NON COHEN ORACLE C.C.C
SH669

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Abstract. The oracle c.c.c. is closely related to Cohen forcing. During an iteration we can "omit a type"; i.e. preserve "the intersection of a given family of Borel sets of reals is empty" provided that Cohen forcing satisfies it. We generalize this to other cases. In §1 we replace Cohen by "nicely" definable c.c.c., do the parallel of the oracle c.c.c. and end with a criterion for extracting a subforcing (not a complete subforcing, $<\!!$) of a given nicely defined one such that the subforcing satisfies the oracle.

§ 0. Introduction

This answers a question from [Sh:b, Ch.IV] (the chapter dealing with the oracle c.c.c.) asking to replace Cohen by e.g. random. Later we will deal with the parallel for oracle proper and for the case $\bar{\varphi}_\alpha$ is a (definition of a) nep forcing from [Sh:630]. An application will appear in a work with Tomek Bartoszynski [BrSh:926].

How do we use this framework? We start with a universe satisfying $\diamondsuit_{\aleph_1}$ and probably $2^{\aleph_1} = \aleph_2$ and choose $\langle S^*_i : i < \omega_2\rangle, S^*_i \subseteq S^* \subseteq \omega_1$ such that $S^*_i/\mathcal{D}_{\omega_1}$ is strictly increasing and for every $i < \omega_2, \diamondsuit_{S^*_{i+1}\backslash S^*_i}$ holds and for simplicity $S^*_i \subseteq S^*_{i+1}$ where $\mathcal{D}_{\omega_1}$ is the club filter on ω_1. We choose by induction on $i < \omega_2$, a c.c.c. forcing $\mathbb{P}_i$ of cardinality $\aleph_1$, a sequence $\bar{M}^i = \langle M^i_\alpha : \alpha \in S^*_i\rangle$ of countable models $\subseteq (\mathcal{H}(\aleph_1), \in)$ of some version of ZFC; without loss of generality transitive and a 1-commitment mainly connected to, for each $\alpha \in S_i$, a $\mathbb{P}_i$-name $\underset{\sim}{\nu}^i_\alpha$ which is, e.g. random over M^i_α (and the commitment is that if, $j > i, \mathbb{P}'_j \cong \mathbb{P}_j/\mathbb{P}_i$ is represented (i.e. replaced by an isomorphism copy) such that it has set of elements $\subseteq \omega_1, G \subseteq \mathbb{P}'_j$ is generic over $\mathbf{V}^{\mathbb{P}_i}$, then for a club of $\alpha \in S^*_i, \underset{\sim}{\nu}^i_\alpha \in \mathbf{V}^{\mathbb{P}_i}$ is random also over $M^i_\alpha[G]$ which naturally is defined as $M^i_\alpha[G \cap \alpha]$. They are increasing in the relevant sense and the work at limit stages is done by the general claims here. In stage i, by bookkeeping we are given a task connected with a $\mathbb{P}_i$-name $\underset{\sim}{X}_i$, we have some freedom in choosing $\mathbb{P}_{i+1}$, usually $\mathbb{P}_{i+1} = \mathbb{P}_i * \underset{\sim}{\mathbb{Q}}_i$. So, working in $\mathbf{V}^{\mathbb{P}_i}$, $\mathbb{Q}_i$ has to satisfy a 0-commitment on S^*_i, and we like it to satisfy that task, usually connected with $X_i \subseteq \mathbb{R}^{V[\mathbb{P}_i]}$, say $X_i = \underset{\sim}{X}_i[G_{\mathbb{P}_i}]$. We essentially have to choose $\bar{M}^{i+1}$ such that $\bar{M}^{i+1} \restriction S^*_i = \bar{M}^i$ but we have freedom (in addition to choosing $\underset{\sim}{\mathbb{Q}}_i$) in choosing $\langle M^{i+1}_\alpha : \alpha \in S^*_{i+1}\backslash S^*_i\rangle$ and a 0-commitment on $S^*_{i+1}\backslash S^*_i$. Also the reals generic for the chosen forcing notion as well as M^{i+1}_α for $\alpha \in S^*_{i+1}\backslash S^*_i$ can be chosen considering

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X_i. E.g. M^{i+1}_α can be the Mostowski Collapse of some countable $M \prec (\mathcal{H}(\aleph_2), \in)$ to which $\mathbb{P}_i, \bar{M}^i$ and $\underset{\sim}{X}_i$ belongs.

Really this corresponds to the omitting type as in [Sh:e, XI]. This was originally part of [Sh:630], particularly close to the so called faking; on more general treatments, including replacing $\aleph_0$ by a larger cardinal, see [Sh:895], [Sh:F1000]. We thank the referee and (in 2008-2010) Jakob Kellner, Martin Goldstern, Heike Mildenberger and Wolfgang Wohofsky for corrections and comments.

§ 1. Definitions and preliminaries

{12.1}

Hypothesis 1.1.

(a) we assume CH, moreover $\diamondsuit_{S^*}$ where $S^* \subseteq \{\delta < \omega_1 : \delta \text{ limit}\}$ is stationary

(b) let ZFC_* be a large enough subset of ZFC satisfied by $(\mathcal{H}(\aleph_2), \in)$ (or just $(\mathcal{H}(\chi_*), \in)$ for some χ_* with minor changes in the proof.

{1.1a}

Definition/Notation 1.2. 1) $\bar{M}$ denotes an oracle, i.e., a sequence of the form $\langle M_\delta : \delta \in S\rangle, M_\delta$ a transitive countable model of ZFC_* satisfying $\delta + 1 \subseteq M_\delta$ and $S \subseteq S^*$ is stationary satisfying: for every $X \subseteq \omega_1$, the set $\{\delta \in S^* : X \cap \delta \in M_\delta\}$ is stationary. For such $\bar{M}$ let $S_{\bar{M}} = S$.
2) $\mathcal{D}$ denotes a normal filter on ω_1 usually extending $\mathcal{D}_{\bar{M}}$ which is defined in 1.6(1) below (of course, the default value is $\mathcal{D}_{\bar{M}}$, see 1.8(1)).
3) For a countable forcing $\mathbb{P}$, a wide $\mathbb{P}$-name is a Borel function giving for every $G \subseteq \mathbb{P}$ an object, i.e. the "input" of such a Borel function are $\langle$truth value$(p \in G) : p \in \mathbb{P}\rangle$, the "output" is normally in $\{x \in \mathcal{H}(\aleph_1) : \mathrm{rk}(x) < \gamma\}$ for some $\gamma < \omega_1$. So if $p \in \mathbb{P} \Rightarrow p \in \mathbb{P}'$ then any wide $\mathbb{P}$-name is still a wide $\mathbb{P}'$-name hence a $\mathbb{P}'$-name but it is natural to restrict ourselves, e.g. to the case $\mathbb{P} \subseteq_{\mathrm{ic}} \mathbb{P}'$, see below.
4) $\mathbb{P} \subseteq_{\mathrm{ic}} \mathbb{P}'$ <u>when</u>: $\mathbb{P}, \mathbb{P}'$ are forcing notions, i.e. quasi-orders, $\mathbb{P} \subseteq \mathbb{P}'$ and if $p, q \in \mathbb{P}$ then they are incompatible in $\mathbb{P}$ iff they are incompatible in $\mathbb{P}'$.
5) $\mathbb{P} \lessdot \mathbb{P}'$ <u>iff</u> $\mathbb{P} \subseteq_{\mathrm{ic}} \mathbb{P}'$ and every predense subset of $\mathbb{P}$ is a predense subset of $\mathbb{P}'$ (equivalently demands this for maximal antichains).
6) For a forcing notion $\mathbb{P}$ let

(a) $\mathbb{L}_{\omega_1,\omega}(\mathbb{P})$ be the set of $\mathbb{L}_{\omega_1,\omega}$ sentences in propositional logic considering the members of $\mathbb{P}$ as propositional variables

(b) for a generic $\mathbf{G} \subseteq \mathbb{P}$ and $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbb{P})$ let $\psi[\mathbf{G}]$ be the truth value (so if $\psi = p \in P$ we have $\psi[\mathbf{G}] =$ true iff $p \in G$, etc. (see [Sh:f, Ch.IX])

(c) let $p \Vdash_{\mathbb{P}} \psi$ means $p \Vdash_{\mathbb{P}} \psi[\underset{\sim}{G}] = \text{true}$ for $p \in \mathbb{P}, \psi \in \mathbb{L}_{\omega_1,\omega}(\mathbb{P})$

(d) let $\hat{\mathbb{P}}$ be the following quasi order
<u>set elements</u>: $\{\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbb{P})$: for some $p \in \mathbb{P}$ we have $p \Vdash \psi[\underset{\sim}{G}] = \text{true}\}$
<u>quasi order</u>: $\psi_1 \le \psi_2$ iff $\Vdash_{\mathbb{P}}$ "if $\psi_2[\underset{\sim}{G}] = \text{true}$ then $\psi_1[\underset{\sim}{G}] = \text{true}$".

{1.1C}

Observation 1.3. *1) So $\mathbb{P} \subseteq \hat{\mathbb{P}}$ as quasi orders, in fact, $\mathbb{P}$ is a dense subset of $\hat{\mathbb{P}}$ (so no real difference as forcing notion).*
2) If $\mathbb{P}_1 \lessdot \mathbb{P}_2$ then $\hat{\mathbb{P}}_1 \lessdot \hat{\mathbb{P}}_2$.
3) Assume $M \subseteq \mathcal{H}(\aleph_1)$ is a countable transitive model of $ZFC_*, M \models$ "$\mathbb{P}$ is a forcing notion, the pair $(\bar{\varphi}, \underset{\sim}{\eta})$ is as in 1.11(1) below and ν_α is a $(\mathbb{Q}_{\bar{\varphi}}, \underset{\sim}{\eta})$-generic over M". <u>Then</u>

(a) *$\mathbb{P}' = (\mathbb{P} * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})/(\underset{\sim}{\eta} = \nu)$ is a forcing; note notion in $M[\nu]$*

(b) *if $\mathbb{P} \subseteq_{ic} \mathbb{P}'$ and every predense suset of $\mathbb{P}$ which belongs to M is a predense subset of $\mathbb{P}'$ <u>then</u> $\Vdash_{\mathbb{P}'}$ "$\underset{\sim}{G} \cap \mathbb{P}$ is a subset of $\mathbb{P}$ generic over M and some $\underset{\sim}{G}'' \subseteq \mathbb{P} * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}}$ generic over M extends $\underset{\sim}{G} \cap \mathbb{P}$ and satisfies $\underset{\sim}{\eta}[G''] \equiv \nu$"*

(c) *$\mathbb{P} \subseteq_{ic} \mathbb{P}' \subseteq (\mathbb{L}_{\omega_1,\omega}(\mathbb{P}))^{M[\nu]}$; note: if $M_\alpha \models$ "$\mathbb{P}$ uncountable" use $(\mathbb{L}_{|P|^+,\omega}(\mathbb{P}))^{M[\omega]}$ or do it outside M, see explanation*

(d) *if in addition $M \models$ "$\mathbb{P}_1 \lessdot \mathbb{P}_2$" <u>then</u> $(\mathbb{P}_1 * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^M \lessdot (\mathbb{P}_2 * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^M$ and $(\mathbb{P}_1 * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^M/(\underset{\sim}{\eta} = \nu) \lessdot (\mathbb{P}_2 * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}}))^M/(\underset{\sim}{\eta} = \nu)$.*

Discussion 1.4. How $\mathbb{P}'$ is interpreted as $\subseteq \mathbb{L}_{\omega_1,\omega}(\mathbb{P})^{M[\nu]}$.

We should assume $M_\alpha \models$ "P is a c.c.c." or use $(\mathbb{L}_{|\mathbb{P}|^+,\omega}(\mathbb{P}))^{M_\alpha}$ which is $\mathbb{L}_{\omega_2,\omega}(\mathbb{P})^{\mathbf{V}}$

- $(*)$ if $\mathbf{G} \subseteq \mathbf{P}$ is generic over $\mathbb{P}$ and ν is $(\mathbb{Q}_\varphi, \underset{\sim}{\eta})$-generic for $M[\mathbf{G}]$ then there is
  - (α) there is a unique $\mathbf{G}^+ \subseteq (\mathbb{P} * \mathbb{Q}_{\bar{\varphi}})^M/(\underset{\sim}{\eta} = \nu)$ such that it is generic for $(\mathbb{P} * \mathbb{Q}_\varphi), M$
  - (β) $\mathbf{G} \subseteq \mathbf{G}^+$
  - (γ) $\mathbf{G} \cap \mathbb{Q}^M_\varphi = G_\nu$
  - (δ) there is a sequence of Borel functions $\langle \mathbf{B}_p : p \in (\mathbb{P}*\mathbb{Q}_{\bar{\varphi}})^M \rangle$ with infinite $\mathbf{P}$, in fact, in M and output a truth value $G^+ = \{p \in (P * \mathbb{Q}_{\bar{\varphi}})^M : B_p(\ldots, \text{truth value}(r \in \underset{\sim}{G}), \ldots)_{r \in P} = \text{truth}$, (in fact uniformly in ν, but not used)
- $(*)$ we can identify $p = (p_0, p_1) \in (\mathbb{P} * \mathbb{Q}_{\bar{\varphi}})^M$ with $B_p(\ldots,$ truth value$(r \in \underset{\sim}{G}), \ldots)_{r \in P}$.

2) The meaning of "$\underset{\sim}{\eta} \in {}^\omega\omega$ is generic for $\mathbb{Q}_{\bar{\varphi}}$ mean then for every $r \in \mathbb{Q}_{\bar{\varphi}}$ we can compute in a Borel way truth value$(r \in \underset{\sim}{G})$ from the value of η (in our case ν).
3) For random reals: if we interpret as the family of closed sets then $\underset{\sim}{\nu} \in (\nu$ a closed subset r of ${}^\omega 2)$ is O.K.
4) In the case of random $(p, \underset{\sim}{q}) \in (\mathbb{P} * \mathbb{Q}_\varphi)^M/\underset{\sim}{\eta} = \nu$ is interpreted as (now $\underset{\sim}{F}^q_n : \underset{\sim}{I}^q_n = \{T \cap {}^{n\geq}2 : T$ a perfect subtree of ${}^{\omega>}2\}$

$$p \wedge \bigwedge_n (\bigwedge_{r \in \underset{\sim}{I}_n} (r \to \nu {\restriction} n \in \underset{\sim}{F}^q_n(r))).$$

Remark 1.5. In 1.3(2)(c), note that

- (α) $\mathbb{P}_1 \lessdot \mathbb{P}_2$ by the assumption as this is upward absolute from M as M is transitive
- (β) the conclusion holds in $M[\nu]$ which is a generic extension of M hence in $\mathbf{V}$ as above.

We first give the old definitions from [Sh:f, IV]

{12.2} **Definition 1.6.** 1) $\mathcal{D}_{\bar{M}}$ is
$\{X \subseteq \omega_1 : \text{for some } Y \subseteq \omega_1 \text{ we have} : Y \cap \delta \in M_\delta \Rightarrow \delta \in S_{\bar{M}} \cap X\}$.
2) A forcing notion $\mathbb{P}$ of cardinality $\leq \aleph_1$ satisfies the $(\bar{M}, \mathcal{D})$-c.c. iff $S_{\bar{M}} \in \mathcal{D}^+$ and for some (equivalently any) one to one $f : \mathbb{P} \to \omega_1$ the set S defined below belongs to $\mathcal{D}$ and $\mathbb{P}$ has minimal element $\emptyset_{\mathbb{P}}$ where:

$$S := \big\{\delta : \begin{array}{l} \text{if } \delta \in S_{\bar{M}} \text{ and } X \in M_\delta \text{ and } \{y \in \mathbb{P} : f(y) < \delta \text{ and } f(y) \in X\} \\ \text{is predense in } \mathbb{P} \restriction \{y \in \mathbb{P} : f(y) < \delta\} \text{ \underline{then} } X \text{ is predense in } \mathbb{P} \end{array} \big\}$$

3) If $\mathcal{D} = \mathcal{D}_{\bar{M}}$ we may write "$\bar{M}$-c.c.". Recall that $\mathcal{D}^+ = \{A \subseteq \omega_1 : \omega_1 \backslash A \notin \mathcal{D}\}$.

4) Let $\bar{M}^1 \leq \bar{M}^2$ if $\bar{M}^\ell = \langle M^\ell_\delta : \delta \in S_\ell \rangle$ and $\{\delta : \delta \in S_1 \backslash S_2$ or $\delta \in S_1 \cap S_2$ and $M^1_\delta \neq M^2_\delta\}$ is not stationary; let $\bar{M}^1 \leq_{\mathcal{D}} \bar{M}^2$ be defined similarly (i.e. the set is $= \emptyset \bmod \mathcal{D}$); we call E a witness when $E \in \mathcal{D}$ is disjoint to those sets.
4A) Let $\bar{M}^1 \leq^* \bar{M}^2$ when $\bar{M}^\ell = \langle M^\ell_\delta : \delta \in S_\ell \rangle$ and $\{\delta : \delta \in S_1 \backslash S_1$ or $\delta \in S_1 \cap \S_2$ and $M^1_\delta \subseteq M^2_\delta\}$ is not stationary; let $\bar{M}^1 \leq^*_{\mathcal{D}} \bar{M}^2$ be defined similarly.

5) A forcing notion $\mathbb{P}$ satisfies the $(\bar{M},\mathcal{D})$-c.c. iff: $|\mathbb{P}| \le \aleph_0$ or for every $X \subseteq \mathbb{P}$ of cardinality $\le \aleph_1$ there is $\mathbb{P}_1 \lessdot \mathbb{P}$ of cardinality $\aleph_1$ which includes X and satisfies the $(\bar{M},\mathcal{D})$-c.c.

Remark 1.7. For the $\bar{M}$-c.c. the order $\le^*$ from 1.6(4A) is natural, but here it is easier to use $\le$ from 1.6(4).

Fact 1.8. 1) $\mathcal{D}_{\bar{M}}$ is a normal filter on ω_1. {12.3}
2) The $\bar{M}$-c.c. implies the c.c.c., and if $\mathcal{D}_{\bar{M}} \subseteq \mathcal{D}$ (or just there is a normal filter $\mathcal{D}' \supseteq \mathcal{D}_{\bar{M}} \cup \mathcal{D}$) then the $(\bar{M},\mathcal{D})$-c.c.c. implies the c.c.c. and if $\mathcal{D}_2 \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_{\bar{M}}$ are normal filters, then the $(\bar{M},\mathcal{D}_1)$-c.c. implies the $(\bar{M},\mathcal{D}_2)$-c.c.
3) We can find $\langle S^*_\zeta : \zeta < \omega_2\rangle$ such that $S^*_\zeta \subseteq S^*, \zeta < \xi \Rightarrow S^*_\zeta \subseteq S^*_\xi \mod \mathcal{D}_{\bar{M}}, S^*_\zeta \subseteq S^*_{\zeta+1}$ and $S^*_{\zeta+1}\backslash S^*_\zeta \in \mathcal{D}^+_{\bar{M}}$, moreover $\zeta < \xi < \omega_2 \Rightarrow S^*_\zeta \backslash S^*_\xi$ is countable.
4) If $\bar{M}^1 \le \bar{M}^2$ and the forcing notion $\mathbb{P}_2$ satisfies the $(\bar{M}^2,\mathcal{D})$-c.c. and $\mathbb{P}_1 \lessdot \mathbb{P}_2$, then $\mathbb{P}_1$ satisfies the $(\bar{M}^1,\mathcal{D})$-c.c.
5) Like part (4) when $\bar{M}^1 \le^* \bar{M}^2$.

Proof. See [Sh:f, Ch.IV], but for the reader's convenience we prove part (4).
4),5) Without loss of generality $\mathbb{P}_2$ has cardinality $\aleph_1$ and even its set of elements is ω_1. As $\mathbb{P}_1 \lessdot \mathbb{P}_2$ there is a function $f : \mathbb{P}_2 \to \mathbb{P}_1$ such that

$(*)_1$ $q \in \mathbb{P}_2 \wedge f(q) \le_{\mathbb{P}_1} p \in \mathbb{P}_1 \Rightarrow p, q$ are compatible in $\mathbb{P}_2$.

Let $g : \mathbb{P}_2 \times \mathbb{P}_2 \to \mathbb{P}_2$ be such that

$(*)_2$ if $p, q \in \mathbb{P}_2$ are compatible then $g(p,q)$ is a common upper bound and $p, q \in \mathbb{P}_1 \Rightarrow g(p,q) \in \mathbb{P}_1$.

So there is a club E^1 of ω_1 which is closed under f, g so

$(*)_3$ if $\delta^1 \in E \cap \mathrm{Dom}(\bar{M}^1)$ and $\mathcal{I} \subseteq \mathbb{P}_1 \cap \delta$ is predense in $\mathbb{P}_1 \restriction \delta$ then $\mathcal{I}$ is predense in $\mathbb{P}_2 \restriction \delta$.

[Why? If $q \in \mathbb{P}_2 \cap \delta$ then $f(q) \in \mathbb{P}_1 \cap \delta$ so by the assumption on $\mathcal{I}, f(q)$ is compatible with some $r_1 \subseteq \mathcal{I} \in \mathbb{P}_1 \cap \delta$, so there is $r_2 \in \mathbb{P}_1 \restriction \delta$ above $f(q)$ and r_1, e.g. $g(f(q), r_1)$. By the definition of f the conditions r_2, q are compatible in $\mathbb{P}_2$ hence $g(r_2, q)$ is a common upper bound of them in $\mathbb{P}_2 \restriction \delta$]

$(*)_4$ there is $E_2 \in \mathcal{D}$ such that if $\delta \in E_2$ and $\mathcal{I} \subseteq \mathcal{P}_1 \restriction \delta$ and $\mathcal{I} \in M^2_\delta$ is predense in $\mathbb{P}_2 \restriction \delta$ then it is predense in $\mathbb{P}_2$.

[Why? As we are assuming that $\mathbb{P}_2$ satisfies the $(\bar{M}^2,\mathcal{D})$-c.c.]

$(*)_5$ if $\mathcal{I}$ is a predense subset of $\mathbb{P}_2$ and is $\subseteq \mathbb{P}_1$ then $\mathcal{I}$ is a predense subset of $\mathbb{P}_1$.

[Why? As $\mathbb{P}_1 \lessdot \mathbb{P}_2$ is assumed.]

$(*)_6$ there is $E_3 \in \mathcal{D}$ such that $\delta \in E_\delta \Rightarrow M^1_\delta \subseteq M^2_\delta$.

[Why? As $\bar{M}^1 \le \bar{M}^2$ implies $\bar{M}^1 \le^* \bar{M}^2$ and see the definition.]
Putting together $(*)_3, (*)_4, (*)_5, (*)_6$ we are done. $\square_{1.8}$

{12.3d}

Definition 1.9. 1) We define when "$\bar{\varphi}$ is an absolute definition of a c.c.c. forcing notion with generic $\underset{\sim}{\eta}$ (for ZFC$_*$) say with a parameter, a real and countable ordinal means that for any transitive model N of ZFC , to which the parameters belong $\mathbb{Q} = \mathbb{Q}^n_{\bar{\varphi}}$ is a forcing notion, the property "$p \in \mathbb{Q}_{\bar{\varphi}}, p \le_{\mathbb{Q}_{\bar{\varphi}}} q, p, q \in \mathbb{Q}_{\bar{\varphi}}$ are incompatible", "$\mathcal{I}$ is a countable predense subset" are preserved in forcing extensions[1] of N.
2) We add with the generic $\underset{\sim}{\eta}$ when $\underset{\sim}{\eta}$ is a $\mathbb{Q}_{\bar{\varphi}}$-name, the parameter including the relevant information of a member of ${}^\omega 2$ or of $\alpha(0) < \omega$ and the generic is reconstructible from $\underset{\sim}{\eta}$.
3) We say "$\bar{\varphi}$ is a nep forcing problem with the generic $\underset{\sim}{\eta}$" as in [Sh:630]. See below.

{12.3A} *Remark* 1.10. 1) Note that below when $\mathbb{Q}_{\bar{\varphi}_\alpha}$ is the older case ([Sh:f, IV]) we just preserve every predense set, so in M_α (in the cases the "commitment", see below, is obeyed) the forcing is countable.
2) We may forget to mention this case as it is by now easy.

{12.4} **Definition 1.11.** 1) We say $\mathcal{Y} = (S, \bar{\Phi}, \underset{\sim}{\bar{\eta}}, \bar{\nu}) = (S^{\mathcal{Y}}, \bar{\Phi}^{\mathcal{Y}}, \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}, \bar{\nu}^{\mathcal{Y}})$ is a 0-<u>commitment</u> for $\bar{M}$ <u>iff</u> for some $E \in \mathcal{D}_{\bar{M}}$:

- (a) $S \subseteq S^*, S \in \mathcal{D}^+_{\bar{M}}$
- (b) $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\alpha : \alpha \in S\rangle, \bar{\Phi} = \langle \bar{\varphi}_\alpha : \alpha \in S\rangle$ and if $\alpha \in S \cap E$ then
  $\bar{\varphi}_\alpha \in M_\alpha$ and
  $M_\alpha \models$ "$\bar{\varphi}_\alpha$ is an absolute definition of a c.c.c. forcing notion called $\mathbb{Q}_\alpha = \mathbb{Q}_{\bar{\varphi}_\alpha}$ with generic real $\underset{\sim}{\eta}_\alpha$" so $M_\alpha \models (\Vdash_{\mathbb{Q}_{\bar{\varphi}_\alpha}}$ "$M_\alpha[\underset{\sim}{G}_{\mathbb{Q}_{\bar{\varphi}_\alpha}}] = M_\alpha[\underset{\sim}{\eta}_\alpha]$"); note, absolute here means that forcing extensions of M_α, preserve predensity of countable sets (in the sense of M_α), preserve order and its negation and preserves incompatibility
- (c) $\bar{\nu} = \langle \nu_\alpha : \alpha \in S\rangle$ where $\nu_\alpha \in {}^\omega\omega$ and for every $\alpha \in S \cap E$ the real ν_α is $(\mathbb{Q}_\alpha, \underset{\sim}{\eta}_\alpha)$-generic over M_α or ν_α is $(M_\alpha, \mathbb{Q}^{\bar{\varphi}_\alpha}, \underset{\sim}{\eta}_\alpha)$-generic sequence which means that for some $G \subseteq \mathbb{Q}^{\bar{\varphi}_\alpha}$ generic over M_α we have $\nu_\alpha = \underset{\sim}{\eta}_\alpha[G]$.
- (d) we can allow $\Vdash_{Q_{\bar{\varphi}_\alpha}}$ "$\underset{\sim}{\eta}_\alpha : \beta^1_{\bar{\varphi}_\alpha} \to \beta^2_{\bar{\varphi}_\alpha}$" for some ordinals $\beta^1_{\bar{\varphi}_\alpha}, \beta^2_{\bar{\varphi}_\alpha}$ from M_α and so $\nu_\alpha : \beta^1_{\bar{\varphi}_\alpha} \to \beta^2_{\bar{\varphi}_\alpha}$.

We shall ignore $\bar{M}$ if clear from the context. We can replace $\bar{M}$ by $(\bar{M}, \mathcal{D})$ if above $E \in \mathcal{D}, S \in \mathcal{D}^+$; similarly below.
1A) A forcing notion $\mathbb{P}$ of cardinality $\le \aleph_1$ satisfies the 0-commitment $\mathcal{Y} = (S^{\mathcal{Y}}, \bar{\Phi}^{\mathcal{Y}}, \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}, \bar{\nu}^{\mathcal{Y}})$ for an $\aleph_1$-oracle $\bar{M}$ (we may suppress this if clear from the context) <u>when</u> $\mathbb{P}$ is a forcing notion and for any one-to-one mapping $h : \mathbb{P} \to \omega_1$ for some $E \in \mathcal{D}_{\bar{M}}$ for every $\alpha \in E$ we have

- (e) every predense subset $\mathcal{I}$ of $\{p \in \mathbb{P} : h(p) < \alpha\}$ for which $\{h(p) : p \in \mathcal{I}\} \in M_\alpha$ is a predense subset of $\mathbb{P}$
- (f) if $\alpha \in S \cap E$ then $\Vdash_{\mathbb{P}}$ "the real ν_α is a $(\mathbb{Q}_\alpha, \underset{\sim}{\eta}_\alpha)$-generic real over $M_\alpha[\alpha \cap h''G_{\mathbb{P}}]$"; moreover
- $(f)^+$ letting $\mathbb{P}^h = h(\mathbb{P})$ the forcing notion $\mathbb{P}^h_\alpha = \mathbb{P}^h_1{\restriction}\alpha$ belongs to M_α and $(\mathbb{P}^h_\alpha * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}^\alpha})^{M_\alpha}/(\underset{\sim}{\eta}_\alpha = \nu)$ is $\subseteq_{\rm ic} \hat{\mathbb{P}}^h$ and, moreover, any predense subset of it from $M_\alpha[\nu_\alpha]$ is predense in $\hat{\mathbb{P}}'$, (in fact this implies clause (d), i.e. any predense subset of $\mathbb{P}^h_\alpha$ from M_α is predense in $\mathbb{P}^h$)

[1] may restrict ourselves to the relevant forcing

(g) so we get for the old case: if $\alpha \in S \cap E$ and $\mathbb{Q}_\alpha$ is a singleton (hence $\nu_\alpha \in M_\alpha$, a degenerated case), this actually follows from (d)

(h) the old case, i.e. $\mathbb{Q}_{\bar{\varphi}}$ = singleton occurs for a set of α's from $\mathcal{D}^+_{\bar{M}}$ (nec?); we may add $M_\alpha \models |\alpha| = \aleph_0$ most naturally in (e), but not used so far.

2) Let $\mathbb{P} \in \mathcal{H}(\aleph_2)$ be an $\bar{M}$-c.c. forcing notion. We say that $\mathcal{Y} = (S, \underset{\sim}{\bar{\Phi}}, \underset{\sim}{\bar{\eta}}, \underset{\sim}{\bar{\nu}}) = (S^{\mathcal{Y}}, \underset{\sim}{\bar{\Phi}}^{\mathcal{Y}}, \underset{\sim}{\bar{\eta}}^{\mathcal{Y}}, \underset{\sim}{\bar{\nu}}^{\mathcal{Y}})$ is a 1-commitment on $\mathbb{P}$ for $\bar{M}$ if: for any $\bar{N}$ satisfying $(*)_1$ below, the clauses (a)-(d) of $(*)_2$ below hold

$(*)_1$ $\bar{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ is increasing continuous, $N_\alpha \prec (\mathcal{H}(\aleph_2), \in)$ is countable, $\bar{N} \restriction (\alpha+1) \in N_{\alpha+1}$ and $\{\bar{M}, \mathbb{P}\} \subseteq N_\alpha$ for some $\alpha < \omega_1$ and $h \in N_0$ is a one-to-one mapping from $\mathbb{P}$ into ω_1

$(*)_2$ (a) $S \subseteq \text{Dom}(\bar{M}) \subseteq S^*, S \in \mathcal{D}^+_{\bar{M}}$

(b) $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\alpha : \alpha \in S \rangle, \bar{\Phi} = \langle \underset{\sim}{\bar{\varphi}}_\alpha : \alpha \in S \rangle$ so $(\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha)$ is a $\mathbb{P}$-name of a pair as in 1.11(1)(b), both are wide names over $\mathbb{P}$

(c) $\underset{\sim}{\bar{\nu}} = \langle \underset{\sim}{\nu}_\alpha : \alpha \in S \rangle$ and $\underset{\sim}{\nu}_\alpha$ a wide $\mathbb{P}$-name of a real

(d) the set of the $\alpha \in S$ satisfying the following belongs to $\mathcal{D}_{\bar{M}} + S$: $\underset{\sim}{\bar{\varphi}}_\alpha \in M_\alpha$, $\text{MosCol}_{N_\alpha}(N_\alpha) \in M_\alpha$, and letting[2] $h_\alpha = \text{MosCol}_{N_\alpha}$ we have $\mathbb{P}'_\alpha = h_\alpha(\mathbb{P}^{N_\alpha}) \in M_\alpha$ so e.g. the set of members of $\mathbb{P}'_\alpha$ is $\{h_\alpha(x) : x \in \mathbb{P} \cap N_\alpha\}$, we have $M_\alpha \models$ "$\bar{\varphi}$ is a wide $\mathbb{P}'_\alpha$-name of an absolute definition of a c.c.c. forcing with generic real $\underset{\sim}{\eta}_\alpha$" and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\nu}_\alpha$ is $(\mathbb{Q}_{\underset{\sim}{\bar{\varphi}}_\alpha}, \underset{\sim}{\eta}_\alpha)$-generic over $M_\alpha[\alpha \cap h``(\underset{\sim}{G}_{\mathbb{P}})]$"

$(d)^+$ moreover if $\mathcal{I} \in M_\alpha$ is a predense subset of $\mathbb{P}_\alpha$ if $M_\alpha \models \psi(\bar{p}, \underset{\sim}{\eta}) \in \mathbb{L}_{\infty,\omega}$ propositional sentence in the propositional variable $\bar{p}$, listing $\mathbb{P}'_\alpha$ and the $\underset{\sim}{\eta}(i), i < \ell g(\eta)$ such that $\Vdash_{\mathbb{P}'_\alpha * \mathbb{Q}_{\bar{\varphi}}}$ "$\psi(\bar{p}, \underset{\sim}{\eta}) = \text{true}$", <u>then</u> $\Vdash_{\mathbb{P}}$ "$\psi(h^{-1}_\alpha(\bar{p}), \underset{\sim}{\nu}) = \text{true}$.

Hence in particular if $\mathcal{I} \in M_\alpha$ is a predense subset of $\mathbb{P}'_\alpha$ then $h^{-1}_\alpha(\mathcal{I})$ is a predense subset of $\mathbb{P}$.

For transparency the reader may concentrate below on the case $\langle (\bar{\varphi}_\alpha, \underset{\sim}{\eta}_\alpha) : \alpha \in S \rangle \in \mathbf{V}$.

3) Let $IS = \{(\mathbb{P}, \mathcal{Y}, \bar{M}) : \mathbb{P} \in \mathcal{H}(\aleph_2)$ is an $\bar{M}$-c.c. forcing notion and $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ for $\bar{M}\}$.

We shall omit $\bar{M}$ if clear from the context. We can replace $\bar{M}$ by $(\bar{M}, \mathcal{D})$ naturally and write $\text{IS}_{\mathcal{D}}$, but the claims are the same.

4) For $(\mathbb{P}^\ell, \mathcal{Y}^\ell, \bar{M}^\ell) \in IS\,(\ell = 1,2)$ let $(\mathbb{P}^1, \mathcal{Y}^1, \bar{M}^1) \leq^* (\mathbb{P}^2, \mathcal{Y}^2, \bar{M}^2)$ means $\bar{M}^1 \leq \bar{M}^2, \mathbb{P}^1 \lessdot \mathbb{P}^2$ and for some $E \in \mathcal{D}_{\bar{M}^1}$ we have

(a) $S^{\mathcal{Y}^1} \cap E \subseteq S^{\mathcal{Y}^2} \cap E$,

(b) $\underset{\sim}{\bar{\Phi}}^{\mathcal{Y}^1} \restriction (S^{\mathcal{Y}_1} \cap E) = \underset{\sim}{\bar{\Phi}}^{\mathcal{Y}^2} \restriction (S^{\mathcal{Y}^1} \cap E)$,

(c) $\underset{\sim}{\bar{\eta}}^{\mathcal{Y}^1} \restriction (S^{\mathcal{Y}^1} \cap E) = \underset{\sim}{\bar{\eta}}^{\mathcal{Y}^2} \restriction (S^{\mathcal{Y}^1} \cap E)$ and

(d) $\underset{\sim}{\bar{\nu}}^{\mathcal{Y}^1} \restriction (S^{\mathcal{Y}^1} \cap E) = \underset{\sim}{\bar{\nu}}^{\mathcal{Y}^2} \restriction (S^{\mathcal{Y}^1} \cap E)$.

We call E a witness to $(\mathbb{P}^1, \mathcal{Y}^1, \bar{M}^1) \leq^* (\mathbb{P}^2, \mathcal{Y}^2, \bar{M}^2)$. Note that it does not matter if we demand $E \in \mathcal{D}_{\bar{M}^2}$ or $E \in \mathcal{D}_{\bar{M}^1}$.

[2]this holds for a set of α's which belongs to $\mathcal{D}_{\bar{M}}$

We point out the connection between 0-commitment and 1-commitments.

{12.4A} **Fact 1.12.** 1) If $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ for $\bar{M}$, so $\mathbb{P}$ is an $\bar{M}$-c.c. forcing notion of cardinality $\leq \aleph_1$, then $\Vdash_{\mathbb{P}}$ "$\mathcal{Y}[\underset{\sim}{G}_{\mathbb{P}}] = (S^{\mathcal{Y}}, \bar{\underset{\sim}{\Phi}}^{\mathcal{Y}}[\underset{\sim}{G}_{\mathbb{P}}], \bar{\underset{\sim}{\eta}}^{\mathcal{Y}}[\underset{\sim}{G}_{\mathbb{P}}], \bar{\underset{\sim}{\nu}}^{\mathcal{Y}}[\underset{\sim}{G}_{\mathbb{P}}])$ is a 0-commitment" so we call it $\mathcal{Y}[\underset{\sim}{G}_{\mathbb{P}}]$. Note that each $\underset{\sim}{\eta}_\alpha[\underset{\sim}{G}_{\mathbb{P}}]$ is still a name.
2) If $\mathbb{P} = \{\emptyset\}$ (the trivial forcing) then: $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ iff $\mathcal{Y}$ is a 0-commitment.
3) If $\langle \bar{M}^i : i < \zeta \rangle$ is $\leq$-increasing, $\zeta < \omega_2$ and $\mathrm{Dom}(\bar{M}^i) \backslash S$ is not stationary while $S \subseteq S^*$, then there is $\bar{M}$ satisfying $\mathrm{Dom}(\bar{M}) = S$ such that $i < \zeta \Rightarrow \bar{M}^i \leq \bar{M}$.
4) Assume $\bar{M}^2$ is oracle, $S \subseteq \bar{M}$ belongs to $\mathcal{D}^+_{\bar{M}^2}$ and $\bar{M}^1 = \bar{M}^2 \restriction S$. If $\mathcal{Y}^2 = (S^2, \bar{\Phi}^2, \bar{\underset{\sim}{\eta}}^2, \bar{\nu}^2)$ is a 0-commitment for $\bar{M}^2$ and $S \subseteq S^2$ then $\mathcal{Y}^1 := (S, \bar{\Phi}^2 \restriction S, \bar{\underset{\sim}{\eta}}^2 \restriction S, \bar{\nu}^2 \restriction S)$ is a 0-commitment for $\bar{M}^1$ and any forcing notion satisfying $\mathcal{Y}^2$ satisfies $\mathcal{Y}^1$. If $\mathcal{Y}^2 = (S^2, \bar{\underset{\sim}{\Phi}}^2, \bar{\underset{\sim}{\eta}}^2, \bar{\underset{\sim}{\nu}}^2)$ is a 1-commitment on $\mathbb{P}$ for $\bar{M}^2$ and $S \subseteq S^2$ then $(S, \bar{\underset{\sim}{\Phi}}^2 \restriction S, \bar{\underset{\sim}{\eta}}^2 \restriction S, \bar{\underset{\sim}{\nu}} \restriction S)$ is a 1-commitment on $\mathbb{P}$ for $\bar{M}^1$.
5) If a forcing notion $\mathbb{P}$ satisfies the 0-commitment $\mathcal{Y}$ for the $\aleph_1$-oracle $\bar{M}$ and $S' = \{\alpha \in S^{\mathcal{Y}} : \mathbb{Q}_{\bar{\varphi}_\alpha}$ is a singleton (i.e., $\bar{\varphi}_\alpha = \bar{\varphi}^{\mathcal{Y}}_\alpha$ and α is of the old case for $\mathcal{Y}$)$\}$ and $S' \neq \emptyset \bmod \mathcal{D}_{\bar{M}}$ then

- (a) $\mathbb{P}$ satisfies the $(\bar{M} \restriction S')$-c.c. which is defined as in [Sh:f, IV]
- (b) if $S'' \subseteq S'$ and $S'' \neq \emptyset \bmod D_{\bar{M}}$ and $M_\delta \models$ "$X_\delta \subseteq {}^\omega 2$ is not meagre" for every $\delta \in S''$ then $\bigcup\limits_{\delta \in S''} X_\delta$ is not meagre in $\mathbf{V}[\underset{\sim}{G}_{\mathbb{P}}]$.

As a warm-up (see [Sh:630] for more)

{12.4.xad} **Claim 1.13.** *1) Assume*

- (a) *M is a countable transitive model of ZFC$_*$, $M \models$ "$\mathbb{P}_1$ is a forcing notion"*
- (b) *$M \models$ "$\bar{\varphi}$ is an absolute definition of a c.c.c. forcing notion $\mathbb{Q}^{\bar{\varphi}}$ with generic $\underset{\sim}{\eta} : \beta_1 \to \beta_2$" so $\beta_1, \beta_2 \in \mathrm{Ord}^M \subseteq \omega_1 \cap M$*
- (c) *ν is a $(M, \mathbb{Q}^{\bar{\varphi}}, \underset{\sim}{\eta})$-generic sequence, recalling it means that there is $G \subseteq (\mathbb{Q}^{\bar{\varphi}})^M$ generic over M such that $\nu = \underset{\sim}{\eta}[G]$.*

Then we can find a countable $\mathbb{P}_2$ such that:

- (α) *$\mathbb{P}_1 \subseteq_{ic} \mathbb{P}_2$ and every $\mathcal{J} \in M$ which is predense in $\mathbb{P}_1$ is predense in $\mathbb{P}_2$*
- (β) *$\Vdash_{\mathbb{P}_2}$ "ν is $(M', \mathbb{Q}^{\varphi}, \underset{\sim}{\eta})$-generic sequence where $M' = M[G_{\mathbb{P}_2} \cap \mathbb{P}_1]$".*

2) Similarly for $\bar{\varphi}$ defining a nep forcing.

Proof. 1) In M we can define $\mathbb{P}^+ = \mathbb{P}_1 * (\underset{\sim}{\mathbb{Q}}^{\bar{\varphi}})^{M[\underset{\sim}{G}_{\mathbb{P}_1}]}$, now as $\mathbb{Q}^{\bar{\varphi}}$ is absolutely c.c.c., we know that $q \mapsto (\emptyset, q)$ is a complete embedding of $(\mathbb{Q}^{\bar{\varphi}})^M$ into $\mathbb{P}^+$. So if $G_* \subseteq (\mathbb{Q}^{\bar{\varphi}})^M$ is generic over M such that $\nu = \underset{\sim}{\eta}[G_*]$ then let $\mathbb{P}_2 = \{(p, \underset{\sim}{q}) \in \mathbb{P}_1 * (\underset{\sim}{\mathbb{Q}}^{\bar{\varphi}})^{[\underset{\sim}{G}_{\mathbb{P}_1}]} : (p, \underset{\sim}{q})$ is compatible with $(\emptyset, q')$ for every $q' \in G_*\}$. Now check.
2) See [Sh:630]. $\square_{1.13}$

§ 2. The iteration theorem

{12.5}

Crucial Claim 2.1. *In IS, any $\leq^*$-increasing ω-chain has an upper bound.*

Remark 2.2. 1) The ω-limit is the crucial one not the ω_1-limit. Actually for the ω_1-limit we take the union and we preserve what we need by using the square (and having done something toward it in earlier limits or stages of cofinality $\aleph_0$).
2) When is the union not an upper bound? If, e.g., for some $S' \subseteq S^{\mathcal{Y}}, S' \in \mathcal{D}^+_{\bar{M}}$ and for every $\alpha \in S'$ the forcing notion $\bar{\varphi}^{\mathcal{Y}}_\alpha$ is random real forcing, we have in particular to preserve $\{\nu_\alpha : \alpha \in S'\}$ is non-null, but the union normally adds a Cohen.

Proof. So assume $(\mathbb{P}_n, \mathcal{Y}^n, \bar{M}^n) \in IS$ and $(\mathbb{P}_n, \mathcal{Y}^n, \bar{M}^n) \leq^* (\mathbb{P}_{n+1}, \mathcal{Y}^{n+1}, \bar{M}^{n+1})$ for $n < \omega$, let $\bar{M}$ be such that $\bar{M} \geq \bar{M}^n$ for each n; so let $E_n \in \mathcal{D}_{\bar{M}}$ witness both, see the last line of Definition 1.11(4) and Definition 1.6(4). For simplicity assume that above any $p \in \mathbb{P}_n$ there are two incompatible elements (in $\mathbb{P}_n$), and $0 \in \mathbb{P}_\emptyset$ is minimal in all $\mathbb{P}_n$, i.e. is $\emptyset_{\mathbb{P}_n}$. Without loss of generality the set of elements of $\mathbb{P}_n$ is $X_n \subseteq \omega_1$ and $\omega_1 \backslash \bigcup\limits_{n<\omega} \mathbb{P}_n$ has cardinality $\aleph_1$ and let X_ω be such that $\bigcup\limits_{n<\omega} \mathbb{P}_n \subseteq X_\omega \subseteq \omega_1$ and $|X_\omega \backslash \bigcup\limits_{n<\omega} \mathbb{P}_n| = \aleph_1$; this notation helps in a future use, also there we replace ω by a (countable) ordinal of cofinality $\aleph_0$. For transparency assume: if $n < m$ and $\mathbb{P}_m \models$ "$p \leq q$" and $q \in \mathbb{P}_n$ then $p \in \mathbb{P}_n$.

We can define functions $F_n, F_{n,m}, F_{n,m,\ell}$ (when $n < m < \omega, \ell < \omega$) such that[3]

$\boxplus_1$ $(a)_n$ if $p, q \in \mathbb{P}_n$ are compatible then $F_n(p,q) \in \mathbb{P}_n$ is a common upper bound

$(b)_{n,m}$ if $n < m$ and $p \in \mathbb{P}_m$, then $\langle F_{n,m,\ell}(p) : \ell < \omega\rangle$ is a maximal antichain of $\mathbb{P}_m$, such that for each ℓ:
<u>either</u> $p, F_{n,m,\ell}(p)$ are incompatible (in $\mathbb{P}_m$)
<u>or</u> p is compatible in $\mathbb{P}_m$ with every $q \in \mathbb{P}_n$ which is above $F_{n,m,\ell}(p)$.
Note that necessarily for some ℓ the second possibility occurs.

Let

$\boxplus_2$ (a) E is the set of $\delta \in E$ such that
- δ is closed under $F_n, F_{n,m}, F_{n,m,\ell}$
- $\langle \mathbb{P}_n{\restriction}\delta, F_n{\restriction}\delta, F_{n,m}{\restriction}\delta, F_{n,m,\ell}{\restriction}\delta, n, m, \ell\rangle \in M_\delta$
- $\langle X_\delta \cap \varepsilon : \varepsilon \leq \omega\rangle \in M_\delta$
- $\mathrm{otp}(X_\omega \cap \delta \backslash \bigcup\limits_{n<\omega} \mathbb{P}^n) = \delta$

(b) let $E_\omega = \cap\{E_n : n < \omega\} \cap E$ and let E'_ω be the closure of E_ω so $E_\omega \subseteq \omega_1$ is stationary and E'_ω is a club of ω_1.

We would like to define a forcing notion $\mathbb{P}_\omega$ with universe X_ω, and a 1-commitment $\mathcal{Y}^\omega$, and functions $F_\omega, F_{n,\omega,\ell}$ satisfying the natural requirements $(a)_\omega, (b)_{n,\omega}$. First, let $S^{\mathcal{Y}^\omega} = \bigcup\limits_{n<\omega} S^{\mathcal{Y}^n} \cap E_\omega$, and for $\alpha \in S^{\mathcal{Y}^\omega}$ the triple $(\underset{\sim}{\bar{\varphi}}{}^{\mathcal{Y}^\omega}_\alpha, \underset{\sim}{\eta}{}^{\mathcal{Y}^\omega}_\alpha, \underset{\sim}{\nu}{}^{\mathcal{Y}^\omega}_\alpha)$ is $(\underset{\sim}{\bar{\varphi}}{}^{\mathcal{Y}^{n(\alpha)}}_\alpha, \underset{\sim}{\eta}{}^{\mathcal{Y}^{n(\alpha)}}_\alpha, \underset{\sim}{\nu}{}^{\mathcal{Y}^{n(\alpha)}}_\alpha)$

[3] in older version we also have:

$(c)_{n,m}$ if $n < m, p \in \mathbb{P}_m, q \in \mathbb{P}_n$ <u>then</u> $q \leq F_{n,m}(p,q) \in \mathbb{P}_n$ and if there is r such that $q \leq_{\mathbb{P}_n} r$ and r, p are incompatible in $\mathbb{P}_m$ then $p, F_{n,m}(p,q)$ are incompatible in $\mathbb{P}_n$.

where $n(\alpha) = \text{Min}\{n : \alpha \in S^{\mathcal{Y}^n}\}$. Defining $\mathbb{P}_\omega, F_\omega, F_{n,\omega,\ell}$ is harder, so we first define AP, a set of approximations to them. A member t of AP has the form $(\delta^t, \mathbb{P}^t, \Gamma^t, F^t_\omega, F^t_{n,\omega,\ell})_{n<\omega,\ell<\omega}$ satisfying:

$\boxplus_3$ (α) $\delta^t \in E'_\omega$

(β) $\mathbb{P}^t$ is a forcing notion with set of elements $\subseteq X_\omega \cap \delta^t$ and $\supseteq \delta^t \cap \bigcup_n \mathbb{P}_n$ and $0 \leq_{\mathbb{P}^t} p$ for every $p \in \mathbb{P}^t$

(γ) $\mathbb{P}^t \restriction (X_n \cap \delta^t) = \mathbb{P}_n \restriction (X_n \cap \delta^t)$

(δ) if $p, q \in \mathbb{P}^t$ are compatible in $\mathbb{P}^t$ then $F^t_\omega(p,q)$ is a common upper bound in $\mathbb{P}_t$ such that $n < \omega \wedge \{p,q\} \subseteq \mathbb{P}_n \Rightarrow F^t_\omega(p,q) = F_n(p,q)$

(ε) if $p \in \mathbb{P}^t, n < \omega$ then $\langle F^t_{n,\omega,\ell}(p) : \ell < \omega\rangle$ is a maximal antichain of $\mathbb{P}_n$, the members are $< \delta^t$, and for each ℓ, <u>either</u> $p, F^t_{n,\omega,\ell}(p)$ are incompatible in $\mathbb{P}^t$ <u>or</u> $(\forall q \in \mathbb{P}_n \cap \delta^t)(\mathbb{P}_n \models$ "$F^t_{n,\omega,\ell}(p) \leq q$" $\Rightarrow p, q$are compatible in $\mathbb{P}^t$) and for at least one ℓ the second case occurs

(ζ) if $m < \omega$ and $p \in \mathbb{P}^t \cap \mathbb{P}_m \backslash \bigcup_{k<m} \mathbb{P}_\ell$ then $F^t_{n,\omega,\ell}(p) = F_{n,m,\ell}(p)$ for $n < m, \ell < \omega$

(η) Γ^t is a sequence $\langle \bar{p}^t_\zeta : \zeta < \zeta^t\rangle, \zeta^t < \omega_1$ and $\bar{p}^t_\zeta$ is a sequence of length ω of members of $\mathbb{P}^t$ which form a predense subset of $\mathbb{P}^t$

(θ) if $p \in \mathbb{P}^t$ and $n < m < \omega$ and $r \in \mathbb{P}_n \cap \delta^t$ and $[r \leq r' \in \mathbb{P}_n \cap \delta^t \Rightarrow r', p$ are compatible in $\mathbb{P}^t]$, <u>then</u> the set $\{F^t_{m,\omega,\ell}(p) : \ell < \omega$ and p is compatible with $F^t_{m,\omega,\ell}(p)$ in $\mathbb{P}^t\}$ satisfies: if $r \leq q \in \mathbb{P}_n$ then in $\mathbb{P}_m$ the condition q is compatible with some member of this set

(ι) if $\zeta < \zeta^t$ and $n < \omega$ <u>then</u>: $\{F^t_{n,\omega,\ell}(p^t_{\zeta,k}) : k < \omega, \ell < \omega$ and $p^t_{\zeta,k}, F^t_{n,\omega,\ell}(p^t_{\zeta,k})$ are compatible in $\mathbb{P}^t\}$ is a predense subset of $\mathbb{P}_n$. Note that trivially, this subset is predense in $\mathbb{P}_n \cap \delta^t$; similarly in clause (κ)

Moreover,

(κ) if $p^* \in \mathbb{P}^t$ and $n < \omega$ and $\zeta < \zeta^t$ <u>then</u> the following set is predense in $\mathbb{P}_n$:

$$\mathcal{I}^t_{\zeta,n,p^*} := \Big\{r' \in \mathbb{P}_n \cap \delta^t : \begin{array}{ll} (i) & r', p^* \text{ incompatible in } \mathbb{P}^t \text{ \underline{or}} \\ (ii) & \text{for some } k < \omega \text{ and } p' \text{ we have } p^* \leq_{\mathbb{P}^t} p', p^t_{\zeta,k} \leq_{\mathbb{P}^t} p' \\ & \text{and } (\forall r'')[r' \leq r'' \in \mathbb{P}_n \cap \delta^t \rightarrow (\{r'', p'\} \\ & \text{has an upper bound in } \mathbb{P}^t)]\Big\}. \end{array}$$

* * *

We define the (natural) partial order $\leq^*$ on AP as follows: for $t, s \in$ AP we let $t \leq^* s$ <u>iff</u>:

$\boxplus_4$ (i) $\delta^t \leq \delta^s$

(ii) $\mathbb{P}^t \subseteq \mathbb{P}^s$

(iii) $F^t_\omega \subseteq F^s_\omega$

(iv) $F^t_{n,\omega,\ell} \subseteq F^s_{n,\omega,\ell}$ for $n, \ell < \omega$

(v) Γ^t is an initial segment of Γ^s.

$\square$

Fact A: $AP \neq \emptyset$.

Proof. Easy: choose $\delta \in E_\omega$, let $\delta^t = \delta, \mathbb{P}^t = \big(\bigcup\limits_{n<\omega} \mathbb{P}_n\big) \restriction \delta, F^t_\omega(p,q) = F_{\mathbf{n}(p,q)}(p,q)$ where $\mathbf{n}(p,q) = \text{ Min}\{n : p \in \mathbb{P}_n \text{ and } q \in \mathbb{P}_n\}$.
For $n < \omega, p \in \mathbb{P}^t$ let $\langle F^t_{n,\omega,\ell}(p) : \ell < \omega\rangle$ be $\langle F_{n,m,\ell}(p) : \ell < \omega\rangle$ for the first $m > n$ such that $p \in \mathbb{P}_m$.

Lastly, $\Gamma^t =$ the empty sequence. It is easy to check that $t \in$ AP, as required. $\square$

Fact B: If $t \in$ AP and $\delta^t < \delta \in E_\omega$ and δ is an old case for some $\mathcal{Y}_n$, then there is s satisfying $t \leq^* s \in$ AP with $\delta^s \geq \delta, \zeta^s = \zeta^t$.

Proof. Without loss of generality $t, \langle \mathbb{P}_n \restriction \delta : n < \omega\rangle, \langle \delta \cap X_n : n \leq \omega\rangle$ belong to M_δ.

Note

(∗) any $\mathcal{J} \in M_\delta$ which is a predense subset of $\mathbb{P}_n \restriction \delta$ is a predense subset of $\mathbb{P}_n$ and $n < m \Rightarrow \mathbb{P}_n \restriction \delta \lessdot \mathbb{P}_m \restriction \delta$, and of course $\mathbb{P}_n{\restriction}\delta \in M_\delta$.

Let $A = \mathbb{P}^t \subseteq X_\omega \cap \delta^t, B = \bigcup\limits_{n<\omega} \mathbb{P}_n \cap \delta$. We define a forcing notion $\mathbb{Q}$, with set of elements $\subseteq A \times B$ identifying $(p, 0)$ with p and $(0, q)$ with q. Now $(p, q) \in A \times B$ belongs to $\mathbb{Q}$ iff : $p = 0$ or $q = 0$ or there are $r \in A \cap B$ and $n = \mathbf{n}(p, q)$ such that: $\mathbb{P}_n \models$ "$r \leq q$", and $(\forall r')[r \leq r' \in \mathbb{P}_n \cap \delta^t \to r', p$ compatible in $\mathbb{P}^t]$; we call such r a witness and n a possible value for $\mathbf{n}(p,q)$ and may say r, n witness this. The order on $\mathbb{Q}$ is $(p,q) \leq (p',q') \Leftrightarrow p \leq_{\mathbb{P}^t} p'$ and $\bigvee\limits_n (q \leq_{\mathbb{P}_n} q')$.

Now note

(α) $\mathbb{Q} \restriction A = \mathbb{P}^t$

(β) $\mathbb{Q} \restriction B = \bigcup\limits_{n<\omega} \mathbb{P}_n \restriction \delta$

(γ) If $(p,q) \in \mathbb{Q}, m = \mathbf{n}(p,q)$ hence $q \in \mathbb{P}_m \restriction \delta$ and $\mathbb{P}_m \restriction \delta \models$ "$q \leq q'$" and $\mathbb{P}^t \models$ "$p' \leq p$", then $(p', q') \in \mathbb{Q}$ and $\mathbb{Q} \models$ "$(p,q) \leq (p,q')$".

[Why? As r is a witness and m a possible value for $\mathbf{n}(p', q')$.]

(δ) if $(p,q) \in \mathbb{Q}$ and $n = \mathbf{n}(p,q) \leq m < \omega$, then for some q_1 we have: $(p,q) \leq (p,q_1) \in \mathbb{Q}$ and $\mathbf{n}(p,q_1) = m$, or at least m is a possible value for $\mathbf{n}(p,q_1)$.

[Why? Let $r \in \mathbb{P}_{n(p,q)}$ be a witness. By clause (θ) of the Definition of AP the set $\mathcal{J} = \{F^t_{m,\omega,\ell}(p) : \ell < \omega$ and p is compatible with $F^t_{m,\omega,\ell}(p)$ in $\mathbb{P}^t\}$ is predense above r in $\mathbb{P}_m$. Now $\mathbb{P}_n \models r \leq q$ hence $\mathbb{P}_m \models r \leq q$ so for some $\ell, F^t_{m,\omega,\ell}(p) \in \mathcal{J}$ is compatible with q in $\mathbb{P}_m$ so there is $q_1 \in \mathbb{P}_m \cap \delta$ such that $\mathbb{P}_m \models q \leq q_1 \wedge F^t_{m,\omega,\ell}(p) \leq q_1$. So $(p,q_1) \in \mathbb{Q}$ as witnessed by m and $r' = F^t_{m,\omega,\ell}(p)$, is as required.]

(ε) $\mathbb{P}_n \restriction \delta \lessdot \mathbb{Q}$.

[Why? Recall that $\mathbb{Q}_1 \lessdot \mathbb{Q}_2$ iff $\mathbb{Q}_1 \subseteq \mathbb{Q}_2$ and for every $p_2 \in \mathbb{Q}$ there is $p_1 \in \mathbb{Q}_1$ such that $p_1 \le p_1' \in \mathbb{Q}_1 \Rightarrow p_1', p_2$ have a common upper bound in $\mathbb{Q}_2$; we shall use this criterion. Let $(p^0, q^0) \in \mathbb{Q}$, of course, we can replace this pair by any larger one, so by clause (δ) above without loss of generality some $m \in [n, \omega)$ is a possible value for $\mathbf{n}(p^0, q^0)$ so we have $q^0 \in \mathbb{P}_m \restriction \delta$, hence recalling that $\mathbb{P}_n \restriction \delta \lessdot \mathbb{P}_m \restriction \delta$ there is $q^1 \in \mathbb{P}_n \restriction \delta$ such that:

$$(\forall r \in \mathbb{P}_n)(\mathbb{P}_n \restriction \delta \models q^1 \le r \Rightarrow r, q^0 \text{ compatible in } \mathbb{P}_m \restriction \delta).$$

Assume $q^1 \le r \in \mathbb{P}_n \restriction \delta$. So r, q^0 are compatible in $\mathbb{P}_m \restriction \delta$ hence has a common upper bound $q^2 \in \mathbb{P}_m \restriction \delta$.
In particular $q^0 \le q^2 \in \mathbb{P}_m \restriction \delta$ so by clause (γ) we have $(p^0, q^2) \in \mathbb{Q}$ and $(p^0, q^0) \le^{\mathbb{Q}} (p^0, q^2)$; also $r = (0, r) \le (p^0, q^2)$ as $r \le q^2$ together $r, (p^0, q^0)$ are compatible in $\mathbb{Q}$, so $[q^1 \le r \in \mathbb{P}_n \restriction \delta \Rightarrow (p^0, q^0), r = (0, r)$ are compatible in $\mathbb{Q}]$. As $(p^0, q^0) \in \mathbb{Q}$ was arbitrary we are done.]

(ζ) if $p_1, p_2 \in \mathbb{P}^t$ are incompatible in $\mathbb{P}^t$ <u>then</u> they are incompatible in $\mathbb{Q}$.

[Why? Look at the order of $\mathbb{Q}$].

(η) if $\zeta < \zeta^t$ then $\bar{p}^t_\zeta$ is a maximal antichain in $\mathbb{Q}$.

[Why? Let $(p^*, q^*) \in \mathbb{Q}$ and we shall prove that it is incompatible in $\mathbb{Q}$ with some $(p^t_{\zeta,k}, 0)$ with $k < \omega$. Let $n < \omega$ be a possible value of $\mathbf{n}(p^*, q^*)$ so $q^* \in \mathbb{P}_n \restriction \delta$ and there is a witness $r^* \le q^*, r^* \in \mathbb{P}_n \restriction \delta^t$ for $(p^*, q^*) \in \mathbb{Q}$.
By clause (κ) in the definition of $t \in AP$ we know that for some $r \in \mathbb{P}_n \cap \delta^t$ we have:

(i) $r \in \mathcal{I}^t_{\zeta,n,p^*}$
(ii) q^*, r are compatible in $\mathbb{P}_n$

As q^*, r are compatible and $r^* \le q^*$ also r^*, r are compatible in $\mathbb{P}_n$ hence in $\mathbb{P}_n \cap \delta^t$, so by the demand on r^*, we have: r, p^* are compatible in $\mathbb{P}^t$. So in clause (κ) of the definition of AP, in the definition of $\mathcal{I}^t_{\zeta,n,p^*}$ for our r subclause (i) fails hence subclause (ii) holds so there are k, p' as in subclause (ii) there. Also let $q^1 \in \mathbb{P}_n \restriction \delta$ be a common upper bound of q^*, r. So r witnesses that $(p', q^1) \in \mathbb{Q}$ with n a possible value of $n(p', q^1)$. Clearly it is above (p^*, q^*) and above $p^t_{\zeta,k}$ so we are done.]

Let $\delta^s = \delta$. Clearly $\mathbb{Q} \in M_\delta$ and $M_\delta \models$ "$|\mathbb{Q}| = |\delta|$" so as $X^* \cap \delta \backslash \bigcup_{n<\omega} \mathbb{P}_n$ has order type δ and $\mathbb{P}^t$ is bounded in it, there is $f \in M_\delta$ such that $f : \mathbb{Q} \to X_\omega \cap \delta$ is a one to one (into or even onto), extending $\mathrm{id}_A \cup \mathrm{id}_B$, and define $\mathbb{P}^s$ such that f is an isomorphism from $\mathbb{Q}$ onto $\mathbb{P}^s$. We can define $F^s_\omega, F^s_{n,\omega,\ell}$ $(n, \ell < \omega)$ extending $F^t_\omega, F^t_{n,\omega,\ell}$ as required, e.g., $F^s_{n,\omega,\ell}((p, q)) = F_{n,m,\ell}(q)$ for some $m > n$ such that $q \in \mathbb{P}_m$ except when $q = 0$ then $F^s_{n,\omega,\ell}((p, 0)) = F^t_{n,\omega,\ell}(p)$. Now it is easy to check clause (θ) of the definition of $s \in$ AP. Also clauses $(\iota), (\kappa)$ hold because the construction is made in M_δ recalling $(*)$ above since the construction is made in M_δ.

Lastly, let $\Gamma^s = \Gamma^t$.

Fact C: If $t^i \in \mathrm{AP}$ and $t^i \le^* t^{i+1}$ for $n < \omega$ then there is t such that $i < \omega \Rightarrow t^i \le^* t \in AP$ and $\delta^t = \bigcup_{i<\omega} \delta^{t^i}$ and $\zeta^t = \bigcup_{n<\omega} \zeta^{t^n}$.

[Why? Just let δ^t, ζ^t be as above, as E'_ω is a cub of ω_1 and $\delta^{t^i} \in E'_\omega$ necessarily $\delta^t \in E'_\omega$. Next, $\mathbb{P}^t = \bigcup_{i<\omega} \mathbb{P}^{t^k}, F^t_\omega = \bigcup_{i<\omega} F^{t^i}_\omega, F^t_{m,\omega,\ell} = \bigcup_{i<\omega} F^{t^i}_{m,\omega,\ell}$ and $p^t_{\zeta,k} = p^{t^i}_{\zeta,k}$ for every i large enough. Now check.]

Main Fact D: Assume $t \in \mathrm{AP}$, $\delta^t \in E_\omega := \bigcap_{n<\omega} E_n \cap E, t \in M_{\delta(t)}$ and $\delta(t) := \delta^t \in \bigcup_{n<\omega} S^{\mathcal{Y}_n}$. Then there is $s \in AP$ such that

- (a) $t \le^* s$
- (b) $\underset{\sim}{\nu}_{\delta(t)}$ is actually a wide $\mathbb{P}^s$-name (i.e. all the countably many conditions appearing in its definition belong[4] to $\bigcup_{n<\omega} \mathbb{P}_n \cap \delta^s \subseteq \mathbb{P}^s$):
- (c) for any forcing notion $\mathbb{P}'$ if $\mathbb{P}^s \subseteq_{ic} \mathbb{P}'$, and for each $\zeta < \zeta^s$ the sequence $\bar{p}^s_\zeta$ is a maximal antichain of $\mathbb{P}'$, then $\Vdash_{\mathbb{P}'}$ "there is $G' \subseteq \mathbb{Q}^{M_{\delta(t)}}_{\underset{\sim}{\bar{\varphi}}_{\delta(t)}}[\underset{\sim}{G}]$ generic over $M_{\delta(t)}[\underset{\sim}{G}_{\mathbb{Q}}]$ such that $(\underset{\sim}{\eta}_{\delta(t)}[\underset{\sim}{G}_{\mathbb{Q}}])[G'] = \underset{\sim}{\nu}_{\delta(t)}[G]$".

$\square$

Proof. Let δ be such that

$(*)_1$
- (a) $\delta \in E_\omega$
- (b) $\delta > \delta(t)$
- (c) $M_{\delta(t)}, \underset{\sim}{\nu}_{\delta(t)} \in M_\delta$ and $n(*) = \min\{n : \delta^t \in S^{\mathcal{Y}^n}\}$
- (d) δ is of the old case in $\mathcal{Y}_{n(*)}$ (is not really used).

We stipulate the $\emptyset_{\mathbb{Q}_{\underset{\sim}{\bar{\varphi}}_{\delta(t)}}}$ depends on $\delta(t)$ and not on the value of $\underset{\sim}{\bar{\varphi}}_{\delta(t)}$ so we write $\emptyset_{\bar{\varphi}_{\delta(t)}}$. We shall work in M_δ and we shall choose $s \in M_\delta$, as required with $\delta^s = \delta$, so the real task is to choose $\mathbb{P}^s$,

$(*)_2$
- (a) let $\mathbb{P}_{t,n} = \mathbb{P}_n \restriction \delta^t$ for $n < \omega$
- (b) $\mathbb{P}_{t,\bar{\varphi}} = \mathbb{P}^t * \mathbb{Q}_{\bar{\varphi}}$ in the sense of $M_{\delta(t)}$ when $\bar{\varphi} \in M_\delta$ is a possible value of $\underset{\sim}{\bar{\varphi}}_{t(\delta)}$, so $\mathbb{P}_{t,\bar{\varphi}}$ belongs to $M_{\delta(t)}$
  note: $\underset{\sim}{\bar{\varphi}}$ slightly depends on $\underset{\sim}{G}_{p_s,\varphi_{n(*)}}$ but each possible value $\in M_{\delta(t)}$
- (c) $\mathbb{P}_{s,n} = \mathbb{P}_n \restriction \delta$
- (d) $\mathbb{P}_{s,\omega} = \cup\{\mathbb{P}_{s,n} : n < \omega\}$.

So

$(*)_3$
- (a) $\langle \mathbb{P}_{t,n} : n < \omega \rangle \hat{}\, \langle \mathbb{P}^t \rangle \in M_{\delta(t)}$
- (b) $\langle \mathbb{P}_{s,\varepsilon} : \varepsilon \le \omega \rangle \in M_\delta$.

[Why? Clause (a) holds by "$\delta^t \in E_\omega$" see $\boxplus_2$, and clause (b) holds because $\delta \in E_\omega$, see $(*)_1(a)$.]

Let

[4] in fact, for some n, all those conditions are from $\mathbb{P}_n \cap \delta^s$

$(*)_4$ (a) the set of elements of $\mathbb{P}_{t,\bar\varphi}$ be $A \subseteq ((X_\omega \cap \delta^t) \cup (X_\omega \cap \delta \setminus \bigcup\limits_{n<\omega} X_n)), A \in M_\delta$ and

(b) $B = \bigcup\limits_{n<\omega} \mathbb{P}_{s,n}$.

We define a forcing notion $\mathbb{Q}$:

(A) $p \in \mathbb{Q}$ iff p has the form (p_1, p_2) such that for some $n, \bar\varphi$
- (a) $p_2 \in \mathbb{P}_{s,\omega}$
- (b) $n \in [n(*), \omega)$ is minimal such that $p_2 \in \mathbb{P}_n \restriction \delta$ so let $\mathbf{m}(p) = \mathbf{m}(p_1, p_2) = n$
- (c) $p_1 = (p'_1, p''_1), p'_1 \in \mathbb{P}^t$ and one of the following holds
  - $\bullet_1$ $p_2 = \emptyset_{\mathbb{P}_s}$ and $p''_1 = \emptyset_{\bar\varphi_{\delta(t)}}$
  - $\bullet_2$ p_2 forces $\underset{\sim}{\bar\varphi}$ is $\bar\varphi$ and p''_1 is a $\mathbb{P}^t$-name of a member of $\mathbb{Q}_{\bar\varphi}$ in $M_{\delta(t)}$ so $\bar\varphi = \bar\varphi_{p_2} = \bar\varphi_{(p_1,p_2)}$
- (d) if $\mathbf{G} \subseteq \mathbb{P}_n \restriction \delta$ is generic over M_δ and $p_2 \in \mathbf{G}$ and $\bar\varphi = \underset{\sim}{\bar\varphi}_{\delta(t)}[\mathbf{G}], \nu = \underset{\sim}{\nu}_\delta[\mathbf{G}]$ then there is $\mathbf{G}' \subseteq \mathbb{P}_{t,\bar\varphi}$ generic over $M_{\delta(t)}$ such that
  - (α) $\mathbf{G} \cap \mathbb{P}_{t,n} \subseteq \mathbf{G}'$
  - (β) $\underset{\sim}{\eta}_{\delta(t)}[\mathbf{G}'] = \nu$
  - (γ) $p_1 \in \mathbf{G}'$

(B) $\mathbb{Q} \models$ "$(p_1, p_2) \le (q_1, q_2)$ iff (a) + (b) where
- (a) if (p_1, p_2) satisfies $\bullet_1$ of clause (A)(c) then $\mathbb{P}^+_t \models$ "$p'_1 \le q'_1$"
- (b) if not, then $\mathbf{m}(p_1, p_2) \le \mathbf{m}(q_1, q_2)$ and $\mathbb{P}_t * \mathbb{Q}_{\bar\varphi} \models$ "$p_1 \le q_1$" and $\mathbb{P}_n \models$ "$p_2 \le q_2$" for n large enough.

So

$(*)_5$ (a) $\mathbb{Q}$ is a forcing notion, i.e. a quasi order

(b) $\mathbb{Q}$ belongs to M_δ.

[Why? Least trivial is to show transitivity of $\le_{\mathbb{Q}}$, we split the proof by cases and it is obvious.]

We define a function $\mathbf{j}$ from $\mathbb{P}^t$ into $\mathbb{Q}$:

$(*)_6$ for $p \in \mathbb{P}^t$ let $\mathbf{j}(p) = ((p, \emptyset_{\bar\varphi_{\delta(t)}}), \emptyset_{\mathbb{P}_{s,\omega}})$ where $\emptyset_{\bar\varphi_{\delta(t)}}, \emptyset_{\mathbb{P}_{s,\omega}}$ is the minimal condition in the forcing notions $\mathbb{P}_{\bar\varphi_{t(\delta)}}, \mathbb{P}_{s,\omega}$ respectively

$(*)_7$ $\mathbf{j}$ is a $\subseteq_{\rm ic}$-embedding of $\mathbb{P}^t$ into $\mathbb{Q}$.

[Why? First, clearly $\mathbf{j}$ is one to one. Second, for $p \in \mathbb{P}^t$ why $\mathbf{j}(p) \in \mathbb{Q}$? We have to check clauses (a)-(d) in (A) above. Now clauses (a),(b),(c) are obvious, as for (d), let $n, \mathbf{G}$ and then $\bar\varphi, \nu$ be as there. Note that ν is generic for $(\underset{\sim}{\eta}, \mathbb{Q}_{\bar\varphi})$ over $M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{t,n}]$ and let $\mathbb{P}'_t = \mathbb{P}^t_{\ge p'_1}/(\mathbf{G} \cap \mathbb{P}_{t,n})$.

We know also that $M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{t,n}] \models$ "$\mathbb{Q}_{\bar\varphi} \lessdot (\mathbb{P}^t * \mathbb{Q}_{\bar\varphi})/(\mathbf{G} \cap \mathbb{P}_{n,t}) = (\mathbb{P}^t/(\mathbf{G} \cap \mathbb{P}_{t,n})) * \underset{\sim}{\mathbb{Q}}_{\bar\varphi}$" and recalling $\mathbb{Q}_{\bar\varphi}$ is absolutely c.c.c., moreover $M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{t,n}] \models$ "$\mathbb{Q}_{\bar\varphi} \lessdot \mathbb{P}'_t * \underset{\sim}{\mathbb{Q}}_{\bar\varphi}$".

Now there is $\mathbf{G}' \subseteq \mathbb{Q}^{M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{n,t}]}_{\bar\varphi}$ generic over $M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{n,t}]$ such that $\underset{\sim}{\eta}[\mathbf{G}'] = \nu$ hence there is $\mathbf{G}'' \subseteq \mathbb{P}'_t * \underset{\sim}{\mathbb{Q}}_{\bar\varphi}$ generic over $M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{n,t}]$ extending $\mathbf{G}'$. Together $\mathbf{G}''$ is as required.

Third, $\mathbb{P}^t \models$ "$p \le q$" $\Leftrightarrow \mathbb{Q} \models$ "$\mathbf{j}(p) \le \mathbf{j}(q)$" by the definitions of the order in $\mathbb{Q}$, i.e. clause (B) above and the definition of the order in $\mathbb{P}^+_t = \mathbb{P}^t * \mathbb{Q}_{\underset{\sim}{\bar\varphi}_{\delta(t)}}$.

Lastly, to prove the $\subseteq_{\rm ic}$-embedding, toward contradiction assume that $p, q \in \mathbb{P}^t$ are compatible in $\mathbb{P}^t$ but some $r \in \mathbb{Q}$ is a common $\leq_{\mathbb{Q}}$-upper bound of $\mathbf{j}(p), \mathbf{j}(q)$, let $r = (r_1, r_2) = ((r'_1, r''_1), r_2)$. By the definition of $\leq_{\mathbb{Q}}$ we have $\mathbb{P}^t \models$ "$p \leq r'_1$" and $\mathbb{P}^t \models$ "$q \leq r'_2$", so $p, q \in \mathbb{P}^t$ are compatible in $\mathbb{P}^t$ so we are done proving $(*)_7$.]

$(*)_8$ (a) $\mathbf{j} \in M_\delta$

(b) $\mathbb{Q} \backslash$ Rang$(\mathbf{j})$ is dense in $\mathbb{Q}$.

[Why clause (a)? Think.

Why clause (b)? Assume that $p \in \mathbb{P}^t$ so $\mathbf{j}(p) = ((p, \emptyset_{\bar{\varphi}_\delta}), \emptyset_{\mathbb{P}_{s,\omega}})$. Now $\mathbb{P}_{t,n(*)} \lessdot \mathbb{P}^t$ hence there is $p_0 \in \mathbb{P}_{t,n(*)}$ such that $[p_0 \leq p'_0 \in \mathbb{P}_{t,n(*)} \Rightarrow p'_0, p$ are compatible in $\mathbb{P}^t]$ and without loss of generality $p^*_0 \neq \emptyset_{\mathbb{P}_n}$. Let $\bar{\varphi}$ and p_2 be such that $p_2 \in \mathbb{P}_{s,n}$ is above p_0 and forces $\underset{\sim}{\bar{\varphi}}_{\delta(t)} = \bar{\varphi}$. Consider the triple $((p, \emptyset_{\bar{\varphi}_\delta}), p_2)$, it suffices to prove that it belongs to $\mathbb{Q}$, as $p_2 \neq \emptyset_{\mathbb{P}_n}$ and the demand $\mathbf{j}(p) \leq ((p, \emptyset_{\bar{\varphi}_\delta}), p_2)$ holds trivially.

Now in clause (A) above, subclauses (a),(b),(c) are obvious and as for subclause (d) we prove more than needed. Let $n \in [n(*), \omega)$ and $\mathbf{G} \subseteq \mathbb{P}_n \restriction \delta = \mathbb{P}_{s,n}$ generic over M_δ be given such that $p_2 \in \mathbf{G}$, so as $p_2 \in \mathbf{G}_{s,n(*)}$ and $p_2 \Vdash$ "$\underset{\sim}{\bar{\varphi}}_{\delta(t)} = \bar{\varphi}$" clearly $\bar{\varphi} = \underset{\sim}{\bar{\varphi}}_{t(\delta)}[\mathbf{G}]$ and let $\nu = \underset{\sim}{\nu}_\delta[\mathbf{G}]$. We should find $\mathbf{G}'$ as there.

Recall $\mathbf{G} \cap \mathbb{P}_{t,n}$ is a subset of $\mathbb{P}_{t,n}$ generic over $M_{\delta(t)}$ and $M_{\delta(t)} \models$ "$\mathbb{P}_{t,n} \lessdot \mathbb{P}^t$ and every member of $\mathbb{P}_{t,n}$ compatible with p_0 is compatible with p".

As $p_0 \in \mathbf{G}$ and $\mathbf{G} \in \mathbb{P}_{t,n}$ is directed and p_0 belongs to it. Clearly $M_{t,\delta}[\mathbf{G} \cap \mathbb{P}_{t,n}] \models$ "$p \in \mathbb{P}^t / (\mathbf{G} \cap \mathbb{P}_{t,n})$". As ν is generic for $(\underset{\sim}{\eta}_{\bar{\varphi}_\delta}, M_{\delta(t)}[G \cap \mathbb{P}_{t,n}])$ so $M_{t(\delta)}[\mathbf{G} \cap \mathbb{P}_{t,n}]^{[\nu]}$ is a generic extension of $M_{t(\delta)}[\mathbf{G} \cap \mathbb{P}_{t,n}]$ and it belongs to $M_\delta[\mathbf{G}]$ which is a generic extension of M_δ.

In $M_{\delta(t)}$ let $\mathbb{P}' = (\mathbb{P}^*_{t,n} * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^{M_{\delta(t)}}$ and $\mathbb{P}'' = (\mathbb{P}^t * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^{M_{\delta(t)}}$ so we know

$(*)$ in $M_{\delta(t)}$

(a) $\mathbb{P}' \lessdot \mathbb{P}''$

(b) $(p_0, \emptyset_{\mathbb{Q}_{\bar{\varphi}}}) \in \mathbb{P}'$

(c) $(p, \emptyset_{\bar{\varphi}}) \in \mathbb{P}''$

(d) if $(p_0, \emptyset_{\mathbb{Q}_{\bar{\varphi}}}) \leq r \in \mathbb{P}'$ then $r, (p, \emptyset_{\mathbb{Q}_{\bar{\varphi}}})$ are compatible in $\mathbb{P}''$.

Now $\mathbf{G} \cap \mathbb{P}_{t,n}$ is a subset of $\mathbb{P}_{t,n}$ generic over $M_{\delta(t)}$ and ν is $(\underset{\sim}{\eta}_{\bar{\varphi}}, \mathbb{Q}_{\bar{\varphi}})$-generic over $M_{\delta(\ell)}[\mathbf{G} \cap \mathbb{P}_{t,n}]$. Hence there is $\mathbf{H} \subseteq \mathbb{Q}^{M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{t,n}]}$ generic over $M_{\delta(t)}[\mathbf{G} \cap \mathbb{P}_{t,n}]$ such that $\underset{\sim}{\eta}_{\bar{\varphi}}[\mathbf{H}] = \nu$.

So $\mathbf{H}' = (\mathbf{G} \cap \mathbb{P}_{t,n}) * \mathbf{H}$ is a subset of $\mathbb{P}'$ generic over $M_{\delta(t)}$. As $p_0 \in \mathbf{G} \cap \mathbb{P}_{t,n} \subseteq \mathbf{H}'$ by $(*)(a) + (d)$ there is $\mathbf{G}' \subseteq \mathbb{P}''$ extending $\mathbf{H}'$ generic over $M_{\delta(t)}$ such that $p \in \mathbf{G}'$ hence $G \cap \mathbb{P}_{t,n} \subseteq \mathbf{G}'$ and note that $\mathbb{P}'' = \mathbb{P}^+_{t,\bar{\varphi}}$ so we are done.]

$(*)_9$ We identify $p \in \mathbb{P}^+_t$ with $\mathbf{j}(p)$.

Next

$\oplus_1$ (a) let gen$(\mathbb{Q}) = \{\mathbf{G} : \mathbf{G}$ is a directed subset of $\mathbb{Q}$ such that $\mathbf{G} \cap \mathbb{P}^t$ is generic over $M_{\delta(t)}$ and $\mathbf{G} \cap \mathbb{P}_{s,k}$ is generic over M_δ for each $k < \omega\}$

(b) $\text{gen}^+(\mathbb{Q}) = \{\mathbf{G} \subseteq \mathbb{Q} : \mathbf{G}$ is directed, $\mathbf{G} \cap \mathbb{P}_{s,n}$ generic over M_δ for $n < \omega, \mathbf{G} \cap \mathbb{P}^t$ is generic over $M_{\delta(t)}$, and letting $\bar{\varphi} = \underset{\sim}{\bar{\varphi}}_\delta[\mathbf{G} \cap \mathbb{P}_{s,n(*)}]$ for some ν we have $\underset{\sim}{\nu}_\delta[\mathbf{G} \cap \mathbb{P}_{s,n(*)}] = \nu$ and $\underset{\sim}{\eta}_{\bar{\varphi}}[\mathbf{G} \cap \mathbb{P}^+_{t,\bar{\varphi}}] = \nu\}$

Now

$\oplus_{1.1}$ note that $\text{gen}^+(\mathbb{Q}) = \{\mathbf{G} \in \text{gen}(\mathbb{Q})$: letting $\bar{\varphi} = \underset{\sim}{\bar{\varphi}}_\delta[\mathbf{G} \cap \mathbb{P}_{s,n(*)}]$ we have $\underset{\sim}{\nu}_\delta[\mathbf{G} \cap \mathbb{P}_{s,n(*)}]$ is $(\underset{\sim}{\eta}, \mathbb{Q}_\varphi)$-generic for $M_{\delta(t)}[\mathbf{G} \cap P_{t,\varphi}]\}$

$\oplus_{1.2}$ if $\mathbf{G} \in \text{Gen}^+(\mathbb{Q})$ and $\bar{\varphi} = \underset{\sim}{\bar{\varphi}}_\delta[\mathbf{G} \cap \mathbb{P}_{s,n(*)}]$ then $\mathbf{G} \cap \mathbb{P}^+_{t,\bar{\varphi}}$ is a subset of $\mathbb{P}^+_{t,\bar{\varphi}}$ generic over $M_{\delta(t)}$.

A major point is

$\oplus_2$ if $p \in \mathbb{Q}$ then there is: $\mathbf{G} \in \text{gen}^+(\mathbb{Q})$ to which p belongs.

[Why? So let $p = (p_1, p_2), p_1 = (p'_1, p''_2)$ and by $(*)_8$ without loss of generality $p \notin \text{Rang}(g)$ and so $p_2 \Vdash \underset{\sim}{\bar{\varphi}}_\delta = \bar{\varphi}$, so $n = \mathbf{n}(p_1, p_2) = \min\{n < \omega : n \geq n(*)$ and $p_2 \in \mathbb{P}_{s,n}\}$ is well defined.

Next, let $\langle \mathcal{I}_{1,k} : k \in [n, \omega) \rangle$ list the dense open subsets of $\mathbb{P}_{t,\bar{\varphi}}$ from $M_{\delta(t)}$ and let $\langle (j_k, \mathcal{I}_{2,k}) : k \in [n, \omega) \rangle$ list the pairs $(j, \mathcal{I})$ such that $j < \omega$ and $\mathcal{I} \in M_\delta$ is a dense open subset of $\mathbb{P}_{s,k}$ and without loss of generality $j_k \leq k$ and let $\mathcal{I}^+_{2,k} = \{q \in \mathbb{P}_{s,\omega} : q$ is above some $q \in \mathcal{I}_{2,k}\}$. By induction on $k \in [n, \omega)$ we choose q_k such that

$\oplus_{2.1}$ (a) $q_k = (q_{k,1}, q_{k,2}) = ((q'_{k,1}, q'_{k,2}), q_{k,2}) \in \mathbb{Q}$ hence

- $q_{k,1} \in \mathbb{P}_{t,\bar{\varphi}}$
- so $q'_{k,1} \in \mathbb{P}^t, q''_{k,2}$ is a $\mathbb{P}^t$-name of a member of $\underset{\sim}{\mathbb{Q}}_\varphi$
- $q_{k,2} \in \mathbb{P}_{s,k}$

(b) if $k = n$ then $q_k = p$

(c) if $k = m + 1 > n$, then $\mathbb{Q} \models$ "$q_m \leq q_k$"

(d) if $k = m + 1 > n$ and m is odd, then $q_{k,1} \in \mathcal{I}_{1,m}$.

Clearly if we succeed then $\{q \in \mathbb{Q} : q' \leq q_k\}$ for some $k < \omega$ is as required (noting that for every n the set $\{p \in \mathbb{P}_{n(*)} : p$ forces a value to $\underset{\sim}{\nu}_\delta \restriction n\} \in \{I_{2,k} : k < \omega\}$) so it suffices to carry the induction.

For $k = n$ there is nothing to do.

Let $k = m + 1 > n$ and assume q_m has already been chosen. Let $\mathbf{G}_m$ be a subset of $\mathbb{P}_{s,m}$ generic over M_δ such that $q_{2,m} \in \mathbf{G}_m$

$\oplus_{2.2}$ $M_\delta[\mathbf{G}_m] \Vdash$ "the forcing notion $((\mathbb{P}_{t,k}/(\mathbf{G}_m \cap \mathbb{P}_{t,m})) * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^{M_{t(\delta)}[\mathbf{G}_m \cap \mathbb{P}_{t,m}]}$ is a complete subforcing $(\mathbb{P}^t/(\mathbf{G}_m \cap \mathbb{P}_{t,m})) * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^{M_{t(0)}[\mathbf{G}_m \tilde{\cap} \mathbb{P}_{t,m}]}$.

Let $\nu := \underset{\sim}{\nu}_\delta[\mathbf{G}_m] = \underset{\sim}{\nu}_\delta[\mathbf{G}_m \cap \mathbb{P}_{s,n(*)}]$.

Recalling $q_m \in \mathbb{Q}$, there is $\mathbf{G}'_m$ such that

$\oplus_{2.3}$ $\mathbf{G}'_m$ is a subset of $\mathbb{P}^+_{t,\bar{\varphi}} = (\mathbb{P}^t * \underset{\sim}{\mathbb{Q}}_{\bar{\varphi}})^{M_{\delta(t)}}$ generic over $M_{\delta(t)}$ such that $\underset{\sim}{\eta}_{\bar{\varphi}}[\mathbf{G}'_m] = \nu, \mathbf{G}_m \cap \mathbb{P}_{t,m} \subseteq \mathbf{G}'_m$ and $q_{m,s} \in \mathbf{G}'_m$.

So there is $q_{k,1}$

$\oplus_{2.4}$ $q_{k,1} = (q'_k, q''_{k,1}) \in \mathbf{G}'_m$ is above $q_{m,1}$ and belongs to $\mathbf{G}'_m$.

By $\oplus_{2.2}$ we have

$\oplus_{2.5}$ $(\mathbb{P}_{t,k} * \underset{\sim}{\mathbb{Q}}_{\bar\varphi})^{M_{t(\delta)}} \cap \mathbf{G}'_m$ is a subset of $(\mathbb{P}_{t,k} * \underset{\sim}{\mathbb{Q}}_{\bar\varphi})^{M_{\delta(t)}}$ generic over $M_{t(\delta)}$ and still for it $\underset{\sim}{\eta}_{\bar\varphi}$ is interpreted as ν.

Hence there is a condition $(r'_1, r''_1) \in (\mathbb{P}_{t,k} * \underset{\sim}{\mathbb{Q}}_{\bar\varphi})^{M_{t(\delta)}} \cap \mathbf{G}'_m$ such that

$\oplus_{2.6}$ (a) $r'_1 \in \mathbb{P}_{t,k}/(\mathbf{G}_m \cap \mathbb{P}_{t,m})$
(b) if $r'_1 \le r \in \mathbb{P}_{t,k}$ then $r, q_{k,1}$ are compatible in $\mathbb{P}^t$,

moreover

$\oplus_{2.7}$ (a) $(r'_1, r''_1) \in (\mathbb{P}_{t,k} * \underset{\sim}{\mathbb{Q}}_{\bar\varphi})^{M_{t(\delta)}}$
(b) if $(r'_1, r''_1) \le (r'_2, r''_2) \in (\mathbb{P}_{t,k} * Q_{\bar\varphi})^{M_{t(\delta)}}$ <u>then</u> $(r'_2, r''_2), (q'_{k,1}, q''_{k,1})$ are compatible in $((\mathbb{P}^t/(\mathbf{G}_m \cap \mathbb{P}_{t,m})) * \underset{\sim}{\mathbb{Q}}_{\bar\varphi})^{M_{t(\delta)}[\mathbf{G}_m]}$.

So noting $\mathbf{G}_k$ is directed and $r'_1, q_{m,2} \in \mathbf{G}_k$ clearly there is $q_{k,2}$ such that

$\oplus_{2.8}$ (a) $q_{k,2} \in \mathbf{G}_k \subseteq \mathbb{P}_{s,k}$
(b) $r'_1 \le q_{k,2}$ in $\mathbb{P}_{s,k}$
(c) $q_{m,2} \le q$ in $\mathbb{P}_{s,k}$
(d) $q_{k,2} \in \mathcal{I}^+_{2,m}$
(e) $q_{k,2} \Vdash_{\mathbb{P}_{2,k}}$ "$\underset{\sim}{\nu}_\delta$ satisfies r''_1".

We next show that

$\oplus_{2.9}$ $q_k = (q_{k,1}, q_{k,2})$ is as required.

Now why does clause $\oplus_{2.1}(a)$, i.e. why $q_k \in \mathbb{Q}$? Now in the definition (A) of $\mathbb{Q}$ clauses (a),(b),(c) obviously holds, and if we succeed to prove (d) there then clauses (c),(d) of $\oplus_{2.1}$ are obvious as $\oplus_{2.4}$ and $\oplus_{2.8}(d)$ respectively and clause (b) of $\oplus_{2.1}$ holds emptily, so it is enough to prove (d) of (A).

So why does (A)(d) hold? Let $\mathbf{G}_k \subseteq \mathbb{P}_{s,k}$ be generic over M_δ such that $q_{k,2} \in \mathbf{G}_k$ and let $\nu = \underset{\sim}{\nu}_\delta[\mathbf{G}_k]$. Now we work in $M_\delta[\mathbf{G}_k]$, so $M_{\delta(t)}[\mathbf{G}_k \cap \mathbb{P}_{t,k}]$ is a generic extension of $M_{\delta(t)}$, both are countable and $q''_{k,1}[\mathbf{G}_k \cap \mathbb{P}_{t,k}]$ is a member of $\mathbb{Q}_{\bar\varphi}^{M_{\delta(t)}[\mathbf{G}_k \cap \mathbb{P}_{t,k}]}$ which ν satisfies, i.e. there is a generic subset $\mathbf{H}$ of $\mathbb{Q}^{M_{\delta(t)}}[\mathbf{G}_k \cap \mathbb{P}_{t,k}]_{\bar\varphi_\delta}$ to which it belongs and $\underset{\sim}{\eta}_{\bar\varphi_\delta}[\mathbf{H}_1] = \nu$.

So $\mathbf{H}^+ = (\mathbf{G}_k \cap \mathbb{P}_{t,k}) * \mathbf{H}$ is a subset of $(\mathbb{P}_{t,k} * \underset{\sim}{\mathbb{Q}}_{\bar\varphi_\delta})^{M_{\delta(t)}}$ generic over $M_{\delta(t)}$. Recalling $\oplus_{2.2} + \oplus_{2.7}$ there is $\mathbf{H}'$ a subset of $\mathbb{P}_{t,\bar\varphi} = (\mathbb{P}^t * \underset{\sim}{\mathbb{Q}}_{\bar\varphi_\delta})^{M_{\delta(t)}}$ generic over $M_{\delta(t)}$ which extends $\mathbf{H}^+$. So clearly $\mathbf{G}'_k = \mathbf{H}' \cap \mathbb{P}_{t,\bar\varphi}$ is as required in (A)(d), so indeed $q_k \in \mathbb{Q}$ so indeed we have carried the induction, but as said after $\oplus_{2.1}$ it suffices to carry the induction for proving $\oplus_2$ so we are done proving $\oplus_2$.]

$\oplus_3$ We identify $p \in \mathbb{P}^t$ with $\mathbf{j}(p)$ and $q \in \mathbb{P}_{s,\omega}$ with $(\emptyset, q)$ pedantically replace $\mathbb{Q}$ by the quasi order $\mathbb{Q}'$ with set of elements $\{p : p \in \mathbb{P}^t \text{ or } p = (\mathbb{P}^t, q), q \in \mathbb{Q}\}$ and $\mathbb{Q}' \models$ "$r_1 \le r_2$" iff $\mathbb{P}^t \models$ "$r_1 \le r_2$" or $r_1 \in \mathbb{P}^t, r_2 = (\mathbb{P}^t, q_2), \mathbb{Q} \models$ "$\mathbf{j}(r_1) \le q_2$" or $r_1 = (\mathbb{P}^t, q_1), r_2 = (\mathbb{P}^t, q_2), \mathbb{Q} \models$ "$q_1 \le q_2$"

$\oplus_4$ in M_δ let $\mathbb{Q}_*$ be the forcing generated freely in M_δ for our requirements (see [Sh:f, Ch.IX]), i.e.
(a) let Ω be the set of $\psi \in M_\delta$ such that: ψ is a propositional $\mathbb{L}_{\omega_1,\omega}$-sentence generated by the set $\mathbb{Q}$ of propositional variables

(b) for $\psi \in \Omega$ and $\mathbf{G} \subseteq \Omega$ we define the truth value $\psi[\mathbf{G}]$ by induction of ψ such that: if $\psi \in \mathbb{Q}$ then $\psi[\mathbf{G}] = \text{truth} \Leftrightarrow \psi \in \mathbf{G}$ the connecting act as usual

(c) let $\mathbb{Q}^+$ set of elements be the set of $\psi \in \Omega$ such that there is $\mathbf{G} \in \text{gen}^+(\mathbb{Q})$ such that the truth value $\psi[\mathbf{G}]$ is true, i.e. $\mathbf{G}$ is a model of ψ

(d) $\mathbb{Q}^t \models \psi_1 \le \psi_2$ iff $(\forall \mathbf{G} \in \text{gen}(\mathbb{Q}))$ [if $\psi_2[\mathbf{G}]$ is true then also $\psi_1[\mathbf{G}]$ is true]

[Why? Because there is no condition $((p'_1, p''_1), p_2) \in \mathbb{Q}$ that could force the opposite.] $\square_{2.1}$

Discussion 2.3. On the one hand $M_{\delta(t)}$ know $\mathbb{P}_{t,n} = \mathbb{P}_n \restriction \delta(t)$ is countable but have the commitment about $\underset{\sim}{\nu}_{\delta(t)}$. On the other hand M_δ knows $\mathbb{P}_{s,n-1}$ but have no commitment (generic old).

So $M_\delta \models$ "$\mathbb{Q}$ is countable" and for $\mathbb{Q}^t$ we do not have to think about a commitment in δ.

Now

$(*)_1$ $\mathbb{Q}_* \in M_\delta$.

[Why? Formally as $M_\delta \models$ "$\mathcal{P}(\omega)$ exist", the class of all relevant sentences exist and is of cardinality continuum in the sense of M exist. Think of $M_\delta^{\text{Levy}(\aleph_0, 2^{\aleph_0})}$ and use absoluteness (the danger is that the ordinal depth $> M_\delta \cap$ Ord, but by the above this does not occur). So definition of $\mathbb{Q}_*$ in M_δ is $\{\psi$: as above $+ \Vdash_{\text{Levy}(\aleph_0, 2^{\aleph_0})}$ "there is $\mathbf{G}\}$".]

Next

$(*)_2$ $M_\delta \models$ "$\mathbb{P}_{s,n} \lessdot \mathbb{Q}_*$".

[Why? Let $\mathcal{I} \in M_\delta$ be a predense subset of $\mathbb{P}_{s,n}$ and let $\psi \in \mathbb{Q}_*$. So there is $\mathbf{G} \in \text{gen}^+(\mathbb{Q})$ such that $\psi[\mathbf{G}] = \text{true}$; of course $\mathbf{G} \in \text{gen}(\mathbb{Q})$. By the definition of $\mathbf{G} \in \text{gen}(\mathbb{Q})$ necessarily $\mathbf{G} \cap \mathcal{I} \neq \emptyset$; let $p \in \mathbf{G} \cap \mathcal{I}$, and let $\psi_1 = \psi \wedge p$, so $\psi_1 \in \mathbf{G}$ hence $\psi_1 \in \mathbb{Q}_*$. Also obviously $\mathbb{Q}_* \models$ "$\psi \le \psi_1$" and $\mathbb{Q}_* \models$ "$p \le \psi_1$" so ψ, r are compatible in $\mathbb{Q}_*$, so we are done.]

$(*)_3$ $M_\delta \models$ "if $\mathcal{I} \in M_{\delta(t)}$ is predense in $\mathbb{P}^t$ then $\mathcal{I}$ is predense in $\mathbb{Q}_*$".

[Why? Similarly.]

$(*)_4$ $\mathbb{Q} \subseteq_{\text{ic}} \mathbb{Q}_*$ (ignoring separability) moreover $\Vdash_{\mathbb{Q}^+}$ "$\underset{\sim}{\mathbf{G}}_{\mathbb{Q}_*^+} \cap \mathbb{Q}$ is directed".

[Why? Assume $p, q \in Q, \psi \in \mathbb{Q}^+, \mathbb{Q}^+ \models$ "$p \le \psi \cap q \le \psi$" and we shall find a common upper bound $r \in \mathbb{Q}$ of p, q which is compatible with ψ in $\mathbb{Q}^+$, this clearly suffices. As $\psi \in \mathbb{Q}^+$ there is $\mathbf{G} \in \text{Gen}^+(\mathbb{Q})$ such that $\psi[\mathbf{G}] = \text{true}$. As $\mathbb{Q}^+ \models$ "$p \le \psi \wedge q \le \psi$" clearly $p, q \in \mathbf{G}$. Let $n < \omega$ be large enough such that $p_2, q_2 \in \mathbb{P}_{r,n}$ and $n(*) \le n$. We continue as in the proof of $\oplus_2$.]

$(*)_5$ (a) if $\psi_1, \psi_2 \in \mathbb{Q}_*$ then $\psi_1 \wedge \psi_2$ is the $\le_{\mathbb{Q}_*}$-lub of $\psi_1, \psi_2, \psi_1 \wedge \psi_2 \in \mathbb{Q}_*$

(b) $\psi_1, \psi_2 \in \mathbb{Q}_*$ are incompatible in $\mathbb{Q}_*$ if $\psi_1 \wedge \psi_2 \notin \mathbb{Q}_*$

(c) if $\psi, \neg\psi \in \mathbb{Q}_*$ then they are incompatible and every $\psi \in \mathbb{Q}_*$ is compatible with at leaset one of them

(d) if $\psi \in \mathbb{Q}_*, \neg\psi \notin \mathbb{Q}_*$ then every $\varphi \in \mathbb{Q}_*$ is compatible with ψ

(e) $\psi, \neg\neg\psi$ are equivalent (both belong or does not belong to $\mathbb{Q}_*$, etc.)

(f) like (a),(b) for $\langle \psi_\ell : i < \alpha \rangle$ for $\alpha < \omega_1$.

[Why? Obvious.]

$(*)_6$ if $\mathbf{G} \subseteq \mathbb{Q}_*$ is generic over M_δ <u>then</u> for $\psi \in \mathbb{Q}_*$ we have $\psi \in \mathbf{G} \Leftrightarrow \psi[\mathbf{G} \cap \mathbb{Q}] =$ truth.

[Why? We prove this by induction on ψ.

<u>Case 1</u>: $\psi \in \mathbb{Q}_*$
Obvious.

<u>Case 2</u>: $\psi = \neg\varphi$
By $(*)_5(c), (d)$.

<u>Case 3</u>: $\psi = \bigwedge\limits_{i<\alpha} \psi_i$ so $M_\delta \models$ "$|\alpha| = \aleph_0$"
By $(*)_5(e)$.

$(*)_7$ if $\mathbf{G} \subseteq \mathbb{Q}_*$ is generic over M_δ <u>then</u> $\mathbf{G}' := \mathbf{G} \cap \mathbb{Q} \in$ Gen(Q).

[Why? First $G'(\subseteq \mathbb{Q}_*)$ is directed by $(*)_4$.

Second, if $\mathbb{Q} \models$ "$p \le q$" then $\mathbb{Q}^+ \models$ "$p \le q$" hence $q \in \mathbf{G}' \Rightarrow p \in \mathbf{G}'$ so $\mathbf{G}'$ is downward directed. Third, if $n < \omega$ and $\mathcal{I} \in M_\delta, \mathcal{I} \subseteq \mathbb{P}_{s,n}$ is predense <u>then</u> $\mathcal{I} \cap \mathbf{G}' \neq \emptyset$ by $(*)_2$. Fourth, $\mathbf{G}' \cap \mathbb{P}^t$ is generic over $M_{\delta(t)}$ by $(*)_3$.]

$(*)_8$ let $\psi_* \in \Omega$ says that $\underset{\sim}{\nu}_{\delta(t)}[G \cap P_{s,n(*)}]$ is $(\underset{\sim}{\eta}_{\bar{\varphi}}, \mathbb{Q}_{\bar{\varphi}})$-generic over $M_{\delta(t)}[G \cap \mathbb{P}^t]$ when $\bar{\varphi} = \underset{\sim}{\bar{\varphi}}_\delta[G \cap \mathbb{P}_{s,n(*)}]$

$(*)_9$ $\psi_* \in \mathbb{Q}_*$.

[Why? First, $\underset{\sim}{\nu}_\delta, \underset{\sim}{\bar{\varphi}}, \underset{\sim}{\eta}$ are defined by $\aleph_0$ maximal antichain. Second, Gen$^+(\mathbb{Q}) \neq \emptyset$ and Gen$^+(\mathbb{Q}) \subseteq$ Gen$(\mathbb{Q})$.]

$(*)_{10}$ let $\mathbb{Q}^+ = \mathbb{Q}_* \restriction \{\psi : \psi_* \le_{\mathbb{Q}_*} \psi\}$.

Now all should be clear.

Lastly

$(*)_{11}$ if $\mathbf{G}$ is a subset of $\mathbb{Q}^+$ generic over M_δ <u>then</u> $\mathbf{G}' := \mathbf{G} \cap \mathbb{Q} \in$ Gen$^+(\mathbb{Q})$.

[Why? By $(*)_7, (*)_8, (*)_9$.]

We define $s = (\delta^s, \mathbb{P}^s, \Gamma^s, F^s_\omega, F^s_{n,\omega,\ell})_{\ell<\omega}$ as follows:

$\oplus_5$ (a) δ^s is δ, chosen in $(*)_1$ above

(b) $\mathbb{P}^s$ is $\mathbb{Q}^+$ defined in $\oplus_4$ above

(c) $\Gamma^s = \Gamma^t \cup \{\bar{p} \in M_\delta : \bar{p}$ is a sequence listing a predense subset of $\mathbb{Q}^+\}$

(d) F^s_ω is a two-place function from $\mathbb{Q}$ to $\mathbb{Q}$ such that
- $F^s_\omega(p,q)$ is $F^t_\omega(p,q)$ if $p, q \in \mathbb{P}^t$
- $F^s_\omega(p,q) = F_n(p,q)$ <u>when</u> $p, q \in \mathbb{P}_{s,n}$

- $F^s_\omega(p,q)$ if $p,q \in \mathbb{Q}$ is a common $\leq_{\mathbb{Q}}$-upper bound of p,q if $(p,q \in \mathbb{Q})$ and $\{p,q\} \nsubseteq \mathbb{P}^t$, $\bigwedge_{n<\omega} \{p,q\} \nsubseteq \mathbb{P}_{s,n}$ and there is a common $\leq_{\mathbb{Q}}$-upper bound

(note: the first and second case are not contradictory by clause (γ) of $\boxplus_3$, the definite of AP)

(e) $F^s_{n,\omega,\ell}$ is a function from $\mathbb{Q}$ to $\mathbb{Q}$ as defined for $n < \omega < \ell < \omega$ such that

- $F^s_{n,\omega,\ell}(p) = F^t_{n,\omega,\ell}(p)$ if $p \in \mathbb{P}^t$
- $F^s_{n,\omega,\ell}(p) = F_{n,m,\ell}(p)$ if $n < m < \omega, p \in \mathbb{P}_{s,m}$
- if $p \in \mathbb{Q} \backslash \mathbb{P}^t \backslash \cup \{\mathbb{P}_{t,n} : n < \omega\}$ and $n < \omega$ then $\langle F^s_{n,\omega,\ell}(p) : \ell < \omega\rangle$ is a sequence of members of $\mathbb{P}_{s,n}$ as in clause (ε) of $\boxplus_3$.

No problem we can choose such s. Now we have to check that s is as required

$\oplus_6$ $s \in$ AP.

[Why? Check the clauses in $\boxplus_3$.]

$\oplus_7$ $t \leq^* s$, i.e. clause (a) of Fact D.

[Why? Check the clauses in $\boxplus_4$.]

$\oplus_8$ $\underset{\sim}{\nu}_{\delta(t)}$ a wide $\mathbb{P}^s$-name.

[Why? As $\underset{\sim}{\nu}_{\delta(t)}$ is a $\mathbb{P}_{s,n}$-name and $\{\mathbb{P}_{s,n}, \underset{\sim}{\nu}_{\delta(t)}\} \in M$ and clause (c) of $(*)_7$.]

$\oplus_9$ clause (c) of Fact D.

[Why? By absoluteness from M_δ to v.]

Fact E: If in Main fact D, $\mathbb{Q}_{\underset{\sim}{\bar{\varphi}}^t_\delta}$ is a singleton (hence $\nu_\delta \in M_\delta$ so the main fact is trivial) then there is $s \in$ AP such that $t \leq^* s$ and $\bar{p} \in M_\delta$ is an ω-sequence listing a predense subset of $\mathbb{P}^t$ then $\bar{p}$ appears in the sequence Γ^s. [Why? Easy.] So we can choose $t_\varepsilon \in AP$ by induction on $\varepsilon < \omega_1$ such that t_ε is $\leq^*$-increasing continuous, $\delta^{t_{\varepsilon+1}} > \delta^{t_\varepsilon}$, and if $t_\varepsilon \in M_{(\delta^{t_\varepsilon})}, \delta^{t_\varepsilon} \in \bigcap_{n<\omega} E_n \cap E \cap \bigcup_{n<\omega} S^{\mathcal{Y}_n}$ then $t_{\varepsilon+1}$ is gotten by Fact D. No problem to carry this ($\varepsilon = 0$ by Fact A, $\varepsilon = \varepsilon_1 + 1$ by Fact D if possible and by Fact B if not; lastly, if ε is a limit ordinal, use Fact C).

Now let $\mathbb{P}_\omega = \bigcup_{\varepsilon<\omega_1} \mathbb{P}^{t_\varepsilon}$ and $\mathcal{Y}^\omega$ has already been defined; now check the requirements.

{12.6} **Definition 2.4.** Let $\bar{C}^* = \langle C^*_\delta : \delta < \omega_2$ a limit ordinal$\rangle$ (and $C^*_\alpha = \emptyset$ otherwise) be a square sequence and $\bar{X}^* = \langle X^*_i : i < \omega_1\rangle$ be an increasing sequence of subsets of $\omega_1, |X^*_i \backslash \bigcup_{j<i} X^*_j| = \aleph_1, X^*_{\omega_1} = \bigcup_{i<\omega_1} X^*_i$.

Let $\alpha \leq \omega_2$. We say that $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i, \bar{M}_i) : i < \alpha\rangle$ is a $(\bar{C}^*, \bar{X}^*)$-iteration (we omit $\bar{M}^i$ and write $(\bar{M}, \bar{C}^*, \bar{X}^*)$-iteration if $i < \alpha \wedge \alpha < \omega_2 \Rightarrow \bar{M}^i \leq \bar{M}$ or an $\bar{M}$-iteration when $\bar{C}^*, \bar{X}^*$ are clear from context) when:

(a) $(\mathbb{P}_i, \mathcal{Y}_i, \bar{M}^i) \in IS$ is $<^*$-increasing and $\mathrm{Dom}(\bar{M}_i) = S^{\mathcal{Y}_i}$

(b) f_i is a one to one function from $\mathbb{P}_i$ onto $X^*_{\text{otp}(C^*_i)}$, and let $(\mathbb{P}'_i, \mathcal{Y}'_i)$ be such that f_i maps $(\mathbb{P}_i, \mathcal{Y}_i)$ to $(\mathbb{P}'_i, \mathcal{Y}'_i)$

(c) if $j \in \text{acc}(C^*_i)$ then $f_j \subseteq f_i$

(d) if $\text{cf}(i) = \aleph_0$ and $i = \sup \text{acc}(C^*_i)$ then $(\mathbb{P}'_i, \mathcal{Y}'_i)$ is gotten from $\langle(\mathbb{P}'_j, \mathcal{Y}'_j) : j \in \text{acc}(C^*_i)\rangle$ as in the proof of 2.1 (for our $\bar{M}_i$ using $\langle X^*_j : j \in \text{acc}(C^*_i)\rangle, X^*_{\text{otp}(C^*_i)}$ instead of $\langle X_n : n < \omega\rangle, X_\omega$ so $\text{acc}(C^*_i)$ replace ω and we generate $\bar{t}_i = \langle t^i_\alpha : \alpha < \omega_1\rangle$ and by it define $(\mathbb{P}'_i, \mathcal{Y}'_i)$ hence can choose $f_i, (\mathbb{P}_i, \mathcal{Y}_i)$)

(e) in clause (d), further assume:

- $\delta = \text{otp}(C^*_i), j_* \in C^*_i$ minimal such that $\delta \in \text{Dom}(\bar{M}_{j_*})$
- $M = M^{j_*}_\delta, (\bar{\varphi}, \underset{\sim}{\eta}, \underset{\sim}{\nu}) = (\bar{\varphi}_\delta, \underset{\sim}{\eta}_\delta, \underset{\sim}{\nu}_\delta)[\mathcal{Y}'_{j_*}]$
- if $j \in C^*_i \backslash j_*$ then $\Vdash_{\mathbb{P}'_j}$ "$\underset{\sim}{\nu}_\delta$ is $(\mathbb{Q}^{\bar{\varphi}}, \underset{\sim}{\eta}_\delta)$-generic over $M[\underset{\sim}{G}_{\mathbb{P}'_j} \cap \delta]$"
- $\langle(\mathbb{P}'_j, \mathcal{Y}'_j) \restriction \delta : j \in \text{acc}(C^*_i)\rangle \in M$ and for $j_1 < j_2$ from $\text{acc}(C^*_i)\backslash j_*$ the ordinal δ belongs to the club $\{\alpha < \omega_1 : \alpha$ limit closed under the functions F_{j_1} and $F_{j_1,j_2,\ell}$ (see clause (f) below)$\}$.
- Let $t^i_* \in AP$ be defined (using i instead of ω) by $\delta^{t^i_*} = \text{otp}(C^*_i), \mathbb{P}^{t^i_*} = \cup\{\mathbb{P}'_j \restriction \delta : j \in \text{acc}(C^*_i)\}, F^{t^i_*}_i = \cup\{F_{j_1} \restriction \delta \in \text{acc}(C^*_i)\}, F^{t^i_*}_{j_1,i,\ell} = \cup\{F_{j_1,j_2,\ell}\restriction\delta : j_2 \in \text{acc}(C^*_i)\backslash j_1\}$ and let $\Gamma^{t^i_*}$ be <u>empty</u>. <u>Then</u> (recalling $\bar{t}_i$ is from (d))

  (α) $t^i_0 = t^i_*, M^i_\delta = M, (\bar{\varphi}_\delta, \underset{\sim}{\eta}'_\delta, \underset{\sim}{\nu}^t_\delta)[\mathcal{Y}'_i] = (\bar{\varphi}, \underset{\sim}{\eta}, \underset{\sim}{\nu})$

  (β) $\Vdash_{\mathbb{P}'_i}$ "$\underset{\sim}{\nu}_\delta$ is a $(\mathbb{Q}^{\bar{\varphi}}, \underset{\sim}{\eta})$-generic over $M[\underset{\sim}{G}_{\mathbb{P}'_i} \cap \delta]$

(f) F_j is a (partial) two-place function from $X^*_{\text{otp}(C^*_j)}$ to itself such that $F^j(p, q)$ is the $<$-first common upper bound of p and q in $\mathbb{P}'_j$ and if $j_1 \in \text{acc}(C^*_{j_2})$ then we have $\langle F_{j_1,j_2,\ell}(p) : \ell < \omega\rangle$ is a maximal antichain of $\mathbb{P}'_{j_1}$ satisfying: for each ℓ, either $F_{j_1,j_2,\ell}(p)$ is incompatible with p in $\mathbb{P}'_{j_2}$ <u>or</u> p is compatible with r in $\mathbb{P}'_{j_2}$ wherever $\mathbb{P}'_{j_1} \models F_{j_1,j_2,\ell}(p) \le r$.

Claim 2.5. *(iteration at limit) Assume $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i) : i < \zeta\rangle$ is a $(\bar{M}, \bar{C}^*, \bar{X}^*)$-iteration where $\zeta < \omega_2$ is a limit ordinal.* <u>*Then*</u> {12.7}

(a) *we can find $(\mathbb{P}_\zeta, \mathcal{Y}_\zeta, f_\zeta)$ such that $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i) : i < \zeta + 1\rangle$ is an $\bar{M}$-iteration*

(b) *if $S \subseteq S^*, i < \zeta \Rightarrow S^{\mathcal{Y}_i} \subseteq S \bmod \mathcal{D}_{\bar{M}}$,* <u>*then*</u> *we can demand $S^{\mathcal{Y}_\zeta} = S$.*

Remark 2.6. A reader may ask: in the proof when $i = \sup \text{acc}(C^*_i)$, $\text{cf}(i) = \aleph_0$. Why can we fulfill clause (e) of 2.4 when relevant? That is, letting $j_*, \delta, M = M^{j_*}_\delta, (\underset{\sim}{\bar{\varphi}}, \underset{\sim}{\eta}, \underset{\sim}{\nu})$ be as in clause (e) already $\mathbb{P}_i\restriction\delta$ is determined so we have no freedom left, so how can we fulfill the obligation concerning $\underset{\sim}{\nu}$? {12.7y}

The answer is that we should look at the definition of satisfaction of ι-commitment in 1.11, so for notational simplicity consider $\iota = 0$, i.e. 1.11(1A) and assume $\mathbb{P} \subseteq \omega_1$ and h is the identity and E as there. For $\alpha \in E$ note that clause (e) of 1.11(1A) says that $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{G}_{\mathbb{P}} \cap \alpha$ is a subset of $\mathbb{P}'_\alpha := \mathbb{P} \cap \alpha$ generic over M_α". Note that $\mathbb{P}'_\alpha \in M_\alpha$ because $\alpha \in E$ and $M_\alpha \models$ "$|\alpha| = \aleph_0$" hence $M_\alpha \models$ "$\mathbb{P}'_\alpha$ is countable". So $M_\alpha[\underset{\sim}{G}_{\mathbb{P}}]$ is a Cohen extension of M_α, (or is equal to M_α in a degenerate case). So how can we satisfy the demand $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\nu}_\alpha$ is $(Q^{\bar{\varphi}}, \underset{\sim}{\eta})$-generic over $M_\alpha[\underset{\sim}{G}_{\mathbb{P}}]$"? This is not problematic but is not you may say in the next stage; see more in §4 but we shall elaborate. So in clause (e) of 2.4 letting $i = \zeta$ we have $j_*, \delta, \mu, \bar{\varphi}, \underset{\sim}{\eta}, \underset{\sim}{\nu}, \mathbb{Q}^{\bar{\varphi}}, t^i_*$

as there let $\mathbb{P}'_\varepsilon = f_j(\mathbb{P}_j)$ for $j \in C^*_i$ when $\varepsilon =$ otp$(j \cap C^*_j)$ and $\varepsilon_* =$ otp$(j_* \cap C^*_i)$ and let $\varepsilon_{**} =$ otp(C^*_i). Similarly let $\underset{\sim}{\bar\varphi} = f_{j_*}(\underset{\sim}{\bar\varphi})$ so $\underset{\sim}{\bar\varphi}'$ is a $\mathbb{P}_{\varepsilon_*}$-name, hence is a countable object using members of $\mathbb{P}_{\varepsilon_*}\restriction\delta$ only.

We choose $\delta_* > \delta$ from $S[\mathcal{Y}_{j_*,\delta_*}]$ such that $M'_* := M^{\mathcal{Y}_{j_*}}_\delta$ is well defined, $(M, \langle \mathbb{P}'_\varepsilon\restriction\delta : \varepsilon < \epsilon_{**}\rangle) \in M_*$ and $\langle \mathbb{P}'_\varepsilon\restriction\delta_* : \varepsilon < \varepsilon_{**}\rangle$ and $\Vdash_{\mathbb{P}'_\varepsilon}$ "$\underset{\sim}{G}_{\mathbb{P}'_\varepsilon} \cap \delta_*$ is a subset of $\mathbb{P}'_\varepsilon\restriction\delta_*$ generic over M_*". Of course, we can demand $\varepsilon_{**} = \delta$.

Now in M_* we define $\mathbb{P} = \cup\{\mathbb{P}'_\varepsilon\restriction\delta_* : \varepsilon < \varepsilon_{**}\}$. So $M_* \models$ "$\mathbb{P}$ is a countable forcing notion union of the $\lessdot$-increasing sequence $\langle \mathbb{P}'_\varepsilon : \varepsilon < \varepsilon_{**}\rangle, M$ a countable transitive model $\langle \mathbb{P}'_\varepsilon\restriction\delta : \varepsilon < \varepsilon_{**}\rangle$ is $\lessdot$-increasing, if $\varepsilon < \varepsilon_{**}$ then every predense subset $\mathcal{I} \in M$ of $\mathbb{P}'_\varepsilon\restriction\delta, \varepsilon < \varepsilon_{**}$ is predense in $\mathbb{P}'_\varepsilon\restriction\delta_*$ and $\underset{\sim}{\nu}_\delta$ is $(Q^{\underset{\sim}{\bar\varphi}}, \underset{\sim}{\eta})$-generic over $M[G_{\mathbb{P}_\varepsilon\restriction\delta_*}]$ when $\varepsilon \in [\varepsilon_*, \varepsilon_{**}]$.

Now we choose $\mathbb{P}'' \in M_*$ such that $M_* \models$ "$\mathbb{P}$ is an $\lessdot$-extension of $\mathbb{P} * \mathbb{Q}^{\underset{\sim}{\bar\varphi}}/(\underset{\sim}{\eta} = \nu)$ and of each $\mathbb{P}'_\varepsilon\restriction\delta_*, \varepsilon < \varepsilon_{**}$", as in the proof of Fact D.

Lastly, $t^i_0 = t'_*, \delta_{t^i_1} = \delta_*$ and continue the proof of Fact D to choose t^i_ε.

Proof. If cf$(\zeta) = \aleph_0$ we use 2.1 but taking care of clause (e) of Definition 2.4, this just dictates to us how to start the induction there (as is done by "Main Fact D" from inside the proof of 2.1). Note that if $\zeta >$ sup acc(C^*_ζ) we still use 2.1, just our work is easier as we do not have to take care of clause (e). If cf$(\zeta) = \aleph_1$, then by the square bookkeeping (see clause (e) in Definition 2.4) our work is done (using $f_\zeta = \cup\{f_\xi : \xi \in$ acc$(C^*_\zeta)\}$). $\square_{2.5}$

{12.7A} **Claim 2.7.** *1) Assume*

- (a) *$\mathcal{Y} = (S, \bar\Phi, \underset{\sim}{\bar\eta}, \underset{\sim}{\bar\nu})$ is a 1-commitment on the forcing notion $\mathbb{P} \in \mathcal{H}(\aleph_2)$ for $\bar M$, an oracle with domain $S_{\bar M}$*
- (b) *$G_\mathbb{P} \subseteq \mathbb{P}$ is generic over $\mathbf{V}, \bar\nu^0 = \langle \nu^0_\alpha : \alpha \in S\rangle$ where $\nu^0_\alpha = \underset{\sim}{\nu}_\alpha[G_\mathbb{P}]$, $\bar M^1 = \bar M[G_\mathbb{P}] = \langle M_\delta[f''(G_\mathbb{P})] : \delta \in S_{\bar M}\rangle$ for some one to one function f from $\mathbb{P}$ into ω_1*
- (c) *in $\mathbf{V}[G_\mathbb{P}], \mathcal{Y}^1 = (S^1, \bar\Phi^1, \underset{\sim}{\bar\eta}^1, \bar\nu^1)$ is a 0-commitment, $S \subseteq S^1 \bmod \mathcal{D}_{\bar M[G_\mathbb{P}]}, \bar\Phi^1 \restriction (S \cap S^1) = \bar\Phi \restriction (S \cap S_1), \underset{\sim}{\bar\eta}^1 \restriction (S \cap S_1) = \underset{\sim}{\bar\eta} \restriction (S \cap S_1), \bar\nu^1 \restriction (S \cap S_1) = \bar\nu^0 \restriction (S \cap S_1)$ and $(S^1, \bar\Phi^1, \underset{\sim}{\bar\eta}^1) \in \mathbf{V}$*
- (d) *in $\mathbf{V}[G_\mathbb{P}], \mathbb{Q}$ is a forcing notion satisfying the 0-commitment $\mathcal{Y}^1$ for $\bar M^1$.*

<u>Then</u> *for some $\mathbb{P}$-name $\underset{\sim}{\mathbb{Q}}$ and 1-commitment $\mathcal{Y}^2$ we have:*

- (α) $(\mathbb{P}, \mathcal{Y}, \bar M) \le^* (\mathbb{P} * \underset{\sim}{\mathbb{Q}}, \mathcal{Y}^2, \bar M)$
- (β) $S^{\mathcal{Y}^2} = S^1, \Phi^{\mathcal{Y}^2} = \bar\Phi^1, \underset{\sim}{\bar\eta}^{\mathcal{Y}_2} = \underset{\sim}{\bar\eta}^1, \underset{\sim}{\bar\nu}^{\mathcal{Y}_2}[G_\mathbb{P}] = \bar\nu^1$
- (γ) $\underset{\sim}{\mathbb{Q}}[G_\mathbb{P}] = \mathbb{Q}$.

2) If for every $G_\mathbb{P} \subseteq \mathbb{P}$ generic over $\mathbf{V}$ there are $\mathbb{Q}$ satisfying some ψ_1 and $(S^1, \bar\Phi^1, \underset{\sim}{\bar\eta}^1, \bar\nu^1) \in \mathbf{V}[G_\mathbb{P}]$ as above satisfying some ψ_2, <u>*then*</u> *we can demand*

- (δ) *$\Vdash_\mathbb{P}$ "$\underset{\sim}{\mathbb{Q}}[G_\mathbb{P}], \mathcal{Y}^2$ as above satisfy ψ_1, ψ_2 respectively".*

3) We may allow $\langle (\bar\varphi_\alpha, \underset{\sim}{\eta}_\alpha) : \alpha \in S^1\rangle$ be a sequence of $\mathbb{P}$-names and even $(\mathbb{P} * \underset{\sim}{\mathbb{Q}})$-names.

Proof. Straight. $\square$

{12.7F}

Definition 2.8. For a pair $(\bar{\varphi}, \underset{\sim}{\eta})$ as in Definition 1.11, we say $Z \subseteq {}^\omega 2$ is positive for $(\bar{\varphi}, \underset{\sim}{\eta})$ <u>when</u>: for χ large enough, the set $\{N : N \prec (\mathcal{H}(\chi), \in)$ is countable, $(\bar{\varphi}, \underset{\sim}{\eta}) \in N$ and there is $\nu \in Z$ which is $(N, \mathbb{Q}^{\bar{\varphi}}, \underset{\sim}{\eta})$-generic$\}$ is stationary, equivalently not empty.

{12.8}

Claim 2.9. *(iteration in successor case: increasing the commitment).*
Assume

- (a) *$\langle (\mathbb{P}_i, \mathcal{Y}_i, f_i) : i < \zeta \rangle$ is an $\bar{M}$-iteration and $\zeta = \xi + 1, S^{\mathcal{Y}_\xi} \subseteq S \subseteq S^*, S \subseteq$* $\mathrm{Dom}(\bar{M})$
- (b) *$\langle (\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha) : \alpha \in S \backslash S^{\mathcal{Y}_\xi} \rangle$ is as required in Definition 1.11.*

Lastly

- (c) *$\underset{\sim}{Z}_\alpha \subseteq {}^\omega 2$ is a $\mathbb{P}_\xi$-name of a positive set for $(\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha)$ for every such α.*

<u>*Then*</u> *we can find $(\mathbb{P}_\zeta, \mathcal{Y}_\zeta, f_\zeta)$ such that*

- (i) *$\langle (\mathbb{P}_i, \mathcal{Y}_i, f_i) : i < \zeta + 1 \rangle$ is an $\bar{M}$-iteration*
- (ii) *$\mathbb{P}_\zeta = \mathbb{P}_\xi, S^{\mathcal{Y}_\zeta} = S$,*
- (iii) *$(\underset{\sim}{\bar{\varphi}}^{\mathcal{Y}_\zeta}_\alpha, \underset{\sim}{\eta}^{\mathcal{Y}_\zeta}_\alpha) = (\underset{\sim}{\bar{\varphi}}_\alpha, \underset{\sim}{\eta}_\alpha)$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\nu}^{\mathcal{Y}_\zeta}_\alpha \in \underset{\sim}{Z}_\alpha$" when $\alpha \in S \backslash S^{\mathcal{Y}_\xi}$.*

Proof. Straight. □

{12.9}

Claim 2.10. *(iteration at successor: increasing the forcing)*
Suppose

- (a) *assume in claim 2.9*
- (b) *$\underset{\sim}{\mathbb{Q}}$ is a $\mathbb{P}$-name satisfying, for every $G \subseteq \mathbb{P}$ generic over $\mathbf{V}$, the following:*
  - (i) *$\underset{\sim}{\mathbb{Q}}[G]$ is a c.c.c. forcing notion of cardinality $\aleph_1$*
  - (ii) *$\{\delta \in S_{\bar{M}} : \mathbb{P} \restriction \delta \in M_\delta$ and $G \cap M_{\delta[\nu_\delta]}$ is a generic subset of $\mathbb{R}^{\mathcal{Y}}_\delta * Q_{\underset{\sim}{\bar{\varphi}}_\delta}/(\underset{\sim}{\eta}_\delta = \nu_\delta)$ over $M_\delta[\nu_\delta]$ which $\mathcal{D}_{\bar{M}}$-almost always occurs and $\underset{\sim}{\mathbb{Q}}[G] \restriction \delta \in M_\delta[G \cap \delta]$ and there is $\mathbb{R}^* \in M_\delta$ such that $R^{\mathcal{Y}}_\delta \lessdot \mathbb{R}^*$ and $R^*_\delta * \underset{\sim}{\mathbb{Q}}^{M_\delta[\mathbb{R}_\delta]}_{\bar{\varphi}} \lessdot \mathbb{P} * Q\} \in \mathcal{D}_{\bar{M}[G]}$.*

<u>*Then*</u> *we can find $(\mathbb{P}^+, \mathcal{Y}^+)$ such that $(\mathbb{P}, \mathcal{Y}) \leq^* (\mathbb{P}^+, \mathcal{Y}^+) \in IS$ and the $\mathbb{P}$-name $\mathbb{P}^+/\underset{\sim}{G}_{\mathbb{P}}$ is equivalent to $\underset{\sim}{\mathbb{Q}}$.*

Proof. Straight.

Now we draw an easy conclusion: consistently $2^{\aleph_0} = \aleph_2$ and for each ideal defined naturally by $(\bar{\varphi}, \underset{\sim}{\eta})$ where $\bar{\varphi}$ absolutely defines a c.c.c. forcing $Q_{\bar{\varphi}}$ and $\underset{\sim}{\eta}$ is a wide name for a read <u>then</u> for some set of $\aleph_1$ reals is positive for this ideal and more. □

§ 3. Conclusions

{12.10}

Conclusion 3.1. Assume $(\bar{C}^*, \bar{X}^*)$ is as in 2.4. Let Φ be a set of definitions of forcing notions with some real parameters, and $\langle S^*_i : i < \omega_2\rangle$ is as in 1.8 for $\mathcal{D}_{\bar{M}}$ for some $\bar{M}$.

We can find $\langle(\mathbb{P}_i, \mathcal{Y}_i, f_i, \bar{M}^i) : i < \omega_2\rangle$ such that

- (a) it is an $(\bar{C}^*, \bar{X}^*)$-iteration (for this we have to allow $\alpha = \omega_2$ in Definition 2.4)
- (b) $\mathbb{P} = \{\mathbb{P}_i : i < \omega_2\}$ is a c.c.c. forcing notion of cardinality $\aleph_2$ (so in $\mathbf{V}^{\mathbb{P}}, 2^{\aleph_0} \leq \aleph_2$) and except in degenerated cases equality holds)
- (c) $S^{\mathcal{Y}_i} = S^*_i$ from 1.8(3)
- (d) if in $\mathbf{V}^{\mathbb{P}_i}$ we have $(\bar{\varphi}, \eta)$ is a case of Φ as in 1.11, moreover $\Vdash_{\mathbb{P}_i}$ "$\{\delta \in S^*_{i+1} \backslash S^*_i : M^{i+1}_\delta[\underset{\sim}{f''_i}(G_{\mathbb{P}_i})]$" $\models$ "$(\bar{\varphi}, \underset{\sim}{\eta})$ as required in 1.11"$\} \in \mathcal{D}^+_{\bar{M}^{i+1}}$ (even less with more bookkeeping) and $Z \subseteq ({}^{\omega}2)^{\mathbf{V}^{\mathbb{P}}}$ is positive for $(\bar{\varphi}, \underset{\sim}{\eta})$, <u>then</u>
  - (α) $\{\delta \in S^{\mathcal{Y}_{i+1}} \backslash S^{\mathcal{Y}_i} : (\bar{\varphi}_\delta, \underset{\sim}{\eta}_\delta)/G_{\mathbb{P}_i} = (\bar{\varphi}, \underset{\sim}{\eta})$ and $\underset{\sim}{\nu}_\delta[G_{\mathbb{P}_i}] \in Z\} \in \mathcal{D}^+_{\bar{M}}$: in fact the set is forced to include such old set (from $\mathbf{V}$) by this we can get
  - (β) for some $j > i, \delta \in S^{\mathcal{Y}_{j+1}} \backslash S^{\mathcal{Y}_j} \Rightarrow (\bar{\varphi}_\delta, \underset{\sim}{\eta}_\delta) = (\bar{\varphi}, \underset{\sim}{\eta}), \underset{\sim}{\nu}_\delta[G_{\mathbb{P}_i}] \in Z$
- (e) if H is a pregiven function such that for every $i < \omega_2$ and $(\mathbb{P}, \mathcal{Y}, \bar{M})$ satisfying the demands on $(\mathbb{P}_i, \mathcal{Y}_i, \bar{M})$: we have $(\mathbb{P}, \mathcal{Y}_{\bar{M}}) \leq^* H(\mathbb{P}, \mathcal{Y}, \bar{M}) \in IS$ such that $H(\mathbb{P}, \mathcal{Y}, \bar{M})$ satisfies the demands from (a) + (c) on $(\mathbb{P}_{i+1}, \mathcal{Y}_{i+1}, \bar{M}^{i+1})$, <u>then</u> we can demand $(\exists^{\aleph_2} j)[(\mathbb{P}_{j+1}, \mathcal{Y}_{j+1}, \bar{M}^{j+1}) = H(\mathbb{P}_j, \mathcal{Y}_j, \bar{M}^j)]$; moreover, if $S_H \subseteq \{\delta < \omega_2 : \text{ cf}(\delta) = \aleph_1\}$ is stationary we can demand $\{j \in SH : (\mathbb{P}_{j+1}, \mathcal{Y}_{j+1}, \bar{M}^{j+1}) = H(\mathbb{P}_j, \mathcal{Y}_j, \bar{M}^j)\}$ is stationary.
  (Of course, we can promise this for $\aleph_2$ such functions)
- (f) similarly for $S_H \subseteq \{\delta < \omega_2$: cf$(\delta) = \aleph_0, \delta \notin \cup\{C^*_i : i\}\}$ but the domain is a sequence $\langle(\mathbb{P}^n, \mathcal{Y}^n, \bar{M}^n) : n < \omega\rangle, S^{\mathcal{Y}_n} = S_{i_n}, \langle i_n : n < \omega\rangle$ increases to i.

Proof. Put together the previous claims. (Concerning clause (f) without loss of generality $\{i < \omega_1 : \text{otp}(C^*_i) = 0\}$ is stationary) so in those stages we have no influence of clause (f) of 2.4; anyhow the influence of 2.4(f) is minor. □

{12.11}

Discussion 3.2. We discuss here some possible extensions. We can add a version of the conclusion without the oracles, etc.

{12.10a}

Claim 3.3. *Assume $\langle S_i : i < \omega_2\rangle$ is a sequence of pairwise almost disjoint stationary subsets of ω_1, each with diamond and $i < j \Rightarrow S_i \subseteq S^+_j \bmod \mathcal{D}_{\omega_1}$, so $S^+_i \subseteq \omega_1$ and $S_i \cap S^+_i = \emptyset$ and $\langle S^+_i/\mathcal{D}_{\omega_1} : i < \omega_2\rangle$ is increasing*

<u>Then</u> in the following game between the bookkeeper and the forcer, the bookkeeper has a winning strategy.

A play lasts ω_2 moves, before the α-th move a sequence $\langle(\mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i, \underset{\sim}{\bar{M}}^i, \mathcal{Y}_i) : i < \alpha\rangle$ is defined such that

- (a) *$\mathbb{P}_i$ a c.c.c. forcing notion of cardinality $\aleph_1$, say $\subseteq \mathcal{H}_{<\aleph_1}(\aleph_2)$*
- (b) *$\underset{\sim}{\mathbb{Q}}_i$ is a $\mathbb{P}_i$-name of a forcing notion of cardinality $\leq \aleph_1$, say $\subseteq \omega_1$*
- (c) *$\mathbb{P}_i$ is $\lessdot$-increasing*
- (d) *$\mathbb{P}_{i+1}, \mathbb{P}_i * \underset{\sim}{\mathbb{Q}}_i$ are isomorphic over $\mathbb{P}_i$*
- (e) *$\underset{\sim}{\bar{M}}^i d$ is a $\mathbb{P}_i$-name of an S_i-oracle*

(*f*) $\mathcal{Y}_i$ *is a* $\mathbb{P}_i$*-name of an* S_i*-commitment.*

In the i-th move:

(*a*) *the bookkeeper chooses* $\mathbb{P}_i$ *and a* $\mathbb{P}_i$*-name* $(\bar{N}^i, \underset{\sim}{\mathcal{Y}}^+)$ *of an* S_i^+*-oracle and* 0*-commitment*

(*b*) *the forcer chooses* $\underset{\sim}{\mathbb{Q}}_i$ *and* $(\underset{\sim}{\bar{M}}^i, \underset{\sim}{\mathcal{Y}}_i), \mathbb{P}_i$*-names such that* $\underset{\sim}{\mathbb{Q}}_i$ *satisfies* $(\bar{N}^i, \underset{\sim}{\mathcal{Y}}_i^+)$ *and* $(\underset{\sim}{\bar{M}}^i, \underset{\sim}{\mathcal{Y}}_i)$.

In the end the bookkeeper wins if

$$i < j < \omega_2 \Rightarrow \mathbb{P}_j/\mathbb{P}_i \text{ satisfies } (\underset{\sim}{\bar{M}}^i, \underset{\sim}{\mathcal{Y}}_i).$$

Proof. Similar to earlier proofs. □

We give an easy criterion for existence of forcing notion satisfying a given 0-commitment and a (not complete) sub-forcing of given nicely definable one.
The following uses more from [Sh:630].

Claim 3.4. *Assume* {12.12}

(*a*) $(\mathbb{P}, \leq, \leq_n)_{n<\omega}$ *is a definition of a forcing notion satisfying condition A of Baumgartner with* $\leq_n$ *as witness and* ZFC_*^- *says this, in a way preserved by suitable forcing*

(*b*) $\mathcal{Y} = (S, \bar{\Phi}, \underset{\sim}{\eta}, \bar{\nu})$ *is a 0-commitment, so* $\bar{\Phi} = \langle \bar{\varphi}_\alpha : \alpha \in S\rangle$

(*c*) $\mathbb{P}$ *is absolutely nep such that for each* $\alpha \in S^{\mathcal{Y}}$ *it is* $\leq_n$*-purely* $I_{\mathbb{Q}^{\bar{\varphi}_\alpha}}$*-preserving, i.e.*

(∗) *if* M *is a* $\mathbb{P}$*-candidate and a* $\mathbb{Q}^{\bar{\varphi}_\alpha}$*-candidate,* $p \in \mathbb{P}^M, n < \omega$ *and* $q \in (\mathbb{Q}^{\bar{\varphi}_\alpha})^M$ *then for some* p', η, ν *we have* $p \leq_n p' \in \mathbb{P}, p'$ *is* $\langle M, \mathbb{P}\rangle$*-generic and* ν *is* $(\mathbb{Q}^{\bar{\varphi}_\alpha}, \underset{\sim}{\eta}_\alpha)$*-generic over* M *satisfying* q *(see* [Sh:630]*) and* $p' \Vdash_{\mathbb{P}}$ *"*ν *is* $(\mathbb{Q}_{\bar{\varphi}_\alpha}, \underset{\sim}{\eta}_\alpha)$*-generic over* $M\langle \underset{\sim}{G}_{\mathbb{P}} \cap P^M\rangle$*".*

Then there is a c.c.c. forcing notion $\mathbb{P}' \subseteq \mathbb{P}$ *(not necessarily* $\mathbb{P}' \lessdot \mathbb{P}$*) satisfying the 0-commitment* $\mathcal{Y}$ *such that* $\Vdash_{\mathbb{P}'}$ *"for a club of* $\delta < \omega_1, \varphi(\underset{\sim}{\nu}, \underset{\sim}{\eta}_\delta^*)$*".*

Remark 3.5. 1) Why the φ_δ's? We hope it helps, for example in the following; suppose we are given $f : \mathbb{R} \to \mathbb{R}$, we like to force $A \subseteq \mathbb{R}$ which is not in $I_{\mathbb{Q}^{\bar{\varphi}_\alpha}}$, see Definition in [Sh:630], and on which the function f is continuous; i.e. to force a continuous $\underset{\sim}{f}^*$ such that $\{\eta \in {}^\omega 2 : \underset{\sim}{f}^*(\eta) = f(\eta)\} \in (I_{\mathbb{Q}}^{ex})^+$, see Definition in [Sh:630]. So not only do we like to find $q \Vdash$ "η_δ is $(\underset{\sim}{\mathbb{Q}}_\delta, \underset{\sim}{\eta}_\delta)$-generic over $M_\delta[\underset{\sim}{G}_{\mathbb{P}}]$" but also $q \Vdash_{\mathbb{P}'}$ "*name*$f^*(\eta_\delta) = f(\eta_\delta)$". This is what $\bar{\varphi}$ says.

Proof. We choose by induction on $\alpha < \omega_1$, a pair $(\mathbb{P}_\alpha, \Gamma_\alpha)$ such that:

(α) $\mathbb{P}_\alpha \subseteq \mathbb{P}$ is countable

(β) Γ_α is a countable family of predense subsets of $\mathbb{P}_\alpha$

(γ) if $\mathcal{I} \in \Gamma_\alpha$ and $p \in \mathbb{P}_\alpha$ and $n < \omega$ then for some q we have $p \leq_n q \in \mathbb{P}_\alpha$ and $\mathcal{I}$ is predense above q in $\mathbb{P}$

(δ) $\mathbb{P}_\alpha$ is increasing continuous in α

(ε) Γ_α is increasing continuous in α.

Case 1: $\alpha = 0$.
Trivial.

Case 2: $\alpha = \beta + 1, \beta$ non-limit or $(\mathbb{P}_\beta, \Gamma_\beta) \notin M_\beta$.
Let $(\mathbb{P}_\alpha, \Gamma_\alpha) = (\mathbb{P}_\beta, \Gamma_\beta)$.

Case 3: α limit.
Let $(\mathbb{P}_\alpha, \Gamma_\alpha) = (\bigcup\limits_{\beta<\alpha} \mathbb{P}_\beta, \bigcup\limits_{\beta<\alpha} \Gamma_\beta)$.

Case 4: $\alpha = \delta + 1$ where δ is a limit ordinal and $(\mathbb{P}_\delta, \Gamma_\delta) \in M_\delta$.

We can find $g \subseteq$ Levy$(\aleph_0, |\mathbb{P}|)^{M_\delta}$, generic over M_δ such that η^*_δ is still $\mathbb{Q}_\delta$-generic over $M_\delta[g]$ (see [Sh:630],§6).

In $M_\delta[g]$ we define $\mathbb{P}^+_\delta = \{p : M_\delta[g] \models p \in \mathbb{P}$ and $\mathcal{I} \in \Gamma_\alpha \Rightarrow \mathcal{I}$ predense above $p'\}$ using the induction hypothesis, as in $M_\delta[g]$ the set Γ_δ is countable, so:

(∗) for every $p \in \mathbb{P}_\delta$ and $n < \omega$ there is $p' \in \mathbb{P}^+_\delta$ such that $\mathbb{P} \models p \leq_n p'$.

Again by [Sh:630],§6 for every $n < \omega$ and $p \in \mathbb{P}^+_\delta$, there is $q_{p,n} \in \mathbb{P}$ such that $p \leq_n q_{p,n} \in \mathbb{P}, q_{p,n}$ is $(M_\delta[g], \mathbb{Q})$-generic and $q_{p,n} \Vdash_{\mathbb{P}}$ "ν_δ is a $(\mathbb{Q}_\delta, \underset{\sim}{\eta}_\delta)$-generic real over $M_\delta[g][\underset{\sim}{G}_{\mathbb{P}}]$".

Let $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta \cup \{q_{p,n} : p \in \mathbb{P}^+_\delta$ and $n < \omega\}$ and $\Gamma_{\delta+1} = \Gamma_\delta \cup \{\mathcal{I}_\delta\}$ where $\mathcal{I}_\delta = \{q_{p,n} : p \in \mathbb{P}^+_\delta$ and $n < \omega\}$. $\square_{3.4}$

§ 4. Relatives and closing remarks

The comments below try to connect [Sh:669], [Sh:895], [Sh:F699], [Sh:F1000]. A related work is [Sh:895], the main difference is that there the continuumis forced to be λ^+ and normally $\lambda > \aleph_1$. Now if there is $\lambda = \aleph_1$, still the construction is more general. Here, e.g.

(∗) if the forcing notion $\mathbb{P}$ satisfies the o-obligation $\mathcal{Y}$ then $\Vdash_{\mathbb{P}}$ "$({}^\omega 2)^{\mathbf{V}}$ is non-meagre".

[Why? Without loss of generality $|\mathbb{P}| \le \aleph_1$ hence without loss of generality $\mathbb{P} \subseteq \omega_1$. So if $\underset{\sim}{B}$ is a $\mathbb{P}$-name of countable union of no-where-dense subsets of ${}^\omega 2$, then for some $\delta \in S_{\mathcal{Y}}$ we have $\underset{\sim}{B} \in M_\delta$ and $\mathbb{P}'_\delta := \mathbb{P} \cap \delta \in M_\delta$ and $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{G}_{\mathbb{P}} \cap \delta$ is a subset of $\mathbb{P}'_\delta$ which is generic over M_δ".

So $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{G}_{\mathbb{P}} \cap \delta$ is generic for $(\mathbb{P}'_\delta, M_\delta)$" hence $({}^\omega 2)^{M_\delta}$ is non-meagre in $M_\delta[G_{\mathbb{P}} \cap \delta]$ hence has member $\rho \notin \underset{\sim}{B}$".

By absoluteness $\Vdash_{\mathbb{P}}$ "$({}^\omega 2)^{\mathbf{V}} \nsubseteq \underset{\sim}{B}^{\mathbf{V}[\mathbb{P}]}$", so we are done.]

Now [Sh:895], for the case $\lambda = \aleph_1$, this proof does not work. In fact we can avoid this but it requires some care.
We now describe one version of [Sh:895].

Definition 4.1. 1) A 2-commitment base is a sequence $\mathbf{p} = \langle (\bar{P}_\delta, <_\delta) : \delta \in S \rangle$ such that

- $S \subseteq \omega_1$ is stationary,
- $\le_\delta$ is a well ordering of δ
- for $\delta \in S, \bar{\mathbb{P}}_\delta = \langle (\mathbb{P}_{\delta,\gamma} : \gamma < \gamma_\delta \rangle$ is a $\lessdot$-increasing sequence of forcing notions
- $\mathbb{P}_{\delta,\gamma}$ is $\subseteq \delta$.

2) Above $\mathbf{p}$ is positive for the normal filter $\mathcal{D}$ on ω_1 when:

- if $\langle \mathbb{P}_\gamma : \gamma < \gamma(*) \rangle$ is a $\lessdot$-increasing sequence of c.c.c. forcing notion with a set of elements $\subseteq \alpha(*), \alpha(*), \gamma(*) < \omega_1$ and $\bar{u} = \langle u_\alpha : \alpha < \omega_1 \rangle$ is $\subseteq$-increasing continuous sequence of countable sets of ordinals $< \alpha(*)$ with union $\alpha(*)$ the following set $\in \mathcal{D}^+$

$$\begin{aligned} S_{\mathbf{p},\bar{u}} = \{\delta \in S_{\mathbf{p}} : \ & \mathrm{otp}(u_\delta) \le \ \mathrm{otp}(\delta, <_{\mathbf{p}_\delta}) \text{ and letting} \\ & h_\delta \text{ be the unique order preserving function from } u_\delta \\ & \text{onto an initial segment of } (\delta, <_{\mathbf{p},\delta}) \text{ then} \\ & \gamma \in u_\delta \cap \gamma(*) \Rightarrow h''(\mathbb{P}_\gamma \cap u_\delta) = \mathbb{P}_{\mathbf{p},\delta,h(\gamma)}\}. \end{aligned}$$

3) A 2-commitment is a pair$(\mathbf{p}, \mathbf{q})$ such that

- $\mathbf{p} = \langle (\bar{\mathbb{P}}_\delta, <_\delta) : \delta \in S \rangle$ is a 2-commitment base
- $\mathbf{q} = \langle \mathbb{P}^+_\delta : \delta \in S \rangle$
- $\bar{\mathbb{P}}^+_\delta = \langle \mathbb{P}^+_{\delta,\gamma} : \gamma < \gamma(*) \rangle$
- $\mathbb{P}_{\delta,\gamma} \lessdot \mathbb{P}^+_{\delta,\gamma}$ for $\gamma < \delta \in S$.

4) Let $\gamma(*) < \omega_2$. We say then $\langle \mathbb{P}_\gamma : \gamma < \gamma(*) \rangle$ satisfies $(\mathbf{p}, \mathbf{q}, \mathcal{D})$ <u>when</u>

- $\mathcal{D}$ a normal filter on ω_1 such that $S_{\mathbf{p}} \in \mathcal{D}$
- if $\langle u_\alpha : \alpha < \omega_1 \rangle$ is as above

then for some $\mathcal{W} \in \mathcal{D}$ for every $\delta \in \mathcal{W} \cap S_{\mathbf{p},\bar{u}}$ letting h_δ be as there then for some function $h \supseteq h_\delta, \gamma \in \gamma(*) \cap u_\delta \Rightarrow bbP_{\mathbf{q},\delta,\gamma} \subseteq$ Rang$(h) \cap h^{-1}(\mathbb{P}_{\mathbf{q},\delta,\gamma}) \lessdot \mathbb{P}_\gamma$.

Remark 4.2. 1) This definition fits [Sh:F699].
2) Replacing $\aleph_0$ by μ-see [Sh:F1000].
3) We can translate to proof here is a winning strategy $\mathbf{st}_\delta$ is [Sh:895] frame.
4) We may replace the sequence $\bar{P}_\delta$ by a tree, then we can demand success on a club (or member of the filter). Necessary if we like to demand "no δ such that $\underset{\sim}{G}_{\mathbf{p}} \cap \delta$ is generic for $(\mathbb{P} \cap \delta, M_\delta), M_\delta \models$ "$|\delta| = \aleph_0$".
5) A natural question, continuing §3 is: for transparency assume $\mathbf{V} \models$ GCH and for $\ell = 1, 2$ let $\mathbf{Q}_\ell$ be a set of pairs $(\bar{\varphi}, \underset{\sim}{\eta})$ so $\bar{\varphi}$ from $\mathbf{V}$, we ask: is there a generic extension $\mathbf{V}^{\mathbb{P}}$ of $\mathbf{V}$ such that:

- if $(\bar{\varphi}, \underset{\sim}{\eta}) \in \mathbf{Q}_1$ then there is a $(\bar{\varphi}, \underset{\sim}{\eta})$-positive set of cardinal $\aleph_0$
- if $(\bar{\varphi}, \underset{\sim}{\eta}) \in \mathbf{Q}_2$ then there is no $(\bar{\varphi}, \underset{\sim}{\eta})$-positive set of cardinal $\aleph_1$.

We surely can phrase sufficient conditions, but can we phrase sufficient and necessary conditions?
6) More complicated when we have $\underset{\sim}{\bar{\varphi}}$ above, so, i.e. allow real parameter in $\underset{\sim}{\bar{\varphi}}$, so we will have $2^{\aleph_0}$ such cases rather than $\aleph_1$.

§ 5. Private Appendix

Question on $\oplus_5$ in the proof of Fact D, from pg.18:

Martin and Wolfgang would like to know if $\mathbb{Q}_*$ is necessary. They think that every filter $G \subseteq \mathbb{Q}$ which is $\mathbb{Q}$-generic over M_δ will satisfy the following (and therefore belong to $\text{gen}^+(\mathbb{Q})$):

- $G \cap \mathbb{P}^+_{t,\bar{\varphi}}$ is generic over $M_{\delta(t)}$ (the same holds for $G \cap \mathbb{P}^t$)
- $G \cap \mathbb{P}_{s,n}$ is generic over M_δ (for each $n < \omega$)
- $\underset{\sim}{\nu}_\delta[G \cap \mathbb{P}_{x,n(*)}] = \underset{\sim}{\eta}[G \cap \mathbb{P}^+_{t,\bar{\varphi}}]$.

Discussion 5.1. (based on 2010.3.18 remark) after coments of Hecke
1) In Definition 1.11(2) in $(*)_2(d)$ should we separate for obligations of kind 1.
2) In the proof of $(*)_8$ in Main Fact D inside the proof of 2.1, the paragraph on "why clause (b)?" Heike thought n not $n(*)$.

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