap B_{\delta_2}| \)

\[(e)\] \( \diamond S \).

Then for some \( \langle B'_\delta : \delta \in S' \rangle \) we have

\[(\alpha)\] \( S' \subseteq S \)

\[(\beta)\] \( B'_\delta \subseteq \delta \) has order type \( \theta \)

\[(\gamma)\] for \( \delta_1 \neq \delta_2 \) from \( S' \) we have \( \sigma > |B'_{\delta_1} \cap B'_{\delta_2}| \)

\[(\delta)\] if \( Z \subseteq \lambda \) is unbounded in \( \lambda \) then for stationarily many \( \delta \in S' \) we have \( B_\delta \subseteq Z \)

\[(\varepsilon)\] \( S' \) is stationary

\[(\zeta)\] if \( F : \lambda \rightarrow [\lambda]^{< \kappa}, \kappa = \text{cf}(\kappa), \kappa^{++} < \lambda \) then for stationarily many \( \delta \in S' \) the set \( B_\delta \) is \( F \)-free.

**2.5 Remark.** 1) If clauses \((a)-(d)\) of 1.4 hold in \( \mathbf{V} \), then \((a)-(e)\) holds after we force with \( \text{Levy}(\lambda, 2^{< \lambda}) \), note that if \( \lambda = 2^{< \lambda} \) this is equivalent to adding a Cohen subset to \( \lambda \).

2) We can add (in 2.4 see proof below):

\[(\delta)\] if \( Z_\varepsilon \subseteq \lambda = \text{sup}(Z_\varepsilon) \) for \( \varepsilon < \theta \) then for stationarily many \( \delta \in S' \) we have: for every \( \varepsilon < \theta \), the \( \varepsilon \)-th member of \( B_\delta \) belong to \( Z_\varepsilon \)

\[(\beta)_1\] if \( \delta \in S \Rightarrow \text{cf}(\delta) = \text{cf}(\theta) \) then \( B_\delta \) is unbounded in \( \delta \)

\[(\beta)_2\] if \( \delta \in S \Rightarrow \text{cf}(\delta) = \theta_1 \neq \text{cf}(\theta) \) so \( \theta_1 < \theta \) then \( B_\delta \) has order type \( \theta \times \theta_1 \) and is unbounded in \( \delta \).
Proof. Without loss of generality $\text{otp}(B_\delta) = \theta$.

Let $\bar{Z} = \langle Z_\delta : \delta \in S \rangle$ be such that $Z_\delta \subseteq \delta$ and for every $Z \subseteq \lambda$ the set \{\delta \in S : Z \cap \delta = Z_\delta \} is stationary, such a sequence exists as $\diamondsuit_S$ holds.

Now we choose $B'_\delta \subseteq \delta$ by induction on $\delta$ such that $B'_\delta \neq \emptyset \Rightarrow \text{otp}(B'_\delta) = \theta$. We let $B'_\delta$ be $Z_\delta' = \{ \alpha \in Z_\delta : \text{otp}(Z_\delta \cap \alpha) \in B_\delta \}$ when $\text{otp}(Z_\delta) = \theta$ & $\langle \forall \delta' \in S \cap \delta | [Z_\delta' \cap B'_\delta] < \sigma \rangle$ and let $B'_\delta$ be $\emptyset$ otherwise. Let $S' = \{ \delta \in S : B'_\delta \neq \emptyset \}$ and we shall prove that $\langle B'_\delta : \delta \in S' \rangle$ is as required.

Clauses ($\alpha$), ($\beta$), ($\gamma$) are obvious and clause ($\varepsilon$) follows from clause ($\delta$), so let us prove clause ($\delta$).

Let $Z \subseteq \lambda$ be unbounded. So $C_Z = \{ \delta < \lambda : \delta = \text{sup}(Z \cap \delta) = \text{otp}(Z \cap \delta) \}$ is a club of $\lambda$ and let $S_Z = \{ \delta \in S : Z \cap \delta = Z_\delta \}$. By the choice of $Z$ clearly $S_Z$ is a stationary subset of $\lambda$, so also $S_Z \cap C_Z$ is a stationary subset of $\lambda$. Let $S'_Z = \{ \delta \in S_Z \cap C_Z : B'_\delta = Z_\delta \}$, so it is enough to prove that $S'_Z$ is a stationary subset of $\lambda$, we shall prove more:

\[(*) S'_Z = S_Z \cap C \setminus S'_Z \text{ is not a stationary subset of } \lambda.\]

Toward contradiction assume $S'_Z$ is stationary.

Now for every $\delta \in S'_Z$, clearly $Z'_\delta$ is a subset of $Z_\delta = Z \cap \delta$ of order type $\theta$, but $B'_\delta \neq Z'_\delta$ hence $(\exists \alpha_1 \in S \cap \delta) ([Z'_\delta \cap B'_\alpha_1] \geq \sigma)$, so we choose such $\alpha_1 \in S \cap \delta$. So for some stationary $S'' \subseteq S$ and $\alpha^*$ we have $(\forall \delta \in S'' \cap \alpha^*) ([\alpha_1 \in S \cap \delta] [Z'_\delta \cap B'_\alpha_1] \geq \sigma)$. Now $\delta \in S''$ implies $\sigma \subseteq [Z'_\delta \cap B'_\alpha^*]$ hence for some $A_\delta^* \in [B'_\alpha^*]^\sigma$ we have $A_\delta^* \subseteq Z'_\delta$. As $|[B'_\alpha^*]| \leq \theta^\sigma < \lambda = \text{cf}(\lambda)$, possibly shrinking $S''$ for some $A^*$ we have $\delta \in S'' \Rightarrow A_\delta^* = A^*$.

Now easily $\delta \in S'' \Rightarrow B_\delta \supseteq \{ \text{otp}(\gamma \cap Z) : \gamma \in A^* \}$ which has cardinality $\sigma$, so $\delta \in S'' \Rightarrow \sigma \subseteq |B_\delta|$, contradiction.

Lastly, clause ($\zeta$) follows from clause ($\delta$) by 2.1 as $\lambda$ is regular or alternatively if $F : \lambda \rightarrow [\lambda]^{\leq \kappa}$, by [Ha61] some unbounded $Z \subseteq \lambda$ is $F$-free so by clause ($\delta$) there are stationarily many $\delta \in S$ such that $B'_\delta$ is $F$-free. \(\square_{2.4}\)

Proof of 2.5(2). Without loss of generality $(\forall \delta \in S)(\text{cf}(\delta) = \theta_1)$ for some $\theta_1$.

Let $\delta^* \in [\theta, \theta^+)$, $\text{cf}(\delta^*) = \theta_1$ for what we state in 2.5 we have $\delta^*$ is $\theta$ if $\theta_1 = \theta$ and is $\theta \times \theta_1$ if $\theta_1 < \theta$. Let $h : \delta^* \rightarrow \theta$ be one to one onto. Let $(\langle Z_{\delta, \varepsilon} : \varepsilon \leq \delta^* \rangle : \delta \in S)$ be such that for every sequence $\bar{Z} = \langle Z_\varepsilon : \varepsilon \leq \delta^* \rangle$ satisfying $Z_\varepsilon \subseteq \lambda$ the set \{\delta \in S : (\forall \varepsilon \leq \delta^*)(Z_\varepsilon \cap \delta = Z_{\delta, \varepsilon})\} is stationary, it exists as $\diamondsuit_S$ holds. Moreover, we can find $(C_\delta : \delta \in S)$ such that $C_\delta$ is a closed unbounded subset of order type $\delta^*$, let $C_{\delta} = \{ \gamma_{\delta, \varepsilon} : \varepsilon < \delta^* \}$ and if $Z_\varepsilon \subseteq \lambda$ for $\varepsilon < \delta^*$ then the following subset of $\delta$ is stationary

\[
\{ \delta \in S : Z_{\delta, \varepsilon} = Z_{\delta, \varepsilon} \cap \delta \text{ for } \varepsilon < \delta^* \text{ and } \text{otp}(Z_{\delta, \varepsilon} \cap \gamma_{\delta, \varepsilon}) = \gamma_{\delta, \varepsilon} \text{ for } \varepsilon < \zeta < \delta^* \text{ and } \gamma_{\delta, \varepsilon} \text{ is closed under pr (a pairing function)} \}.
\]
Let
\[ Z_\delta^* = \{ \alpha : \text{for some } \varepsilon < \delta^* \text{ and } \beta \text{ we have} \]
\[ \gamma_{\delta,\varepsilon} < \alpha < \gamma_{\delta,\varepsilon+1}, \beta \in A_\delta, \otp(\beta \cap A_\delta) = h(\varepsilon) \]
and \( \alpha \) is the \( \text{pr}(\gamma_{\delta,\varepsilon}, \beta) \)-th member of \( Z_{\delta,\varepsilon}^* \).

Now check. \( \square_{2.5} \)

### 2.6 Conclusion: If \( \text{CON}(\text{ZFC} + \exists \text{ supercompact}) \), then \( \text{CON}(\text{CH} + \text{ there is a } T_3\text{-topological space } X \text{ with clopen basis, even compact, with } \aleph_{\omega+1} \text{ nodes such that if we divide } X \text{ to countably many parts, at least one contains a closed copy of the Cantor discontinuum, } \omega^2) \).

**Proof.** By 2.2 + 2.3 we get a universe with \( \text{GCH} \) and \( \sigma = \aleph_0, \theta = \aleph_1, \kappa = \aleph_2, \lambda = \aleph_{\omega+1} \) satisfying the set theoretic requirements of 1.2. So as the Cantor discontinuum satisfies clauses (D), (E) of 1.2 and the demand in 1.2(2) we are done by 1.2. \( \square_{2.6} \)

* * *

Lastly, we start to resolve the connection between the various statements around. Now \([\text{HJSh 249}]\) continue and strengthen \([\text{Sh 108}], [\text{Sh 88a}]\) (and \([\text{HJSh 697}]\) continue them). We show that by a “small nice forcing” (not involving extra large cardinals assumption) we can get the result of \([\text{HJSh 249}]\) used above from the one in \([\text{Sh 108}], [\text{Sh 88a}]\). (See also \([\text{Sh 652, §5}]\) on the semi-additive colouring involved, i.e. it is proved that consistently there is a colouring of the kind appearing in the analysis (there, or see the proof of 2.7 below)). On \( I[\lambda] \) see \([\text{Sh 108}], [\text{Sh 88a}], [\text{Sh 420, §1}]\). However, there is a price, our “small nice forcing” has to violate \( \text{G.C.H.} \) quite strongly.

### 2.7 Claim. Assume

(a) \( \text{cf}(\mu) = \kappa < \mu \) and \( (\forall \alpha < \mu)(|\alpha|^{\kappa} < \mu) \) and \( \lambda = \mu^+ \)

(b) \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa^+ \} \) is stationary, \( S \notin I[\lambda] \) and

(c) \( 2^{\kappa^+} < \lambda \) and \( \kappa = \kappa^{<\kappa} \).

**Then for some forcing notion \( \mathbb{Q} \) we have:**

(\( \alpha \)) \( \mathbb{Q} \) is \( (\prec \kappa)\)-complete, \( |\mathbb{Q}| = \kappa^+ \) and \( \mathbb{Q} \) is \( \kappa^+-\text{c.c.} \)

(\( \beta \)) in \( \text{V}^{\mathbb{Q}} \), for some stationary \( S' \subseteq S \) there is a sequence \( \langle A_\delta : \delta \in S' \rangle \) such that each \( A_\delta \) is an unbounded subset of \( \delta \) of order type \( \kappa^+ \) and \( \delta_1 \neq \delta_2 \in S' \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \kappa \).
Proof. Let \( \mu = \sum_{i < \kappa} \lambda_i \) where \( \lambda_i < \mu \) is increasing continuous with \( i, \lambda_0 > \kappa \). Choose \( \bar{A} = \langle A_\alpha : \alpha \in S \rangle \), with \( A_\alpha = \{ \gamma_{\alpha, \varepsilon} : \varepsilon < \kappa^+ \} \) being any unbounded subset of \( \alpha \) of order type \( \kappa^+ \) and \( \gamma_{\alpha, \varepsilon} \) increasing with \( \varepsilon \).

We can find \( \bar{a}^\alpha = \langle a_i^\alpha : i < \kappa \rangle \) for \( \alpha < \mu^+ \) such that

\[(*)_1 \ \alpha = \bigcup_{i < \kappa} a_i^\alpha, a_i^\alpha \text{ is increasing continuous in } i, |a_i^\alpha| \leq \lambda_i \]

\[(*)_2 \text{ if } \alpha \in a_i^\beta \text{ then } a_i^\alpha \subseteq a_i^\beta.\]

Without loss of generality

\[(*)_3 \ A_\alpha \subseteq a_0^\alpha.\]

Let \( c : [\mu^+]^2 \to \kappa \) be \( c\{\alpha, \beta\} = \min\{i : \alpha \in a_i^\beta\} \) for \( \alpha < \beta < \lambda^+ \) so

\[\forall \alpha < \beta < \gamma \Rightarrow c\{\alpha, \gamma\} \leq \max\{c\{\alpha, \beta\}, c\{\beta, \gamma\}\}.\]

For \( \alpha \in S \) let \( c_\alpha : [\kappa^+]^2 \to \kappa \) be defined by:

for \( \varepsilon < \zeta < \kappa^+ \) we let

\[c_\alpha\{\varepsilon, \zeta\} = c\{\gamma_{\alpha, \varepsilon}, \gamma_{\alpha, \zeta}\}.\]

Let \( \mathcal{C} = \{ c_\alpha : \alpha \in S \} \) so \( c_\alpha \in ([\kappa^+]^2)\kappa \), hence \( |\mathcal{C}| \leq 2^{\kappa^+} \). Let for \( c \in \mathcal{C}, S_c = \{ \alpha \in S : c_\alpha = c \} \), so \( \langle S_c : c \in \mathcal{C} \rangle \) is a partition of \( S \) to \( \leq 2^{\kappa^+} < \mu^+ \) sets hence necessarily for some \( c_* \in \mathcal{C} \) we have

\[(*)_4 \ S_{c_*} \notin I[\lambda] \text{ and in particular is stationary.}\]

We fix \( c_* \). We define a forcing notion \( Q \):

\[(A) \ p \in Q \text{ iff } p = (u^p, \xi^p) \text{ where } u^p \in [\kappa^+]^{<\kappa} \text{ and } \xi^p < \kappa \text{ and } \text{Rang}(c_* \upharpoonright [u^p]^2) \subseteq \xi^p \]

\[(B) \ Q \models p \leq q \text{ iff: } (p, q \in Q \text{ and }) \]

\[(i) \ u^p \subseteq u^q \]

\[(ii) \ \xi^p \leq \xi^q \]

\[(iii) \text{ for every } \beta \in u^p \text{ and } \alpha \in (u^q \setminus u^p) \cap \beta \text{ we have } c_*\{\alpha, \beta\} \geq \xi^p.\]
Now

\(\{\star\}_5\)

(a) \(\mathbb{Q}\) is a \((< \kappa)\)-complete partial order of cardinality \(\kappa^+\)

(b) \(\mathbb{Q}' = \{p \in \mathbb{Q} : u^p\) has a maximal element\}\) is a dense subset of \(\mathbb{Q}'\)

(c) if \(\alpha < \kappa^+\) then \(\mathbb{Q}'_\alpha = \{p \in \mathbb{Q} : u^p\) has a maximal element and 
\[
\max(u') > \alpha
\] is a dense subset of \(\mathbb{Q}'\).

[Why? As \(\kappa = \kappa^+\) clearly |\(\mathbb{Q}\)| = \(\kappa^+\) and \(\mathbb{Q}\) is closed under union of length \(< \kappa\), together we have Clause (a), as for clause (b), for any \(p \in \mathbb{Q}\) choose \(j \in (\sup(u^p) + 1, \kappa^+)\) and define \(q = (u^q, \xi^q)\) by \(u^q = u^p \cup \{j\}\) and \(\xi^q = \sup\{\xi^p\} \cup \text{Rang}(c, [u^q]^2)\) + 1 clearly \(p \leq q \in \mathbb{Q}\) (clause (iii) of (B) is empty) and \(u^q\) has a last member \(j\). Clause (c) has the same proof except that we choose \(j > \alpha\)]

\(\{\star\}_6\)

\(\mathbb{Q}\) satisfies the \(\kappa^+-\text{c.c.}\).

[Why? Assume toward contradiction that \(\langle p_i : i < \kappa^+\rangle\) are pairwise incompatible. Without loss of generality \(p_i \in \mathbb{Q}'\). As \(\kappa = \kappa^+\) without loss of generality \(\langle u^p_i : i < \kappa^+\rangle\) is a \(\Delta\)-system with heart \(u^*\). Also without loss of generality \(\xi^{p_i} = \xi^*\). So \(C = \{\delta < \kappa^+ : u^{p_\alpha} \setminus u^*\) is disjoint to \(\delta\) and \(\langle \forall j < \delta (u^{p_j} \subseteq \delta)\}\} is a club of \(\kappa^+\). For \(\delta \in C\) let \(\varepsilon_\delta = \text{Min}(u^{p_\delta} \setminus \delta)\) and \(\zeta_\delta = \max(u^{p_\delta})\) so \(\delta \leq \varepsilon_\delta \leq \zeta_\delta\). Now assume \(\alpha < \beta\) are from \(C\), and \(p_\alpha, p_\beta\) is incompatible. Why is \(q = (u^{p_\alpha} \cup u^{p_\beta}, \xi)\) not a common upper bound where we let \(\xi = \sup(\{\xi^*\} \cup \text{Rang}(c, [u^{p_\alpha} \cup u^{p_\beta}]^2)) + 1\)? As \(q \in \mathbb{Q}\) and as \(u^{p_\alpha} \cap \alpha = u^{p_\beta} \cap \beta, u^{p_\alpha} \subseteq \beta\) and \(\xi^* = \xi^{p_\alpha} = \xi^{p_\beta} \leq \xi^q\) clearly \(p_\alpha \leq q\), hence necessarily \(\neg(p_\beta \leq q)\) so clause (iii) of (B) fails, i.e. for some \(\gamma_2 \in u^{p_\beta}\) and \(\gamma_1 \in u^q \cap \gamma_2 \setminus u^{p_\beta}\) (hence \(\gamma_1 \in u^{p_\alpha} \setminus \alpha\) and \(\gamma_2 \in u^{p_\beta} \setminus \beta\) we have \(c_\varepsilon(\gamma_1, \gamma_2) < \xi^{p_\beta} = \xi^*\). But \(\varepsilon_\alpha \leq \gamma_1\) and \(\varepsilon_\alpha < \gamma_1 \Rightarrow c_\varepsilon(\varepsilon_\alpha, \gamma_1) \leq \xi^{p_\alpha} = \xi^*\) and \(\gamma_2 \leq \zeta_\beta\) and \(\gamma_2 < \zeta_\beta \Rightarrow c_\varepsilon(\gamma_2, \zeta_\beta) \leq \xi^{p_\beta} = \xi^*\). Hence by \(\Xi\) above necessarily \(c_\varepsilon(\varepsilon_\alpha, \zeta_\beta) < \xi^*\).

So for \(\delta \in S_{c_\varepsilon}, \langle \gamma_{\delta, \varepsilon} : i < \delta \rangle\) is strictly increasing hence with limit \(\delta\) and for each \(i \in C\), \(\gamma_{\delta, \xi_i}\) is above \(\{\gamma_{\delta, \xi} : j < i\}\) but \(< \delta\) and

\[
\gamma_{\delta, \xi_i} < \xi^* \Rightarrow \gamma_{\delta, \xi} \in a_{\xi^*}^{\gamma_{\delta, \xi}}.
\]

By [Sh 108] it follows that \(S \in I[\lambda]\) (or directly, for every \(\gamma < \lambda\), \(|\{\gamma_{\delta, \xi} : j \in C \cap i^* : \delta \in S, i^* \in C, \gamma_{\delta, \xi} = \gamma\}| < \lambda\) as for each \(i < \kappa^+\) (and \(\gamma\)) we have \(|a_{\xi^*}^{\gamma}||i^*| \leq (\lambda_{\xi^*})^{|i^*|} \leq \mu\) possibilities); contradiction to \(\{\star\}_4\). So \(\mathbb{Q}\) satisfies the \(\kappa^+-\text{c.c.}\)]

Now clearly for every \(i < \kappa^+\) there is \(p_i \in \mathbb{Q}'\) such that \(i < \max(u^{p_i})\), hence (by \(\{\star\}_6\)), for some \(i^* < \kappa^+\) we have \(p_{i^*} \models Q \ "W_1 = \{i : p_i \in G \land \text{cf}(i) = \kappa\}"\) is stationary in \(\kappa^+\). Let \(p_{i^*} \in G \subseteq \mathbb{Q}\) with \(G\) generic over \(V\) and \(W_1 = W_1[G]\). Let
$C = \{ \delta < \kappa^+: (\forall i < \delta)[\sup(u^p_i) < \delta]\}$, it is a club of $\kappa^+$. Let $W_2 = C \cap W_1$ and for $i \in S_2$ let $\varepsilon_i = \text{Min}(u^p_i \setminus i), \zeta_i = \max(u^p_i)$. Now

\[ (*_7) \text{ if } i \in W_2 \text{ and } \xi < \kappa, \text{ then } \{ j \in W_1 \cap i : c_\ast(\varepsilon_j, \varepsilon_i) < \xi \} \text{ has cardinality } < \kappa. \]

[Why? By density argument for some $q \in G$ we have $p_i \leq q$ and $\xi^q > \xi$. Now if $j \in W_1 \cap i \setminus u^q$ then $p_j \in G$ hence for some $q^+ \in G \subseteq \mathcal{Q}$ we have $q \leq q^+ \& p_j \leq q^+$, so $\varepsilon_j \in u^{q^+} \cap \varepsilon_i$ and as $q \leq q^+$ by the definition of $\leq^\mathcal{Q}$, necessarily $c_\ast(\varepsilon_i, \varepsilon_j) \geq \xi^q > \xi$, as asserted.] Now for $\delta \in S_\ast$ define $A'_\delta = \{ \gamma_{\delta, \varepsilon} : \varepsilon \in W_2 \}$.

\[ (*_8) \text{ if } \delta_1 \neq \delta_2 \text{ are from } S_\ast, \text{ then } A'_{\delta_1} \cap A'_{\delta_2} \text{ has cardinality } < \kappa. \]

[Why? Without loss of generality $\delta_1 < \delta_2$, let $\varepsilon(\ast) \in S_2$ be such that $\delta_1 < \gamma_{\delta_2, \varepsilon(\ast)}$. Assume toward contradiction that $A = A'_{\delta_1} \cap A'_{\delta_2}$ has cardinality $\geq \kappa$. Recall (by $(*_3)$) that $\beta \in A \Rightarrow c(\beta, \delta_1) = 0$; now letting $\xi^* = c(\delta_1, \gamma_{\delta_2, \varepsilon(\ast)}) < \kappa$ we get by $\mathbb{V}$ that $\beta \in A \Rightarrow c(\beta, \gamma_{\delta_2, \varepsilon(\ast)}) \leq \max\{c(\beta, \delta_1), c(\delta_1, \gamma_{\delta_1, \varepsilon(\ast)})\} = \max\{0, \xi^*\} = \xi^*$. So $A^- = \{ \varepsilon < \kappa^+ : \gamma_{\delta_2, \varepsilon} \in A \}$ has cardinality $\kappa$ and $\varepsilon \in A^- \Rightarrow c_\ast(\varepsilon, \varepsilon(\ast)) \leq \xi^*, \text{ contradicting } (*_7)\].

So we are done. $\square_{2.7}$

2.8 Claim. Assume

(A)\,(i) $\lambda > \kappa > \theta > \sigma \geq \aleph_0$ and $\kappa = \kappa^\lambda$

(B)\,$_1$ $\mathcal{A} \subseteq |\lambda|^\theta$ and $A_1 \neq A_2 \in \mathcal{A} \Rightarrow |A_1 \cap A_2| < \sigma$.

Then for some forcing notion $\mathcal{Q}$ and $\mathcal{Q}$-name $\mathcal{A}'$ of a subset of $\mathcal{A}$ we have:

(a) $\mathcal{Q}$ is a strategically ($< \kappa$)-complete forcing notion (hence add no new sequence of length < $\kappa$)

(b) $\mathcal{Q}$ is $\kappa^+$-c.c. forcing notion of cardinality $\lambda^{< \kappa}$

(c) in $\mathbb{V}^\mathcal{Q}$, clauses (A)i, (B)$_1$ above still hold for $\mathcal{A}$ hence for $\mathcal{A}'$ and $\mathcal{A}'$ satisfies also (B)$_2$ from 1.2, i.e. $\mathcal{A}$ is ($< \kappa$)-free

(d) if $\lambda, \kappa, \mathcal{A}$ satisfies clause (C) of 1.2 in $\mathbb{V}$, then $\lambda, \kappa, \mathcal{A}$ satisfies clause (C) in $\mathbb{V}^\mathcal{Q}$

(e) like clause (d) for (C)* from 1.4
Proof. Let $\mathcal{A} = \{A_\zeta : \zeta < \lambda^*\}$ with no repetitions.
Let $\mathcal{Q}$ be the set of $p = (v, v_\ast) = (v^p, v^\ast_\ast)$ such that:

(a) $v_\ast \subseteq v \in [\lambda^*]^{<\kappa}$

(b) there is a list $\bar{\zeta} = \langle \zeta(\varepsilon) : \varepsilon < \varepsilon^* \rangle$ of $v_\ast$ such that for every $\varepsilon < \varepsilon^*$ we have

$A_{\zeta(\varepsilon)} \cap \bigcup_{\xi < \varepsilon} A_{\zeta(\xi)}$ has cardinality $< \sigma$; we call $\langle \zeta(\varepsilon) : \varepsilon < \varepsilon^* \rangle$ a witness, also

the list $\bar{\zeta}$ and the the well ordering on $v^\ast_\ast$ it induces are called witnesses.

The order is defined by

\[ p \leq q \iff (\alpha) \quad v^p_\ast \subseteq v^q_\ast \quad \text{and} \quad \]
\[ v^p \setminus v^p_\ast \subseteq v^q \setminus v^q_\ast \quad (\beta) \]
\[ \text{every } \bar{\zeta} \text{ witnessing } p \in \mathcal{Q} \text{ can be end-extended to } \bar{\zeta}' \text{ witnessing } q \in \mathcal{Q}. \quad (\gamma) \]

Define $\mathcal{Q}$-names $Y = \bigcup \{v^p_\ast : p \in G_\mathcal{Q}\}$ and $\mathcal{A}' = \{A_\zeta : \zeta \in Y\}$. Now

(\ast)_1 $\mathcal{Q}$ is a partial order

(\ast)_2 $|\mathcal{Q}| = (\lambda^*)^{<\kappa} \leq (\lambda^{<\beta})^{<\kappa} = \lambda^{<\kappa}$

(\ast)_3 any increasing continuous sequence of members of $\mathcal{Q}$ of length $< \kappa$ has a least upper bound.

Hence

(\ast)_4 $\mathcal{Q}$ is strategically $(< \kappa)$-complete.

For $p \in \mathcal{Q}$ let $u^p = \bigcup \{A_\zeta : \zeta \in v^p\}$

(\ast)_5 for $p \in \mathcal{Q}$ we have $u^p \in [\lambda]^{<\kappa}$ and $p \leq q \Rightarrow u^p \subseteq u^q$.

Let $\mathcal{Q}' = \{p \in \mathcal{Q} : \text{if } \zeta < \lambda^* \text{ and } |A_\zeta \cap u^p| \geq \sigma \text{ then } \zeta \in v^p\}$; compare with the proof of 1.7. For $p \in \mathcal{Q}$ let $v^p_\ast = \{\zeta < \lambda^* : |A_\zeta \cap u^p| \geq \sigma\}$, so:

(\ast)_6 (a) $v^p \subseteq v^p_\ast$ and $p \in \mathcal{Q} \Rightarrow |v^p_\ast| \leq \kappa$ and

(b) if $(\forall \alpha < \kappa)[|\alpha|^\sigma < \kappa]$ then $p \in \mathcal{Q} \Rightarrow |v^p_\ast| < \kappa$, and

(c) if $p \in \mathcal{Q}$ then $(p \in \mathcal{Q'}) \Rightarrow (v^p_\ast = v^p)$ and

(d) $\mathcal{Q}'$ is a dense subset of $\mathcal{Q}$ if $(\forall \alpha < \kappa)[|\alpha|^\sigma < \kappa]$

[Why? E.g. for clause (d), let $p \in \mathcal{Q}$ we choose by induction on $\varepsilon \leq \sigma^+ (< \kappa)$

a condition $p_\varepsilon$ such that: $p_0 = p, v^p_\ast = v^p_\varepsilon, p_\varepsilon$ is increasing continuous with $\varepsilon$
and \( v^{p+1} = \{ \zeta < \lambda^+ : \zeta \in v^p \text{ or just } |A_\zeta \cap u^p| \geq \sigma \} \). There are no problems and \( p_{\sigma^+} \) as required as \( |A_\zeta \cup u^{p+1}| \geq \sigma \Rightarrow \) for some \( \varepsilon < \sigma^+\), \( |A_\zeta \cup u^{p+1}| \geq \sigma \Rightarrow \) for some \( \varepsilon < \sigma^+, \zeta \in v^{p+1} \subseteq u^{p+1} \).

\[ (\ast)_7 \text{ if } p \in \mathcal{Q}' \), \( \zeta \in \lambda^+ \setminus v^p \) or just \( p \in \mathcal{Q}, \zeta \in \lambda^+ \setminus v^{p_0} \) then \( p' = (v^p \cup \{ \zeta \}, v_p^p \cup \{ \zeta \}) \) and \( p'' = (v^p \cup \{ \zeta \}, v_p^p) \) are in \( \mathcal{Q} \) (even \( p \in \mathcal{Q}' \Rightarrow p' \in \mathcal{Q}' \)) and are \( \geq p \).

We say \( p_0, p_1 \in \mathcal{Q} \) are isomorphic if \( \text{otp}(v^{p_0}) = \text{otp}(v^{p_1}) \), \( \text{otp}(v^p) = \text{otp}(v^{p_1}) \), and \( \text{OP}_{v^p_1, u^p_0} \) maps \( v^p_0 \) onto \( v^p_1 \), \( \text{OP}_{u^p_1, u^p_0} \) maps \( u^p_0 \) onto \( u^p_1 \) and for \( \zeta \in v^p_0, \alpha \in u^p_0 \) we have \( \alpha \in A_\zeta \iff \text{OP}_{u^p_1, u^p_0}(\alpha) \in A_{\text{OP}_{v^p_1, v^p_0}(\zeta)} \).

\[ (\ast)_8 \mathcal{Q} \text{ satisfies the } \kappa^+ \text{-c.c.} \]

[Why? Let \( p_\alpha \in \mathcal{Q} \) for \( \alpha < \kappa^+ \). Let \( v_\alpha = \bigcup_{\beta < \alpha} v_\beta^p \) and \( u_\alpha = \bigcup \{ A_\zeta : \zeta \in v_\alpha \} \) so \( v_\beta^p \subseteq u_\alpha \) for \( \beta < \alpha \) and \( \{ v_\alpha : \alpha < \kappa^+ \}, \{ u_\alpha : \alpha < \kappa^+ \} \) are increasing continuous. As \( v_\alpha^p \in [\lambda]^\leq \kappa \), we can find stationary \( S \subseteq \{ \delta < \kappa^+ : \text{cf}(\delta) = \kappa \} \) and \( v \) such that \( \alpha \in S \Rightarrow v_\alpha^p \cap v^p = v \). Similarly without loss of generality \( \alpha \in S \Rightarrow u_\alpha \cap u = u \). Without loss of generality for \( \alpha, \beta \in S \) the conditions \( p_\alpha, p_\beta \) are isomorphic, the isomorphisms being the identity \( v \) and \( u_\alpha \cap u = v_\alpha \) for some \( \alpha, \beta \leq \alpha \leq \beta \).

(Why? If not by the isomorphism of \( p_\alpha \) and \( p_\beta \) we can find \( \zeta_1 \in v^{p_\alpha} \setminus v^{p_\beta}, \zeta_2 \in v^{p_\beta} \setminus v^{p_\alpha} \) such that \( \zeta_2 = \text{OP}_{v^{p_\beta}, v^{p_\alpha}}(\zeta_1) \) and \( \zeta \in \{ \zeta_1, \zeta_2 \} \) and \( A_{\zeta_1} \cap u = A_{\zeta_2} \cap u \), so \( |A_{\zeta_1} \cap u| \geq |A_{\zeta_1} \cap A_{\zeta_2}| \geq \sigma \) hence \( \zeta_2 \in v^{p_{\zeta_1}} \) hence \( \zeta_2 \in v \) so \( \zeta_2 \in v^{p_\beta} \setminus v^{p_\alpha} \) hence \( \zeta_1 = \zeta_2 \in v^{p_\beta} \cap v^{p_\beta} \) contradiction.

Hence \( \zeta \in v^{p_\beta} \setminus v^{p_\beta} \Rightarrow |A_{\zeta} \cap u| < \sigma \) (otherwise \( A_{\zeta} \cap u \subseteq u_\beta \cap u^{p_\beta} = u \) hence \( |A_{\zeta} \cap u| \geq \sigma \) and get a contradiction by the previous statement) and \( \zeta \in v^{p_\beta} \setminus v^{p_\alpha} \Rightarrow |A_{\zeta} \cap v^{p_\alpha}| < \sigma \) (similar proof). Now define a two-place relation \( <^* \) on \( v^p \):

\[ \zeta_1 <^* \zeta_2 \iff \zeta_1 <_\alpha \zeta_2 \text{ (so } \zeta_1, \zeta_2 \in v^{p_\alpha}) \]

or \( \zeta_1 \in v^{p_\alpha} \& \zeta_2 \in v^{p_\beta} \setminus v^{p_\alpha} \]

or \( \{ \zeta_1, \zeta_2 \} \subseteq v^{p_\beta} \setminus v^{p_\alpha} \& \zeta_1 <_\beta \zeta_2 \)

Easily \( <^* \) is a well order of \( v^p \) (as \( \zeta \in v^{p_\beta} \setminus v^{p_\alpha} \Rightarrow |A_{\zeta} \cap u^{p_\alpha}| < \sigma \), and it is a witness. So \( q \in \mathcal{Q} \). Does \( p_\alpha \leq q \)? Clauses (\( \alpha \), (\( \beta \)) are very straight and for clause (\( \gamma \)), as \( p_\alpha, p_\beta \) are isomorphic for any given witness \( <^1 \), a well ordering of \( v_\alpha^p \), we can find \( <^2 \), a witness for \( p_\beta \) which is a well ordering of \( v_\beta^p \), and is conjugate to \( <^1 \); now use \( <^1, <^2 \) as we use \( <^*, <^\beta \) above. So really \( p_\alpha \leq q \). Similarly \( p_\beta \leq q \).]
(\ast)_9 \models_{\mathbb{Q}} "\mathcal{A}' = \{ A_{\zeta} : \zeta \in \bigcup \{ v^p_\mathbb{Q} : p \in G_{\mathbb{Q}} \} \} is \ (< \kappa )\text{-free}".$

[Why? Read the definitions of $\mathbb{Q}$ and of being \ (< \kappa )\text{-free}, remembering that forcing with $\mathbb{Q}$ add no new sets of ordinals \ (< \kappa \) as it is strategically \ (< \kappa )\text{-complete}.

\begin{align*}
(\ast)_{10} & \text{ if } p, q \in \mathbb{Q} \text{ are compatible, then they have an upper bound } r \in \mathbb{Q} \text{ such that } \\
v^r = v^p \cup v^q
\end{align*}

(\ast)_{11} \text{ if } \mathcal{A} \text{ satisfies clause (C) of 1.2 then } \mathcal{A}' \text{ satisfies it in } V^\mathbb{Q}.$

[Why? Assume $p^* \in \mathbb{Q}, p^* \models_{\mathbb{Q}} "F : \lambda \to [\lambda]^{< \kappa} \text{ is a counterexample}". As $\mathbb{Q}$ satisfies the $\kappa^+\text{-c.c.} \text{ and as increasing the } F(\alpha) \text{ is O.K., without loss of generality each } F(\alpha) \text{ is an object from } V \text{ so for some function } F : \lambda \to [\lambda]^{< \kappa} \text{ from } V \text{ we have } F = F. \text{ As we can increase each } F(\alpha), \text{ without loss of generality } \zeta \in v^{p^*} \Rightarrow A_{\zeta} \subseteq \bigcap \alpha F(\alpha). \text{ As } V, \mathcal{A} \text{ satisfies clause (C) there are } \zeta \text{ and } A \in [A_{\zeta}]^{\theta} \text{ which is } F\text{-free, by the previous sentence } \zeta \notin v^{p^*}. \text{ Define } q = (v^q, v^q_\mathbb{Q}), v^q = v^{p^*} \cup \{ \zeta \}, v^q_* = v^{p^*} \cup \{ \zeta \}. \text{ It is easy to prove } p^* \leq q \in \mathbb{Q}, \text{ the point being } |A_{\zeta} \cap \bigcup \{ A_{\xi} : \xi \in v^{p^*}_\mathbb{Q} \}| < \sigma \text{ which holds as } \zeta \notin v^{p^*}_\mathbb{Q}, \text{ and } q \text{ forces that } A \in [A_{\zeta}]^{\theta} \text{ is as required concerning } F.\] 

\[\square_{2.8}\]

\[2.9 \text{ Observation. Assume that } \kappa = \kappa^{< \kappa} < \lambda \text{ and } S \subseteq \lambda \text{ stationary. Then for some } \kappa^+\text{-c.c.}, \text{ strategically } \kappa\text{-complete forcing notion } \mathbb{Q} \text{ of cardinality } \lambda^{< \kappa}, \text{ we have } \models_{\mathbb{Q}} "S \text{ is the union of } \leq \kappa \text{ sets each not reflecting any } \delta \text{ of cofinality } < \kappa".\]

Proof. Straightforward. [Used in (C) \Rightarrow (D) of the proof of 3.2 below.]

\[\square_{2.9}\]

So putting together the claims above we can conclude, e.g.

\[2.10 \text{ Conclusion If } (\ast) \text{ below holds, then there is a forcing notion } \mathbb{P} \text{ of cardinality } 2^\mu = \mu^+ \text{ not adding sequences of length } < \kappa, \text{ not collapsing cardinals } \leq \mu^+ \text{ (or } > 2^\mu), \text{ not changing cofinalities such that in } V[\mathbb{P}] \text{ the cardinals } (\sigma < \theta < \kappa = \kappa^{< \kappa}, 2^\kappa \leq \mu) \text{ satisfies the assumption of 1.2; also its conclusion and } (C)^* \text{ of 1.7 where }\]

\[\ast) \quad \sigma = \text{cf}(\sigma), \theta = \sigma^+ < \kappa = \kappa^{< \kappa} < \mu, \mu \text{ strong limit singular of cofinality } \sigma \text{ such that } \{ \delta < \mu^+ : \text{cf}(\delta) = \sigma^+ \} \notin I[\lambda].\]
§3 Equi-consistency

Let $\omega^2$ denote here the Cantor discontinuum.
The following theorem clarifies the consistency strength of the problem to a large extent. We can hardly expect a stronger kind of result as long as inner models for supercompacts have not been discovered. Concentrating on $\omega^2$ is for historical reason; we can replace $\aleph_0$ by $\mu$. Also, using the same claims we can replace $\lambda > \beth_2$ by other restrictions. Note that 3.7 continues [Sh 460, §3], [HJSh 249]. The claims will give more, naturally. However, a real problem is:

3.1 Problem: What occurs if we demand GCH?

3.2 Theorem. The following are equi-consistent with ZFC + $\kappa = \text{cf}(\kappa) > 2^{\aleph_0}$.

(In fact we get more than equiconsistency: the model for one statement is gotten from another by (set) forcing. Moreover, the forcing notions we use are from a very restricted family where $\kappa$ is involved in its definition. We use only forcing notions which preserves the cardinals and cofinalities $\leq (2^{\aleph_0})^+$ and even $\leq \kappa$ and do not change the value of $2^{\aleph_0}$, in fact finite composition of $\kappa$-complete ones and c.c.c. of cardinality $\leq 2^{\aleph_0}$ ones; so we can add $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} = \aleph_2$ or $2^{\aleph_0} = \aleph_{\omega^3 + \omega + 3}$ or whatever, to all clauses simultaneously) letting $\mu = 2^{<\kappa}$:

(A) $[\omega^2] = (A)(\omega^2)$ there is a compact Hausdorff space $X$ such that $X \rightarrow_{w} (\omega^2)^1_2$ but no subspace with $\leq \mu$ points has this property

(B) $[\omega^2] = (B)(\omega^2)$ there is a compact Hausdorff space $X$ with clopen basis such that $X \rightarrow_{w} (\omega^2)^1_{< \text{cf}(2^{\aleph_0})}$ but no subspace with $\leq \mu$ points has this property

(C) there are $\lambda, S, \bar{f}$ such that

(a) $S \subseteq \lambda$ is stationary, $\lambda > \mu$ is regular
(b) $\bar{f} = \langle f_\delta : \delta \in S \rangle$
(c) $f_\delta$ is a one-to-one function from $A \subseteq \omega^2$ of cardinality $2^{\aleph_0}$ to $\delta$
(d) if $\delta_1 \neq \delta_2$ then $\{ \eta \in \omega^2 : f_{\delta_1}(\eta) = f_{\delta_2}(\eta) \}$ has scattered closure (in the topological space $\omega^2$)

\[^6\text{of course, if, e.g., } \kappa = (2^{\aleph_0})^+ \text{ this holds.}\]
(D) there are $\lambda, S, \bar{A}$ such that

(a) $S \subseteq \lambda$ is stationary, $\lambda > \mu$ is regular
(b) $\bar{A} = \langle A_\delta : \delta \in S \rangle$
(c) $A_\delta$ is a subset of $\delta$ of cardinality $2^{\aleph_0}$
(d) for $\delta_1 \neq \delta_2$ from $S$ we have $A_{\delta_2} \cap A_{\delta_1}$ is finite
(e) $\{ A_\delta : \delta \in S \}$ is $\kappa$-free, that is, for any $u \in [S]^{<\kappa}$ there is a sequence $\langle B_\delta : \delta \in u \rangle$ such that $B_\delta \in [A_\delta]^{<\aleph_0}$ and $\langle A_\delta \setminus B_\delta : \delta \in u \rangle$ are pairwise disjoint
(f) if $F : \lambda \to [\lambda]^{<\kappa}$ then for some $\delta \in S$ the set $A_\delta$ is $F$-free.

3.3 Remark. 1) Note that we can easily add clauses sandwiched between two existing ones. We can also add the parallel statement on $[Y]^{<\cf(2^{\aleph_0})}$, see 3.12, 3.13, 3.8.
2) We can add the case of regular spaces (i.e. $T_3$) or work as in 1.7.
3) Clearly most of the proof of most arrows in the proof (of 3.2) have little to do with the properties of the topological space $\omega^2$; still mainly 3.14 does, so

3.4 Question: With what can we replace the space $\omega^2$ (but see 4.17(2))? We make some definitions and prove some claims before proving 3.2. One of them (3.13) depends on §4, also 3.7, 3.8 which are not explicitly needed. The following definition is used in 3.7. To see the point of this definition look at Example 3.6 below and part (2) of the definition.

3.5 Definition. 1) For a cardinal $\kappa$ and $I_0, I_1$ such that $I_\ell \subseteq \{(a, b) : a, b \subseteq \kappa$ are disjoint\} (normally $\kappa = \cup\{ a \cup b : (a, b) \in I_0 \cup I_1 \}$) so we may forget to mention $\kappa$) and cardinal $\theta$ we say that a cardinal $\lambda$ is $(I_0, I_1, \theta)$-approximate or $(\kappa, I_0, I_1, \theta)$-approximate if we can find $\bar{P} = \langle P_\alpha : \alpha \in C \rangle$ such that

(i) $C$ a club of $\lambda$
(ii) $P_\alpha \subseteq [\alpha]^{<\theta}$ for $\alpha \in C$ and $|P_\alpha| \leq \Min(C \setminus (\alpha + 1))$
(iii) for any 1-to-1 function $f$ from $\kappa$ to $\lambda$, for some $\alpha \in C$ at least one of the following holds

(a) for some $c \in P_\alpha$ and $(a, b) \in I_1$ we have $\langle \forall i \in a \rangle (f(i) \in c)$ and $\langle \forall i \in b \rangle [f(i) \geq \alpha]$
(b) for some $(a, b) \in I_0$ we have

(a) $\langle \forall i < \kappa \rangle (f(i) < \alpha \rightarrow i \in a)$
(b) $\langle \forall i < \kappa \rangle [i \in b \rightarrow \alpha < f(i) < \Min(C \setminus (\alpha + 1))]$. |
2) If \( \ell \) is a function from \( \mathcal{P}(\kappa) \) to \( \mathcal{P}(\kappa) \) and \( K \subseteq \mathcal{P}(\kappa) \) and

\[
I_1[\kappa, \ell, K] = I_1[\ell, K] = I_1 = \{(a, b) : a \subseteq \kappa, b \in K \text{ and } b \subseteq \ell(a)\}
\]

\[
I_0[\kappa, \ell, K] = I_0[\ell, K] = I_0 = \{(a, b) : a \subseteq \kappa, b \in K \text{ and } a \cap b = \emptyset\}
\]

then we may say \( \lambda \) is \((K, \ell, \theta)\)-approximate or \((\kappa, K, \ell, \theta)\)-approximate instead of \( \lambda \) is \((I_0, I_1, \theta)\)-approximate.

3) We may replace \( \kappa \) by another set of this kind call the domain of the tuple (understood from \( I_0, I_1 \)). We may write this set before \( I_0, i.e. \) in the place of \( \kappa \) for clarification.

4) We may replace \((I_0, I_1, \theta)\) by \((I, \theta)\) if \( I \) is a set of pairs \((I_0, I_1)\) such that \( \langle \mathcal{P}_\alpha : \alpha \in C \rangle \) satisfies the requirement above for all the triples \((I_0, I_1, \theta)\) such that \((I_0, I_1) \in I \) (not necessarily all pairs have the same domain \( A \)). Similarly, \( K \) stands for a set of tuples \((\kappa, K, \ell, \theta)\) or in short \((\kappa, K, \ell)\) when \( \theta \) is understood from the context or even \((K, \ell)\) as in part (2). (We may even vary \( \theta \)).

Concerning 4.4 below

3.6 Examples: 1) Let \( C \) be a Cantor set (say \( \omega_2 \)), \( c^C \) is the (topological) closure operation on subsets of \( C \)

\[ K^C = \{A \subseteq C : A \text{ is closed perfect hence uncountable}\} \]

and \( I^C_\ell = I_\ell[C, c^C, K^C] \) for \( \ell = 0, 1 \); see Definition 3.5(2).

2) Let \( R \) be the real line, \( c^R \) be the (topological) closure operation on subsets of \( R \)

\[ K^R = \{A \subseteq R : A \text{ is closed perfect uncountable, bounded (from below and above)}\} \]

and \( I^R_\ell = I_\ell(R, c^R, K^R) \) for \( \ell = 0, 1 \).

3.7 Lemma. Assume

(a) \( \lambda > \chi \geq \kappa \geq \theta \) and \( \sigma \) are infinite cardinals,

(b) \( \ell \) is a partial function from \( [\lambda]^<\theta \) to \( K \subseteq [\lambda]^\kappa \)

(c) \( K \) is a set of triples \((\kappa, K^*, c^*)\) with \( K^* \subseteq \mathcal{P}(\kappa) \), \( c^* \) a function from \( [\kappa]^<\theta \)

to \( \mathcal{P}(\kappa) \) as in Definition 3.5(2) above (for \( \theta \))

(d) if \( b \in K \), then for some \((\kappa, K^*, c^*) \in K \) and one to one function \( f \) from \( \kappa \)

into \( b \), we have:

(\( \alpha \)) \( b' \in K^* \Rightarrow \{f(\alpha) : \alpha \in b'\} \in K \)

(\( \beta \)) \( a', b' \subseteq \kappa \text{ and } c^*(a') = b' \Rightarrow c\{f(\alpha) : \alpha \in a'\} \supseteq \{f(\alpha) : \alpha \in b'\} \)
(e) for every $A \subseteq [\lambda]^{\leq \chi}$ we can find a $[K, \sigma]$-colouring $c$ of $A$, where:

for any $A \subseteq \lambda$, $c$ is a $[K, \sigma]$-colouring of $A$ means that $c$ is a function from $A$ to $\sigma$ such that $a \in K \& a \subseteq A \Rightarrow \text{Rang}(c \upharpoonright a) = \sigma$

(f) for every $\mu$, if $\chi < \mu \leq \lambda$ then $\mu$ is $(K, \theta)$-approximate.

Then there is $[K, \sigma]$-colouring $c$ of $\lambda$.

Proof. See after the proof of 4.14 below. (The reader may prefer to read first §4 up to the proof of 3.7, 3.13).

3.8 Conclusion: 1) Assume

(a) every cardinal $\mu, 2^{\aleph_0} < \mu \leq \lambda$ is $(C, K^C, c^C, \aleph_1)$-approximate (using the notation of 3.6(1))

(b) $X$ is a Hausdorff topological space.

Then $X \not\rightarrow \langle \text{Cantor set} \rangle_{\frac{1}{2}}$ moreover $X \not\rightarrow \langle \text{Cantor set} \rangle_{2^{\aleph_0}}$, see Definition 3.12 below.

2) We can replace in part (1), $C$ by $\mathbb{R}$.

Proof. By 3.7 (and 3.6).

3.9 Claim. The forcing notions in 1.2 and in 2.8 satisfies, e.g., the condition $*_{\kappa^+}^\sigma$; see below Definition 3.10(1A).

Proof. Included in the proof of 1.2, 2.8, respectively.

3.10 Definition. 1) Let $D$ be a normal filter on $\mu^+$ to which $\{\delta < \mu^+: \text{cf}(\delta) = \mu\}$ belongs. A forcing notion $\mathbb{Q}$ satisfies $*_{D}^\epsilon$ where $\epsilon$ is a limit ordinal $< \mu$, if player I has a winning strategy in the following game $*_{D}^\epsilon[\mathbb{Q}]$ defined as follows:

Player I — if $\zeta \neq 0$ he chooses $\langle q^\zeta_i : i < \mu^+ \rangle$ such that $q^\zeta_i \in \mathbb{Q}$

and $\langle \forall \xi < \zeta, (\forall D, i < \mu^+) \rho^\xi_i \leq q^\zeta_i \rangle$ and he chooses a function $f_\zeta : \mu^+ \rightarrow \mu^+$ such that for the $D$-majority of $i < \mu^+$, $f_\zeta(i) < i$;

if $\zeta = 0$ let $q^\zeta_i = 0$; $f_i = 0$ is identically zero.

Proof included in the proof of 1.2, 2.8, respectively.
Player II — he chooses \( \langle p^\xi_i : i < \mu^+ \rangle \) such that \( (\forall D) q^\xi_i \leq p^\xi_i \) and \( p^\xi_i \in Q \).

The Outcome: Player I wins provided that for some \( E \in D \); if \( \mu < i < j < \mu^+ \), \( i, j \in E \) and \( \bigwedge_{\xi < \epsilon} f_\xi(i) = f_\xi(j) \) then the set \( \{ p^\xi_1 : \xi < \epsilon \} \cup \{ p^\xi_2 : \xi < \epsilon \} \) has an upper bound in \( Q \).

1A) If \( D \) is \( D_\mu^* =: \{ A \subseteq \mu^+ : \text{for some club } E \text{ of } \mu^+ \text{ we have } i \in E \text{ and } \text{cf}(i) = \mu \Rightarrow i \in A \} \) we may write \( \mu \) instead of \( D \) (in \( *_D \) and in the related notions defined below and above). Usually we assume \( D_\mu^* \subseteq D \).

2) We may allow the strategy to be non-deterministic, e.g. choose not \( f_\xi \) just \( f_\xi / D \).

3) We say a forcing notion \( Q \) is \( \varepsilon \)-strategically complete if for the following game, \( \times^\varepsilon_Q \) player I has a winning strategy.

A play last \( \varepsilon \) moves. In the \( \xi \)-th move:

Player I - if \( \xi \neq 0 \) he chooses \( q_\xi \in Q \) such that \( (\forall \xi < \xi) p_\xi \leq q_\xi \) if \( \xi = 0 \) let \( q_\xi = \emptyset \).

Player II - he chooses \( p_\xi \in Q \) such that \( q_\xi \leq p_\xi \).

The Outcome: In the end Player I wins provided that he always has a legal move.

3.11 Lemma. If \( \mu = \mu^\lambda \) and \( \varepsilon \) is a limit ordinal, then the property “\( Q \) is \( (< \mu) \)-strategically complete and has \( \ast^\varepsilon_Q \)” is preserved by \( (< \mu) \)-support iteration.

Proof. See [Sh 546] and history there; in each coordinate we preserve that the sequence of conditions is increasingly continuous and on each stationary \( S \subseteq \{ \delta < \mu^+ : \text{cf}(\delta) = \mu \} \) on which the pressing down function is constant the conditions form a \( \Delta \)-system.

\( \square_{3.11} \)

We can also consider

3.12 Definition. 1) We say \( X^* \rightarrow [Y^*]^\theta_\delta \) if \( X^*, Y^* \) are topological spaces and for every \( h : [X^*]^\alpha \rightarrow \theta \) there is a closed subspace \( Y \) of \( X^* \) homeomorphic to \( Y^* \) such that for some \( \alpha < \theta, \alpha \notin \text{Rang}(h \upharpoonright [Y]^\alpha) \) is not \( \theta \).

2) If we omit the “closed” we shall write \( \rightarrow_w \) instead of \( \rightarrow \) and \( \rightarrow, \rightarrow_w \) denote the negations. [Compare with 3.2, 3.7.]

3.13 Claim. 1) Assume \( X \) is a Hausdorff space with \( \lambda \) points. Assume further \( X \rightarrow [\omega_2]^\theta_\delta \) and \( \mu \geq 2^{\aleph_0} \) but no subspace \( X^* \) of \( X \) with \( \leq \mu \) points satisfies \( X^* \rightarrow [\omega_2]^\theta_\delta \) and \( \mu = \mu^{\aleph_0} \). Then

\( (*) \) we can find a regular \( \kappa \in (\mu, \lambda] \), a stationary \( S \subseteq \kappa \) and a sequence \( \bar{f} = \langle f_\alpha : \alpha \in S \rangle \) such that:

(\( i \)) \( \text{Dom}(f_\alpha) \subseteq \omega_2 \) has cardinality \( 2^{\aleph_0} \)
(ii) \( f_\alpha \) is one-to-one and is a homeomorphism from \( \omega^2 \mid \text{Dom}(f_\alpha) \) onto \( X \mid \text{Rang}(f_\alpha) \) if \( \lambda \) is the set of points of \( X \)

(iii) if \( \alpha \neq \beta \) are from \( S \), then \( \{ \eta \in \text{Dom}(f_\alpha) : f_\alpha(\eta) \in \text{Rang}(f_\beta) \} \) has scattered closure in \( \omega^2 \)

(iv) for a club of \( \delta \in S \) we have \( \text{Rang}(f_\alpha) \subseteq \bigcup_{\beta \in \alpha \cap S} \text{Rang}(f_\beta) \).

2) Similarly for \( \rightarrow_w \) and for \( \mathbb{R} \) instead \( \omega^2 \).

We shall prove it later (after the proof of 4.14).

3.14 Observation: There is a c.c.c. forcing notion \( Q \) of cardinality \( 2^{\aleph_0} \) such that:

\[ \models_Q \text{"there is } h : \omega^2 \rightarrow \omega \text{ such that :} \]

\[ (\alpha) \quad \text{if } C(\in V) \text{ is closed scattered then each } \]
\[ C \cap h^{-1}\{n\} \text{ is finite, and} \]

\[ (\beta) \quad \text{if } A \subseteq (\omega^2) \text{ is uncountable (and from } V) \]
\[ \text{then } |A \cap h^{-1}\{n\}| = |A| \text{ for each } n". \]

Proof. Let \( p \in Q \) be \( (f^p, C^p) \) where \( f^p \) is a finite function from \( \omega^2 \) to \( \omega \) and \( C^p \) is a finite family of closed scattered subsets of \( \omega^2 \).

The order is:

\[ p \leq q \text{ if and only if } f^p \subseteq f^q, C^p \subseteq C^q \text{ and } C \in C^p \& \eta \in C \cap \text{Dom}(f^q) \& \eta \neq \nu \& \nu \in C \cap \text{Dom}(f^q) \setminus \text{Dom}(f^p) \Rightarrow f^q(\eta) \neq f^q(\nu). \]

Clearly

\[ (\ast)_1 \text{ } Q \text{ is a forcing notion of cardinality } 2^{\aleph_0} \]

\[ (\ast)_2 \text{ } Q \text{ satisfies the c.c.c.} \]

[why? let \( p_\alpha \in Q \) for \( \alpha < \omega_1 \), let \( \text{Dom}(f_\alpha) = \{ \eta_{\alpha,\ell} : \ell < \ell_\alpha \}, C^{p_\alpha} = \{ C_{\alpha,k} : k < k_\alpha \} \) both lists with no repetitions and let \( m_\alpha = \min\{ m : \langle \eta_{\alpha,\ell} \upharpoonright m : \ell < \ell_\alpha \rangle \text{ is with no repetitions} \}. \] Without loss of generality \( m_\alpha = m(\ast), \ell_\alpha = \ell(\ast), k_\alpha = k(\ast), \eta_{\alpha,\ell} \upharpoonright m(\ast) = \nu_\ell \). By \( \Delta \)-system lemma without loss of generality for some \( \ell(\ast) \leq \ell(\ast) \) we have:

\[ (\alpha) \quad \ell < \ell(\ast) \Rightarrow \langle \eta_{\alpha,\ell} : \alpha < \omega_1 \rangle \text{ is with no repetitions} \]
\((\beta)\) \(\alpha < \omega_1 \& \ell \in [\ell(**), \ell(*)] \Rightarrow \eta_{\alpha, \ell} = \eta_{\ell} \)

\((\gamma)\) \(\{\eta_{\alpha, \ell} : \alpha < \omega_1, \ell < \ell(**)\} \) is with no repetitions.

Now as each \(C_{\alpha, k}\) is closed and scattered it is necessarily countable so without loss of generality

\[
\alpha < \beta < \omega_1 \& \ell < \ell(**) \Rightarrow \eta_{\beta, \ell} \notin \bigcup_{k < k(\ast)} C_{\alpha, k}. 
\]

We now choose by induction on \(\ell \leq \ell(**)\) sets \(A_\ell, B_\ell \in [\omega_1]^{\aleph_1}\), decreasing with \(n\) such that

\[
\alpha \in A_{\ell+1} \& \beta \in B_{\ell+1} \& \alpha < \beta \rightarrow \eta_{\alpha, \ell} \notin \bigcup_{k < k(\ast)} C_{\beta, k}.\]

This is straight: let \(A_0 = \omega_1 = B_0\); let \(A_\ell, B_\ell\) be given. Clearly for some \(\alpha^*_k \in A_\ell\) the set \(\{\eta_{\alpha, \ell} : \alpha \in A_\ell \setminus \alpha^*_k\}\) is \(\aleph_1\)-dense in itself, i.e. \((\forall \alpha \in A_\ell \setminus \alpha^*_k)(\forall n < \omega)(\exists \beta \in A_\ell)(\eta_{\beta, \ell} \uparrow n = \eta_{\alpha, \ell} \uparrow n)\). Let \(T_\ell = \{\eta_{\alpha, \ell} \uparrow n : \alpha \in A_\ell \setminus \alpha^*_k\) and \(n < \omega\}\), it is a subtree of \(\omega^+ 2\) and \(\text{lim}(T_\ell)\) is a perfect subset of \(\omega^2\). So for each \(\beta \in B_\ell\) for some \(\nu^\ell_{\beta} \in T_\ell\) we have \((\forall \rho \in \bigcup_{k} C_{\beta, k})(\nu^\ell_{\beta} < \rho)\)

so for some \(\nu_\ell \in T_\ell\) we have \(B_{\ell+1} =: \{\beta \in B_\ell : \nu^\ell_{\beta} = \nu_\ell\}\) is uncountable and let \(A_{\ell+1} =: \{\alpha \in A_\ell : \nu_\ell \not\prec \eta_{\alpha, \ell}\}\).

For \(\alpha < \beta, \alpha \in A_{\ell(**)}, \beta \in B_{\ell(**)}\), we have \(p_\alpha, p_\beta\) are compatible.\]

\((*)_3\) if \(A \subseteq \omega^2\) is uncountable and \(n < \omega\) then \(\mathcal{J}_{A,n} =: \{p : \text{for some } \eta \in A, \text{dom}(f^p(\eta)) = n\}\) is dense open.

[Why? Let \(p \in \mathbb{Q}\). Now as \(C^* =: \bigcup\{\mathcal{C} : \mathcal{C} \in \mathcal{P}\}\) is closed and scattered hence countable clearly for some \(\eta \in A\) we have \(\eta \notin C^*\) so \(q = (f^p \cup \{(\eta, n)\}, \mathcal{P})\) satisfies \(p \leq q \in \mathbb{Q} \cap \mathcal{J}_{A,n}\).]

\((*)_4\) for each \(\eta \in \omega^2\) the set

\(\mathcal{J}_\eta = \{p : \eta \in \text{dom}(f^p)\}\) is dense open

[why? being open is trivial; as for density for \(p \in \mathbb{Q}\) let \(n = \sup(\text{rang}(f^p)) + 1\) and without loss of generality \(p \notin \mathcal{J}_\eta\) hence \(\eta \notin \text{dom}(f^p)\), now letting \(q = (f^p \cup \{(\eta, n)\}, \mathcal{P})\) we have \(p \leq q \in \mathcal{J}_\eta\).]

\((*)_5\) for each closed scattered \(C\), the set \(\mathcal{J}_C = \{p : C \in \mathcal{P}\}\) is dense open

[why? immediate as \(p \in \mathbb{Q} \Rightarrow p \leq (f^p, \mathcal{P} \cup \{C\}) \in \mathbb{Q}\).]

Let \(ur = \bigcup\{f^p : p \in G\}\), it is a \(\mathbb{Q}\)-name.
\( f \) is a function from \( \omega^2 \) to \( \omega \) and for each closed scattered \( C \in V, f \upharpoonright C \) is one to one except on a finite set.

[Why? For any \( p \in Q \) there is \( q \) such that \( p \leq q \in Q \) & \( C \in \mathcal{C}^q \), now \( q \leq r \in Q \Rightarrow f^r \upharpoonright (C \setminus \operatorname{Dom}(f^q)) \) is one to one; so \( q \models \text{"} f \upharpoonright (C \setminus \operatorname{Dom}(f^q)) \) is one to one, so as \( \operatorname{Dom}(f^q) \) is finite we are done.]

\((\ast)_7\) \( \models Q \) "\( A \cap f^{-1}\{n\} \) has cardinality \( |A| \) for \( A \in V, A \subseteq \omega^2, A \) uncountable".

[Why? As in \( V \) we can find pairwise disjoint \( A_i \subseteq A \) for \( i < |A|, |A_i| = |A| \) and apply \((\ast)_3\).]

Together we are done. \( \square_{3.14} \)

**Proof of Theorem 3.2.**

\((B)^+ \Rightarrow (B)^{[\omega^2]}\)

Trivial (special case).

\((A)^+ \Rightarrow (A)^{[\omega^2]}\)

Trivial (a special case).

\((B)^+ \Rightarrow (A)^+\)

Trivial (stronger demands).

\((B)^{[\omega^2]} \Rightarrow (A)^{[\omega^2]}\)

Trivial (stronger demands).

\((A)^{[\omega^2]} \Rightarrow (C)\)

By 3.13 for \( \theta = 2, \mu = 2^{\kappa^\kappa} \).

\((C) \Rightarrow (D)\)

Forcing by \( \operatorname{Levy}(\kappa, 2^{\kappa^\kappa}) \) change nothing so without loss of generality \( \kappa = \kappa^{<\kappa} \). Let \( \lambda, S, \tilde{f} = \langle f_\alpha : \alpha \in S \rangle \) be as in clause (C) of 3.2. Next let \( \mathbb{Q} \) be the forcing notion from 3.14 which is a c.c.c. forcing notion of cardinality \( 2^{\kappa_0} \), so we get the conclusion of 3.14 so let \( h \) be as there and let \( g : (\omega^2)^V \rightarrow 2^{\kappa_0} \) be one to one; without loss of generality \( (2^{\kappa_0})^\omega \) divide \( \alpha \) for every \( \alpha \in S \). For each \( \alpha \in S \) let

\[
A_\alpha =: \{ (2^{\kappa_0}) \times f(\eta) + g(\eta) : \eta \in \operatorname{Dom}(f_\alpha) \text{ and } h(\eta) = 0 \}.
\]

we get: \( A_\alpha \subseteq \omega^2, |A_\alpha| = 2^{\kappa_0} \) and for \( \alpha \neq \beta \) from \( S, A_\alpha \cap A_\beta \) is finite. So clauses (a)-(d) of clause (D) from Theorem 3.2 holds. Then we force by \( \operatorname{Levy}(\lambda, 2^{<\lambda}) \), nothing changes but we get \( \diamond_S \). By 2.4 without loss of generality clause (f) of (D) of 3.2 holds. By 2.8 without loss of generality we have the \( \kappa \)-freeness (i.e., clause
(e) of (D) of 3.2 which is equivalent to (B)2 of 1.2 while clause (f) of (D) of 3.2 (= clause (C) of 1.2) is preserved by clause (d) of the conclusion of 2.8.

\((D) \Rightarrow (B)^+ \text{ and } (D) \Rightarrow (A)^+[2^\omega]\)

We do it by forcing but for the proof any \(\kappa\) such that \(\aleph_1 \leq \text{cf}(\kappa) = \kappa, 2^{\aleph_0} < \lambda\) can serve. We can force by \(\text{Levy}(\kappa, 2^{\aleph_0})\), so without loss of generality \(\kappa = \kappa^{<\kappa}\).

First assume that \(\kappa > 2^{\aleph_0}\) (as in the main case) and we restrict ourselves to spaces \(Y^*\) with a basis of cardinality \(< \kappa\) which is no restriction if \(\kappa > \aleph_2\) or if we are proving just \((D) \Rightarrow (A)^+[2^\omega]\), then we can use a product of forcing instead of iteration. Now any strategically \(< \kappa\)-complete \(\kappa^+\text{-c.c.}\) forcing notion preserves (D), we do not use this in this first case, but still note it. By forcing by \(\text{Levy}(\lambda, 2^{<\lambda})\) (see 4.3) without loss of generality \(\diamondsuit\) for the \(S\) of clause (D), this will be preserved for any forcing notion \(P\) if \(P\) has density \(< \lambda\), which holds in our case.

Let \(\{Y_i^*: i < i^*\}\) list the topological spaces as in clause \((B)^+\) with set of points \(2^{\aleph_0}\) or \(i^* = 1\) & \(Y_0^* = \omega, 2\), depending on what we are proving. For each \(i < i^*\), let \(Q_i\) be the forcing from 1.2, the assumption of 1.2 holds by (D) and the assumptions on \(Y_i^*\) and \(X_i^*\) be the \(Q_i\)-name of the topological space which \(Q_i\) produces. Let \(Q\) be the product of \(\{Q_i: i < i^*\}\) with support \(< \kappa\). Now \(Q\) is \(\kappa\)-complete. Hence by our present assumption no new relevant space \(Y^*\) is added by forcing by \(Q\).

Why is \(X_i^*\) as required, i.e., \(X_i^* \rightarrow (Y_i^*)^{1}_{\text{cf}(2^{\aleph_0})}\) also in \(V^Q\)? Forcing by \(Q\) add more “colouring” \(c\), i.e., functions from \(X_i^*\), i.e., \(\lambda\) into some ordinal \(< 2^{\aleph_0}\). However, the proof of 1.2 can be repeated for this case.

Second, consider the general case.

Now we use iterated forcing \(\langle P_j, Q_i: j \leq i(*)\rangle\) with \(< \kappa\)-support, each satisfying the \(\ast^{\kappa^+}\) version of \(\kappa^+\text{-c.c.}\) and for simplicity \(< \kappa\)-strategically complete (see 3.9). Now let each \(Q_i\) be as in 1.2 for some \(Y_i^*\) a \(P_i\)-name of a topological space as in 1.2) and it forces an example \(X_i^*\). With suitable bookkeeping (if \(\kappa > 2^{\aleph_0}\) is easier) we finish as those iterations preserve “\(< \kappa\)-strategic completeness hence no new set of ordinals of cardinality \(< \kappa\) and (the strong version of) \(\kappa^+\text{-c.c.}”\) is preserved, see 3.11.

Still we have to prove that the example \(X_i^*\) we force to satisfy “\(X_i^* \rightarrow (Y_i^*)^{1}_{\text{cf}(2^{\aleph_0})}\) if \(\sigma < \text{cf}(2^{\aleph_0})\)” has this property not only in \(V^{P_i+1}\) but also in \(V^{P_i(\sigma)}\). For this we repeat the relevant part of the proof of 1.2 noting the explicit way the \(Q_i\)’s; this will be presented in full in [Sh:F567].

For self-containment we recall (really [Sh:g, II,2.2] and see [Sh 108], [Sh 88a]).

3.15 Claim. Assume \(\kappa\) is strongly compact and \(\chi = \text{cf}(\chi) \leq \text{cf}(\mu) < \kappa < \lambda = \text{cf}(\lambda) = \mu^+\) (so \(\lambda = \lambda^{<\kappa}\) and \((\forall \alpha < \mu)(|\alpha|^{<\kappa} < \mu))\) and \(a \subseteq \text{Reg} \cap \mu\setminus \kappa, \mu = \)
sup(a), |a| = cf(µ) and \( \tilde{f} = \langle f_\alpha : \alpha < \lambda \rangle \) is a \( < J_{a}^{b} \)-increasing cofinal sequence in \( II_a \).

Then for some \( \mu_0 \in (\chi, \kappa) \), we have \( cf(\mu_0) = cf(\mu) < \mu, (\forall \alpha < \mu_0)(|\alpha|^{\chi} < \mu_0) \) and if \( P = \text{Levy}(\chi, \mu_0) \ast \text{Levy}(\mu_0^{++}, \kappa) \), in \( V^P \) we have

(a) \( S_{\mu_0}^{\mu^{+}} = \{ \delta < \mu^{+} : V \models cf(\delta) = \mu_0^{+} \} \subseteq \{ \delta < \mu^{+} : cf(\delta) = \chi^{+} \} \) does not belong to \( I[\mu^{+}] \),

(b) \( \text{bad}(\tilde{f}) \supseteq S^{*} = \{ \delta < \mu^{+} : cf(\delta) = \mu_0^{+}, \tilde{f} \upharpoonright \delta \text{ has a } \langle J_{a}^{b} \rangle \text{-lub} \}
\)

\( f \in II_a \) such that \( \theta \in a \Rightarrow cf(f(\theta)) < cf(\delta) \) is a stationary subset of \( \mu^{+} \)

(c) forcing by \( P \), preserve \( \mu_0^{+} \) is a cardinal and the stationarity of subsets of \( S_{\mu_0}^{\mu^{+}} \) (from \( V \)) and preserve “\( \delta \in S^{*} \) is not a good point in \( \tilde{f} \)” and \( \tilde{f} = \langle f_\alpha : \alpha < \lambda \rangle \) is \( < J_{a}^{b} \)-increasing cofinal in \( II_a \); if \( \mu = \kappa^{+}\omega, a = \{ \mu^{+n} : n \in (0, \omega) \} \)

we get the result for \( \prod_{n<\omega} \mathcal{R}_n/J_{\omega}^{b} \).

**Proof.** The choice of \( \mu_0 \) (and clause (a)) is a main point in [Sh 108], [Sh 88a]. Now \( S^{*} = \cup\{ S_{\mu_0}^{\mu^{+}} : \mu_0^{+} < \kappa, cf(\mu_0) = cf(\mu) < \mu_0 \} \) is stationary by [Sh:g, 2.2.5.6] using \( (\ast)' \) not \( (\ast) \), so for some \( \mu_0^{+}, S_{\mu_0^{+}}^{\mu^{+}} \) is stationary, and (c) is obvious. Of course, (b) \( \Rightarrow \)

(a) by [Sh:g, I]. \( \square_{3.15} \)
§4 Decomposing families of almost disjoint functions

Let \((I, J)\) be a pair of ideals on say \(\theta = \text{Dom}(I, J)\) such that \(I \subseteq J\) and we consider a family \(\mathcal{F}\) of functions each from some \(A \in J^+\) to \(\lambda\) which are almost disjoint in the sense that

\[\forall f \neq g \text{ from } \mathcal{F} \text{ then } \{i < \theta : i \in \text{Dom}(f) \cap \text{Dom}(g) \text{ and } f(i) = g(i)\} \in I.\]

A decomposition is a representation of \(\mathcal{F}\) as \(\bigcup_{\alpha} \mathcal{F}_\alpha\) such that the \(\mathcal{F}_\alpha\) are pairwise disjoint, “small” and \(f \in \mathcal{F}_\alpha \Rightarrow \{i \in \text{Dom}(f) : f(i) \notin \bigcup \{\text{Rang}(g) : g \in \mathcal{F}_\beta \text{ for some } \beta < \alpha\}\}\) is “small”. We try to prove that if such decomposition does not exist, then there are “transparent” counterexamples.

This helps the equiconsistency in §3 and continue [Sh 161], [HJSh 249], [Sh:g, II, §6], [Sh 460, 3.9].

The reader can concentrate on the case that \(Y\) is a singleton.

4.1 Definition. 1) Let \(\mathcal{Y}\) denote a set of pairs of the form \((I, J)\) where

(a) \(I \subseteq J\) are ideals over a common set called \(\text{Dom}(I, J) = \text{Dom}(I) = \text{Dom}(J)\)

or just

(b) \(\emptyset \in I \subseteq J \subseteq \mathcal{P}(\text{Dom}(I, J))\) and \([A \subseteq B \in I \Rightarrow A \in I], [A \subseteq B \in J \Rightarrow A \in J]\), \(\text{Dom}(I, J) \notin J\).

Let \(I^+ = \mathcal{P}(\text{Dom}(I)) \setminus I\) and \(J^+ = \mathcal{P}(\text{Dom}(J)) \setminus J\).

Let \(\kappa(\mathcal{Y}) = \sup\{|\text{Dom}(I, J)| : (I, J) \in \mathcal{Y}\}\). We call \(\mathcal{Y}\) standard if for each \((I, J) \in \mathcal{Y}\), the set \(\text{Dom}(I, J)\) is a cardinal; we call \(\mathcal{Y}\) disjoint if \((\text{Dom}(I, J) : (I, J) \in \mathcal{Y})\) is a sequence of pairwise disjoint sets.

2) \(\text{NFr}_1(\lambda, \mathcal{Y})\) means that for some \(\lambda^* > \lambda\) we have \(\text{NFr}_1(\lambda^*, \lambda, \mathcal{Y})\) which means that \(\lambda \geq |\mathcal{Y}| + \kappa(\mathcal{Y})\) and there are \(\langle \mathcal{F}_{(I,J)} : (I, J) \in \mathcal{Y}\rangle\) exemplifying it which means:

(a) \(\mathcal{F}_{(I,J)} \subseteq \{f : f \text{ a function, } \text{Dom}(f) \in J^+\}\)

(b) if \(f \neq g \in \mathcal{F}_{(I,J)}\) then
\[\{x : x \in \text{Dom}(f) \cap \text{Dom}(g) \text{ but } f(x) = g(x)\}\] belongs to \(I\)

(c) \(\lambda \geq |\bigcup \{\text{Rang}(f) : f \in \mathcal{F}_{(I,J)} \text{ and } (I, J) \in \mathcal{Y}\}|\)

(d) \(\lambda < \lambda^* = \sum\{|\mathcal{F}_{(I,J)}| : (I, J) \in \mathcal{Y}\}\).

2) \(\text{NFr}_2(\lambda, \mathcal{Y})\) means that \(\lambda\) is regular > \(|\mathcal{Y}| + \kappa(\mathcal{Y})\) and there is \(\langle f_\delta : \delta \in S\rangle\) such that
(a) $S \subseteq \lambda$ is stationary and is the disjoint union of $\langle S_{(I,J)} : (I,J) \in \mathcal{Y} \rangle$

(b) $\text{Dom}(f_\delta) \in J^+$ and $\text{Rang}(f_\delta) \subseteq \delta$ for each $\delta \in S_{(I,J)}$

(c) $\delta_1 \neq \delta_2 \in S_{(I,J)} \Rightarrow \{ x : x \in \text{Dom}(f_{\delta_1}) \cap \text{Dom}(f_{\delta_2}) \text{ and } f_{\delta_1}(x) = f_{\delta_2}(x) \} \in I$.

3) We omit $N$ from $\text{NFr}$ in parts (1) and (2) for the negation. If $\mathcal{Y} = \{ (I,J) \}$ we may write just $(I,J)$.

4.2 Fact: 1) $\text{NFr}_1(\lambda, \mathcal{Y})$ is preserved by increasing $\mathcal{Y}$ to $\mathcal{Y}'$ when $|\mathcal{Y}'| + \kappa(\mathcal{Y}') \leq \lambda$. Also $\text{NFr}_2(\lambda, \mathcal{Y})$ is preserved by increasing $\mathcal{Y}$ to $\mathcal{Y}'$ if $|\mathcal{Y}'| + \kappa(\mathcal{Y}') < \lambda$. Similarly if $\text{NFr}_1(\lambda^*, \mathcal{Y}, \lambda), \lambda^* \geq \lambda_1 \geq \lambda_1 \geq \mathcal{Y}_1 + \kappa(\mathcal{Y})$ and $\mathcal{Y}_1 \supseteq \mathcal{Y}$ then $\text{NFr}_1(\lambda^*, \lambda, \mathcal{Y}_1)$. 2) $\text{NFr}_1(\lambda, \mathcal{Y})$ is equivalent to $\text{NFr}_1(\lambda^+, \lambda, \mathcal{Y})$ which is equivalent to $\exists (I,J) (\in \mathcal{Y})$ $\text{NFr}_1(\lambda^+, \lambda, (I,J))$. 3) If $\lambda^*$ is regular or at least $\text{cf}(\lambda^*) > |\mathcal{Y}|$ then $\text{NFr}_1(\lambda^*, \lambda, \mathcal{Y})$ iff there is $(I,J) \in \mathcal{Y}$ such that $\text{NFr}_1(\lambda^*, \lambda, \{ (I,J) \})$. 4) $\text{NFr}_2(\lambda, \mathcal{Y})$ implies $\text{NFr}_2(\lambda^+, \mathcal{Y})$. 5) $\text{NFr}_2(\lambda, \mathcal{Y})$ iff there is $(I,J) \in \mathcal{Y}$ such that $\text{NFr}_2(\lambda, \{ (I,J) \})$ and $|\mathcal{Y}| < \lambda$.  

Proof. Check.

4.3 Claim. 1) Assume that $\text{NFr}_2(\lambda, \{ (I,J) \})$ and let $\bar{f} = \{ f_\delta : \delta \in S \}$ exemplifies it and $\tau^{++} < \lambda, \tau \geq |\text{Dom}(I,J)|$ and for simplicity $\kappa = \text{Dom}(I,J)$. If $\Diamond_S$ then we can find $E$ a club of $\lambda$ and $\{ f'_\delta : \delta \in S \cap E \}$ exemplifying $\text{NFr}_2(\lambda, \{ (I,J) \})$ such that

(*) if $F : \lambda \rightarrow [\lambda]^{\leq \tau}$ then for some $\delta \in S \cap E$ the set $\text{Rang}(f'_\delta)$ is $F$-free (i.e. $\alpha \neq \beta \in \text{Rang}(f'_\delta) \Rightarrow \beta \notin F(\alpha)$).

2) The forcing of adding a Cohen subset of $\lambda$ (i.e. $(\lambda^{>2}, \triangleleft)$) preserve “$\bar{A}$ exemplifies $\text{NFr}_2(\lambda, \{ (I,J) \})$” (as it preserves “$S$ is stationary”), add no bounded subsets to $\lambda$ and forces $\Diamond_S$.

Proof. 1) As in the proof of 2.3.

Let $\bar{h} = \{ h_\delta : \delta \in S \}$ be such that $h_\delta : \delta \rightarrow [\delta]^{\leq \tau}$ and for every $h : \lambda \rightarrow [\lambda]^{\leq \tau}$ the set $\{ \delta \in S : h_\delta = h \upharpoonright \delta \}$ is a stationary subset of $\lambda$; such $\bar{g}$ exists as we assume $\Diamond_S$. Let $E = \{ \delta < \lambda : \tau^{++} \times \omega \text{ divide } \delta \}$ it is a club of $\lambda$ and for $\delta \in S \cap E$ we define the function $g_\delta : \tau^{++} \rightarrow [\tau^{++}]^{\leq \tau}$ by
\[ g_\delta(\beta) = \{ \gamma < \tau^{++} : \text{for some } \varepsilon_1, \varepsilon_2 < \text{Dom}(I, J) \text{ we have} \]
\[ \tau^{++} \times f_\delta(\varepsilon_1) + \gamma \in h_\delta(\tau^{++} \times f_\delta(\varepsilon_2) + \beta) \}. \]

Note that \(|g_\delta(\beta)| \leq \tau\) as \(h_\delta(\tau^{++} \times f_\delta(\varepsilon_2) + \beta)\) has cardinality \(\leq \tau\) and the number relevant of \(\varepsilon_1, \varepsilon_2\) is \(\leq |\text{Dom}(I, J)| = \kappa \leq \tau\). So by [Ha61] there is an unbounded subset \(Z_\delta\) of \(\kappa^{++}\) such that \(\beta_1 \neq \beta_2 \in Z_\delta \Rightarrow \beta_1 \notin g_\delta(\beta_2)\).

Let \(Z_\delta = \{ \gamma_{\delta, \varepsilon} : \varepsilon < \tau^{++} \}\), with \(\gamma_{\delta, \varepsilon}\) increasing with \(\varepsilon\). Now for \(\delta \in S \cap E\) we define \(f'_\delta : \text{Dom}(I, J) \rightarrow \delta\) by
\[ f'_\delta(\varepsilon) = \tau^{++} + f_\delta(\varepsilon) + \gamma_{\delta, \varepsilon}. \]

Now clearly \(f'_\delta\) is a function from \(\kappa = \text{Dom}(I, J)\) into \(\lambda\), in fact, it is into \(\delta\) as \(\text{Rang}(f_\delta) \subseteq \delta\) \& \((\tau^{++} \times \omega)|\delta\). Also for \(\delta_1 \neq \delta_2\) from \(S \cap E\)
\[ \{ \varepsilon < \kappa : f'_{\delta_1}(\varepsilon) = f'_{\delta_2}(\varepsilon) \} = \{ \varepsilon < \kappa : \tau^{++} \times f_{\delta_1}(\varepsilon) + \gamma_{\delta_1, \varepsilon} = \tau^{++} \times f_{\delta_2}(\varepsilon) + \gamma_{\delta_2, \varepsilon} \} \subseteq \{ \varepsilon < \kappa : f_{\delta_1}(\varepsilon) = f_{\delta_2}(\varepsilon) \} \in I. \]

Lastly, if \(\delta \in S \cap E\) and \(\varepsilon_1 \neq \varepsilon_2 < \kappa\) and \(f'_\delta(\varepsilon_1) \in h_\delta(f'_\delta(\varepsilon_2))\) then
\[ \tau^{++} \times f_\delta(\varepsilon_1) + \gamma_{\delta, \varepsilon_1} = f'_\delta(\varepsilon_1) \in h_\delta(f'_\delta(\varepsilon_1)) = h_\delta(\tau^{++} \times f_\delta(\varepsilon_2) + \gamma_{\delta, \varepsilon_2}) \subseteq \bigcup_{\varepsilon < \kappa} h_\delta(\tau^{++} \times f_\delta(\varepsilon) + \gamma_{\delta, \varepsilon}). \]

so \(\gamma_{\delta, \varepsilon_1} \in g_\delta(\gamma_{\delta, \varepsilon_2})\) by the definition of \(g_\delta\), but \(\gamma_{\delta, \varepsilon_1}, \gamma_{\delta, \varepsilon_2}\) are distinct members of \(Z_\delta\), contradiction to its choice. By the choice of \(\langle h_\delta : \delta \in S \rangle\), for every \(F : [\lambda] \rightarrow [\lambda]^{\leq \tau}\) for stationary many \(\delta \in S\) (hence \(\delta \in S \cap E\)0 we have \(h_\delta = h | \delta\) \& \(\delta \in S \cap E\) hence \(\text{Rang}(f'_\delta)\) is \(F\)-free.

2) Straight.

\[ \square_{4.3} \]

Now we give sufficient conditions for the existence of decomposition which implies easily (in the cases needed see later) the existence of suitable colouring. The reader may concentrate on the case \(\mathcal{Y}\) is a singleton.
4.4 The Decomposition Claim. Assume:

(a) $\mathcal{Y}$ is as in Definition 4.1(1)

(b) $\lambda > \mu \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$

(c) for no regular $\kappa \in (\mu, \lambda]$ do we have $\text{NFr}_2(\kappa, \mathcal{Y})$

(d) $\text{cl}$ is a function from $[\lambda]^{\leq \mu}$ to $[\lambda]^{\leq \mu}$

(e) for $A, B \in [\lambda]^{\leq \mu}$ we have $A \subseteq \text{cl}(A)$ and $A \subseteq B \Rightarrow \text{cl}(A) \subseteq \text{cl}(B)$

(f) $\mathcal{P} \subseteq [\lambda]^{\leq \mu}$ has cardinality $8 \leq \lambda$ or at least has a dense $9$ such subfamily and satisfies:

for every $A \in \mathcal{P}$ there are a pair $(I, J) \in \mathcal{Y}$, a set $\mathcal{U} \in J^+$ and a one to one function $f : \mathcal{U} \to A$ such that:

$(\alpha)_{f, A, I}$ if $\mathcal{U}' \subseteq \mathcal{U}$ & $\mathcal{U}' \in I^+$ then for some $A' \in \mathcal{P}$ we have $A' \subseteq A \cap \text{cl}(\{f(i) : i \in \mathcal{U}'\})$

$(\beta)_{f, A, I}$ there are $\mathcal{U}' \subseteq \mathcal{U}$, $\mathcal{U}' \in I^+$ for $\alpha < \alpha^*$ for some $\alpha^* \leq \mu$ such that for any $\mathcal{U}' \subseteq \mathcal{U}$, $\mathcal{U}' \in I^+$ for some $\alpha < \alpha^*$ we have $\mathcal{U}' \subseteq \mathcal{U}'$ or at least $A \cap \text{cl}(\{f(i) : i \in \mathcal{U}'\}) \subseteq A \cap \text{cl}\{f(i) : i \in \mathcal{U}'\}$

(1) if $A \in \mathcal{P}$ then $\text{cl}(A) = A$.

Then

$\text{Dec}(\lambda, \mathcal{P}, \mu, \mathcal{Y})$: for every $\chi > \lambda$ and $x \in \mathcal{H}(\chi)$ there is a sequence $(M_\alpha : \alpha < \lambda)$ such that:

(i) $M_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi)$

(ii) $\mu \cup \{\mathcal{Y}, \lambda, \mu, x\} \subseteq M_\alpha$ and $\|M_\alpha\| = \mu$

(iii) $\bigcup_{\alpha < \lambda} M_\alpha$ includes $\lambda$

(iv) Assume $A \in \mathcal{P}$ and define $\alpha(A) = \text{Min}\{\alpha \leq \lambda : \text{if } \alpha < \lambda \text{ then for some } (I, J) \in \mathcal{Y} \text{ and } \mathcal{U} \in J^+ \text{ and } f : \mathcal{U} \to A \text{ which is one-to-one, we have } \{f, \mathcal{U}\} \in M_\alpha \text{ hence } \text{Rang}(f) \subseteq M_\alpha \text{ and } \{i \in \mathcal{U} : f(i) \in \bigcup_{\beta \leq \alpha} M_\beta\} \in J^+\}$. Then $\alpha(A) < \lambda$ and for some $(I, J), \mathcal{U}, f$ which are witnesses to $\alpha(A) = \alpha$

\footnote{no real harm in adding $\text{cl}(A) = \text{cl}(\text{cl}(A))$

\footnote{used in 4.7}
we have\(^{11}\):
\[ \{ i \in \mathcal{U} : f(i) \in \bigcup_{\beta < \alpha} M_\beta \} \in J \]

(v) For any pregiven \( \sigma = \text{cf}(\sigma) \leq \mu \) we can demand \( M_\alpha = \bigcup_{\varepsilon < \sigma} M_{\alpha, \varepsilon} \)
increasing with \( \varepsilon \) and \( \langle M_{\alpha, \zeta} : \zeta \leq \varepsilon \rangle \in M_{\alpha, \varepsilon + 1} \) and \( \mu \cup \{ \mathcal{U}, \lambda, \mu, x \} \subseteq M_{\alpha, \varepsilon} \).

Now below we shall prove that Claim 4.4 follows from the following variant (we change (d), (e), (f)).

4.5 Claim. Assume

(a)' \( \mathcal{U} \) is as in Definition 4.1(1)

(b)' \( |X| = \lambda > \mu \geq |\mathcal{U}| + \kappa(\mathcal{U}) \)

(c)' for no regular \( \kappa \in (\mu, \lambda] \) do we have \( \text{NFr}_2(\kappa, \mathcal{U}) \)

(d)' \( \mathcal{T} = \langle \mathcal{T}_t : t \in T \rangle, T \) is a partial order of cardinality\(^{12} \leq \lambda \) or at least density\(^{13} \leq \lambda \); we consider the \( \mathcal{T}_t \)'s as indexed sets such that \( t \neq s \implies \mathcal{T}_s \cap \mathcal{T}_t = \emptyset \) though they may have common members, so \( f \in \mathcal{T}_{t(f)} \)

(e)' for each \( t \in T \) each member \( f \in \mathcal{T}_t \) is a one-to-one function such that for some \( (I, J) = (I_f, J_f) \in \mathcal{U} \) we have \( \text{Dom}(f) \in J^+, \text{Rang}(f) \subseteq X \)

(f)' if \( t \in T \) and \( f \in \mathcal{T}_t \), then there is a subset \( T[f] \) of \( T \) of cardinality \( \leq \mu \) such that \( T[f] \) is a cover of \( T_{<f>} \) which means \( (\forall s \in T_{<f>})(\exists t \in T[f])(s \leq_T t) \) where for \( f \in \mathcal{T}_t \) we let

\[ T_{<f>} =: \{ r \in T : \text{and for some } g \in \mathcal{T}_r \text{ we have } (I_g, J_g) = (I_f, J_f) \text{ and } \{ i : i \in \text{Dom}(f), i \in \text{Dom}(g) \text{ and } f(i) = g(i) \} \in I_f^+ = I_g^+ \}. \]

THEN \( \text{Dec}(\lambda, \mathcal{T}, \mu, \mathcal{U}) : \) for every \( \chi > \lambda \) and \( x \in \mathcal{H}(\chi) \) there is a sequence \( \langle M_\alpha : \alpha < \lambda \rangle \) such that:

(i) \( M_\alpha \prec (\mathcal{H}(\chi), \in, <^*_\chi) \)
(ii) \( \mu \cup \{ \mathcal{U}, \lambda, \mu, x \} \subseteq M_\alpha \) and \( \| M_\alpha \| = \mu \)
(iii) \( \bigcup_{\alpha < \lambda} M_\alpha \) includes \( \lambda \)

\(^{11}\)Moreover for some \( X \in M_\alpha \) of cardinality \( \leq \mu \) (so \( X \subseteq M_\alpha \)) we have \( \{ i \in \mathcal{U} : f(i) \in X \setminus \bigcup_{\beta < \alpha} M_\beta \} \in J^+ \), and the parallel in 4.5.

\(^{12}\)Light assumption by 4.2(4)

\(^{13}\)I.e., there is \( T' \subseteq T \) satisfying \( |T'| \leq \lambda \) and \( (\forall s \in T)(\exists t \in T')(s \leq_T t) \)
(iv) if \( s \in T \), then for some \( t, s \leq_T t \in T \) and for some \( \alpha < \lambda \) and \( g \in F_t \) we have
\[
\begin{align*}
(\alpha) & \quad \{ i \in \text{Dom}(g) : g(i) \in \bigcup_{\beta < \alpha} M_\beta \} \in J_g \\
(\beta) & \quad t, g \in M_\alpha \text{ hence } \text{Rang}(g) \subseteq M_\alpha
\end{align*}
\]

(v) for any pregiven \( \sigma = \text{cf}(\sigma) \leq \mu \) we can demand \( M_\alpha = \bigcup_{\varepsilon < \sigma} M_{\alpha, \varepsilon} \) where
\[
\langle M_{\alpha, \varepsilon} : \varepsilon < \sigma \rangle \text{ is increasing, } \mu \cup \{ Y, \lambda, \mu, M_* \} \subseteq M_{\alpha, \varepsilon}, M_\alpha = \bigcup_{\varepsilon < \sigma} M_{\alpha, \varepsilon}, \langle M_{\alpha, \zeta} : \zeta \leq \varepsilon \rangle \in M_{\alpha, \varepsilon}+1 \text{ and } \langle M_{\beta : \beta < \alpha} \rangle \in M_{\alpha, \varepsilon}.
\]

Before proving 4.5 we deduce 4.4 from it and prepare the ground.

Proof of 4.4 from 4.5. Clearly without loss of generality \( \mathcal{V} \) is disjoint and let the set of elements of \( T \) be \( \mathcal{P} \) and for \( A \in T \) we let
\[
\mathcal{F}_A = \left\{ f : \text{for some } (I, J) \in \mathcal{V}, \text{ we have } \text{Dom}(f) \in J^+ \text{ and,} \right. \\
\left. f \text{ is one to one into } A \text{ and clauses } (\alpha)_{f, A, I}, (\beta)_{f, A, I} \text{ of } (f) \text{ of 4.4 hold} \right\}
\]

and for any \( f \in \mathcal{F}_A \) let \( (I_f, g_f) \) be as in the definition of \( \mathcal{F}_A \).

We define the partial order \( \leq_T \) on \( T \) by: \( A_1 \leq_T A_2 \) iff \( A_2 \subseteq A_1 \). We have to check that the assumptions in 4.5 holds, now clauses \((a)', (b)', (c)'\) are the same as \((a), (b), (c)\) of 4.4, and clauses \((d)', (e)'\) are obvious. As for clause \((f)'\) it follows by clause \((f)\) and the definition of \( \langle \mathcal{F}_A : A \in \mathcal{P} \rangle \).

[Why? We are given in \((f)'\) the objects \( t \in T, f \in F_t \) so \( t = A \in \mathcal{P} \) and let \( (I, J) = (I_f, J_f) \in \mathcal{V} \) and \( \mathcal{V} = \text{Dom}(f) \in J^+ \), see the definition of \( T \). Now as \( f \in \mathcal{F}_A \) by the definition of \( \mathcal{F}_A \) let \( \langle \mathcal{V}'_{\alpha : \alpha < \alpha^*} \rangle \) be as in subclause \((\beta)\) of clause \((f)\) of 4.4. For each \( \alpha < \alpha^* \) choose \( A'_{\alpha} \) as in subclause \((\alpha)_{f, A, I}\) of clause \((f)\) of 4.4 for \( \mathcal{V}' = \mathcal{V}'_{\alpha} \) so in particular \( A'_{\alpha} \subseteq \mathcal{P} \) and \( A'_{\alpha} \subseteq A \cap \text{cl}\{ f(i) : i \in \mathcal{V}'_{\alpha} \} \). Let us choose \( T[f] = \{ A'_{\alpha : \alpha < \alpha^*} \} \), so obviously \( T[f] \in [\mathcal{P}]^{\leq \mu} = [T]^{\leq \mu} \) as \( \alpha^* \leq \mu \). Let us check that \( T[f] \) is as required in clause \((f)'\) of 4.5. The main point is checking that \( T[f] \) is a cover of \( T(f); \) let \( r \in T(f) \), i.e., let \( r = A' \) and (by the definition of \( T(f) \) inside 4.5), there is \( g \in F_r \) such that \( (I_g, J_g) = (I_f, J_f) \) satisfying \( \mathcal{V}' = \{ i : i \in \text{Dom}(f) \} \) and \( i \in \text{Dom}(g) \) and \( f(i) = g(i) \in I_f^+ \), so \( \mathcal{V}' \in I^+ \). So (by clause \((\beta)_{f, A, I}\) from \((f)\) of 4.4) for some \( \alpha < \alpha^* \) we have \( A \cap \text{cl}\{ f(i) : i \in \mathcal{V}'_{\alpha} \} \subseteq A \cap \text{cl}\{ f(i) : i \in \mathcal{V}' \} \); by the choice of \( T[f] \) we have \( A'_{\alpha} \in T[f] \), let \( s = A'_{\alpha} \).

Now \( s \in T[f] \) and (by the choice of \( A'_{\alpha} \) clearly \( s = A'_{\alpha} \subseteq A \) which means \( r \leq s \). Lastly, \( T[f] \) has cardinality \( \leq |\alpha^*| \leq \mu \). So clause \((f)'\) of 4.5 holds.]
Finally let $\chi$ be large enough and $x \in \mathcal{H}(\chi)$. So by 4.5 there is a sequence $\langle M_\alpha : \alpha < \lambda \rangle$ for our $\langle \mathcal{F}_A : A \in T \rangle, x, \chi$ as required there. It is enough to show that $\langle M_\alpha : \alpha < \lambda \rangle$ is as required in the conclusion of 4.4. Now clauses (i), (ii), (iii) and (v) of the conclusion of 4.4 are just like clauses (i), (ii), (iii) and (v) of the conclusion of 4.5, so we should check only clause (iv). So assume $A \in \mathcal{P}$ and let $\alpha(A)$ be as defined there. By clause (iv) of the conclusion of 4.5 applied to $(I, J) \in \mathcal{Y}$ and $U_1 \in J^+, U_0 \in J$ then $|U_1 \setminus U_0| = \mu$. So $t \in \mathcal{P}, t \subseteq A, Rang(g) \subseteq t \subseteq s = A, Dom(g) \subseteq M_\alpha$ and Rang$(g) \in M_\alpha$. So $\alpha$ is as required.

4.6 Observation. 1) If in Claim 4.4 we add as an assumption clause (g) stated below and $\theta \leq \mu$, then we can find a function $h$ from $\lambda$ to $\mu$ such that for every $A \in \mathcal{P}$ we have $\theta = \text{Rang}(f \upharpoonright A)$; where

(h) if $(I, J) \in \mathcal{Y}$ and $U \in J^+$ then $|U| = \mu$.

2) Assume

(a)$'' \mathcal{Y}$ as in Definition 4.1(1)
(b)$'' \lambda > \mu \geq |\mathcal{Y}| + \kappa(\mathcal{Y})$
(c)$'' \mathcal{P} \subseteq [\lambda]^{\leq \mu}$
(d)$''$ the conclusion of 4.4 holds
(e)$''$ as (g) above.

Then we can find $h : \lambda \to \mu$ such that $A \in \mathcal{P} \Rightarrow \mu = \text{Rang}(f \upharpoonright A)$.

Proof. 1) For each $\alpha < \lambda$ let $B_\alpha = M_\alpha \cap \lambda \setminus \{M_\beta : \beta < \alpha\}$, and choose $h_\alpha : B_\alpha \to \mu$ such that

$A \in M_\alpha \land A \in [\lambda]^\mu \land |A \cap B_\alpha| = \mu \Rightarrow \text{Rang}(h_\alpha \upharpoonright (A \cap B_\alpha)) = \mu.$

Let $h : \lambda \to \mu$ extend every $h_\alpha$.
2) Similarly.

Below we think of the functions from $\mathcal{F}$ as say continuous embedding.

\footnote{if $J$ is not an ideal we should say: if $(I, J) \in \mathcal{Y}, U_1 \in J^+, U_0 \in J$ then $|U_1 \setminus U_0| = \mu$.}
4.7 Claim. 1) In 4.4 (i.e. if its assumption so its conclusion holds), we have 
\((A)_\theta \Rightarrow (B)_\theta\) where 
\[(A)_\theta\] if \(\mathcal{P}' \subseteq \mathcal{P}\) has cardinality \(\leq \mu\), then we can find \(h : \cup\{A : A \in \mathcal{P}'\}\) to \(\theta\) such that \(A \in \mathcal{P}' \Rightarrow \theta = \operatorname{Rang}(h \upharpoonright A) = \theta\) 
\[(B)_\theta\] we can find \(h : \lambda \to \theta\) such that \(A \in \mathcal{P} \Rightarrow \theta = \operatorname{Rang}(h \upharpoonright A)\) provided that we add in clause (f) of 4.4 the statement 
\[(\gamma)_\theta\] in (\(\beta\)) we can add: 
\[\text{if } \mathcal{U}_1 \in \mathcal{J} \text{ then for some } \alpha < \alpha^*, \operatorname{cl}\{f(i) : i \in \mathcal{U}_\alpha'\} \text{ is disjoint to } \{f(i) : i \in \mathcal{U}\}.\]

2) In 4.5 we can conclude \((A)_\theta \Rightarrow (B)_\theta\) when 
\[(A)_\theta\] if \(\mathcal{T}' \subseteq \mathcal{T}, |\mathcal{T}'| \leq \mu\) and \(G\) is a function with domain \(\cup\{\mathcal{F}_t : (\exists s \in \mathcal{T}') (s \leq_T t)\}\) such that \(G(f) \in \mathcal{J}_f\), then we can find a function \(h\) and \((t_s, f_s) : s \in \mathcal{T}')\) such that for every \(s \in \mathcal{T}'\) we have \(s \leq_T t_s\) and \(f_s \in \mathcal{F}_{t_s}\) and \(\theta = \{\langle h(f_s(i)) : i \in \operatorname{Dom}(f_s) \rangle \setminus G(f_s)\}\) 
\[(B)_\theta\] we can find a function \(h : \lambda \to \theta\) as in \((A)_\theta\) for \(\mathcal{T}' = \mathcal{T}\) provided that we strengthen in clause (f) of 4.5 
\[\text{for every } s \in \mathcal{T}, (\forall g \in \mathcal{F}_s)(\forall \mathcal{U} \in \mathcal{J})(\exists t \in \mathcal{T}[f]) (s \leq_T t) \& (\forall h \in \mathcal{F}_t)(\forall i \in \operatorname{Dom}(g))(h(i) \notin \operatorname{Rang}(g \upharpoonright \mathcal{U})].\]

3) In part (1) and in part (2) we can replace \((\gamma)_\theta\) by 
\[(\gamma)_\theta\] if \(\mathcal{U} \in \mathcal{J}^+\) and \(\mathcal{U}_1 \in \mathcal{J}\) then \(|\mathcal{U} \setminus \mathcal{U}_1| = \mu.\)

Proof. 1) Recall that \(A \in \mathcal{P} \Rightarrow \operatorname{cl}(A) = A\). Let \(\{A_{\alpha, \zeta}^* : \zeta < \zeta_\alpha \leq \mu\}\) list \(\{A \in \mathcal{P} : \alpha(A) = \alpha\}\) and let \((I_\zeta^\alpha, J_\zeta^\alpha), \mathcal{U}_{\zeta, \alpha}, f_\zeta^\alpha\) witness \(\alpha(A_i) = \alpha\). So by the assumption of 4.4, clause (f)(\(\gamma\)) appearing only in 4.7(1) there is \(A_{\alpha, \zeta}' \in \mathcal{P} \cap M_{\zeta}\) such that \(A_{\alpha, \zeta}' \subseteq \operatorname{cl}\{f_\zeta^\alpha(i) : i \in \mathcal{U}_\zeta\} \cap \mathcal{U} \setminus \mathcal{U}_1\) and \(f_\zeta^\alpha(i) \in M_\alpha \setminus \bigcup_{\beta < \alpha} M_\beta\) \(\subseteq A_{\alpha, \zeta}'\). Clearly \(A_{\alpha, \zeta}' \in \mathcal{P}\) and \(\alpha(A_{\alpha, \zeta}') = \alpha\) and we apply clause \((A)_\theta\) to \(\mathcal{P}_{\zeta} = \{A_{\alpha, \zeta}' : \zeta < \zeta_\alpha\}\) getting \(h_\alpha : \bigcup_{\zeta < \zeta_\alpha} A_{\alpha, \zeta}' \to \theta\) so without loss of generality \(h_\alpha : \lambda \cap M_{\zeta} \setminus \bigcup_{\beta < \alpha} M_\beta \to \theta.\)

Now \(h = \bigcup_{\alpha < \lambda} h_\alpha\) is as required.

2) FILL!
3) Similar. \(\square_{4.7}\)

4.8 Remark. In part (1) of 4.7 we can omit clause \((\gamma)\) if we replace \((A)_\theta\) by
\((A) _{\lambda} ^{\eta} \) if \( \mathcal{P} ^{\prime} \subseteq \mathcal{P} \) has cardinality \( \leq \mu \) and \( \langle A _{\alpha, \zeta} ^{*}, I _{\zeta} ^{\alpha}, J _{\zeta} ^{\alpha}, \mathcal{U} _{\zeta} ^{\alpha}, f _{\zeta} ^{\alpha} \rangle : \zeta < \zeta _{\alpha} ^{*} \) is as in the proof of 4.7, then for some function \( h \) with domain \( \bigcup \{ A _{\alpha, \zeta} : \zeta < \zeta _{\alpha} \} \setminus \{ B _{\alpha, \zeta} : \zeta < \zeta _{\alpha} \} \) we have \( B _{\alpha, \zeta} \subseteq \mathcal{U} _{\zeta} ^{\alpha}, B _{\alpha, \zeta} \in J _{\alpha} \), \( \text{Rang}(h | (\text{Rang}(f _{\zeta} ^{\alpha} | \\mathcal{U} _{\zeta} ^{\alpha}) \cap \text{Dom}(h))) = 0 \).

The following is close to [Sh 161, §3] (or see [Sh 523, §3] or [EM]).

4.9 Definition. 1) We say \( \Gamma = (S, \lambda) \) is a full \( (\lambda, \mu) \)-set if:

(a) \( S \neq \emptyset \) is a set of finite sequences of ordinals
(b) \( S \) is closed under initial segments
(c) \( \bar{\lambda} = (\lambda _{\eta} : \eta \in S) \) and \( \lambda _{<\eta} = \lambda \)
(d) for each \( \eta \in S \), the set \( \{ \alpha : \eta ^{\text{cf}}(\alpha ) \in S \} \) is empty or the regular \( \text{cf}(\lambda _{\eta}) \)
(e) \( \lambda _{\eta} > \mu \iff \lambda _{\eta} \neq \mu \iff (\exists \alpha)(\eta ^{\text{cf}}(\alpha ) \in S) \iff \eta \in S \setminus S ^{\text{max}} \) where \( S ^{\text{max}} \) is the set of \( \alpha \)-maximal \( \eta \in S \)
(f) \( (\alpha) \)  if \( \lambda _{\eta} > \mu \) is a successor cardinal then \( \alpha < \lambda _{\eta} \Rightarrow \lambda _{\eta} ^{+} \langle \alpha \rangle = \lambda _{\eta} \)
(\( \beta \)  if \( \lambda _{\eta} > \mu \) is a limit cardinal then \( \langle \lambda _{\eta} ^{\text{cf}}(\alpha ) : \alpha < \text{cf}(\lambda _{\eta}) \rangle \)
is (strictly) increasing with limit \( \lambda _{\eta} \).

4.10 Observation/Definition: 1) If \( \Gamma = (S, \lambda) \) is a full \( (\lambda, \mu) \)-set, then from \( S \) and \( \lambda \) we can reconstruct \( \bar{\lambda} \) hence \( \Gamma \), so we may say “\( S \) is a full \( (\lambda, \mu) \)-set” or “\( \lambda = \bar{\lambda}[S] \)”.
Also if \( S \neq \{<\rangle\} \), from \( S \) we can reconstruct \( \lambda \) and \( \mu \).
2) Let \( S ^{\text{max}} = \{ \eta \in S : \lambda _{\eta} = \mu \} \).
3) If \( \eta \in S \) and \( \lambda _{\eta} \neq \mu \) then for every ordinal \( \alpha \) we have \( \alpha < \text{cf}(\lambda _{\eta}) \iff \eta ^{\text{cf}}(\alpha ) \in S \).

4.11 Fact/Definition: 1) If \( S \) is a full \( (\lambda, \mu) \)-set and \( \eta \in S \) let \( S ^{<\eta>} = : \{ \nu : \eta ^{\text{cf}}(\nu) \in S \} \), it is a full \( (\lambda _{\eta}, \mu) \)-set.
2) If for each \( \alpha < \text{cf}(\lambda), S _{\alpha} \) is a full \( (\lambda _{\alpha}, \mu) \)-set and \( \lambda _{\alpha} = \lambda _{0} \& \lambda = \lambda _{0} ^{+} \) or \( \langle \lambda _{\alpha} : \alpha < \text{cf}(\lambda) \rangle \) is (strictly) increasing with limit \( \lambda, \lambda _{0} \geq \mu \), then \( S = \{<\rangle \} \cup \bigcup _{\alpha < \text{cf}(\lambda)} \{ \langle \alpha \rangle ^{\text{cf}}(\eta) : \eta \in S _{\alpha} \} \) is a full \( (\lambda, \mu) \)-set.
3) For a full \( (\lambda, \mu) \)-set \( S \) and \( \eta \in S \), let \( \eta ^{+} = \langle \eta(\ell) : \ell < k \rangle ^{\langle \eta(k) + 1 \rangle} \) if \( \ell \gamma(\eta) = k + 1 \) but \( <\rangle^{\gamma} \) will be used though not well defined.

Proof. Straightforward.
4.12 Definition. 1) We define by induction on \( \lambda \) the following. For a set \( X \) of cardinality \( \lambda, \chi \) large enough and \( x \in \mathcal{H}(\chi) \) we say \( \bar{N} \) is a full \( \mu \)-decomposition of \( X \) for \( \mathcal{H}(\chi), x \) (or \( (\lambda, \mu) \)-decomposition) if for some full \( (\lambda, \mu) \)-set \( S \) we have

\[
(*) \quad \bar{N} \text{ is an } S\text{-decomposition of } X \text{ inside } \mathcal{H}(\chi) \text{ for } x, \text{ which means that for (the uniquely determined } \langle \lambda_\eta : \eta \in S \rangle \text{ letting } \lambda_{<\eta} = \lambda \text{ we have: }
\]

(a) \( \bar{N} = \langle (N_\eta, N_\eta^+) : \eta \in S \rangle \)
(b) \( N_\eta < N_\eta^+ < (\mathcal{H}(\chi), \varepsilon, <^*) \), but not necessarily \( N_\eta \in N_\eta^+ \)
(c) \( \{X, x\} \in N_\eta^+ \) and \( \ell < \ell g(\eta) \Rightarrow \{N_\eta|\ell, N_\eta^+|\ell\} \in N_\eta^+ \)
(d) \( \|N_\eta^+\| = \lambda_\eta^+ = |(N_\eta^+ \setminus N_\eta) \cap X| \) and \( \lambda_\eta^+ \subseteq N_\eta^+ \)
(e) if \( \lambda_{<\eta} > \mu \), then \( \langle N_{<\eta} : \alpha < \text{ cf}(\lambda_{<\eta}) \rangle \) is \( \prec \)-increasing continuous with union containing \( N_{<\eta}^+ \)
(f) \( N_{<\eta}^+ = N_{<\eta^+} \) for \( \alpha < \text{ cf}(\lambda_{<\eta}) \) and \( N_{<0} = N_{<\eta} \) has cardinality \( \mu \)
(g) for each \( \alpha < \text{ cf}(\lambda_{<\eta}(S)) \) the sequence \( \langle (N_{<\eta} : \eta \in S^{<\alpha}) : \eta \in S^{<\alpha} \rangle \) is a \( (\lambda, \mu) \)-decomposition of \( X \cap N_{<\eta}^+ \) for \( \mathcal{H}(\chi) \) and \( x' =: \langle x, \eta, \alpha, \alpha^+ \rangle \).

2) We say \( \bar{N} \) is a full \( (\lambda, \mu, \sigma) \)-decomposition of \( X \) for \( \mathcal{H}(\chi), x \) if \( \sigma = \text{ cf}(\sigma) \leq \mu \) and in addition

\( (h) \) for each \( \eta \in S \setminus S^{\max} \) there is a sequence \( \langle N_{\eta, \varepsilon} : \varepsilon \leq \sigma \rangle \) which is \( \prec \)-increasing continuous, \( N_{\eta, 0} = N_\eta \), for each \( \varepsilon < \sigma \) we have \( \langle N_{\eta, \varepsilon} : \varepsilon \leq \sigma \rangle \in N_{\eta, 0^+} \) and \( N_{\eta}^+ = N_{\eta, \sigma} \) [alternatively the objects we demand \( \in N_{\eta, \sigma} \) in clauses (c) and (h)] and in (c) we add \( \ell < \ell g(\eta) \Rightarrow \langle N_{\eta|\ell, \varepsilon} : \varepsilon \leq \sigma \rangle \in N_{\eta}^+ \).

3) We can write \( \langle N_\eta : \eta \in S \cup \{<\eta^+\} \rangle \) instead of \( \langle (N_\eta, N_\eta^+) : \eta \in S \rangle \) by clause (f) so in particular \( N_{<\eta^+} = N_{<\eta}^+ \).

4.13 Definition. 1) Let \( X, \lambda, \mu, \mathcal{Y}, \mathcal{F} \) be as in 4.5 so \( T = \text{ Dom}(\mathcal{F}) \). We say \( \bar{N} \) is good for \( (x, X, \mathcal{Y}, T, \mathcal{F}) \) if:

(a) \( \bar{N} \) is a full \( (\lambda, \mu) \)-decomposition of \( X \) for \( \mathcal{H}(\chi) \) and \( x' =: \langle x, X, \lambda, \mu, \mathcal{Y}, T, \mathcal{F} \rangle \); let
\[
\bar{N} = \langle (N_\eta, N_\eta^+) : \eta \in S \rangle \text{ and } \bar{\lambda} = \bar{\lambda}^{[S]}
\]
(b) if \( s \in T \), then for some \( t \in T, s \leq_T t \) and for some \( \eta \in S^{\max} \) (i.e., \( \lambda_\eta = \mu \)) there is \( f \in \mathcal{F}_t \) and so \( (I_f, J_f) \in \mathcal{Y}, \mathcal{U}_f \subseteq J_f^+ \) and \( f : \mathcal{U}_f \rightarrow \text{ Rang}(g) \) witnessing it, such that:

\( (*)_1 \) \( \{i \in \mathcal{U}_f : f(i) \in \cup\{N_\nu : \nu \leq x, \eta \text{ and } \nu \in S^{\max}\} \} \) belongs to \( J_f \)
2) We may omit $x$ if clear from the context.

4.14 The Main Claim. Under the assumption of 4.5, for $x \in \mathcal{H}(\chi)$, $\sigma = \text{cf}(\sigma) \leq \mu$ and $\chi$ large enough there is a full $(\lambda, \mu, \sigma)$-decomposition of $X$ for $\chi, x$ good for $(X, \mathcal{Y}, T, \mathcal{F})$.

Proof. By induction on $\lambda = |X|$ for all possible $(T, \mathcal{F})$ without loss of generality $|T| = \lambda$.

Case 1: $\lambda = \mu$.

Trivial.

Case 2: $\lambda = \text{cf}(\lambda) > \mu$.

Choose $\langle N_\alpha : \alpha < \text{cf}(\lambda) \rangle$ such that the set $x^* = \{x, X, \mathcal{F}, \mu, \lambda, f \mapsto T_{<f}, f \mapsto T[f]\}$ belongs to $N_0, N_\alpha < (\mathcal{H}(\chi), \in, \chi)$, $N_\alpha$ is $\in$-increasing continuous, $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$, each $N_\alpha$ has cardinality $< \lambda$ and $N_\alpha \cap \lambda$ is an initial segment if $\alpha > 0$ and $||N_0|| = \mu, \mu \subseteq N_0$. For $t \in T$ let $\alpha(t) =: \text{Min}\{\alpha : \text{for some } f \in \bigcup_{s \geq t} \mathcal{F}_s \text{ and } (I_f, J_f) \in \mathcal{Y} \text{ (as in 4.5 clause (e))} \}$ we have $\{i : i \in \text{Dom}(f) \text{ and } f(i) \in N_\alpha\} \subseteq J^+$. Let $S = \{\beta < \lambda : \text{for some } t \in T \text{ we have } \beta = \alpha(t)\} \subseteq \lambda$. For each $\beta \in S$ choose $t_\beta \in T$ and $s_\beta$ satisfying $t_\beta \leq_T s_\beta$ such that $\beta = \alpha(t_\beta)$ and $f_\beta \in \mathcal{F}_{s_\beta}$ witness this. Let $\mathcal{U}_\beta = \{i \in \text{Dom}(f_\beta) : f_\beta(i) \in N_\beta\}$ and let $f_{J_\beta} =: f_\beta \upharpoonright \mathcal{U}_\beta$ and let $(I_\beta, J_\beta) = (I_{f_\beta}, J_{f_\beta})$ so $\mathcal{U}_\beta \subseteq J_\beta^+$ and $\text{Rang}(f_\beta) \subseteq N_\beta \cap \lambda$. Now without loss of generality $f_\beta \in N_{\beta+1}$ (hence $s_\beta, I_\beta, J_\beta \in N_{\beta+1}$) as all the requirements on $f_\beta$ have parameters in $N_{\beta+1}$ so we could have chosen $f_\beta$ in $N_{\beta+1}$. By renaming, without loss of generality $X = \lambda$.

Assume toward contradiction that $S$ is stationary. Now as $\mathcal{Y} \in N_0, ||\mathcal{Y}| \leq \mu < \lambda$ clearly $\mathcal{Y} \subseteq N_0$ hence for some $y \in \mathcal{Y}$ the set $S_y = \{\beta \in S : (I_\beta, J_\beta) = y\}$ is stationary. Let $y = (I^*, J^*)$ and $S_y' = \{\beta \in S_y : N_\beta \cap \lambda = \beta\}$, clearly it is stationary. It suffices to show that $\langle f_{J'} : \delta \in S_y' \rangle$ exemplifies $\text{NFr}_2(\lambda, \mathcal{Y})$ contradicting assumption (e)' from 4.5. Clearly $S_y' \subseteq \lambda$ and $\delta \in S_y' \Rightarrow \text{Dom}(f_{J'}) \in J^+$ and $\text{Rang}(f_{J'}) \subseteq \delta$, i.e., let us prove clause (c) of Definition 4.1(2) holds. If not, for some $\delta_1 < \delta_2$ in $S_y'$ we have $B =: \{i : i \in \text{Dom}(f_{J_1}), i \in \text{Dom}(f_{J_2})\}$
and $f'_2(i) = f'_2(i) \in I^+$, hence $t_{\delta_2} \in T(f'_1)$ (see 4.5, clause \((f')\)) hence by an assumption there is $t_{\delta_2}$ such that $t_{\delta} \leq_T t_{\delta_2} \in T[f'_1]$. But $x^*, \mathcal{F}, f_{\delta}$ belong to $N_{\delta + 1} < N_{\delta}$ hence $T[f_{\delta}] = N_{\delta + 1}$ but $T[f_{\delta}]$ has cardinality $\leq \mu$ (see clause \((f')\) of 4.5) hence $T[f_{\delta}] \subseteq N_{\delta + 1}$ but $t_{\delta_2} \in T[f_{\delta}]$ so $t_{\delta_2} \in N_{\delta + 1}$ hence $\mathcal{F}_{t_{\delta_2}} \in N_{\delta + 1}$ hence there is $f' \in \mathcal{F}_{t_{\delta_2}} \cap N_{\delta + 1}$ contradicting the demand $\alpha(t_{\delta_2}) = \delta_2$. So in Definition 4.1(2) only “$S$ is a stationary subset of $\lambda$” may fail, but something has to fail. So $S$ is not stationary.

Let $E$ be a club of $\lambda$ disjoint to $S$ and we can find $\hat{N}' = \langle N_{\alpha} : \alpha < \lambda \rangle$ like $\langle N_{\alpha} : \alpha < \lambda \rangle$ such that $E, \hat{N} \in N_{\alpha}'$ so for $\hat{N}', S = \emptyset$. Recall that by the assumption of 4.5, $T$ has cardinality $\leq \lambda$ hence $T \subseteq \cup \{N_{\alpha} : \alpha < \lambda\}$. So for $\alpha \in (0, \lambda)$ for some $\delta \in E$ we have $N_{\alpha}' \cap \lambda = \delta$ so $N_{\alpha}' \cap T = N_{\delta} \cap T, N_{\alpha}' \cap \bigcup T_t \cap \bigcup N_{\beta} = N_{\delta} \cap \bigcup T_t$.

Now for each $\alpha$ we use the induction hypothesis on $X_{\alpha} = X \cap N_{\alpha}' \cap N_{\alpha}'$ and $\langle \mathcal{F}_t^{(\alpha)} : t \in T^{(\alpha)} \rangle$ where $T^{(\alpha)} = \{t \in T : t \in N_{\alpha} \setminus N_{\alpha}^0\}$ and, moreover, for every $f \in \mathcal{F}_t$ the set $\{i \in \text{Dom}(f) : f(i) \in N_{\alpha}'\}$ belongs to $J$ and $\mathcal{F}_t^{(\alpha)} = \{f \upharpoonright \mathcal{U} : \mathcal{U}$ is $\{i \in \text{Dom}(f) : f(i) \in X_{\alpha}\}$ and $f \in \mathcal{F}_t \cap N_{\alpha}' \cap N_{\alpha}'\}$, and $\nu_0 = \langle x^*, \alpha, \hat{N}' \rangle$ so by it we get $\langle N_{\alpha}^0 : \eta \in S^{(\alpha)} \rangle$ and we let $S = \langle \langle \mathcal{U} : \mathcal{U} \subseteq S_{\alpha}, \alpha < \lambda \rangle \rangle$ and $N_{\alpha \nu > \nu} = N_{\nu}^0, N_{\alpha > \nu} = N_{\nu}^0, N_{\alpha} = \bigcup N_{\alpha}^0$.

Note that if $\lambda > \mu^+, |T| > \lambda$, we can still manage\(^\dagger\) but not needed. Also if $\sigma = \text{cf}(\sigma) \leq \mu$, we can guarantee clause (h) of 4.12(2); similarly to Case 3.

**Case 3:** $\lambda$ singular $> \mu$.

Let $\lambda = \sum\limits_{i < \text{cf}(\lambda)} \lambda_i$ and let $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ be increasing continuous, $\lambda_0 > \mu^+ + \text{cf}(\lambda)$. We choose by induction on $\zeta < \mu^+$ a sequence $\langle N_i^{\zeta} : i < \text{cf}(\lambda) \rangle$ such that:

(a) $N_i^{\zeta}$ is $\prec$-increasing continuous in $i$

(b) $\langle \lambda_i : i < \text{cf}(\lambda) \rangle, X, \lambda, \mu, \mathcal{F}$ all belong to $N_0^\zeta$

(c) $\lambda_i \subseteq N_i^{\zeta}$ and $\|N_i^{\zeta}\| = \lambda_i$ if $(i, \zeta) \neq (0, 0)$ and $\mu \subseteq N_0^\zeta, \|N_0^\zeta\| = \mu$

(d) for each $i, \langle N_i^{\zeta} : \zeta \leq \mu^+ \rangle$ is $\prec$-increasing continuous

(e) $\langle \langle N_i^{\zeta} : i < \text{cf}(\lambda) \rangle : \varepsilon \leq \zeta \rangle \in N_i^{\zeta + 1}$

For each $i < \text{cf}(\lambda)$ and $\zeta < \mu^+$ and $(I, J) \in \mathcal{U}$ let $\mathcal{F}_{(I, J)}^{\zeta, i}$ be a maximal family of functions $f \in \{f \upharpoonright \mathcal{U} : \mathcal{U} \in J^+, f \in \bigcup T_t \}$ and $\mathcal{F} \subseteq \text{Dom}(f)$ such that

\(^\dagger\)We should strengthen the induction hypothesis: instead $X$ we have $X_0 \subseteq X_1$ such that $|X_0| = \lambda$, and continues naturally.
Rang$(f) \subseteq X \cap N_i^\zeta$ and $f \neq g \in \mathcal{F}(I,J)$ implies $\{i : i \in \text{Dom}(f), i \in \text{Dom}(g) \text{ and } f(i) \neq g(i)\} \in I$. Without loss of generality $\mathcal{F}(I,J) \subseteq N_0^{\zeta+1}$ and by 4.2(4) and assumption 4.5 clause (c) we know $|\mathcal{F}(I,J)| \leq \lambda_i$, so a list of it of length $\leq \lambda_i$ belongs to $N_i^{\zeta+1}$ hence $\mathcal{F}(I,J) \subseteq N_i^{\zeta+1}$. So if $t \in T$ and we define $\alpha(t)$ as in Case 2 for $\langle N^\zeta_\alpha : \alpha \leq \text{cf}(\mu) \rangle$, we get that $\alpha(t)$ is necessarily non-limit. Then let $N_\alpha = N^\mu_\alpha$ if $\alpha \in (0, \text{cf}(\lambda))$ and $N_0 = N_0^0$ and proceed as there (recalling that in Definition 4.12 we have not demanded that $N_\eta \in N_0^+$).

□4.14

**Proof of 4.5.**

Fix $\sigma = \text{cf}(\sigma) \leq \mu$ and by 4.14 we can find a full $(\lambda, \mu, \sigma)$-decomposition of $X$ for $\chi, x$ which is good for $(X, \mathcal{F})$. Let $N = \langle N_\eta : \eta \in S \rangle$ and note that $<_{\text{lex}}$ linearly order $S$ in an order of order type $\lambda$, so let $\langle \eta_\alpha : \alpha < \lambda \rangle$ list $S$ in $<_{\text{lex}}$-increasing order. Choose $M_\alpha = \cap\{N^{+}_\eta : \ell < \ell g(\eta)\} \cap N_\eta$ and check that $(M_\alpha : \alpha < \lambda)$ is as required in 4.5 (reading Definition 4.13).

This completes the proof of 4.4 above, too. □4.4.4.5

**Proof of Lemma 3.7.**

Just\footnote{recall that 3.7 is not used in the proof of 4.18} by 4.14 above and 4.7, we do not elaborate as 3.7, 3.8 are not used in other proofs. Saharon FILL!

**Proof of 3.13.** We use 4.4 + 4.7(1) above.

1) Without loss of generality let $\lambda$ be the set of points of $X$ where $X, \mu$ are given in 3.13, $I = \{A \subseteq \omega^2 : \text{the closure of } A \text{ is countable}\}$, $J$ the following ideal on $\omega^2$

$$\{\mathcal{U} \subseteq \omega^2 : |\mathcal{U}| < 2^{R_0}\}$$

and

$$\mathcal{Y} = \{(I, J)\}.$$
\[ c\ell(A) = \{ \alpha : \alpha \in A \text{ or for some countable } B \subseteq A, \alpha \text{ belongs to the closure of } B \text{ in the topological space } X \text{ and } \\
\text{c\ell}(B) \text{ has cardinality } \leq 2^{\aleph_0} \} \]

(if we like to have \( c\ell(A) = c\ell(c\ell(A)) \), iterate this \( \omega_1 \)-times).

Let us consider the assumptions of 4.4 and 4.7. Now clause (a) holds by the explicit choice of \( Y \) above, as for clause (b), we have \(|\mathcal{Y}| = 1, \kappa(\mathcal{Y}) = 2^{\aleph_0}\) which is \( \leq \mu \) by the assumption of 3.13. Clause (c) is the assumption toward contradiction above, clause (d) (on \( c\ell \)) holds as clearly \( A \in [\lambda]^{\leq \mu} \) implies \( c\ell(A) = \bigcup \{ c\ell(B) : B \in [A]^{\leq \aleph_0} \text{ and } |c\ell(B)| \leq \aleph_0 \} \) and \([A]^{\leq \aleph_0}\) has cardinality \( \mu \) and \( \leq \mu^{\aleph_0} = \mu \) by an assumption of 3.13 and each countable \( B \) contribute at most \( 2^{\aleph_0} \) points. Clause (e) holds by the properties of closure. Lastly, for clause (f) including subclause (\( \gamma \)) which was added in 4.7 we define

\[ \mathcal{P} = \{ A : A \subseteq \lambda \text{ is a subset of } A, \text{ has cardinality continuum and } X \upharpoonright A \text{ is homeomorphic to } \omega^2 \} \]

So for \( A \in \mathcal{P} \) let \( f = f_A \) be a homeomorphism from the topological space \( \omega^2 \) onto the space \( X \upharpoonright A \) and let \( \mathcal{Y}_A = \omega^2 \); we shall show that they are as required in (f) of 4.4. Now for \( A \in \mathcal{P} \), \( X \upharpoonright A \) is a compact space and \( X \) is Hausdorff, hence \( A \) is a closed subset of \( X \). If \( \mathcal{U}' \subseteq \omega^2, \mathcal{U}' \notin \mathcal{I} \), i.e. \( \mathcal{U}' \) is not scattered letting \( \mathcal{U}'' = \{ \nu \in \mathcal{U}' : \text{for no open nb of } \nu \text{ in } \omega^2 \text{ is } \mathcal{U}' \text{ scattered} \} \) and \( f = f_A \), then we have

\[ A' = c\ell\{ f(i) : i \in \mathcal{U}'' \} = \{ f(i) : i \in c\ell_{\omega^2}(\mathcal{U}'') \} \subseteq A \]

is homeomorphic to \( \omega^2 \) hence \( \in \mathcal{P} \), and this proves clause (\( \alpha \)).

Let \( \{ \mathcal{Y}^i_{\alpha} : \alpha < 2^{\aleph_0} \} \) list the countable nonscattered subsets of \( \omega^2 \), it clearly exemplifies clause (\( \beta \)) (of (f) of 4.4).

Lastly, clause (\( \gamma \)) (which does not appear in 4.4 but in 4.7(1)), any perfect subsets of \( \omega^2 \) contain \( 2^{\aleph_0} \) many pairwise disjoint perfect subsets so any member of \( J \) is disjoint to all but \( < 2^{\aleph_0} \) of them.

So as all the assumptions of 4.4 hold so we can apply 4.7(1). There for our \( \theta = \mu \), if \( (A)_{\theta} \) of 4.7(1) holds, then we get \( (B)_{\theta} \) which says \( X \nrightarrow [\omega^2]_{\theta} \) contradicting an assumption. But \( (A)_{\theta} \) of 4.7 holds as we have assumed in 3.13: for every subspace \( X^* \) of \( X \) with \( \leq \mu \) points, \( X^* \nrightarrow (\omega^2)_{\theta} \) and as for (\( \delta \)) there it was checked above.
2) For \( \rightarrow_w \) recall that: if \( X \) is a Hausdorff space, \( A \subseteq X \) and \( X \upharpoonright A \) is a compact space then \( A \) is a closed subset of \( X \). For \( \mathbb{R} \) we just should be more accurate about closure; note that the topological closure of a countable set may have cardinality bigger than \( 2^{\aleph_0} \). For \( A \subseteq X \) let \( c\ell(A) = c\ell(A, X) = \cup (\text{Rang}(f) : f \text{ a one to one}
\text{mapping from } \mathbb{R} \text{ to } X \text{ which is a}
\text{homeomorphism onto } X \upharpoonright \text{Rang}(f) \text{ and such that}
Y_f = \{x \in \mathbb{R} : f(x) \in A\} \text{ is a dense subset of } \mathbb{R}\} \). But for any such \( f_1, f_2 \), if some \( Y \subseteq Y_{f_1} \cap Y_{f_2} \) is countable dense and \( [x \in Y \Rightarrow f_1(y) = f_2(y)] \text{ then } f_1 = f_2 \), so the proof is similar. Alternatively replaced \( \mathbb{R} \) by \([0, 1]_{\mathbb{R}}\).

\( \square_{3,13} \)

As should be clear from the previous part of the paper, \( \text{NFr}_2(\lambda, \mathcal{Y}) \) is closely connected to pcf theory. In particular, on the one hand, §1 uses essentially the cases of \( \text{NFr}_1 \) whose consistency is not clear (i.e. hopefully it will be proved that they are impossible). On the other hand, §2 uses a case of \( \text{NFr}_2 \), say for \( I = [\omega_1]^{<\aleph_0} \). So let us explicate the obvious relation (and the connection to [Sh 460, 3.9]).

The reader may wonder why not finer properties complimentary to the existence of large almost disjoint families were used, as in [Sh 430]; the answer is that here assumption like \( \mu^\sigma = \sigma \) are natural (and limitations on time).

\[\textbf{4.15 Claim.} \]

1) If \( \text{NFr}_1(\lambda^*, \lambda, \mathcal{Y}) \) and \( I^* \) is an ideal on \( \kappa = \kappa(\mathcal{Y}) \), satisfying (*) below, then there is \( \mathcal{F} \subseteq {}^*\lambda \text{ such that } f \neq g \in \mathcal{F} = \{i < \kappa : f(i) = g(i)\} \in I^* \text{ and } |\mathcal{F}| = \lambda^* \) where

\[(*) \text{ if } (I, J) \in \mathcal{Y} \text{ and } A \in J^+ \text{ then for some one-to-one function } h \text{ from } \kappa \text{ into } A \text{ we have } \text{Rang}(h) \notin J \text{ and for every } B \subseteq \kappa \text{ we have } \{\{h(\alpha) : \alpha \in B\} \in I \Rightarrow B \in I^*\}.\]

2) If \( \text{NFr}_1(\lambda^*, \lambda, \{(I^*, J^*)\}) \) or \( (\lambda^*, \lambda, \mathcal{Y}, I^*) \) is as in part (1), and \( 2^\kappa \leq \lambda \) then for some sequence \( \vartheta = (\theta_i : i < \kappa) \) of regular cardinals in \( [2^\kappa, \lambda] \) we have \( \prod_{i<\kappa} \theta_i/I^* \) has true cofinality which is \( \geq \lambda^* \).

3) Assume

(a) \( \text{NFr}_2(\lambda, \mathcal{Y}) \), so \( \lambda \) regular \( > |\mathcal{Y}| \)

(b) \( \mathcal{Y}' \) a family of pairs \( (I, J) \) satisfying \( \kappa(\mathcal{Y}') \leq \kappa \) and: if \( (I, J) \in \mathcal{Y} \), \( h \) is a function from \( \text{Dom}(I, J) \) into a limit ordinal \( \delta \), then for some \( A \in J^+, h''(A) \) is bounded in \( \delta \) and \( (I \upharpoonright A, J \upharpoonright B) \in \mathcal{Y}' \).

Then for some \( \lambda' < \lambda \), we have \( \text{NFr}_1(\lambda, \lambda', \mathcal{Y}') \).

\[\text{Proof. Straight.}\]
4.16 Conclusion. If $\mu$ is a limit cardinal satisfying $\otimes_\mu$ below, then $\lambda = \text{cf}(\lambda) > \mu > \kappa$ implies $\text{Fr}_2(\lambda, ([\mu]^\kappa_\mu, [\mu]^{\kappa}_\kappa))$ where

$\otimes_\mu$ for every $\lambda > \mu$ for some $\theta < \mu$ we have:

if $a \subseteq \text{Reg} \cap \lambda \setminus \mu$ and $|a| < \mu$ then $\text{pcf}_{\text{complete}}(a) \subseteq \lambda$.

Proof. Easy by 4.15.

4.17 Concluding Remark. 1) Of course, we may replace in 3.2 the space $\omega^2$ by many others, e.g. $\mathbb{R}$, or any Hausdorff $Y^*$ space with $2^{\aleph_0}$ points such that for any uncountable $A \subseteq Y^*$, for some countable $B \subseteq A, |\text{cl}_{Y^*}(B)| = 2^{\aleph_0}$ moreover if $Z \subseteq Y^*, |Z| < 2^{\aleph_0}$ for some uncountable $B' \subseteq \text{cl}_{Y^*}(B)$ we have $\text{cl}_{Y^*}(B')$ is disjoint to $Z$.

We can also add variants with $\rightarrow_w$ replacing $\rightarrow$. As long as the space has $\leq 2^{\aleph_0}$ points, the only place we should be concerned is the proof of 3.13, we reconsider the choice of $c\ell$ in the proof. In all cases for an embedding $f$ from $Y \subseteq Y^*$ to $X$, let $c\ell(\text{Rang}(f)) = \{x \in X : \text{for some } y \in Y^*, f \cup \{(y, x)\}$ is an embedding of $Y^* \upharpoonright (\mathbb{N} \cup \{x\})$ to $X \upharpoonright ((\text{Rang}(f)) \cup \{y\})$ and $f^+ = f \cup \{(y, x) : x, y \text{ as above}\}$. The point is that for this choice of $c\ell$, if $Y_1 \subseteq Y_2 \subseteq Y^*, Y_2 \subseteq \text{cl}_{Y^*}(X_1)$ and $f$ embeds $Y_2$ into $X$ with $\text{Rang}(f)$ not necessarily close, then $(f \restriction X_1)^+$ is a function from some $Y_3 \subseteq Y^*$ into $X$ extending $f$.

2) We may like to add to 3.2 the case with continuum many colours that is let $(B_m)_{<\mu}^{\omega^2}$ and $(B_m)_{<\mu}^+$ be defined like $(B)_{<\omega^2}, (B)^+$, replacing $1_{<\mu}^{\omega^2(2^{\aleph_0})}$ by $1_{<\mu}^{\omega^2}$ and we add $(B_m)_{<\omega^2}, (B)^{<\omega^2}$ to the list of equivalent statements. Similarly for (A). More is proved, that is $X \rightarrow (\omega^2)^1_{<\lambda}$ where $X$ has $\lambda$ points (or we get $\lambda$ when we ask for compact $X$). The main point is adopting 1.2 (and 1.7).

For this we add also $(C_m)_{<\omega^2, \omega^2, \omega^2}$ where for $\kappa \geq \theta \geq \sigma$ we let

$(C_m)_{\kappa, \theta, \sigma}$ there are $\lambda, S, \bar{f}$ such that

(a) $S \subseteq \lambda$ is stationary $> \kappa^+, \kappa > \theta \geq \sigma$

(b) $\bar{f} = (f_\delta : \delta \in S)$

(c) $\text{Dom}(f_\delta) = \theta$, each $f_\delta(i)$ is a subset of $\delta \setminus i$ of cardinality $\leq \kappa$ and $\langle \min(f_\delta(i)) : i < \theta \rangle$ is increasing with limit $\delta$ (can ask $i < j < \theta \Rightarrow f(\sup(f_\delta(i))) < \min(f_\delta(j))$)

(d) if $\delta_1 < \delta_2$ are in $S$ then $\{i < \theta : f_{\delta_2}(i) \cap \bigcup_{j < \theta} f_{\delta_1}(j) \neq \emptyset\}$ has cardinality $< \sigma$
(e) if $F_\ell : \lambda \to [\lambda]^{<\kappa}$ for $\ell = 0, 1$ and $F_0(\alpha) \in [\lambda \setminus \alpha]^{<\kappa}$, then for some $\delta \in S$ we have:

(\alpha) $f_\delta$ is $(F_0, F_1)$-free which means:

for $i \neq j < \theta$, the set $F_1(f_\delta(i))$ is disjoint to $F_0(f_\delta(j))$

(\beta) there are $\langle \alpha_i : i < \theta \rangle$ such that $f_\delta(i) = F_0(\alpha_i)$ and $\sup \{ \bigcup f_\delta(i) : j < i \} < \alpha_i$.

Similarly for (D). Why is this O.K.? See below, noting that we get more.

3) As before, $(B_m)^+ \Rightarrow (B_m)[^\omega 2]$ and $(B_m)^+ \Rightarrow (A_m)[^\omega 2]$, also easily $(C) \Rightarrow (C)_2^+ (B_m)^+ \Rightarrow (A)^+, (B_m)[^\omega 2] \Rightarrow (B)[^\omega 2]$ and $(A_m)[^\omega 2] \Rightarrow (A)[^\omega 2]$

(f) if $(F_0, F_1)$ is a pair of functions with domain $\lambda$ and $F_0(i) \in [\lambda \setminus i]^{<\kappa}$.

3A) The forcing in 2.8, with the role of $A_\zeta$ being replaced by $\bigcup_{i<\theta} f_\zeta(i)$ and $A_\zeta^P \subseteq \bigcup_{i<\theta} f_\delta(i)$ such that $i < \theta \Rightarrow |A_\zeta^P \cap f_\delta(i)| \leq 1$ works.

4) Also $\mathbb{2}_3$ implies the consistency of $(B_m)^+ \mathbb{2}_3$.

As before without loss of generality for some $\kappa = \kappa^{<\kappa} \geq \theta = 2^{\aleph_0}, \sigma$ are such that $(C)_{\kappa, \theta, \sigma}$ hold. Now we just need to repeat the proof of 1.2. The asymmetry in clause (d) does not hurt as if $\delta_2 \neq \delta_1$, $A_\delta^P$, $A_{\delta_1}^P$, $A_{\delta_2}^P$ are well defined, then it follows that $|A_\delta^P \cap A_{\delta_1}^P| < \sigma$.

In the crucial point we let $p \ast \Vdash " \zeta : \lambda \to \mu \text{ for some } \mu < \lambda"$. Really less is enough: let $p \ast \Vdash " Z \subseteq \lambda \text{ is unbounded}"$ and we shall find $q$ and $\delta \in S$ such that $p \ast \leq q \in P$ and $q \Vdash " X \upharpoonright A_\delta^P \text{ is a copy of the space } Y \text{ (e.g. } \omega 2 \text{ and } A_\delta^P \subseteq Y"$. How? We define $F_0(\alpha) = \{ \beta : \beta \in [\alpha, \lambda) \text{ and } p \ast \not\Vdash \beta \neq \text{Min}(Z \setminus \alpha) \}$.

$F_1(\alpha) = \cup \{ u_{p, \alpha, i} : i < \kappa \}$ where $\{ p_{\alpha, i} : i < \kappa \}$ is a maximal antichain above $p \ast$ such that $p_{\alpha, i}$ forces $\alpha \notin Z$ or forces $\alpha \notin Z$.

Now we repeat the proof of 1.2, but instead deciding the colour we decide the right member of $Z$.

5) Lastly, we get $(C)^{+\kappa, \theta, \nu}$ from $(C)_{\kappa, \theta, \nu}$. So assume $\lambda > \kappa^+, \kappa > \theta \geq \sigma$ and $\langle A_\delta : \delta \in S \rangle$ are as in $(C)$ and as before (by forcing) without loss of generality $\Diamond S$.

Now we can actually prove $(C)^{\kappa, \theta, \nu}$ for $\lambda$. So we prove
\[ \text{if} \]

\( (\alpha) \quad \lambda > \kappa^+, \kappa > \theta \geq \sigma, \kappa^\sigma < \lambda \)

\( (\beta) \quad J \text{ an ideal on } \theta \text{ such that } (\forall A \in J^+)(\exists a \in J^+)(a \subseteq A) \)

\( (\gamma) \quad S \subseteq \lambda \text{ is stationary, } \bar{f} = \langle f_\delta : \delta \in S \rangle, f_\delta : \theta \to \theta \text{ increasing, } \delta_1 < \delta_2 \Rightarrow \{i < \theta : f_{\delta_1}(i) = f_{\delta_2}(i)\} \in J^+ \)

\( (\delta) \quad \diamond_S. \)

Then \((C)_{\kappa, \theta, \sigma} \text{ as witnessed by } \lambda. \)

So let \(\langle (F_\delta^0, F_\delta^1) : \delta \in S \rangle\) be such that \(F_\delta^\ell : \delta \to [\delta]^{< \kappa} \text{ for } \ell = 0, 1 \text{ be such that: if } F_\ell : \lambda \to [\lambda]^{< \kappa} \text{ for } \ell = 0, 1 \text{ then } S_{(F_0, F_1)} = \{\delta \in S : F_0 \upharpoonright \delta = F_0^\delta \text{ and } F_1 \upharpoonright \delta = F_1^\delta\} \text{ is stationary.} \)

We now choose by induction on \(\delta \in S\) a function \(f_\delta\) such that:

\( (a) \text{ if there is a function } f \text{ with domain } \theta \text{ satisfying the conditions below then } f_\delta \text{ is such a function, otherwise } f_\delta \text{ is constantly } \emptyset \)

\( (\alpha) \quad f(i) \in [\delta]^{< \kappa} \setminus \{\emptyset\} \)

\( (\beta) \quad i < j \Rightarrow \sup(f_\delta(i)) < \min(f_\delta(j)) \)

\( (\gamma) \quad \text{for each } i < \theta \text{ for some } \alpha_i < \delta \text{ we have } F_\delta^0(\alpha_i) = f_\delta(i) \text{ and } \sup(\bigcup_{j<i} f(j)) < \alpha_i \leq \min f(i) \)

\( (\delta) \quad \langle \min(f(i)) : i < \theta \rangle \text{ converge to } \delta \)

\( (\varepsilon) \quad \text{for } i \neq j < \theta \text{ the set } F_\delta^0(f(i)) \text{ and } F_\delta^0(f(j)) \text{ are disjoint} \)

\( (\zeta) \quad \text{if } \delta_1 \in \delta \cap S \text{ then } \{i < \delta : f(i) \cap \bigcup_{j<\theta} f_{\delta_1}(j) \neq \emptyset\} \text{ has cardinality } < \sigma. \)

Let \(S^- = \{\delta \in S : f_\delta \text{ is not constantly } \emptyset\}\) and we suffice to prove that \(\bar{f} = \langle f_\delta : \delta \in S^- \rangle\) is as required. Most clauses hold by the definition and we should check clause \((e)\), so let \(F_0, F_1\) be as there. Let \(S_{F_0, F_1} = \{\delta \in S : F_0 \upharpoonright \delta = F_0^\delta \text{ and } F_1 \upharpoonright \delta = F_1^\delta\}\), so this set is stationary.

For every \(\alpha \in S^* = \{\delta < \lambda : \text{cf}(\delta) = \kappa^+\}\) let \(g(\alpha) = \sup(\alpha \cap F_1(\alpha)) < \alpha \) so \(g\) is constantly \(\alpha(\ast)\) on some stationary \(S^{**} \subseteq S.\)

\(E_0 = \{\delta < \lambda : \text{otp}(S^{**} \cap \delta) = \delta \text{ and } \alpha < \delta \Rightarrow \text{otp}(F_0(\alpha)) < \delta \text{ and } \alpha < \delta \Rightarrow \text{otp}(F_1(\alpha)) < \delta\}. \)

Let \(E_1^* = \{\delta < \lambda : \text{otp}(E_0 \cap \delta) = \delta\}\) and for \(\delta \in E_1 \cap S_{F_0, F_1}\) let \(A'_\delta = \{\alpha \in E_0 : \text{otp}(\alpha \cap E_0) \in A_\delta\}\), so \(A_\delta \subseteq \delta = \text{sup}(A_\delta), \text{otp}(A_\delta) = \theta \text{ and } \delta_1 \neq \delta_2 \in E_1 \cap S_{F_0, F_1} \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \sigma.\)
Let $A_\delta = \{\alpha'_\delta, i < \theta\}$ increasingly and let $\alpha\delta,i = \operatorname{Min}(S^{**}\setminus(\alpha'_\delta + 1))$ so $\alpha\delta,i < \alpha''\delta,i+1$ (even $\alpha\delta,i < \operatorname{Min}(E_1\setminus(\alpha'_\delta + 1)$ and choose $f'_\delta$ a function with domain $\theta$ by

$$f'_\delta(i) = F_0(\alpha\delta,i) = F^\delta_0(\alpha'_i)$$

(the last equality as $F_\ell | \delta = F^\delta_\ell$ as $\delta \in S_{F_0,F_1}$).

Clearly $f'_\delta(i) = F_0(\alpha\delta,i) \subseteq \operatorname{Min}(E_1\setminus(\alpha'_\delta + 1))$ and

$$\gamma \in f'_\delta(i) \Rightarrow F(\gamma) \subseteq \operatorname{Min}(E_1\setminus(\alpha'_\delta + 1)) \leq \alpha'\delta,i+1$$

$$\gamma \in f'_\delta(i) \Rightarrow F(\gamma) \cap \alpha\delta,i \subseteq \alpha(*) < \alpha_0$$

Now $f'_\delta$ satisfies almost all the requirements on $f_\delta$ and if $f'_\delta = f_\delta$ for stationarily many $\delta \in E_1 \cap S_{F_0,F_1}$ we are done. Let $W = \{\delta \in E_1 \cap S_{F_0,F_1} : f'_\delta \neq f_\delta\}$, we shall prove that $W$ is not stationary - this is more than enough.

So for $\delta \in W$ necessarily for some $h(\delta) \in \delta \cap S$ we have

$$w_\delta = \{i < \theta : f'_\delta(i) \cap \bigcup_{j < \theta} f(\delta)(j) \neq 0\}$$

has cardinality $\geq \sigma$, so by Fodor’s lemma for some $\delta(*)$ we have $W_1 = \{\delta \in W : h(\delta) = \delta(*)\}$ is stationary.

Similarly as $\theta^\sigma < \lambda = \operatorname{cf}(\lambda)$ for some $w^* \in [\theta]^\sigma$, $w_2 = \{\delta \in w^* : w^* \subseteq w_\delta\}$ is stationary. As $^\sigma[\bigcup_{j < \theta} f(\delta)(j)]^\sigma$ has cardinality $\kappa^\sigma$ which is $< \lambda$ without loss of generality for some $h^* : w^* \to \bigcup_{j \in \delta} f(\delta)(j)$ the set

$$W_3 = \{\delta \in W_2 : (\forall i \in w^*)(h^*(i) \in f'_\delta(i) \cap \bigcup_{j < \theta} f(\delta)(j))\}$$

is stationary. So if $\delta_1 < \delta_2$ are in $w_3$ the set $\{i < \theta : f'_\delta_1(i) = f'_\delta_2(i)\}$ include $w^*$. But $f'_\delta_1(i) = f'_\delta_2(i)$ implies that $\alpha_i^\delta_1 = \alpha_i^\delta_2$, hence $A\delta_1 \cap A\delta_2$ has cardinality $\geq \sigma$ continuously.

6) $W$ has a $\exists$ clause ($\delta$), we add: $\operatorname{Rang}(f_\delta)$ is bound in $\delta$?

This is equivalent to: for some fixed $\mu < \lambda$, $(\forall \delta)(\operatorname{Rang}(f_\delta) \subseteq \mu)$. Repeating the proof and replacing club of $C \in [\mu]^\mu$ we get clause $(C)_{\kappa,\theta,\sigma}$ witnessing $\lambda$ with $\operatorname{Rang}(f_\delta) \subseteq \mu$. We then get versions of the $(A)$’s and $(B)$’s with $\mu$ points.

(Note one special point: we should rephrase the “weak $\Delta$-system argument, by using it on a tree with two levels.
7) Note that by part (5) we get a stronger version of the topological statements: for any \( \lambda \) (or \( \mu \) in (6)) points there is a close copy of \( ^{\omega}2 \) (or the space \( Y \)) included in it. Of course, if we like the space to be compact this refers only to any set of \( \lambda \) (or \( \mu \)) points among the original ones. Note the Boolean Algebra of clopen sets (when \( Y \) has such a basis) satisfies the c.c.c. (remember in the cases only \( u_{\zeta,2i}^{\mu} \cap u_{\zeta,2i+1}^{\mu} = \emptyset \) is demanded, the Boolean Algebra is free) so we cannot control the set of ultrafilters (= points), but if we allow more disjointness demand we may, but we have not considered it.

4.18 Claim. If \( \mu = \mu^{< \mu} \). Then there is a \( \mu \)-complete \( \mu^+ \)-c.c. forcing notion \( Q \) of cardinality \( 2^\mu \) such that

\[
\Vdash_Q \text{ " there is a function } h : ^\mu \mu \to \mu \text{ such that }
\]

\[
(\alpha) \quad \text{if } C \in V \text{ is a closed subset of } ^\mu \mu \text{ of cardinality } \leq \mu
\]
\[\Rightarrow \alpha < \mu \Rightarrow |C \cap h^{-1}\{\alpha\}| < \mu\]

\[
(\beta) \quad \text{if } A \in V \text{ is a subset of } ^\mu \mu \text{ of cardinality } > \mu
\]
\[\Rightarrow \alpha < \mu \Rightarrow |A \cap h^{-1}\{\alpha\}| = |A|^\mu.\]

Proof. As in the proof of 3.14, it suffices to prove:

(*) Assume that \( i^*, j^* < \mu \) and \( \eta_{\alpha,i} \in ^\mu \mu \) for \( \alpha < \mu^+ \), \( i < i^* \) is with no repetitions and \( C_{\alpha,j} \subseteq ^\mu \mu \) is closed with \( \leq \mu \) points for \( \alpha < \mu^+, j < j^* \). Find \( \alpha < \beta \) such that \( i < i^* \) & \( j < j^* \Rightarrow \eta_{\alpha,i} \notin C_{\beta,j}. \)

Why (*) holds? Assume not. First choose \( \delta^* < \mu^+ \) such that:

(**) if \( \beta < \mu^+ \) and \( \zeta < \mu \) then for some \( \alpha < \delta^* \) we have \( i < i^* \Rightarrow \eta_{\alpha,i} \mid \zeta = \eta_{\beta,i} \mid \zeta. \)

We can find \( \beta \) such that \( \delta^* < \beta < \mu^+ \) and \( \{\eta_{\beta,i} : i < i^*\} \) is disjoint to \( \bigcup_{j < j^*} C_{\delta^*,j}, \)

noting that \( \beta \) exists as \( |\bigcup_{j < j^*} C_{\delta^*,j}| \leq \mu. \) Let \( \zeta^* < \mu \) be large enough such that

\( i < i^* \) & \( j < j^* \Rightarrow \neg(\exists \nu)(\eta_{\beta,i} \mid \zeta < \nu \in C_{\delta^*,j}). \) Lastly, choose \( \alpha < \delta^* \) such that \( i < i^* \Rightarrow \eta_{\alpha,i} \mid \zeta = \eta_{\beta,i} \mid \zeta. \) Now the pair \( (\alpha, \delta^*) \) can serve as \( (\alpha, \beta) \) above.  \( \Box_{4.18} \)
§5 Additions

We can add

5.1 Theorem. The following are equiconsistent with ZFC $+\kappa > \lambda = \text{cf}(\lambda) > \theta$, $\theta = \theta^\theta$, $J$ a family of $< \lambda$ ideals on $\theta^\theta$ such that $J \in J$, $\{A_n : n < \omega\} \subseteq J = \theta \neq \bigcup_n A_n$ letting $\mu = 2^{\kappa^+}$

(C) there are $\lambda, J, S, f$ such that:

(a) $\lambda$ is regular $> 2^{\kappa^+}$, $J \in J$
(b) $S \subseteq S^\lambda_\theta$ is a stationary subset of $\lambda$
(c) $f = \langle f_\delta : \delta \in S \rangle$
(d) $f_\delta : \theta^+ \rightarrow \delta$
(e) if $\delta_1 \neq \delta_2$ are from $S$ then $|A_{\delta_1} \cap A_{\delta_2}| < \theta$, $\{i < \theta : f_{\delta_1}(i) = f_{\delta_2}(i)\} \in J$

(E) one of the following occurs

(α) there is a stationary $S \subseteq S^\lambda_\theta$ which does not belong to $I[\lambda]$
(β) for some $\lambda_1 = \text{cf}(\lambda_1) \in (\theta, \lambda)$ for $i < \theta^+$ and $J \in J$ such that $\lambda = \prod_{i < \theta} \lambda_i / J$.

Proof.

(C) $\Rightarrow$ (E):

If (α) of (E) holds we are done, so assume not, hence for some stationary $S \subseteq \lambda$ we have $S \in I[\lambda]$ and let $\langle a_\alpha : \alpha \in S^* \rangle$ witness it, (so $S^\theta \setminus S^*$ is not stationary, $a_\alpha \leq \alpha$, otp($a_\alpha$) $\leq \theta^+$, [otp($a_\alpha$)] $= \theta^+ \iff \alpha \in S$, $\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$).

[Compare with older works]

Claim. Assume

(a) $\lambda = \mu^+ > \theta > \aleph_0$ are regular cardinal
(b) $S \subseteq S^\lambda_\theta$ is a stationary subset of $\lambda$ which belong to $I[\lambda]$
(c) $f = \langle f_\delta : \delta \in S \rangle$, $f_\delta : \theta \rightarrow \delta$
(d) $J$ an ideal on $\theta$
(e) if $\delta_2 \neq \delta_2$ are from $S$ then $\{i < \theta : f_{\delta_1}(i) = f_{\delta_2}(i)\} \in J$
(f) $\lambda = \mu^+, U_J(\mu) = \mu, 2^\theta < \mu$. 


Then we can find a partition \( \langle A_n : n < \omega \rangle \) of \( \theta \) and sequence \( \langle \lambda_i : i < \theta \rangle \), \( 2^\theta < \lambda_i = \text{cf}(\lambda_i) < \lambda \) such that \( A_n \notin J_h \Rightarrow \lambda = \text{tcf}(\Pi \lambda_i / J) \).

Proof. Let \( \langle a_\alpha : \alpha \in \mathcal{S}^* \rangle \) witness \( S \in I[\lambda] \) (so \( S \setminus S^* \) is not stationary so without loss of generality \( S \subseteq S^* \)).
§1 General spaces: Consistency from strong assumptions.

**Definition 1.1** $X^* \rightarrow (Y^*)_\theta^n$, (having a closed copy of $Y$), monochromatic for a colouring of $n$-tuples by $\theta$ colours, $X^* \rightarrow_w (Y^*)_\theta^n$ (not necessarily a closed copy).

**Theorem 1.2** A sufficient condition for a forcing adding a space $X^*$ such that $X^* \rightarrow (Y^*)_\theta^n < \text{cf}(\theta)$, consisting on conditions on the cardinals

$$((A), (B)_1, (B)_2, (C))$$

and on the space $Y^*$

$$((D), (E))$$

[Saharon: copy and revise to be a proof of 1.5].

**Claim 1.4:** Sufficient pcf conditions for the set theoretic hypothesis of 1.2.

**Observation 1.5:** on beautifying nice scales.

**Claim 1.7:** A variant of 1.2.

**Comments 1.8:** We deal with some variants (e.g. regular spaces $X^*, Y^*$).

**Concluding Remarks 1.8:** Mainly on $T_3$ spaces.

§2 Consistency from supercompacts

**Observation 2.1:** How to deduce $(C)$ from $(C)^+$, a new condition.

**Claim 2.2:** Quoting a “consistency by a supercompact”.

**Claim 2.3:** Sufficient condition of the set theoretic assumption of 1.2.

**Conclusion 2.6:** Getting from a supercompact a universe with $\text{CH} +$ there is a Hausdorff space $X$ with clopen basis such that $X \rightarrow (\text{Cantor discontinuum})_{\delta_0}^1$.

**Claim 2.7:** Upgrading by a small forcing a stationary $S \notin I[\mu^+]$ included in

$$\{ \delta < \mu^+ : \text{cf}(\delta) = \text{cf}(\mu)^+ \}$$

to $\tilde{A} = (A_\delta : \delta \in S' \subseteq S), S'$ stationary, $A_\delta \subseteq \delta = \text{sup}A_0, \delta_1 \neq \delta_2 \Rightarrow |A_{\delta_1} \cap A_{\delta_2}| < \text{cf}(\mu)$.

**Claim 2.8:** Upgrading $\tilde{A}$ as in 2.7 to $\tilde{A} \upharpoonright S'$ which is $\kappa$-free by a $\text{cf}(\mu)^+$-c.c., (< $\text{cf}(\mu)$)-complete forcing notion.

**Observation 2.9:** By forcing we can partition $S$ to non-reflecting subsets.
Conclusion 2.10: Getting the necessary assumptions from non-trivial \( I[\lambda] \).

§3 Equi-consistency

Problem 3.1: What if we assume G.C.H.?

Theorem 3.2: Equi-consistency of several related statements, some are versions of “there is \( X \rightarrow (\omega^2)^1_2 \)” and some relate to pcf statement (and relative to \( I[\lambda] \) non-trivial).

Question 3.4: Phrase such theorems for other spaces.

Definition 3.5: \( (\kappa, I_0, I_1, \theta) \)-approximate.

Example 3.6: On the Cantor discontinuum.

Lemma 3.7: Sufficient conditions for the existence of a \([K, \sigma]\)-colouring of \( \lambda \).

Conclusion 3.8: A sufficient condition on “\( \lambda \) has approximation” for \( X \rightarrow [\text{Cantor set}]^{1}_{2^{\aleph_0}} \).

Claim 3.9: The forcing notions of \( \S 1 \) satisfies a strong \( \kappa^+ \)-c.c.

Definition 3.10: A strong \( \mu^+ \)-c.c. called \( \ast^\epsilon_D \).

Lemma 3.11: “\( \mathbb{Q} \) is \( (< \mu) \)-strategically complete and has \( \ast^\epsilon \)” is preserved by \( (< \mu) \)-support iteration.

Definition 3.12: \( X^* \rightarrow [Y^*]^\kappa_\theta \).

Claim 3.13: From \( X \rightarrow [\omega^2]^{1}_{2^{\aleph_0}} \) to \( \langle f_\alpha : \alpha \in S \rangle \), to help 3.2.

Proof of 3.2:

Observation 3.14: Existence of forcing replacing “countable scattered” by finite.

Claim 3.15: The old claim on \( I[\lambda] \) non-trivial from a strongly compact.

§4 Helping equi-consistency

Definition 4.1: \( NF_{\ell}(\lambda, \mathcal{Y}) \), variant of almost free not free.
Fact 4.2: Basic properties of $\text{NFr}_\ell$.

Claim 4.3: Improving examples for $\text{NFr}$ by forcing (toward freeness).

The Decomposition Claim 4.4: Analyzing $\text{NFr}$.

Claim 4.5: A variant of the previous claim 4.4.

Claim 4.6: Improving 4.4, 4.5.

Claim 4.7: Getting a colouring from decomposition.

Remark 4.8: On a variant of 4.7(1).

Definition 4.9: Defining $(S, \bar{\lambda})$ a full $(\lambda, \mu)$-set.

Observation 4.10: On $\lambda$-set ($\bar{\lambda}$ is computable from $S$).

Fact/Definition 4.11: Analyzing full sets.

Definition 4.12: $\bar{N}$ is a $\mu$-decomposition of $X$ for $\mathcal{H}(\chi)$, $x$.

Definition 4.13: $\bar{N}$, a full $\mu$-decomposition is good for $(X, \mathcal{Y}, \mathcal{F})$.

Claim 4.14: In 4.5, there is a good decomposition.

Proof of 3.13:

Concluding Remarks 4.17:

Claim 4.18: Properties of $\text{NFr}$.

§5 Additions

Theorem 5.1: For an ideal on $\theta^{<b}$ with non-trivial $\aleph_1$-completion.

Claim ?: FILL.

§6 Appendix

Claim ?: Full proof of 1.7.

Claim ?: Full proof of 4.3.
REFERENCES.


