NON COHEN ORACLE C.C.C

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Abstract. The oracle c.c.c. is closely related to Cohen forcing. During an iteration we can “omit a type”; i.e. preserve “the intersection of a given family of Borel sets of reals is empty” provided that Cohen forcing satisfies it. We generalize this to other cases. In §1 we replace Cohen by “nicely” definable c.c.c., do the parallel of the oracle c.c.c. and end with a criterion for extracting a subforcing (not a complete subforcing, \(!\|\) of a given nicely defined one such that the subforcing satisfies the oracle.

§0. Introduction

This answers a question from [Sh:b, Ch.IV] (the chapter dealing with the oracle c.c.c.) asking to replace Cohen by e.g. random. Later we will deal with the parallel for oracle proper and for the case \(\bar{\text{c.c.c.}}\) asking to replace Cohen by e.g. random. Later we will deal with the parallel of the oracle c.c.c. and end with a criterion for extracting a subforcing (not a complete subforcing, \(!\|\) of a given nicely defined one such that the subforcing satisfies the oracle.

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$X_i$. E.g. $M_{i+1}$ can be the Mostowski Collapse of some countable $M \prec (\mathcal{H}(\aleph_2), \in)$
to which $P_{i}, M'$ and $X_i$ belongs.

Really this corresponds to the omitting type as in [Sh:e, XI]. This was originally part of [Sh:630],
particularly close to the so called faking; on more general treatments, including replacing $\aleph_0$
by a larger cardinal, see [Sh:895], [Sh:F1000]. We thank the referee and (in 2008-2010) Jakob Kellner,
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§ 1. Definitions and preliminaries

Hypothesis 1.1.
(a) we assume CH, moreover $\diamondsuit_S$, where $S^* \subseteq \{ \delta < \omega_1 : \delta \text{ limit} \}$ is stationary
(b) let ZFC$_*$ be a large enough subset of ZFC satisfying $(\mathcal{H}(\mathcal{N}_2), \in)$ (or just $(\mathcal{H}(\mathcal{X}_*), \in)$ for some $\mathcal{X}_*$ with minor changes in the proof.

Definition/Notation 1.2. 1) $\bar{M}$ denotes an oracle, i.e., a sequence of the form $(M_\delta : \delta \in S)$, $M_\delta$ a transitive countable model of ZFC$_*$ satisfying $\delta + 1 \subseteq M_\delta$ and $S \subseteq S^*$ is stationary satisfying: for every $X \subseteq \omega_1$, the set $(\delta \in S^* : X \cap \delta \in M_\delta)$ is stationary. For such $M$ let $S_M = S$. 2) $\mathcal{D}$ denotes a normal filter on $\omega_1$ usually extending $\mathcal{D}_M$ which is defined in 1.6(1) below (of course, the default value is $\mathcal{D}_M$, see 1.8(1)).
3) For a countable forcing $\mathbb{P}$, a wide $\mathbb{P}$-name is a Borel function giving for every $G \subseteq \mathbb{P}$ an object, i.e. the “input” of such a Borel function are (truth value($p \in G$) : $p \in \mathbb{P}$), the “output” is normally in $\{ x \in \mathcal{H}(\mathcal{N}_1) : \text{rk}(x) < \gamma \}$ for some $\gamma < \omega_1$. So if $p \in \mathbb{P} \Rightarrow p \in \mathbb{P}'$ then any wide $\mathbb{P}$-name is still a wide $\mathbb{P}'$-name hence a $\mathbb{P}'$-name but it is natural to restrict ourselves, e.g. to the case $\mathbb{P} \subseteq \mathcal{P}'$, see below.
4) $\mathbb{P} \subseteq \mathcal{P}'$ when $\mathbb{P}$ are forcing notions, i.e. quasi-orders, $\mathbb{P} \subseteq \mathbb{P}'$ and if $p, q \in \mathbb{P}$ then they are incompatible in $\mathbb{P}$ iff they are incompatible in $\mathbb{P}'$.
5) $\mathbb{P} \subseteq \mathbb{P}'$ iff $\mathbb{P} \subseteq \mathcal{P}'$ and every predense subset of $\mathbb{P}$ is a predense subset of $\mathbb{P}'$ (equivalently demands this for maximal antichains).
6) For a forcing notion $\mathbb{P}$ let

\[(a) \ L_{\omega_1, \omega}(\mathbb{P}) \text{ be the set of } L_{\omega_1, \omega} \text{ sentences in propositional logic considering the members of } \mathbb{P} \text{ as propositional variables}
\]
\[(b) \text{ for a generic } G \subseteq \mathbb{P} \text{ and } \psi \in L_{\omega_1, \omega}(\mathbb{P}) \text{ let } \psi[G] \text{ be the truth value (so if } \psi = p \in \mathbb{P} \text{ we have } \psi[G] = \text{ true iff } p \in G \text{, etc. (see [Shf, Ch.IX])}
\]
\[(c) \text{ let } p \Vdash_\mathbb{P} \psi \text{ means } p \Vdash_\mathbb{P} \psi[G] = \text{ true} \text{ for } p \in \mathbb{P}, \psi \in L_{\omega_1, \omega}(\mathbb{P})
\]
\[(d) \text{ let } \mathbb{P} \text{ be the following quasi order}
\]
\text{ set elements: } \{ \psi \in L_{\omega_1, \omega}(\mathbb{P}) : \text{ for some } p \in \mathbb{P} \text{ we have } p \Vdash_\mathbb{P} \psi[G] = \text{ true} \}
\text{ quasi order: } \psi_1 \leq \psi_2 \text{ iff } p \Vdash_\mathbb{P} \text{ if } \psi_2[G] = \text{ true then } \psi_1[G] = \text{ true }.
\]

Observation 1.3. 1) So $\mathbb{P} \subseteq \mathbb{P}'$ as quasi orders, in fact, $\mathbb{P}$ is a dense subset of $\mathbb{P}'$ (so no real difference as forcing notion).
2) If $\mathbb{P}_1 \triangleleft_\mathbb{P}_2 \text{ then } \hat{\mathbb{P}}_1 \triangleleft_\mathbb{P}_2.$
3) Assume $M \subseteq \mathcal{H}(\mathcal{N}_1)$ is a countable transitive model of ZFC$_*$, $M \models \"\mathbb{P} \text{ is a forcing notion, the pair } (\check{\varphi}, \check{\gamma}) \text{ as in 1.11(1) below and } \nu_\alpha \text{ is a } (\mathbb{Q}_\varphi, \check{\gamma})$-generic over $M\"$. Then

\[(a) \mathbb{P}' = (\mathbb{P} * \mathbb{Q}_\varphi)/(\check{\gamma} = \nu) \text{ is a forcing; note notion in } M[\nu]
\]
\[(b) \text{ if } \mathbb{P} \subseteq \mathcal{P}' \text{ and every predense suset of } \mathbb{P} \text{ which belongs to } M \text{ is a predense subset of } \mathbb{P}' \text{ then } \mathcal{P} \cap \mathbb{P} \text{ is a subset of } \mathbb{P} \text{ generic over } M \text{ and some } \mathbb{G}' \subseteq \mathbb{P} \cap \mathbb{P} \text{ generic over } M \text{ extends } \mathcal{P} \cap \mathbb{P} \text{ and satisfies } \check{\eta}[\mathbb{G}'] \equiv \nu^\nu
\]
\[(c) \text{ if } M \models \"\mathbb{P} \text{ uncountable} \" \text{ use } (L_{\mathcal{P}[\nu, \omega]}(\mathbb{P}))^{M[\nu]} \text{ or do it outside } M, \text{ see explanation}
\]
\[(d) \text{ if in addition } M \models \"\mathbb{P}_1 \triangleleft_\mathbb{P}_2 \" \text{ then } (\mathbb{P}_1 * \mathbb{Q}_\varphi)^M \triangleleft_\mathbb{P}_2 * \mathbb{Q}_\varphi)^M \text{ and } (\mathbb{P}_1 * \mathbb{Q}_\varphi)^M/(\check{\gamma} = \nu) \triangleleft (\mathbb{P}_2 * \mathbb{Q}_\varphi)^M/(\check{\gamma} = \nu).
\]
Discussion 1.4. How $P'$ is interpreted as $\mathbb{L}_{\omega_1,\omega}^{(P)^M}$. We should assume $M_\alpha = "P"$ is a c.c.c. or use $(L_{P_1,\omega}^{(P)})^{M_\alpha}$ which is $\mathbb{L}_{\omega_2,\omega}^{(P)}$: 

(*) if $G \subseteq P$ is generic over $P$ and $\nu$ is $(\mathbb{Q}_\nu, \eta)$-generic for $M[G]$ then there is 

(a) there is a unique $G^+ \subseteq (P \ast \mathbb{Q}_\nu)^M/\langle \eta = \nu \rangle$ such that it is generic for $(P \ast \mathbb{Q}_\nu)^M$

(b) $G \subseteq G^+$

(c) $G \cap Q_\nu^M = G_\nu$

(d) there is a sequence of Borel functions $\langle B_p : p \in (P \ast \mathbb{Q}_\nu)^M \rangle$ with infinite $P$, in fact, in $M$ and output a truth value $G^+ = \{ p \in (P \ast \mathbb{Q}_\nu)^M : B_p(\ldots, \text{truth value}(r \in G), \ldots)_{r \in P} = \text{truth} \}$, in fact uniformly in $\nu$, but not used

(*) we can identify $p = (p_0, p_1) \in (P \ast \mathbb{Q}_\nu)^M$ with $B_p(\ldots, \text{truth value}(r \in G), \ldots)_{r \in P}$.

2) The meaning of $"\eta \in \omega"$ is generic for $\mathbb{Q}_\nu$ mean then for every $r \in \mathbb{Q}_\nu$ we can compute in a Borel way truth value$(r \in G)$ from the value of $\eta$ (in our case $\nu$).

3) For random reals: if we interpret as the family of closed sets then $\nu \in \langle \text{a closed set } r \text{ of } \omega \rangle$ is O.K.

4) In the case of random $(p, \eta) \in (P \ast \mathbb{Q}_\nu)^M/\eta = \nu$ is interpreted as (now $F^\eta_n : I^\eta_n = \{ T \cap n \geq 2 : T \text{ a perfect subtree of } \omega > 2 \})$

$$p \land \bigwedge_n \bigwedge_{r \in I_n} (r \rightarrow \nu|n \in F^\eta_n(r)).$$

Remark 1.5. In 1.3(2)(c), note that 

(a) $P_1 \ll P_2$ by the assumption as this is upward absolute from $M$ as $M$ is transitive

(b) the conclusion holds in $M[\nu]$ which is a generic extension of $M$ hence in $V$ as above.

We first give the old definitions from [Sh:f, IV]

{12.2}

Definition 1.6. 1) $D_M$ is

$\{ X \subseteq \omega_1 : \text{for some } Y \subseteq \omega_1 \text{ we have }: Y \cap \delta \in M_\delta \Rightarrow \delta \in S_M \cap X \}.$

2) A forcing notion $P$ of cardinality $\leq \aleph_1$ satisfies the $(M, D)$-c.c. if $S_M \in D^+$ and for some (equivalently any) one to one $f : P \rightarrow \omega_1$ the set $S$ defined below belongs to $D$ and $P$ has minimal element $\theta_P$ where:

$$S = \{ \delta : \text{ if } \delta \in S_M \text{ and } X \in M_\delta \text{ and } \{ y \in P : f(y) < \delta \text{ and } f(y) \in X \} \text{ is predense in } P \} \upharpoonright \{ y \in P : f(y) < \delta \} \text{ then } X \text{ is predense in } P \}.$$

3) If $D = D_M$ we may write "M-c.c."

Recall that $D^+ = \{ A \subseteq \omega_1 : \omega_1 \setminus A \notin D \}.$

4) Let $M^1 \leq M^2$ if $M^f = \langle M^1_\delta : \delta \in S_1 \rangle$ and $\{ \delta : \delta \in S_1 \setminus S_2 \text{ or } \delta \in S_1 \cap S_2 \text{ and } M^1_\delta \neq M^2_\delta \}$ is not stationary: let $M^1 \leq D^2 M^2$ be defined similarly (i.e. the set is $= \emptyset \mod D$); we call $E$ a witness when $E \in D$ is disjoint to those sets.

4A) Let $M^1 \leq D^2 M^2$ when $M^f = \langle M^1_\delta : \delta \in S_1 \rangle$ and $\{ \delta : \delta \in S_1 \setminus S_2 \text{ or } \delta \in S_1 \cap S_2 \text{ and } M^1_\delta \neq M^2_\delta \}$ is not stationary: let $M^1 \leq D^2 M^2$ be defined similarly.
5) A forcing notion $\mathbb{P}$ satisfies the $(\mathcal{M}, \mathcal{D})$-c.c. if $|\mathbb{P}| \leq \aleph_0$ or for every $X \subseteq \mathbb{P}$ of cardinality $\leq \aleph_1$ there is $\mathbb{P}_1 < \mathbb{P}$ of cardinality $\aleph_1$ which includes $X$ and satisfies the $(\mathcal{M}, \mathcal{D})$-c.c.

**Remark 1.7.** For the $\mathcal{M}$-c.c. the order $\leq^*$ from 1.6(4A) is natural, but here it is easier to use $\leq$ from 1.6(4).

**Fact 1.8.** 1) $\mathcal{D}_\mathcal{M}$ is a normal filter on $\omega_1$.
2) The $\mathcal{M}$-c.c. implies the c.c.c., and if $\mathcal{D}_\mathcal{M} \subseteq \mathcal{D}$ (or just there is a normal filter $\mathcal{D}' \supseteq \mathcal{D}_\mathcal{M} \cup \mathcal{D}$) then the $(\mathcal{M}, \mathcal{D})$-c.c. implies the c.c.c. and if $\mathcal{D}_2 \supseteq \mathcal{D}_1 \supseteq \mathcal{D}_\mathcal{M}$ are normal filters, then the $(\mathcal{M}, \mathcal{D}_1)$-c.c. implies the $(\mathcal{M}, \mathcal{D}_2)$-c.c.
3) We can find $(S^*_\zeta : \zeta < \omega_2)$ such that $S^*_\zeta \subseteq S^*, \zeta < \zeta \Rightarrow S^*_\zeta \subseteq S^*_\xi$ mod $\mathcal{D}_\mathcal{M}, S^*_\xi \subseteq S^*_\zeta+1$ and $S^*_\zeta+1 \setminus S^*_\zeta \in \mathcal{D}_\mathcal{M}$, moreover $\zeta < \zeta < \omega_2 \Rightarrow S^*_\zeta \setminus S^*_\xi$ is countable.
4) If $M^1 \subseteq M^2$ and the forcing notion $\mathbb{P}_2$ satisfies the $(M^2, \mathcal{D})$-c.c. and $\mathbb{P}_1 \preccurlyeq \mathbb{P}_2$, then $\mathbb{P}_1$ satisfies the $(M^1, \mathcal{D})$-c.c.
5) Like part (4) when $M^1 \subseteq^* M^2$.

**Proof.** See [Sh:f, Ch.IV], but for the reader’s convenience we prove part (4).
4),5) Without loss of generality $\mathbb{P}_2$ has cardinality $\aleph_1$ and even its set of elements is $\omega_1$. As $\mathbb{P}_1 \preccurlyeq \mathbb{P}_2$ there is a function $f : \mathbb{P}_2 \rightarrow \mathbb{P}_1$ such that

$$(*)_1 \ q \in \mathbb{P}_2 \land f(q) \leq \mathbb{P}_1 \ p \in \mathbb{P}_1 \Rightarrow p, q \text{ are compatible in } \mathbb{P}_2.$$ 

Let $g : \mathbb{P}_2 \times \mathbb{P}_2 \rightarrow \mathbb{P}_2$ be such that

$$(*)_2 \ p, q \in \mathbb{P}_2 \text{ are compatible then } g(p, q) \text{ is a common upper bound and}$$

$$p, q \in \mathbb{P}_1 \Rightarrow g(p, q) \in \mathbb{P}_1.$$ 

So there is a club $E^1$ of $\omega_1$ which is closed under $f, g$ so

$$(*)_3 \text{ if } \delta^1 \in E \cap \text{Dom}(\mathcal{M}^1) \text{ and } \mathcal{I} \subseteq \mathbb{P}_1 \cap \delta \text{ is predense in } \mathbb{P}_1 \upharpoonright \delta \text{ then } \mathcal{I} \text{ is predense in } \mathbb{P}_2 \upharpoonright \delta.$$ 

[Why? If $q \in \mathbb{P}_2 \cap \delta$ then $f(q) \in \mathbb{P}_1 \cap \delta$ so by the assumption on $\mathcal{I}, f(q)$ is compatible with some $r_1 \subseteq \mathcal{I} \subseteq \mathbb{P}_1 \cap \delta$, so there is $r_2 \in \mathbb{P}_1 \upharpoonright \delta$ above $f(q)$ and $r_1$, e.g. $g(f(q), r_1)$. By the definition of $f$ the conditions $r_2, q$ are compatible in $\mathbb{P}_2$ hence $g(r_2, q)$ is a common upper bound of them in $\mathbb{P}_2 \upharpoonright \delta$]

$$(*)_4 \text{ there is } E_2 \in \mathcal{D} \text{ such that if } \delta \in E_2 \text{ and } \mathcal{I} \subseteq \mathbb{P}_1 \upharpoonright \delta \text{ and } \mathcal{I} \in M^2_\delta \text{ is predense in } \mathbb{P}_2 \upharpoonright \delta \text{ then it is predense in } \mathbb{P}_2.$$ 

[Why? As we are assuming that $\mathbb{P}_2$ satisfies the $(\mathcal{M}^2, \mathcal{D})$-c.c.] 

$$(*)_5 \text{ if } \mathcal{I} \text{ is a predense subset of } \mathbb{P}_2 \text{ and is } \subseteq \mathbb{P}_1 \text{ then } \mathcal{I} \text{ is a predense subset of } \mathbb{P}_1.$$ 

[Why? As $\mathbb{P}_1 \preccurlyeq \mathbb{P}_2$ is assumed.] 

$$(*)_6 \text{ there is } E_3 \in \mathcal{D} \text{ such that } \delta \in E_3 \Rightarrow M^1_\delta \subseteq M^2_\delta.$$ 

[Why? As $M^1 \subseteq M^2$ implies $M^1 \subseteq^* M^2$ and see the definition.] 

Putting together $(*)_3, (*)_4, (*)_5, (*)_6$ we are done. $\square_{1.8}$
Definition 1.9. 1) We define when $\check{\varphi}$ is an absolute definition of a c.c.c. forcing notion with generic $\eta$ (for ZFC) say with a parameter, a real and countable ordinal means that for any transitive model $N$ of ZFC, to which the parameters belong $Q = Q^*_\alpha$ is a forcing notion, the property "$p \in Q, p \leq Q, q, p, q \in Q$ are incompatible", "$I$ is a countable predense subset" are preserved in forcing extensions\(^1\) of $N$.

2) We add with the generic $\eta$ when $\eta$ is a $Q_\alpha$-name, the parameter including the relevant information of a member of "2 or $\alpha(0) < \omega$ and the generic is reconstructible from $\eta$.

3) We say "$\varphi$ is a nep forcing problem with the generic $\eta$" as in [Sh:630]. See below.

Remark 1.10. 1) Note that below when $Q_{\check{\varphi}_\alpha}$ is the older case ([Sh:f, IV]) we just preserve every predense set, so in $M_\alpha$ (in the cases the "commitment", see below, is obeyed) the forcing is countable.

2) We may forget to mention this case as it is by now easy.

Definition 1.11. 1) We say $\mathcal{Y} = (S, \check{\Phi}, \check{\eta}, \check{\nu}) = (S^\check{\nu}, \check{\Phi}^\check{\nu}, \check{\eta}^\check{\nu}, \check{\nu}^\check{\nu})$ is a 0-commitment for $M$ iff for some $E \in \mathcal{D}_{\check{M}}$:

(a) $S \subseteq S^\check{\nu}, S \in \mathcal{D}_{\check{M}}^+$

(b) $\check{\eta} = \langle \eta_\alpha : \alpha \in S \rangle, \check{\Phi} = \langle \check{\varphi}_\alpha : \alpha \in S \rangle$ and if $\alpha \in S \cap E$ then $\check{\varphi}_\alpha \in M_\alpha$ and $M_\alpha \models \check{\varphi}_\alpha$ is an absolute definition of a c.c.c. forcing notion called $Q_\alpha = Q_{\check{\varphi}_\alpha}$ with generic real $\eta_\alpha$ so $M_\alpha \models (\forall Q_{\check{\varphi}_\alpha} \ "M_\alpha[G_{\check{\varphi}_\alpha}] = M_\alpha[\eta_\alpha])$; note, absolute here means that forcing extensions of $M_\alpha$, preserve predensity of countable sets (in the sense of $M_\alpha$), preserve order and its negation and preserves incompatibility

(c) $\check{\nu} = \langle \nu_\alpha : \alpha \in S \rangle$ where $\nu_\alpha \in {}^\omega \omega$ and for every $\alpha \in S \cap E$ the real $\nu_\alpha$ is ($Q_\alpha, \eta_\alpha$)-generic over $M_\alpha$ or $\nu_\alpha$ is ($M_\alpha, Q_{\check{\varphi}_\alpha}, \eta_\alpha$)-generic sequence which means that for some $G \subseteq Q^{\check{\nu}}_\alpha$ generic over $M_\alpha$ do we have $\nu_\alpha = \eta_\alpha[G]$.

(d) We can allow $\Vdash_{Q_{\check{\varphi}_\alpha}} \ "\eta_\alpha : \beta^1_{\check{\varphi}_\alpha} \rightarrow \beta^2_{\check{\varphi}_\alpha} \ "$ for some ordinals $\beta^1_{\check{\varphi}_\alpha}, \beta^2_{\check{\varphi}_\alpha}$ from $M_\alpha$ and so $\nu_\alpha : \beta^1_{\check{\varphi}_\alpha} \rightarrow \beta^2_{\check{\varphi}_\alpha}$.

We shall ignore $\check{M}$ if clear from the context. We can replace $\check{M}$ by $(\check{M}, D)$ if above $E \in D, S \in D^+$; similarly below.

1A) A forcing notion $P$ of cardinality $\leq \aleph_1$ satisfies the 0-commitment $\mathcal{Y} = (S^\check{\nu}, \check{\Phi}^\check{\nu}, \check{\eta}^\check{\nu}, \check{\nu}^\check{\nu})$ for an $\aleph_1$-oracle $M$ (we may suppress this if clear from the context) when $P$ is a forcing notion and for any one-to-one mapping $h : P \rightarrow \omega_1$ for some $E \in \mathcal{D}_{\check{M}}$ for every $\alpha \in E$ we have

(e) every predense subset $I$ of $\{p \in P : h(p) < \alpha\}$ for which $\{h(p) : p \in I\} \in M_\alpha$ is a predense subset of $P$

(f) if $\alpha \in S \cap E$ then $\Vdash_{P} \ "\text{the real } \nu_\alpha \text{ is a } (Q_\alpha, \eta_\alpha)\text{-generic real over } M_\alpha[\alpha \cap h''G_{\check{\varphi}_\alpha}]\ "$; moreover

(f)\(^+\) letting $P^b = h(P)$ the forcing notion $P^b_\alpha = P^1_\alpha | \alpha$ belongs to $M_\alpha$ and $(P^b_\alpha * Q_{\check{\varphi}_\alpha})M_\alpha/(\eta_\alpha = \nu) \subseteq \check{P}^b_\alpha$ and, moreover, any predense subset of it from $M_\alpha[\nu_\alpha]$ is predense in $P^b_\alpha$, (in fact this implies clause (d), i.e. any predense subset of $P^b_\alpha$ from $M_\alpha$ is predense in $P^b$)

\(^1\)may restrict ourselves to the relevant forcing
(g) so we get for the old case: if \( \alpha \in S \cap E \) and \( Q_\alpha \) is a singleton (hence \( \nu_\alpha \in M_\alpha \), a degenerated case), this actually follows from (d)

(h) the old case, i.e. \( Q_\alpha = \text{singleton occurs for a set of } \alpha \text{’s from } D_M^\alpha \) (nee?)
we may add \( M_\alpha \models \{ \alpha \} = \emptyset \) most naturally in (e), but not used so far.

2) Let \( \mathbb{P} \in \mathcal{H}(\mathbb{N}_2) \) be an \( M \)-c.c. forcing notion. We say that \( \mathcal{Y} = (S, \Phi, \bar{\eta}, \bar{\nu}) = (S^\mathcal{Y}, \Phi^\mathcal{Y}, \bar{\eta}^\mathcal{Y}, \bar{\nu}^\mathcal{Y}) \) is a 1-commitment on \( \mathbb{P} \) for \( M \) if: for any \( \tilde{N} \) satisfying (*)$_1$ below, the clauses (a)-(d) of (*)$_2$ below hold

(*)$_1$ \( \tilde{N} = \langle N_\alpha : \alpha < \omega_1 \rangle \) is increasing continuous, \( N_\alpha \in \langle \mathcal{H}(\mathbb{N}_2), \in \rangle \) is countable, \( \tilde{N} \upharpoonright (\alpha + 1) \in N_{\alpha + 1} \) and \( \{ \mathbb{M}, \mathbb{P} \} \subseteq N_\alpha \) for some \( \alpha < \omega_1 \) and \( h \in N_0 \) is a one-to-one mapping from \( \mathbb{P} \) into \( \omega_1 \)

(*)$_2$
(a) \( S \subseteq \text{Dom}(M) \subseteq \mathcal{Y}, S \in D_M^\mathcal{Y} \)
(b) \( \bar{\eta} = \langle \eta_\alpha : \alpha \in S \rangle \), \( \Phi = \langle \bar{\phi}_\alpha : \alpha \in S \rangle \) so \( \langle \bar{\phi}_\alpha, \eta_\alpha \rangle \) is a \( \mathbb{P} \)-name of a pair as in 1.11(1)(b), both are wide names over \( \mathbb{P} \)
(c) \( \bar{\nu} = \langle \nu_\alpha : \alpha \in S \rangle \) and \( \nu_\alpha \) a wide \( \mathbb{P} \)-name of a real
(d) the set of the \( \alpha \in S \) satisfying the following belongs to \( D_M + S \):
\( \bar{\phi}_\alpha \in M_\alpha \), MosCol$_{N_\alpha}(N_\alpha) \in M_\alpha \), and letting \( h_\alpha = \text{MosCol}_{N_\alpha} \)
we have \( \mathbb{P}_\alpha = h_\alpha(\mathbb{P}^{N_\alpha}) \in M_\alpha \) so e.g. the set of members of \( \mathbb{P}_\alpha \) is \( \{ h_\alpha(x) : x \in \mathbb{P} \cap N_\alpha \} \), we have \( M_\alpha \models \langle \bar{\phi}, \eta \rangle \) is a wide \( \mathbb{P}_\alpha \)-name of an absolute definition of a c.c.c. forcing with generic real \( \eta_\alpha \) and \( \mathbb{P} \models \langle \bar{\phi}_\alpha, \eta_\alpha \rangle \) is \( \mathbb{P}_\alpha \)-generic over \( M_\alpha(\alpha \cap h \langle G \rangle) \)

(d)$_+$ moreover if \( \mathcal{I} \in M_\alpha \) is a predense subset of \( \mathbb{P}_\alpha \) if
\( M_\alpha \models \psi(p, \eta) \in L_{\infty, \omega} \) propositional sentence in the propositional variable \( p \), listing \( \mathbb{P}_\alpha \) and the \( \eta(i), i < \ell_\eta(\eta) \) such that
\( \mathbb{P} \models \psi(h_\alpha(p), \eta) \) true, then
\( \mathbb{P} \models \psi(h_\alpha(p), \eta) \) true.

Hence in particular if \( \mathcal{I} \in M_\alpha \) is a predense subset of \( \mathbb{P}_\alpha \) then \( h^{-1}(\mathcal{I}) \) is a predense subset of \( \mathbb{P} \).

For transparency the reader may concentrate below on the case \( \langle \langle \bar{\phi}_\alpha, \eta_\alpha \rangle : \alpha \in S \rangle \subseteq \mathcal{V} \).

3) Let \( IS = \{ (\mathcal{P}, \mathcal{Y}, M) : \mathcal{P} \in \mathcal{H}(\mathbb{N}_2) \} \) is an \( M \)-c.c. forcing notion and \( \mathcal{Y} \)

is a 1-commitment on \( \mathbb{P} \) for \( M \).

We shall omit \( M \) if clear from the context. We can replace \( M \) by \( (M, \mathcal{D}) \) naturally and write \( IS \), but the claims are the same.

4) For \( \langle \mathcal{P}^\ell, \mathcal{Y}^\ell, M^\ell \rangle \in IS(\ell = 1, 2) \) let \( (\mathcal{P}^1, \mathcal{Y}^1, M^1) \leq^* (\mathcal{P}^2, \mathcal{Y}^2, M^2) \) means \( M^1 \leq M^2 \), \( \mathcal{P}^1 \leq \mathcal{P}^2 \) and for some \( E \in D_{M^1} \) we have

(a) \( S^\mathcal{Y}^1 \cap E \subseteq S^\mathcal{Y}^2 \cap E \),
(b) \( \Phi^\mathcal{Y}^1 \upharpoonright (S^\mathcal{Y}^1 \cap E) = \Phi^\mathcal{Y}^2 \upharpoonright (S^\mathcal{Y}^1 \cap E) \),
(c) \( \bar{\eta}^\mathcal{Y}^1 \upharpoonright (S^\mathcal{Y}^1 \cap E) = \bar{\eta}^\mathcal{Y}^2 \upharpoonright (S^\mathcal{Y}^1 \cap E) \) and
(d) \( \bar{\nu}^\mathcal{Y}^1 \upharpoonright (S^\mathcal{Y}^1 \cap E) = \bar{\nu}^\mathcal{Y}^2 \upharpoonright (S^\mathcal{Y}^1 \cap E) \).

We call \( E \) a witness to \( (\mathcal{P}^1, \mathcal{Y}^1, M^1) \leq^* (\mathcal{P}^2, \mathcal{Y}^2, M^2) \). Note that it does not matter if we demand \( E \in D_{M^1} \) or \( E \in D_{M^2} \).

\[ ^2 \text{this holds for a set of } \alpha \text{’s which belongs to } D_M \]
We point out the connection between 0-commitment and 1-commitments.

**Fact 1.12.** 1) If $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ for $\mathbb{M}$, so $\mathbb{P}$ is an $\mathbb{M}$-c.c. forcing notion of cardinality $\leq \aleph_1$, then $\Vdash_{\mathbb{P}} \text{"} \mathcal{Y}[\mathbb{G}_\mathbb{P}] = (S,\Phi,\phi;\mathbb{G}_\mathbb{P},\eta;\mathbb{G}_\mathbb{P}) \text{"} \text{ is a 0-commitment}”$ so we call it $\mathcal{Y}[\mathbb{G}_\mathbb{P}]$. Note that each $\eta_\alpha[\mathbb{G}_\mathbb{P}]$ is still a name.

2) If $\mathbb{P} = \{\emptyset\}$ (the trivial forcing) then: $\mathcal{Y}$ is a 1-commitment on $\mathbb{P}$ if and only if $\mathcal{Y}$ is a 0-commitment.

3) If $\langle \mathbb{M}^i : i < \zeta \rangle$ is $\leq \omega_2$ and $\text{Dom}(\mathbb{M}^i)\setminus S$ is not stationary while $S \subseteq S^*$, then there is $\mathbb{M}$ satisfying $\text{Dom}(\mathbb{M}) = S$ such that $i < \zeta \Rightarrow \mathbb{M}^i \preceq \mathbb{M}$.

4) Assume $\mathbb{M}^2$ is oracle, $S \subseteq \mathbb{M}$ belongs to $D^{\mathbb{M}_2}_S$ and $\mathbb{M}_1 = \mathbb{M}_2|S$. If $\mathcal{Y}^2 = (S^2,\Phi^2,\eta^2,\nu^2)$ is a 0-commitment for $\mathbb{M}^2$ and $S \subseteq S^2$ then $\mathcal{Y}^1 := (S,\Phi^2,\eta^2|S,\nu^2|S)$ is a 0-commitment for $\mathbb{M}^1$ and any forcing notion satisfying $\mathcal{Y}^2$ satisfies $\mathcal{Y}^1$. If $\mathcal{Y}^2 = (S^2,\Phi^2,\eta^2,\nu^2)$ is a 1-commitment on $\mathbb{P}$ for $\mathbb{M}^2$ and $S \subseteq S^2$ then $(S,\Phi^2,\eta^2|S,\nu^2|S)$ is a 1-commitment on $\mathbb{P}$ for $\mathbb{M}^1$.

5) If a forcing notion $\mathbb{P}$ satisfies the 0-commitment $\mathcal{Y}$ for the $\aleph_1$-oracle $\mathbb{M}$ and $S' = \{\alpha \in S^2 : \varphi_\alpha^2 = \varphi_\alpha^2 \text{ and } \alpha \text{ is of the old case for } \mathcal{Y}\}$ and $S'' \neq \emptyset \mod D_S$ then

\begin{enumerate}
  \item $\mathbb{P}$ satisfies the $(\mathbb{M} \upharpoonright S')$-c.c. which is defined as in [Sh:f, IV]
  \item if $S'' \subseteq S'$ and $S'' \neq \emptyset \mod D_S$ and $M_S \models \text{"}X_\delta \subseteq \bigcup_{\delta \in S''} X_\delta \text{ is not meagre for every } \delta \in S''\text{"}$ then $\mathcal{Y}^1 = (S,\Phi^2,\eta^2|S,\nu^2|S)$ is a 0-commitment for $\mathbb{M}^1$.
\end{enumerate}

As a warm-up (see [Sh:630] for more)

**Claim 1.13.** 1) Assume

\begin{enumerate}
  \item $\mathbb{M}$ is a countable transitive model of ZFC, $M \models \text{"}\mathbb{P}_1$ is a forcing notion”
  \item $\mathbb{M} \models \text{"}\varphi$ is an absolute definition of a c.c.c. forcing notion $\mathbb{Q}^\varphi$ with generic $\eta : \beta_1 \rightarrow \beta_2$ so $\beta_1,\beta_2 \in \text{Ord}^M \subseteq \omega_1 \cap M$.
  \item $\nu$ is a $(\mathbb{M},\mathbb{Q}^\varphi,\eta)$-generic sequence, recalling it means that there is $G \subseteq (\mathbb{Q}^\varphi)^M$ generic over $\mathbb{M}$ such that $\nu = \eta[G]$.
\end{enumerate}

Then we can find a countable $\mathbb{P}_2$ such that:

\begin{enumerate}
  \item $\mathbb{P}_1 \subseteq \mathbb{P}_2$ and every $\mathcal{J} \in \mathbb{M}$ which is predense in $\mathbb{P}_1$ is predense in $\mathbb{P}_2$
  \item $\mathbb{P}_2 \models \text{"}\nu$ is $(\mathbb{M}',\mathbb{Q}^\varphi,\eta)$-generic sequence where $M' = M[\mathbb{G}_{\mathbb{P}_2} \cap \mathbb{P}_1]^\mathbb{P}_2$”.
\end{enumerate}

2) Similarly for $\varphi$ defining a nep forcing.

**Proof.** 1) In $\mathbb{M}$ we can define $\mathbb{P}^+ = \mathbb{P}_1 * (\mathbb{Q}_2)^M[\mathbb{G}_{\mathbb{P}_1}]$, now as $\mathbb{Q}^\varphi$ is absolutely c.c.c., we know that $q \mapsto (\emptyset, q)$ is a complete embedding of $(\mathbb{Q}^\varphi)^M$ into $\mathbb{P}^+$. So if $\mathbb{G}_* \subseteq (\mathbb{Q}_2)^M$ is generic over $\mathbb{M}$ such that $\nu = \eta[\mathbb{G}_*]$ then let $\mathbb{P}_2 = \{(p, q) \in \mathbb{P}_1 * (\mathbb{Q}^\varphi)^{G_{\mathbb{P}_1}^\mathbb{P}_2} : (p, q) \text{ is compatible with } (\emptyset, q') \text{ for every } q' \in \mathbb{G}_*\}$. Now check. 2) See [Sh:630].
§ 2. The iteration theorem

Crucial Claim 2.1. In IS, any $\leq^\ast$-increasing $\omega$-chain has an upper bound.

Remark 2.2. 1) The $\omega$-limit is the crucial one not the $\omega_1$-limit. Actually for the $\omega_1$-limit we take the union and we preserve what we need by using the square (and having done something toward it in earlier limits or stages of cofinality $\aleph_0$).

2) When is the union not an upper bound? If, e.g., for some $S' \subseteq S^3$, $S' \in \mathcal{D}_M^+$ and for every $\alpha \in S'$ the forcing notion $\check{\nu}_\alpha^Y$ is random real forcing, we have in particular to preserve $\{\nu_\alpha : \alpha \in S'\}$ is non-null, but the union normally adds a Cohen.

Proof. So assume $(\mathbb{P}_n, \mathcal{Y}_n, \bar{M}^n) \in IS$ and $(\mathbb{P}_n, \mathcal{Y}_n, \bar{M}^n) \leq^* (\mathbb{P}_{n+1}, \mathcal{Y}^{n+1}, \bar{M}^{n+1})$ for $n < \omega$, let $\bar{M}$ be such that $\bar{M} \geq M^n$ for each $n$; so let $E_n \in \mathcal{D}_M$ witness both, see the last line of Definition 1.11(4) and Definition 1.6(4). For simplicity assume that above any $p \in \mathbb{P}_n$ there are two incompatible elements (in $\mathbb{P}_n$), and $0 \in \mathbb{P}_0$ is minimal in all $\mathbb{P}_n$, i.e. is $\emptyset_{\mathbb{P}_n}$. Without loss of generality the set of elements of $\mathbb{P}_n$ is $X_n \subseteq \omega_1$ and $\omega_1 \setminus \bigcup_{n<\omega} \mathbb{P}_n$ has cardinality $\aleph_1$ and let $X_\omega$ be such that $\bigcup_{n<\omega} \mathbb{P}_n \subseteq X_\omega$ and $|X_\omega| \setminus \bigcup_{n<\omega} \mathbb{P}_n = \aleph_1$; this notation helps in a future use, also there we replace $\omega_1$ by a (countable) ordinal of cofinality $\aleph_0$. For transparency assume: if $n < m$ and $\mathbb{P}_m \models \text{"}p \leq q\text{"}$ and $q \in \mathbb{P}_n$ then $p \in \mathbb{P}_n$.

We can define functions $F_n, F_{n,m}, F_{n,m,\ell}$ (when $n < m < \omega, \ell < \omega$) such that

\[ \mathbb{P}_n \]
\[ (a) \text{ if } p, q \in \mathbb{P}_n \text{ are compatible then } F_n(p, q) \in \mathbb{P}_n \text{ is a common upper bound} \]
\[ (b) \text{ if } n < m \text{ and } p \in \mathbb{P}_m, \text{ then } \langle F_{n,m,\ell}(p) : \ell < \omega \rangle \text{ is a maximal antichain of } \mathbb{P}_m, \text{ such that for each } \ell: \]
\[ \text{either } p, F_{n,m,\ell}(p) \text{ are incompatible (in } \mathbb{P}_m) \]
\[ \text{or } p \text{ is compatible in } \mathbb{P}_m \text{ with every } q \in \mathbb{P}_n \text{ which is above } F_{n,m,\ell}(p). \]

Note that necessarily for some $\ell$ the second possibility occurs.

Let

\[ \mathbb{P}_2 \]
\[ (a) E \text{ is the set of } \delta \in E \text{ such that} \]
\[ \bullet \delta \text{ is closed under } F_n, F_{n,m}, F_{n,m,\ell} \]
\[ \bullet \langle \mathbb{P}_n[\delta], F_n[\delta], F_{n,m}[\delta], F_{n,m,\ell}[\delta], n, m, \ell \rangle \in M_\delta \]
\[ \bullet \langle X_\omega \cap \varepsilon : \varepsilon \leq \omega \rangle \in M_\delta \]
\[ \bullet \text{otp}(X_\omega \cap \delta \setminus \bigcup_{n<\omega} \mathbb{P}_n) = \delta \]
\[ (b) \text{ let } E_\omega = \cap \{E_n : n < \omega\} \cap E \text{ and let } E'_\omega \text{ be the closure of } E_\omega \]
\[ \text{so } E'_\omega \subseteq \omega_1 \text{ is stationary and } E'_\omega \text{ is a club of } \omega_1. \]

We would like to define a forcing notion $\mathbb{P}_\omega$ with universe $X_\omega$, and a 1-commitment $\mathcal{Y}_\omega$, and functions $F_\omega, F_{n,\omega,\ell}$ satisfying the natural requirements $(a)_\omega, (b)_n, (c)_m, (d)_n$. First, let $S^{Y_\omega} = \bigcup S^{Y_n} \cap E_\omega$, and for $\alpha \in S^{Y_\omega}$ the triple $(\check{\nu}_\alpha^{Y_\omega}, \eta_\alpha^{Y_\omega}, \nu_\alpha^{Y_\omega})$ is $(\check{\nu}_\alpha^{Y_n(\alpha)}, \eta_\alpha^{Y_n(\alpha)}, \nu_\alpha^{Y_n(\alpha)})$.

\[ \text{In older version we also have:} \]
\[ (r)_{n,m} \text{ if } n < m, p \in \mathbb{P}_n, q \in \mathbb{P}_n \text{ then } q \leq F_{n,m}(p, q) \in \mathbb{P}_n \text{ and if there is } r \text{ such that } q \leq p_r \text{ and } r, p \text{ are incompatible in } \mathbb{P}_m \text{ then } p, F_{n,m}(p, q) \text{ are incompatible in } \mathbb{P}_n. \]
where \( n(\alpha) = \text{Min}\{n : \alpha \in S^3^n\} \). Defining \( \mathbb{P}_\omega, F_\omega, F_{n,\omega,\ell} \) is harder, so we first define \( AP \), a set of approximations to them. A member \( t \) of \( AP \) has the form \((\delta^t, \mathbb{P}^t, \Gamma^t, F^t_\omega, F^t_{n,\omega,\ell})_{n<\omega, \ell<\omega} \) satisfying:

\begin{enumerate}
  \item \( t \in E^\omega_\omega \)
  \item \( \mathbb{P}^t \) is a forcing notion with set of elements \( \subseteq X_\omega \cap \delta^t \) and \( \succeq \delta^t \cap \bigcup \mathbb{P}_n \) and \( 0 \leq p \) for every \( p \in \mathbb{P}^t \)
  \item \( \mathbb{P}^t \upharpoonright (X_\omega \cap \delta^t) = \mathbb{P}_n \upharpoonright (X_\omega \cap \delta^t) \)
  \item if \( p, q \in \mathbb{P}^t \) are compatible in \( \mathbb{P}^t \) then \( F^t_\omega(p, q) \) is a common upper bound in \( \mathbb{P}_t \) such that \( n < \omega \land \{p, q\} \subseteq \mathbb{P}_n \Rightarrow F^t_\omega(p, q) = F_n(p, q) \)
  \item if \( p \in \mathbb{P}^t, n < \omega \) then \( \langle F^t_{n,\omega,\ell}(p) : \ell < \omega \rangle \) is a maximal antichain of \( \mathbb{P}_n \), the members are \( \delta^t \), and for each \( \ell \), \textbf{either} \( p, F^t_{n,\omega,\ell}(p) \) are compatible in \( \mathbb{P}^t \) \textbf{or} \( \forall q \in \mathbb{P}_n \cap \delta^t)(F_n[p] \models \neg F^t_{n,\omega,\ell}(p) \leq q \) \textbf{or} \( p, q \) are compatible in \( \mathbb{P}^t \) \textbf{and} \( \forall \ell \) \textbf{the second case occurs}
  \item if \( m < \omega \) and \( p \in \mathbb{P}^t \cap \bigcup \mathbb{P}_n \setminus \bigcup_{k<m} \mathbb{P}_k \) then \( F^t_{n,\omega,\ell}(p) = F_{n,m,\ell}(p) \)
  \item \( \Gamma^t \) is a sequence \( (\delta^t_k : \zeta < \zeta^t), \zeta^t < \omega \) and \( \delta^t_k \) is a sequence of length \( \omega \) of members of \( \mathbb{P}^t \) which form a predense subset of \( \mathbb{P}^t \)
  \item if \( p \in \mathbb{P}^t \) and \( n < m < \omega \) and \( r \in \mathbb{P}_n \cap \delta^t \) and \( r \leq r' \in \mathbb{P}_n \cap \delta^t \Rightarrow r', p \) are compatible in \( \mathbb{P}^t \), then the set \( \{F^t_{n,\omega,\ell}(p) : \ell < \omega \} \) is predense in \( \mathbb{P}_m \) the condition \( q \) is compatible with some member of this set
  \item if \( \zeta < \zeta^t \) and \( n < \omega \) then:
    \( \{F^t_{n,\omega,\ell}(p^t_{\zeta,k}) : k < \omega, \ell < \omega \) and \( p^t_{\zeta,k}, F^t_{n,\omega,\ell}(p^t_{\zeta,k}) \) are compatible in \( \mathbb{P}^t \) \)
\end{enumerate}

Moreover,

\begin{enumerate}
  \item if \( p^* \in \mathbb{P}^t \) and \( n < \omega \) and \( \zeta < \zeta^t \) then the following set is predense in \( \mathbb{P}_n \):
    \begin{align*}
      & \{r' \in \mathbb{P}_n \cap \delta^t : \text{(i) } r', p^* \text{ incompatible in } \mathbb{P}^t \text{ or} \hspace{1cm} \text{(ii) for some } k < \omega \text{ and } p' \text{ we have } p^* \leq p^t, p^t_{\zeta,k} \leq p^t, p' \text{ and } (\forall r'')(r' \leq r'' \in \mathbb{P}_n \cap \delta^t \to \{(r''), p'\} \text{ has an upper bound in } \mathbb{P}^t)\}\}
    \end{align*}
\end{enumerate}

We define the (natural) partial order \( \leq^* \) on \( AP \) as follows: for \( t, s \in AP \) we let \( t \leq^* s \) if:

\begin{enumerate}
  \item \( \delta^t \leq \delta^s \)
  \item \( \mathbb{P}^t \subseteq \mathbb{P}^s \)
  \item \( F^t_\omega \subseteq F^s_\omega \)
\end{enumerate}
(iv) \( F^t_{n,\omega,\ell} \subseteq F^s_{n,\omega,\ell} \) for \( n, \ell < \omega \).

(v) \( \Gamma^t \) is an initial segment of \( \Gamma^s \).

\[ \square \]

**Fact A:** \( AP \neq 0 \).

**Proof.** Easy: choose \( \delta \in E_{\omega} \), let \( \delta^t = \delta, \mathbb{P}^t = ( \bigcup_{n<\omega} \mathbb{P}_n ) \upharpoonright \delta, F^t_{n, q} = F^{n, q}_{n, q} \) (\( (669) \) revision:2011-11-07
modified:2011-11-08
witness and \( n \))

\[ \text{Proof.} \] Let \( A \in \mathbb{P}^t \subseteq X_{\omega} \cap \delta^t, B = \bigcup_{n<\omega} \mathbb{P}_n \cap \delta. \] We define a forcing notion \( \mathbb{Q}_t \) with set of elements \( \subseteq A \times B \) identifying \( p, q \) with \( p \) and \( 0, q \) with \( q \). Now \( (p, q) \in A \times B \) belongs to \( \mathbb{Q}_t \) if \( p = 0 \) or \( q = 0 \) or there are \( r \in A \cap B \) and \( n = n(p, q) \) such that: \( \mathbb{P}_n \models \text{"}r \leq q\text{"}, (\forall r') | r \leq r' \in \mathbb{P}_n \cap \delta^t \rightarrow r', p \text{ compatible in } \mathbb{P}^t; \) we call such \( r \) a witness and \( n \) a possible value for \( n(p, q) \) and may say \( r, n \) witness this. The order on \( \mathbb{Q} \) is \( (p, q) \leq (p', q') \Leftrightarrow p \leq p', q \leq q \) and \( \bigvee_{n} (q \leq p_n, q') \).

Now note

(\( \ast \)) any \( J \in M_\delta \) which is a predense subset of \( \mathbb{P}_n \upharpoonright \delta \) is a predense subset of \( \mathbb{P}_n \) and \( n < m \Rightarrow \mathbb{P}_n \upharpoonright \delta \prec \mathbb{P}_m \upharpoonright \delta \), and of course \( \mathbb{P}_n \upharpoonright \delta \in M_\delta \).

Let \( A = \mathbb{P}^t \subseteq X_{\omega} \cap \delta^t, B = \bigcup_{n<\omega} \mathbb{P}_n \cap \delta \). We define a forcing notion \( \mathbb{Q}_t \) with set of elements \( \subseteq A \times B \) identifying \( p, q \) with \( p \) and \( 0, q \) with \( q \). Now \( (p, q) \in A \times B \) belongs to \( \mathbb{Q}_t \) if \( p = 0 \) or \( q = 0 \) or there are \( r \in A \cap B \) and \( n = n(p, q) \) such that: \( \mathbb{P}_n \models \text{"}r \leq q\text{"}, (\forall r') | r \leq r' \in \mathbb{P}_n \cap \delta^t \rightarrow r', p \text{ compatible in } \mathbb{P}^t; \) we call such \( r \) a witness and \( n \) a possible value for \( n(p, q) \) and may say \( r, n \) witness this. The order on \( \mathbb{Q} \) is \( (p, q) \leq (p', q') \Leftrightarrow p \leq p', q \leq q \) and \( \bigvee_{n} (q \leq p_n, q') \).

Now note

(\( \alpha \)) \( \mathbb{Q} \upharpoonright A = \mathbb{P}^t \)

(\( \beta \)) \( \mathbb{Q} \upharpoonright B = \bigcup_{n<\omega} \mathbb{P}_n \upharpoonright \delta \)

(\( \gamma \)) If \( (p, q) \in \mathbb{Q}, m = n(p, q) \) hence \( q \in \mathbb{P}_m \upharpoonright \delta \) and \( \mathbb{P}_m \upharpoonright \delta \models \text{"}q \leq q\text{"} \) and \( \mathbb{P}^t = \text{"}p \leq p\text{", then } (p', q') \in \mathbb{Q} \text{ and } \mathbb{Q} \models \text{"}(p, q) \leq (p', q')\" \).

[Why? As \( r \) is a witness and \( m \) a possible value for \( n(p', q') \).]

(\( \delta \)) if \( (p, q) \in \mathbb{Q} \) and \( n = n(p, q) \leq m < \omega \), then \( (p, q) \leq (p, q), (p, q) \in \mathbb{Q} \) and \( \mathbb{P}(p, q) = m, \text{ or at least } m \) is a possible value for \( n(p', q') \).

[Why? \( \mathbb{P}_n \models r \leq q \) hence \( \mathbb{P}_m \models r \leq q \) \( \text{ so for some } \ell, F^t_{m, \omega, \ell}(p) \in \mathbb{J} \) is compatible with \( q \) in \( \mathbb{P}_m \), so there is \( q_1 \in \mathbb{P}_m \cap \delta \text{ such that } \mathbb{P}_m \models q \leq q_1 \land F^t_{m, \omega, \ell}(p) \leq q_1 \). \( (p, q) \in \mathbb{Q} \text{ as witnessed by } m \) and \( r' = F^t_{m, \omega, \ell}(p) \), is as required.]

(\( \varepsilon \)) \( \mathbb{P}_n \upharpoonright \delta \ll \mathbb{Q} \).
Why? Recall that \( \mathcal{Q}_1 \subseteq \mathcal{Q}_2 \) iff \( \mathcal{Q}_1 \subseteq \mathcal{Q}_2 \) and for every \( p_2 \in \mathcal{Q} \) there is \( p_1 \in \mathcal{Q}_1 \) such that \( p_1 \leq p_1' \in \mathcal{Q}_1 \Rightarrow p_1', p_2 \) have a common upper bound in \( \mathcal{Q}_2 \); we shall use this criterion. Let \( (p^0, q^0) \in \mathcal{Q} \), of course, we can replace this pair by any larger one, so by clause (\( \delta \)) above without loss of generality some \( m \in [\alpha, \omega) \) is a possible value for \( n(p^0, q^0) \) so we have \( q^0 \in \mathcal{P}_m \ \delta \), hence recalling that \( \mathcal{P}_n \ \delta \mathcal{P}_m \ \delta \) there is \( q^1 \in \mathcal{P}_n \ \delta \) such that:

\[
(\forall r \in \mathcal{P}_n) (\mathcal{P}_n \ \delta \vdash q^1 \leq r \Rightarrow r, q^0 \text{ compatible in } \mathcal{P}_m \ \delta).
\]

Assume \( q^1 \leq r \in \mathcal{P}_n \ \delta \). So \( r, q^0 \) are compatible in \( \mathcal{P}_m \ \delta \) hence has a common upper bound \( q^3 \in \mathcal{P}_m \ \delta \). In particular \( q^0 \leq q^3 \in \mathcal{P}_m \ \delta \) so by clause (\( \gamma \)) we have \( (p^0, q^3) \in \mathcal{Q} \) and \( (p^0, q^0) \leq (p^0, q^3) \); also \( r = (0, r) \leq (p^0, q^3) \) as \( r \leq q^3 \) together \( r, (p^0, q^3) \) are compatible in \( \mathcal{Q}_2 \), so \( q^1 \leq r \in \mathcal{P}_n \ \delta \Rightarrow (p^0, q^3), r = (0, r) \) are compatible in \( \mathcal{Q}_2 \). As \( (p^0, q^3) \in \mathcal{Q} \) was arbitrary we are done.

(\( \zeta \)) if \( p_1, p_2 \in \mathcal{P}_4 \) are incompatible in \( \mathcal{P}_4 \text{ then } \) they are incompatible in \( \mathcal{Q} \).

Why? Look at the order of \( \mathcal{Q} \).

(\( \eta \)) if \( \zeta < \zeta ^4 \) then \( \hat{p} \) is a maximal antichain in \( \mathcal{Q} \).

Why? Let \( (p^*, q^*) \in \mathcal{Q} \) and we shall prove that it is incompatible in \( \mathcal{Q} \) with some \( (p^*, q^*) \) with \( k < \omega \). Let \( n < \omega \) be a possible value of \( n(p^*, q^*) \) so \( q^* \in \mathcal{P}_n \ \delta \) and there is a witness \( r^* \leq q^*, r^* \in \mathcal{P}_n \ \delta ^* \) for \( (p^*, q^*) \in \mathcal{Q}_2 \). By clause (\( \kappa \)) in the definition of \( t \in AP \) we know that for some \( r \in \mathcal{P}_n \ \delta ^* \) we have:

(i) \( r \in T^*_{t, \alpha, p^*} \)

(ii) \( q^*, r \) compatible in \( \mathcal{P}_n \)

As \( q^*, r \) are compatible and \( r^* \leq q^* \) also \( r^*, r \) are compatible in \( \mathcal{P}_n \ \delta ^* \), so by the demand on \( r^* \), we have: \( r, p^* \) are compatible in \( \mathcal{P}_4 \). So in clause (\( \kappa \)) of the definition of \( AP \), in the definition of \( T^*_{t, \alpha, p^*} \), for our \( r \) subclause (i) fails hence subclause (ii) holds so there are \( k, p^* = \text{subclause (ii)} \). Also let \( q^1 \in \mathcal{P}_n \ \delta \) be a common upper bound of \( q^*, r \). So \( r \) witnesses that \( (p^*, q^1) \in \mathcal{Q} \) with \( n \) a possible value of \( n(p^*, q^1) \). Clearly it is above \( (p^*, q^*) \) and above \( \hat{p} \) so we are done.

Let \( \delta ^* = \delta \). Clearly \( \mathcal{Q} \in M_5 \) and \( M_5 \models \text{"} |\mathcal{Q}| = |\delta | \text{"} \) so as \( X^* \cap \delta \ \bigcup \mathcal{P}_n \) has order type \( \delta \) and \( \mathcal{P}_4 \) is bounded in it, there is \( f \in M_5 \) such that \( f : \mathcal{Q} \rightarrow X^* \cap \delta \) is a one to one (into or even onto), extending \( \text{id}_A \cup \text{id}_B \), and define \( \mathcal{P}_4 \) such that \( f \) is an isomorphism from \( \mathcal{Q} \) onto \( \mathcal{P}_4 \). We can define \( F^*_w, F^*_{n, w, t}, (n, t < \omega) \) extending \( F^*, F^*_{n, \omega, t} \) as required, e.g., \( F^*_{n, \omega, t}((p, q)) = F^*_{n, \omega, t}(p) \) for some \( m > n \) such that \( q \in \mathcal{P}_m \) except when \( q = 0 \) then \( F^*_{n, \omega, t}(p, 0) = F^*_{n, \omega, t}(p) \). Now it is easy to check clause (\( \theta \)) of the definition of \( \mathcal{S} \in AP \). Also clauses (i), (k) hold because the construction is made in \( M_5 \) recalling (\( * \)) above since the construction is made in \( M_5 \).

Lastly, let \( \Gamma ^* = \Gamma ^4 \).
Fact C: If \( t^i \in \AP \) and \( t^i \leq^* t^{i+1} \) for \( n < \omega \) then there is \( s \) such that \( i < \omega \Rightarrow t^i \leq^* t \in \AP \) and \( \delta^i = \bigcup_{i < \omega} \delta^i \) and \( \zeta^i = \bigcup_{n < \omega} \zeta^n \).

[Why? Just let \( \delta^i, \zeta^i \) be as above, as \( E'_\omega \) is a cub of \( \omega_1 \) and \( \delta^i \in E'_\omega \) necessarily \( \delta^i \in E'_\omega \). Next, \( P^i = \bigcup_{i < \omega} P^i \), \( F^i = \bigcup_{i < \omega} F^i \), \( F^i \), \( F_{m, \omega, \ell} = \bigcup_{i < \omega} F^i \), \( P^i \), \( \bar{P}^i \), \( P^i \), \( p^i_{\ell, k} = p^i_{\ell, k} \) for every \( i \) large enough. Now check.]

**Main Fact D:** Assume \( t \in \AP \), \( \delta^i \in E'_\omega := \bigcap_{n < \omega} E_n \cap E, t \in M_{\delta(t)} \) and \( \delta(t) := \delta^i \in \bigcup_{n < \omega} S^Y_n \). Then there is \( s \in \AP \) such that

\[
\begin{align*}
(a) & \quad t \leq^* s \\
(b) & \quad \psi_{\delta(t)} \text{ is actually a wide } P^\omega \text{-name (i.e. all the countably many conditions appearing in its definition belong to } \bigcup_{n < \omega} P^\omega \cap \delta^s \subseteq P^\omega): \\
(c) & \quad \text{for any forcing notion } P^i \text{ if } P^\omega \subseteq P^i \text{ and for each } \zeta < \zeta^i \text{ the sequence } \bar{P}^i \text{ is a maximal antichain of } P^i, \text{ then } \Vdash_{\bar{P}^i} \text{ "there is } G' \subseteq Q_{\bar{P}^i} \text{ generic over } M_{\delta(t)} \text{ such that } (\eta_{\delta(t)}[G'])[G] = \psi_{\delta(t)}[G]." \\
\end{align*}
\]

**Proof.** Let \( \delta \) be such that

\[
\begin{align*}
(*)_1 & \quad (a) \quad \delta \in E'_\omega \\
& \quad (b) \quad \delta > \delta(t) \\
& \quad (c) \quad M_{\delta(t)}, \psi_{\delta(t)} \in M_{\delta} \text{ and } n(*) = \min\{n : \delta^i \in S^Y_n\} \\
& \quad (d) \quad \delta \text{ is of the old case in } Y_{n(*)} \text{ (is not really used).}
\end{align*}
\]

We stipulate the \( \theta_{Q_{\bar{P}^i}} \) depends on \( \delta(t) \) and not on the value of \( \bar{\psi}_{\delta(t)} \) so we write \( \theta_{\bar{\psi}_{\delta(t)}} \). We shall work in \( M_{\delta} \) and we shall choose \( s \in M_{\delta} \), as required with \( \delta^s = \delta \), so the real task is to choose \( P^s \),

\[
\begin{align*}
(*)_2 & \quad (a) \quad \text{let } P_{t,n} = P_n[\delta^i] \text{ for } n < \omega \\
& \quad (b) \quad P_{t,\varphi} = P^\omega \ast Q_{\varphi} \text{ in the sense of } M_{\delta(t)} \text{ when } \varphi \in M_{\delta} \text{ is a possible value of } \bar{\varphi}_{\delta(t)} \text{, so } P_{t,\varphi} \text{ belongs to } M_{\delta(t)} \\
& \quad \text{note: } \varphi \text{ slightly depends on } G_{P_{t,\varphi}} \text{ but each possible value } \in M_{\delta(t)} \\
& \quad (c) \quad P_{s,n} = P_n[\delta] \\
& \quad (d) \quad P_{s,\omega} = \bigcup\{P_{s,n} : n < \omega\}.
\end{align*}
\]

So

\[
\begin{align*}
(*)_3 & \quad (a) \quad \langle P_{t,n} : n < \omega \rangle^* (P^\omega) \in M_{\delta(t)} \\
& \quad (b) \quad \langle P_{s,\epsilon} : \epsilon \leq \omega \rangle \in M_{\delta}.
\end{align*}
\]

[Why? Clause (a) holds by “\( \delta^i \in E'_\omega \)” see \( \exists_2 \), and clause (b) holds because \( \delta \in E'_\omega \), see \( (*)_1(a) \).]

Let

\[4 \text{in fact, for some } n, \text{ all those conditions are from } P_n \cap \delta^s \]
\((*)_4\) (a) the set of elements of \(P_{t,\varphi}\) be \(A \subseteq ((X_{\omega} \cap \delta^t) \cup (X_{\omega} \cap \delta^{n_0} \cup X_n))\), \(A \in M_5\) and 
\((b)\) \(B = \bigcup_{n < \omega} P_{s,n}\).

We define a forcing notion \(Q\):

\((A)\) \(p \in Q\) if \(p\) has the form \((p_1, p_2)\) such that for some \(n, \varphi\)
\((a)\) \(p_2 \in P_{s,\omega}\)
\((b)\) \(n \in \omega(\varphi), \omega)\) is minimal such that \(p_2 \in P_{n,\delta}\) so let \(m(p) = m(p_1, p_2) = n\)
\((c)\) \(p_1 = (p_1', p_1'')\), \(p_1' \in P_i\) and one of the following holds
\(\bullet_1\) \(p_2 = \emptyset_{P_2}\), and \(p_1'' = 0_{\varphi(t)}\)
\(\bullet_2\) \(p_2\) forces \(\varphi\) is \(\varphi\) and \(p_1''\) is a \(P_i\)-name of a member of \(Q_{\varphi}\)
  in \(M_{\beta(t)}\) so \(\varphi = \varphi_{(p_1, p_2)}\)
\((d)\) if \(G \subseteq P_{n,\delta}\) is generic over \(M_4\) and \(p_2 \in G\) and \(\varphi = \varphi_{(p_1, p_2)}\)
  then there is \(G' \subseteq P_{t,\varphi}\) generic over \(M_{\beta(t)}\) such that
\(\alpha)\) \(G \cap P_{t,\omega} \subseteq G'\)
\(\beta)\) \(g_{(t)}[G'] = \nu\)
\(\gamma)\) \(p_1 \in G'\)

\((B)\) \(Q \models \langle p_1, p_2 \rangle \leq (q_1, q_2)\) if \((a) + (b)\) where
\((a)\) if \((p_1, p_2)\) satisfies \(\bullet_1\) of clause \((A)(c)\) then \(P_i^+ \models \langle p_1', \leq q_1'\rangle\)
\((b)\) if not, then \(m(p_1, p_2) \leq m(q_1, q_2)\) and \(P_i \ast Q_{\varphi} \models \langle p_1 \leq q_1, \text{ and} \rangle\)
\(P_n \models \langle p_2 \leq q_2\rangle\) for \(n\) large enough.

So

\((*)_5\) (a) \(Q\) is a forcing notion, i.e. a quasi order
\((b)\) \(Q\) belongs to \(M_5\).

[Why? Least trivial is to show transitivity of \(\leq_{Q_{\varphi}}\), we split the proof by cases and it is obvious.]

We define a function \(j\) from \(P_i\) into \(Q\):

\((*)_6\) for \(p \in P_i\) let \(j(p) = ((p, \emptyset_{\varphi(t)}, \emptyset_{P_{s,\omega}})\) where \(\emptyset_{\varphi(t)}, \emptyset_{P_{s,\omega}}\) is the minimal condition in the forcing notions \(P_{\varphi(t)}, P_{s,\omega}\) respectively

\((*)_7\) \(j\) is a \(\subseteq_{ic}\)-embedding of \(P_i\) into \(Q\).

[Why? First, clearly \(j\) is one to one. Second, for \(p \in P_i\) why \(j(p) \in Q\)? We have to check clauses \((a) - (d)\) in \((A)\) above. Now clauses \((a), (b), (c)\) are obvious, as for \((d)\), let \(n, G\) and then \(\varphi, \nu\) be as there. Note that \(\nu\) is generic for \((\eta, Q_{\varphi})\) over \(M_{\beta(t)}[G \cap P_{t,\omega}]\) and let \(P_i^t = P_i \ast G \cap P_{t,\omega}\).

We know also that \(M_{\beta(t)}[G \cap P_{t,\omega}] \models \langle Q_{\varphi} \leq (P_i \ast Q_{\varphi})/(G \cap P_{t,\omega}) = (P_i^t)/(G \cap P_{t,\omega}) \ast Q_{\varphi}\rangle\) and recalling \(Q_{\varphi}\) is absolutely c.c., moreover \(M_{\beta(t)}[G \cap P_{t,\omega}] \models \langle Q_{\varphi} \leq P_i^t \ast Q_{\varphi}\rangle\).

Now there is \(G' \subseteq Q_{\varphi} \ast M_{\beta(t)}[G \cap P_{t,\omega}]\) generic over \(M_{\beta(t)}[G \cap P_{t,\omega}]\) such that \(\eta[G'] = \nu\)
  hence there is \(G'' \subseteq P_i^t \ast Q_{\varphi}\) generic over \(M_{\beta(t)}[G \cap P_{t,\omega}]\) extending \(G'\). Together \(G''\) is as required.

Third, \(P_i \models \langle p \leq q \rangle \iff Q \models \langle j(p) \leq j(q) \rangle\) by the definitions of the order in \(Q\), i.e. clause \((B)\) above and the definition of the order in \(P_i^t = P_i \ast Q_{\varphi} \ast \varphi(t)\).
Lastly, to prove the \(C_{ic}\)-embedding, toward contradiction assume that \(p,q \in \mathbb{P}'\) are compatible in \(\mathbb{P}'\) but some \(r \in \mathbb{Q}\) is a common \(\leq_{Q}\)-upper bound of \(j(p),j(q)\), let \(r = (r_1,r_2) = (r_1',r_1''), (r_2',r_2'')\). By the definition of \(\leq_{Q}\) we have \(\mathbb{P}' \models \exists q \leq r_1'\) and \(\mathbb{P}' \models \exists q \leq r_2''\), so \(p,q \in \mathbb{P}'\) are compatible in \(\mathbb{P}'\) so we are done proving \((*)_7\).

\[(*)_8 \quad \begin{align*}
(a) \quad & j \in M_5 \\
(b) \quad & \mathbb{Q}, \text{Rang}(j) \text{ is dense in } \mathbb{Q}.
\end{align*} \]

Why clause (a)? Think.

Why clause (b)? Think. Assume that \(p \in \mathbb{P}'\) so \(j(p) = ((p,\emptyset_{\mathbb{P}'},\emptyset_{\mathbb{P}_{\omega_{\omega}}})\). Now \(\mathbb{P}_{t,n(*)} \in \mathbb{P}'\) hence there is \(p_0 \in \mathbb{P}_{t,n(*)}\) such that \(p_0 \leq p_0' \in \mathbb{P}_{t,n(*)} \Rightarrow p_0', p \text{ are compatible in } \mathbb{P}'\) and without loss of generality \(p_0' \not\in \emptyset_{\mathbb{P}_{\omega_{\omega}}}\). Let \(\varphi\) and \(p_2\) be such that \(p_2 \in \mathbb{P}_{s,n}\) is above \(p_0\) and forces \(\varphi_\delta = \varphi\). Consider the triple \(((p,\emptyset_{\varphi}),p_2)\), it suffices to prove that it belongs to \(\mathbb{Q}\), as \(p_2 \neq \emptyset_{\varphi}\) and the demand \(j(p) \leq ((p,\emptyset_{\varphi}),p_2)\) holds trivially.

Now in clause (A) above, subclauses (a),(b),(c) are obvious and as for subclause (d) we prove more than needed. Let \(n \in [n(*)\omega)\) and \(G \subseteq \mathbb{P}_{n}^\delta\) be such that \(p_2 \in G\), so as \(p_2 \in \mathbb{P}_{s,n}\) and \(p_2 \models \varphi\). As \(\varphi\) is generic over \(M_5\) and \(\mathbb{Q}_{\mathbb{P}_{\omega_{\omega}}}\), it belongs to \(\mathbb{Q}\) as \(p_2 \neq \emptyset_{\varphi}\) and the demand \(j(p) \leq ((p,\emptyset_{\varphi}),p_2)\) holds trivially.

Recall \(G \cap \mathbb{P}_{t,n}\) is a subset of \(\mathbb{P}_{t,n}\) generic over \(\mathbb{M}_{\delta(t)}\) and \(\mathbb{M}_{\delta(t)} \models \langle \mathbb{P}_{t,n} \leq \mathbb{P}' \rangle\) and \(\mathbb{P}' = (\mathbb{P}'_t \ast \mathbb{Q}_{\mathbb{P}})\mathbb{M}_{\delta(t)}\) so we know

\[(*)_8 \quad \begin{align*}
(a) \quad & \mathbb{P}' \leq \mathbb{P}'' \\
(b) \quad & (p_0,\emptyset_{\mathbb{Q}_{\mathbb{P}}}) \in \mathbb{P}' \\
(c) \quad & (p,\emptyset_{\mathbb{P}}) \in \mathbb{P}'' \\
(d) \quad & \text{if } (p_0,\emptyset_{\mathbb{Q}_{\mathbb{P}}}) \leq r \in \mathbb{P}' \text{ then } r, (p,\emptyset_{\mathbb{Q}_{\mathbb{P}}}) \text{ are compatible in } \mathbb{P}''.
\end{align*} \]

Now \(G \cap \mathbb{P}_{t,n}\) is a subset of \(\mathbb{P}_{t,n}\) generic over \(\mathbb{M}_{\delta(t)}\) and \(\nu\) is \((\emptyset_{\varphi},\mathbb{Q}_{\mathbb{P}})\)-generic over \(\mathbb{M}_{\delta(t)}\)[\(G \cap \mathbb{P}_{t,n}\)]. Hence there is \(H \subseteq \mathbb{Q}^{\mathbb{M}_{\delta(t)}[G \cap \mathbb{P}_{t,n}]}\) generic over \(\mathbb{M}_{\delta(t)}[G \cap \mathbb{P}_{t,n}]\) such that \(\emptyset_{\mathbb{P}}[H] = \nu\).

So \(H' = (G \cap \mathbb{P}_{t,n}) \ast H\) is a subset of \(\mathbb{P}'\) generic over \(\mathbb{M}_{\delta(t)}\). As \(p_0 \in G \cap \mathbb{P}_{t,n} \subseteq H'\) by \((*)_8\) there is \(G' \subseteq \mathbb{P}'\) extending \(H'\) generic over \(\mathbb{M}_{\delta(t)}\) such that \(p \in G'\) hence \(G \cap \mathbb{P}_{t,n} \subseteq G'\) and note that \(\mathbb{P}' = \mathbb{P}_{t,n}^+\) so we are done.]

\[(*)_9 \quad \text{We identify } p \in \mathbb{P}_t^+ \text{ with } j(p).\]

Next

\[\oplus_1 \quad \begin{align*}
(a) \quad & \text{let gen}(\mathbb{Q}) = \{G : G \text{ is a directed subset of } \mathbb{Q} \text{ such that } G \cap \mathbb{P}' \\
& \text{is generic over } \mathbb{M}_{\delta(t)} \text{ and } G \cap \mathbb{P}_{s,k} \text{ is generic over } M_5 \text{ for each } k < \omega\}
\end{align*} \]
\[ (b) \quad \text{gen}^+(Q) = \{ G \subseteq Q : G \text{ is directed, } G \cap s \text{ generic over } M_\delta \]
for \( n < \omega, G \cap \mathbb{P}_t \) is generic over \( M_{(t)} \), and letting \[
\bar{\varphi} = \bar{\varphi}_d[G \cap \mathbb{P}_{s,n}] \text{ for some } \nu \text{ we have}
\]
\[ \nu[G \cap \mathbb{P}_{s,n}] = \nu \text{ and } \eta[G \cap \mathbb{P}_{t,\bar{\varphi}}] = \nu \]

Now
\[ \bowtie_{1.1} \text{ note that } \text{gen}^+(Q) = \{ G \in \text{gen}(Q) : \text{letting } \varphi = \bar{\varphi}_d[G \cap \mathbb{P}_{s,n}] \text{ we have}
\]
\[ \nu[G \cap \mathbb{P}_{s,n}] = (\eta, Q_\varphi) \text{-generic for } M_{(t)}[G \cap \mathbb{P}_t, \bar{\varphi}] \}
\]
\[ \bowtie_{1.2} \text{ if } G \in \text{Gen}^+(Q) \text{ and } \bar{\varphi} = \bar{\varphi}_d[G \cap \mathbb{P}_{s,n}] \text{ then } G \cap \mathbb{P}_{t,\bar{\varphi}} \text{ is a subset of } \mathbb{P}_t^+ \]
\text{generic over } M_{(t)}.

A major point is
\[ \bowtie_{2} \text{ if } p \in Q \text{ then there is: } G \in \text{gen}^+(Q) \text{ to which } p \text{ belongs.} \]

[Why? So let \( p = (p_1, p_2), p_1 = (p'_1, p'_2) \) and by \((*)_s\) without loss of generality \( p \notin \text{Rang}(q) \) and so \( p_2 \vdash \bar{\varphi}_d = \bar{\varphi}, \) so \( n = n(p_1, p_2) = \min\{ n < \omega : n \geq n(*) \text{ and } p_2 \in \mathbb{P}_{s,n} \} \) is well defined.

Next, let \( \{ I_{1,k} : k \in [n, \omega]\} \) list the dense open subsets of \( \mathbb{P}_{t,\bar{\varphi}} \text{ from } M_{(t)} \) and let \( \{ (j_k, I_{2,k}) : k \in [n, \omega] \} \) list the pairs \( (j, I) \text{ such that } j < \omega \) and \( I \in M_{(t)} \) is a dense open subset of \( \mathbb{P}_{s,k} \text{ and without loss of generality } j_k \leq k \text{ and let } I_{2,k} = \{ q \in \mathbb{P}_{s,\omega} : q \}
\]
is above some \( q \in I_{2,k} \} \). By induction on \( k \in [n, \omega] \) we choose \( q_k \text{ such that}
\]
\[ \bowtie_{2.1} \quad \begin{cases} (a) & q_k = (q_{k,1}, q_{k,2}) = ((q'_k, q''_k), q_{k,2}) \in Q \text{ hence} \\
& \bullet q_{k,1} \in \mathbb{P}_{t,\bar{\varphi}} \\
& \bullet \text{so } q''_{k,1} \in \mathbb{P}_t, q''_{k,2} \text{ is a } \mathbb{P}_t\text{-name of a member of } Q_\varphi \\
& q_{k,2} \in \mathbb{P}_{s,k} \\
& (b) \quad \text{if } k = n \text{ then } q_k = p \\
& (c) \quad \text{if } k = m + 1 > n, \text{ then } Q \vdash \text{"} q_m \leq q_k \text{"} \\
& (d) \quad \text{if } k = m + 1 > n \text{ and } m \text{ is odd, then } q_{k,1} \in I_{1,m} \text{.} 
\end{cases} \]

Clearly if we succeed then \( \{ q \in Q : q' \leq q_k \} \) for some \( k < \omega \) is as required (noting that for every \( n \) the set \( \{ p \in \mathbb{P}_{s,n} : p \text{ forces a value to } \nu_d \mid n \} \in \{ I_{2,k} : k < \omega \} \) so it suffices to carry the induction.

For \( k = n \) there is nothing to do.

Let \( k = m + 1 > n \) and assume \( q_m \) has already been chosen. Let \( G_m \) be a subset of \( \mathbb{P}_{s,m} \text{ generic over } M_s \) such that \( q_m \in G_m \)
\[ \bowtie_{2.2} \quad M_{(t)}[G_m] \models \text{"the forcing notion } ((G_{m,2}/(G_m \cap \mathbb{P}_{t,m})) \ast Q_\varphi)M_{(t)}(G_m \cap \mathbb{P}_{t,m}) \text{ is a complete subforcing } (\mathbb{P}_t/(G_m \cap \mathbb{P}_{t,m})) \ast Q_\varphi \text{ generic over } M_{(t)}[G_m \cap \mathbb{P}_{t,m}]. \]

Let \( \nu : = \nu_d[G_m] = \nu_d[G_m \cap \mathbb{P}_{s,n}] \).

Recalling \( q_m \in Q \), there is \( G'_m \text{ such that} \]
\[ \bowtie_{2.3} \quad G'_m \text{ is a subset of } \mathbb{P}_{t,\bar{\varphi}} = (\mathbb{P}_t \ast Q_\varphi)M_{(t)} \text{ generic over } M_{(t)} \text{ such that}
\]
\[ \eta[G'_m] = \nu, G_m \cap \mathbb{P}_{t,m} \subseteq G'_m \text{ and } q_{m,s} \in G'_m. \]

So there is \( q_{k,1} \)
\[ \bowtie_{2.4} \quad q_{k,1} = (q'_k, q''_k) \in G'_m \text{ is above } q_{m,1} \text{ and belongs to } G'_m. \]
By $\oplus_{2.2}$ we have

$$
\oplus_{2.5} \quad (\mathbb{P}_{t.k} \ast Q_{P})^{M_{t}(\delta)} \cap G'_{m} \text{ is a subset of } (\mathbb{P}_{t,k} \ast Q_{P})^{M_{\delta}(\delta)} \text{ generic over } \mathbb{P}_{t}(\delta) \text{ and still for it } \eta_{P} \text{ is interpreted as } \nu.
$$

Hence there is a condition $(r'_{1}, r'_{2}) \in (\mathbb{P}_{t,k} \ast Q_{P})^{M_{t}(\delta)} \cap G'_{m}$ such that

$$
\oplus_{2.6} \quad (a) \quad r'_{1} \in \mathbb{P}_{t,k}/(G_{m} \cap \mathbb{P}_{t.m})
$$

(b) if $r'_{1} \leq r \in \mathbb{P}_{t,k}$ then $r, q_{k,1}$ are compatible in $\mathbb{P}'$

moreover

$$
\oplus_{2.7} \quad (a) \quad (r'_{1}, r'_{2}) \in (\mathbb{P}_{t,k} \ast Q_{P})^{M_{t}(\delta)}
$$

(b) if $(r'_{1}, r'_{2}) \leq (r''_{1}, r''_{2}) \in (\mathbb{P}_{t,k} \ast Q_{P})^{M_{t}(\delta)}$ then $(r''_{1}, r''_{2}), (q'_{k,1}, q''_{k,1})$ are compatible in $((\mathbb{P}'/(G_{m} \cap \mathbb{P}_{t,m})) \ast Q_{P})^{M_{t}(\delta)[G_{m}]}$.

So noting $G_{k}$ is directed and $r'_{1}, q_{m,2} \in G_{k}$ clearly there is $q_{k,2}$ such that

$$
\oplus_{2.8} \quad (a) \quad q_{k,2} \in G_{k} \subseteq \mathbb{P}_{s,k}
$$

(b) $r'_{1} \leq q_{k,2}$ in $\mathbb{P}_{s,k}$

(c) $q_{m,2} \leq q$ in $\mathbb{P}_{s,k}$

(d) $q_{k,2} \in \mathbb{I}_{s,m}^{+}$

(e) $q_{k,2} \upharpoonright \mathbb{P}_{s,k}$ satisfies $r''_{1,n}$.

We next show that

$$
\oplus_{2.9} \quad \text{qk} = (q_{k,1}, q_{k,2}) \text{ is as required.}
$$

Now why does clause $\oplus_{2.1}(a)$, i.e. why $q_{k} \in Q$? Now in the definition (A) of $Q$ clauses (a),(b),(c) obviously holds, and if we succeed to prove (d) there then clauses (c),(d) of $\oplus_{2.1}$ are obvious as $\oplus_{2.4}$ and $\oplus_{2.8}(d)$ respectively and clause (b) of $\oplus_{2.1}$ holds emptily, so it is enough to prove (d) of (A).

So why does (A)(d) hold? Let $G_{k} \subseteq \mathbb{P}_{s,k}$ be generic over $\mathbb{M}_{\delta}$ such that $q_{k,2} \in G_{k}$ and let $\nu = \nu_{q}[G_{k}]$. Now we work in $\mathbb{M}_{\delta}[G_{k}]$, so $M_{\delta(\delta)}[G_{k} \cap \mathbb{P}_{t,k}]$ is a generic extension of $M_{\delta(\delta)}$, both are countable and $q'_{k,1}([G_{k} \cap \mathbb{P}_{t,k}])$ is a member of $Q_{P}^{M_{\delta}(\delta)[G_{k} \cap \mathbb{P}_{t,k}]}$ which $\nu$ satisfies, i.e. there is a generic subset $H$ of $Q_{P}^{M_{\delta}(\delta)[G_{k} \cap \mathbb{P}_{t,k}]}$ to which it belongs and $\eta_{P}[H_{1}] = \nu$.

So $H^{+} = (G_{k} \cap \mathbb{P}_{t,k}) \ast H$ is a subset of $(\mathbb{P}_{t,k} \ast Q_{P})^{M_{\delta}(\delta)}$ generic over $M_{\delta(\delta)}$. Recalling $\oplus_{2.2} + \oplus_{2.7}$ there is a subset of $\mathbb{P}_{t,k} = (\mathbb{P}' \ast Q_{\mathbb{P}})\mathbb{P}_{t}(\delta)$ generic over $M_{\delta(\delta)}$ which extends $H^{+}$. So clearly $G_{k}^{'} = H' \cap \mathbb{P}_{t,k}$ is as required in (A)(d), so indeed $q_{k} \in Q$ so indeed we have carried the induction, but as said after $\oplus_{2.1}$ it suffices to carry the induction for proving $\oplus_{2}$ so we are done proving $\oplus_{2}$.]

$$
\oplus_{3} \quad \text{We identify p \in P'} \text{ with j(p) and q \in P}_{s,\omega} \text{ with } (\emptyset, q) \text{ pedantically replace Q by the quasi order } Q' \text{ with set of elements } \{p: p \in P' \text{ or } p = (P', q), q \in Q\} \text{ and Q' } |= \text{ "r}_1 \leq \text{ r}_2" \text{ if } \mathbb{P}' \models \text{ "r}_1 \leq \text{ r}_2" \text{ or } r_1 \in \mathbb{P}', r_2 = (\mathbb{P}', q_2), Q \models \text{ "j(r)_1 \leq j(q)_2" or } r_1 = (\mathbb{P}', q_1), r_2 = (\mathbb{P}', q_2), Q \models \text{ "q}_1 \leq \text{ q}_2".
$$

$$
\oplus_{4} \quad \text{in } \mathbb{M}_{\delta} \text{ let } Q_{\omega,1} \text{ be the forcing generated freely in } \mathbb{M}_{\delta} \text{ for our requirements (see [Shf. Ch.IX]), i.e.} \text{ (a) let } \Omega \text{ be the set of } \psi \in \mathbb{M}_{\delta} \text{ such that: } \psi \text{ is a propositional } \mathbb{P}_{s,\omega}-\text{sentence generated by the set } Q \text{ of propositional variables}
$$
(b) for \( \psi \in \Omega \) and \( G \subseteq \Omega \) we define the truth value \( \psi[G] \) by induction of \( \psi \) such that: if \( \psi \in Q \) then \( \psi[G] = \text{truth} \iff \psi \in G \) the connecting act as usual.

(c) let \( Q^+ \) set of elements be the set of \( \psi \in \Omega \) such that there is \( G \in \gen^+(Q) \) such that the truth value \( \psi[G] \) is true, i.e. \( G \) is a model of \( \psi \).

(d) \( Q^t = \psi_1 \leq \psi_2 \iff (\forall G \in \gen(Q)) [\text{if } \psi_2[G] \text{ is true then also } \psi_1[G] \text{ is true}]. \)

[Why? Because there is no condition \( ((p_1', p_1''), p_2) \in Q \) that could force the opposite.]

\( \square_{2.1} \)

**Discussion 2.3.** On the one hand \( M_{\Omega(t)} \) know \( \mathbb{P}_{t,n} = \mathbb{P}_n | \delta(t) \) is countable but have the commitment about \( \psi_{\delta(t)} \). On the other hand \( M_{\delta} \) knows \( \mathbb{P}_{s,n-1} \) but have no commitment (generic old).

So \( M_{\delta} \models \text{“} Q \text{ is countable”} \) and for \( Q^t \) we do not have to think about a commitment in \( \delta \).

Now

\( (\star)_1 Q_+ \in M_{\delta}. \)

[Why? Formally as \( M_{\delta} \models \text{“} \mathcal{P}(\omega) \text{ exist”} \), the class of all relevant sentences exist and is of cardinality continuum in the sense of \( M \) exist. Think of \( M_{\delta}^{\text{Levy}(\aleph_0, 2^{\aleph_0})} \) and use absoluteness (the danger is that the ordinal depth \( > M_{\delta} \cap \text{Ord} \), but by the above this does not occur). So definition of \( Q_+ \) in \( M_{\delta} \) is \( \{ \psi: \text{as above } + \models_{\text{Levy}(\aleph_0, 2^{\aleph_0})} \text{ “there is } G \} \}. \]

Next

\( (\star)_2 M_{\delta} \models \text{“} \mathbb{P}_{s,n} \preceq Q_+ \text{”}. \)

[Why? Let \( I \in M_{\delta} \) be a predense subset of \( \mathbb{P}_{s,n} \) and let \( \psi \in Q_+ \). So there is \( G \in \gen^+(Q) \) such that \( \psi[G] = \text{true} \); of course \( G \in \gen(Q) \). By the definition of \( G \in \gen(Q) \) necessarily \( G \cap I \neq \emptyset \); let \( p \in G \cap I \), and let \( \psi_1 = \psi \land p \), so \( \psi_1 \in G \) hence \( \psi_1 \in Q_+ \). Also obviously \( Q_+ \models \text{“} \psi \leq \psi_1 \text{”} \) and \( Q_+ \models \text{“} p \leq \psi_1 \text{”} \) so \( \psi, r \) are compatible in \( Q_+ \), so we are done.]

\( (\star)_3 M_{\delta} \models \text{“} \text{if } I \in M_{\delta(t)} \text{ is predense in } \mathbb{P}^t \text{ then } I \text{ is predense in } Q_+ \text{”}. \)

[Why? Similarly.]

\( (\star)_4 Q \subseteq_{\text{inc}} Q_+ \) (ignoring separability) moreover \( \models_{Q^+} \text{ “} G_{Q_+} \cap Q \text{ is directed”}. \)

[Why? Assume \( p, q \in Q, \psi \in Q^+, Q^+ \models \text{“} p \leq \psi \land q \leq \psi' \text{”} \) and we shall find a common upper bound \( r \in Q \) of \( p, q \) which is compatible with \( \psi \) in \( Q^+ \), this clearly suffices. As \( \psi \in Q^+ \) there is \( G \in \Gen^+(Q) \) such that \( \psi[G] = \text{true} \). As \( Q^+ \models \text{“} p \leq \psi \land q \leq \psi' \text{”} \) clearly \( p, q \in G \). Let \( n < \omega \) be large enough such that \( p_2, q_2 \in \mathbb{P}_{r,n} \) and \( n(*) \leq n \). We continue as in the proof of \( \oplus_{2.2} \).

\( (\star)_5 (a) \) if \( \psi_1, \psi_2 \in Q_+ \) then \( \psi_1 \land \psi_2 \) is the \( \leq_{Q_+} \)-lub of \( \psi_1, \psi_2, \psi_1 \land \psi_2 \in Q_+ \).

\( (b) \) \( \psi_1, \psi_2 \in Q_+ \) are incompatible in \( Q_+ \) if \( \psi_1 \land \psi_2 \notin Q_+ \).

\( (c) \) if \( \psi, \neg \psi \in Q_+ \) then they are incompatible and every \( \psi \in Q_+ \) is compatible with at least one of them.
Now all should be clear.

By (669) if $G \subseteq Q_\ast$ is generic over $M_\delta$ then for $\psi \in Q_\ast$ we have $\psi \in G \iff \psi|G \cap Q_\ast = \text{truth}.$

[Why? Obvious.]

(*)if $G \subseteq Q_\ast$ is generic over $M_\delta$ then for $\psi \in Q_\ast$ we have $\psi \in G \iff \psi|G \cap Q_\ast = \text{truth}.$

[Why? We prove this by induction on $\psi$.]

Case 1: $\psi \in Q_\ast$. Obvious.

Case 2: $\psi = -\varphi$
By (669)(c), (d).

Case 3: $\psi = \bigwedge_{i<\alpha} \psi_i$ so $M_\delta \models \lceil |\alpha| = \aleph_0 \rceil$
By (669)(c).

(*)if $\psi \in Q_\ast$ is generic over $M_\delta$ then $G' := G \cap Q \in \text{Gen}(Q)$.

[Why? First $G'(\subseteq Q_\ast)$ is directed by (669)(a).
Second, if $Q \models \lceil p \leq q \rceil$ then $Q' \models \lceil p \leq q \rceil$ then $Q' \models p \in G' \iff p \in G'$ so $G'$ is downward directed. Third, if $n < \omega$ and $I \subseteq M_\delta, I \subseteq P_{s,n}$ is predense then $I \cap G' \neq \emptyset$ by (669)(b). Fourth, $G' \cap P^4$ is generic over $M_{\delta(t)}$ by (669)(d).]

(*)let $\psi_* \in \Omega$ says that $G_{\delta(t)}[G \cap P_{s,n}]$ is $(\rho,\varphi, Q_\varphi)$-generic over $M_{\delta(t)}[G \cap P^4]$ when $\varphi = \bar{\varphi}_G[G \cap P_{s,n}].$

(*) $\psi_* \in Q_\ast.$

[Why? First, $\rho_* \varphi_* \eta_* \xi_*$ are defined by $\aleph_0$ maximal antichain. Second, $\text{Gen}^+(Q) \neq \emptyset$ and $\text{Gen}^+(Q) \subseteq \text{Gen}(Q).$]

(*)let $Q' = Q_\ast[\{ \psi : \psi_* \leq \varphi, \psi \}].$

Now all should be clear.

Lastly

(*)if $G$ is a subset of $Q^+$ generic over $M_\delta$ then $G' := G \cap Q \in \text{Gen}^+(Q)$.

[Why? By (669)(a), (669)(b), (669)(c).]

We define $s := (\delta^*, P^4, \Gamma^*, F^\delta, F^\delta_{n,\omega,\xi})_{t<\omega}$ as follows:

\begin{enumerate}
  \item[(a)] $\delta^*$ is $\delta$, chosen in (669)(a) above
  \item[(b)] $P^4$ is $Q^+$ defined in (669)(b) above
  \item[(c)] $\Gamma^* = \Gamma^t \cup \{ \bar{p} \in M_\delta : \bar{p} \text{ is a sequence listing a predense subset of } Q^+ \}$
  \item[(d)] $F^\delta_\ast$ is a two-place function from $Q$ to $Q$ such that
    \begin{itemize}
      \item $F^\delta_\ast(p, q)$ is $F^\delta_{n,\omega,\xi}(|p, q|)$ if $p, q \in \mathbb{P}^4$
      \item $F^\delta_{n,\omega,\xi}(p, q) = F^\delta_{n,\omega,\xi}(p, q)$ when $p, q \in P_{s,n}$.\end{itemize}
\end{enumerate}
\[ F_n^*(p, q) \text{ if } p, q \in \mathbb{Q} \text{ is a common } \leq_\mathbb{Q} \text{-upper bound of } p, q \text{ if } (p, q \in \mathbb{Q}) \text{ and } \{p, q\} \not\subseteq \mathcal{P}^t, \bigwedge_{n \in \omega} \{p, q\} \not\subseteq \mathcal{P}_{s, n} \text{ and there is a common } \leq_\mathbb{Q} \text{-upper bound} \]

(note: the first and second case are not contradictory by clause (γ) of \( \mathbb{H}_3 \), the definite of AP)

\( c \) \( F_n^* \) is a function from \( \mathbb{Q} \) to \( \mathbb{Q} \) as defined for \( n < \omega < \ell < \omega \) such that

\[ F_n^*(p) = F_i^*(p) \text{ if } p \in \mathcal{P}^t \]

\[ F_n^*(p) = F_n^*(p) \text{ if } n < m < \omega, p \in \mathcal{P}_{s, m} \]

\[ \text{if } p \in \mathbb{Q} \setminus \mathcal{P}^t \cup \{p_{t, n} : n < \omega\} \text{ and } n < \omega \text{ then } (F_n^*(p) : \ell < \omega) \text{ is a sequence of members of } \mathcal{P}_{s, n} \text{ as in clause (} c \text{) of } \mathbb{H}_3. \]

No problem we can choose such \( s \). Now we have to check that \( s \) is as required

\[ \ominus_6 s \in \text{AP.} \]

[Why? Check the clauses in \( \mathbb{H}_3 \).]

\[ \ominus_7 t \leq^* s, \text{ i.e. clause (a) of Fact D.} \]

[Why? Check the clauses in \( \mathbb{H}_3 \).]

\[ \ominus_8 \nu_{s(t)} \text{ a wide } \mathcal{P}^*\text{-name.} \]

[Why? As \( \nu_{s(t)} \) is a \( \mathcal{P}_{s, n} \)-name and \( \{\mathcal{P}_{s, n}, \nu_{s(t)}\} \in M \) and clause (c) of (\( s \))_7.]

\[ \ominus_9 \text{ clause (c) of Fact D.} \]

[Why? By absoluteness from \( M_\delta \) to \( v \).]

**Fact E:** If in Main fact D, \( \mathbb{Q}_{\nu_3}^\sharp \) is a singleton (hence \( \nu_3 \in M_\delta \) so the main fact is trivial) then there is \( s \in \text{AP} \) such that \( t \leq^* s \) and \( p \in M_\delta \) is an \( \omega \)-sequence listing a predense subset of \( \mathcal{P}^t \) then \( p \) appears in the sequence \( \Gamma^* \). [Why? Easy.] So we can choose \( t_c \in \text{AP} \) by induction on \( \varepsilon < \omega_1 \) such that \( t_c \) is \( \leq^* \)-increasing continuous, \( \delta^{\varepsilon + 1} > \delta^\varepsilon \), and if \( t_c \in M_{\langle \delta^\varepsilon \rangle}, \delta^\varepsilon \in \bigcap_{n < \omega} E_n \cap E \cap \bigcup_{n < \omega} S^n \) then \( t_{\varepsilon + 1} \) is gotten by Fact D. No problem to carry this (\( \varepsilon = 0 \) by Fact A, \( \varepsilon = \varepsilon + 1 \) by Fact D if possible and by Fact B if not; lastly, if \( \varepsilon \) is a limit ordinal, use Fact C).

Now let \( \mathbb{P}_{\omega} = \bigcup_{\varepsilon < \omega_1} \mathbb{P}_{\varepsilon} \) and \( \mathcal{Y}_{\omega} \) has already been defined; now check the requirements.

**Definition 2.4.** Let \( \check{C}^* = \langle C^*_\delta : \delta < \omega_2, \text{ a limit ordinal} \rangle \) (and \( C^*_\emptyset = \emptyset \) otherwise) be a square sequence and \( \check{X}^* = \langle X^*_i : i < \omega_1 \rangle \) be an increasing sequence of subsets of \( \omega_1 \setminus I \setminus X^*_i \), \( i < \omega_1 \).

Let \( \alpha < \omega_2 \). We say that \( \langle \langle \mathbb{P}_i, \mathcal{Y}_i, \mathcal{M}_i \rangle : i < \alpha \rangle \) is a \( (\check{C}^*, \check{X}^*)\)-iteration (we omit \( \mathcal{M}^t \) and write \( \langle \mathcal{M}_i, \check{C}^*, \check{X}^* \rangle\)-iteration if \( i < \alpha < \omega_2 \Rightarrow \langle \mathcal{M}_i \rangle \leq \mathcal{M}^t \) or an \( \mathcal{M}\)-iteration when \( \check{C}^*, \check{X}^* \) are clear from context) when:

\[ \text{(a) } \langle \mathbb{P}_i, \mathcal{Y}_i, \mathcal{M}_i \rangle \in IS \text{ is } <^*\text{-increasing and } \text{Dom}(\mathcal{M}_i) = \mathcal{S}^\mathcal{Y}_i. \]
(b) \(f_i\) is a one to one function from \(P_i\) onto \(X^*_\text{otp}(C^*_j)\), and let \((P'_i, Y'_i)\) be such that \(f_i\) maps \((P_i, Y_i)\) to \((P'_i, Y'_i)\).

(c) if \(j \in \text{acc}(C^*_i)\) then \(f_j \subseteq f_i\).

(d) if \(c\ell(i) = 0_0\) and \(i = \sup \text{acc}(C^*_i)\) then \((P'_i, Y'_i)\) is gotten from \((P'_j, Y'_j) : j \in \text{acc}(C^*_i)\) as in the proof of 2.11.1 (for our \(M_i\) using \(X^*_j : j \in \text{acc}(C^*_i)\)), \(X^*_\text{otp}(C^*_i)\) instead of \((X_n : n < \omega), X_\omega\) so \(\text{acc}(C^*_i)\) replace \(\omega\) and we generate \(f_i = (t'_n : \alpha < \omega)\) and by it define \((P'_i, Y'_i)\) hence can choose \(f_i, (P_i, Y_i)\).

(e) in clause (d), further assume:

- \(\delta = \text{otp}(C^*_i), j^* \in C^*_i\) minimal such that \(\delta \in \text{Dom}(\tilde{M}_{j^*})\).
- \(M = M^*_j, (\tilde{\varphi}, \eta, \nu) = (\tilde{\varphi}, \tilde{\eta}, \tilde{\nu})[\nu_i^j]\).
- if \(j \in C^*_i \setminus j^*\) then \(\triangledown_{P_j^{C^*_i}} \varphi\) \(\text{is (Q^2, \eta_0)-generic over } M[G_{P_j} \cap \delta]\)
- \((\nu_i^j) \upharpoonright \delta : j \in \text{acc}(C^*_i) \in M\) and for \(j_1 < j_2\) from \(\text{acc}(C^*_i) \setminus \nu_i^j\), the ordinal \(\delta\) belongs to the club \(\alpha \in \omega : \alpha \text{ closed under the functions } F_{j_1}\) and \(F_{j_1,j_2,\ell}\) (see clause (f) below).
- Let \(t'_\ell \in AP\) be defined (using i instead of \(\omega\)) by \(\delta'^{t'_\ell} = \text{otp}(C^*_i), \nu_i^j = \cup(\nu_i^j \upharpoonright \delta : j \in \text{acc}(C^*_i)), F^{t'_\ell} = \cup(F_{j_1} \upharpoonright \delta \in \text{acc}(C^*_i)), F_{j_1,j_2,\ell}^{t'_\ell} = \cup(F_{j_1,j_2,\ell} \upharpoonright \delta : j_2 \in \text{acc}(C^*_i) \setminus j_1)\) and let \(\Gamma^{t'_\ell}\) be empty. Then (recalling \(t_i\) is from (d))
  \[
  \begin{align*}
  (\alpha) & \quad t_0 = t'_\ell, M^*_j = M, (\tilde{\varphi}, \tilde{\eta}, \tilde{\nu})[\nu_i^j] = (\tilde{\varphi}, \tilde{\eta}, \tilde{\nu}) \\
  (\beta) & \quad \nu_i^j \text{ is a (Q^2, } \eta)\text{-generic over } M[G_{P_j} \cap \delta]
  \end{align*}
  \]

(f) \(F_j\) is a (partial) two-place function from \(X^*_\text{otp}(C^*_j)\) to itself such that \(F^j(p, q)\) is the \(<\)-first common upper bound of \(p\) and \(q\) in \(P'_j\) and if \(j_1 \in \text{acc}(C^*_j)\) then we have \(F_{j_1,j_2,\ell}(p) : \ell < \omega\) is a maximal antichain of \(P'_{j_1}\) satisfying: for each \(\ell\), either \(F_{j_1,j_2,\ell}(p)\) is incompatible with \(p\) in \(P'_j\) or \(p\) is compatible with \(r\) in \(P'_j\) wherever \(P'_j \cap \delta\) replace \(P'_j, (\tilde{\varphi}, \tilde{\eta}, \tilde{\nu})\) replace \(\varphi, \eta, \nu\). Then

\[
\{12.7\}
\]

Claim 2.5. (Iteration at limit) Assume \((P_i, Y_i, f_i) : i < \zeta\) is a \((\mathcal{M}, C^*, X^*)\)-iteration where \(\zeta < \omega_2\) is a limit ordinal. Then

(a) we can find \((P_\zeta, Y_\zeta, f_\zeta)\) such that \((P_\zeta, Y_\zeta, f_\zeta) : i < \zeta + 1\) is an \(\tilde{M}\)-iteration

(b) if \(S \subseteq S^\zeta, i < \zeta \Rightarrow S^{\zeta^0} \subseteq S \text{ mod } D_M\), then we can demand \(S^{\zeta^0} = S\).

Remark 2.6. A reader may ask: in the proof when \(i = \sup \text{acc}(C^*_i)\), \(c\ell(i) = 0_0\). Why can we fulfill clause (e) of 2.4 when relevant? That is, letting \(j_\delta, \delta, M = M^*_j, (\tilde{\varphi}, \tilde{\eta}, \tilde{\nu})\) be as in clause (e) already \(P_i^{C^*_i}\) is determined so we have no freedom left, so how can we fulfill the obligation concerning \(\nu\)?

The answer is that we should look at the definition of satisfaction of \(\iota\)-commitment in 1.11, so for notational simplicity consider \(\iota = 0, i.e. 1.11(1)A\) and assume \(P \subseteq \omega_1\) and \(b\) is the identity and \(E\) as there. For \(\alpha \in E\) note that clause (e) of 1.11(1)A says that \(\triangledown_{P^{C^*_i}} \varphi\) is a subset of \(\mathcal{P}_\alpha := \mathcal{P} \cap \alpha\) generic over \(M_\alpha\). Note that \(\mathcal{P}_\alpha \in M_\alpha\) because \(\alpha \in E\) and \(M_\alpha \models \{ \alpha = 0_0\} \text{ hence } M_\alpha \models \{ \mathcal{P}_\alpha \text{ is countable}\} .\)

So \(M_\alpha[G_{P}]\) is a Cohen extension of \(M_\alpha\) (or is equal to \(M_\alpha\) in a degenerate case). So how can we satisfy the demand \(\triangledown_{\mathcal{P}_\alpha} \varphi\) is \((Q^2, \eta)\)-generic over \(M_\alpha[G_{P}]\)? This is not problematic but is not you may say in the next stage; see more in §4 but we shall elaborate. So in clause (e) of 2.4 letting \(i = \zeta\) we have \(j_\delta, \delta, \mu, \tilde{\varphi}, \tilde{\eta}, \tilde{\nu}, Q^2, t_i^\delta\)
as there let $P'_\varepsilon = f_{j_\varepsilon}(P_j)$ for $j \in C'_\varepsilon$ when $\varepsilon = \operatorname{otp}(j \cap C'_\varepsilon)$ and $\varepsilon = \operatorname{otp}(j \cap C'_\varepsilon)$ and let $\varepsilon_{\ast\ast} = \operatorname{otp}(C'_\varepsilon)$. Similarly let $\varphi = f_{j_\varepsilon}(\varphi')$ so $\varphi'$ is a $P_{\varphi\varepsilon}$-name, hence is a countable object using members of $P_{\varphi\varepsilon}$ only.

We choose $\delta_\varepsilon > \delta$ from $S[\chi_{\varepsilon_\beta_\varepsilon}, \delta]$ such that $M'_{\delta_\varepsilon} := M_{\delta_\varepsilon}^{\varphi'}$ is well defined, $(M, (P'_\varepsilon : \varepsilon < \varepsilon_{\ast\ast}) \in M_\ast$ and $(P'_\varepsilon : \varepsilon < \varepsilon_{\ast\ast})$ and $\varepsilon_\varepsilon$ generic over $M_\ast$. Of course, we can demand $\varepsilon_{\ast\ast} = \delta$.

Now in $M_\ast$ we define $P = \cup\{P'_\varepsilon : \varepsilon < \varepsilon_{\ast\ast}\}$. So $M_\ast \models \text{``$P$ is a countable forcing notion union of the $\varepsilon$-increasing sequence $(P'_\varepsilon : \varepsilon < \varepsilon_{\ast\ast})$, $M$ a countable transitive model $(P'_\varepsilon : \varepsilon < \varepsilon_{\ast\ast})$ is $\varepsilon$-increasing, if $\varepsilon < \varepsilon_{\ast\ast}$ then every predense subset $I$ of $P'_\varepsilon$ and $\varepsilon_{\ast\ast}$ is $\varepsilon_{\ast\ast}$ generic over $M(G_{P_{\varphi\varepsilon}\delta})$ when $\varepsilon \in [\varepsilon, \varepsilon_{\ast\ast}]$}.

Now we choose $P''_\varepsilon \in M_\ast$ such that $M_\ast \models \text{``$P$ is an $\varepsilon$-extension of $P * Q^2 / (\eta = \nu)$ and of each $P'_\varepsilon | \delta_\varepsilon, \varepsilon < \varepsilon_{\ast\ast}$'', as in the proof of Fact D. Lastly, $t_0 = t'_\varepsilon, \delta_\varepsilon = \delta_\varepsilon$ and continue the proof of Fact D to choose $t_\varepsilon$.

Proof. If cf($\zeta$) = $\aleph_0$ we use 2.1 but taking care of clause (e) of Definition 2.4, this just dictates to us how to start the induction there (as is done by “Main Fact D” from inside the proof of 2.1). Note that if $\zeta > \sup \text{acc}(C'_{\varepsilon})$ we still use 2.1, just our work is easier as we do not have to take care of clause (e). If cf($\zeta$) = $\aleph_1$, then by the square bookkeeping (see clause (e) in Definition 2.4) our work is done (using $f_{\xi} = \cup\{f_{\xi} : \xi \in \text{acc}(C'_{\varepsilon})\}$).

Claim 2.7. 1) Assume

(a) $\mathcal{Y} = (S, \Phi, \eta, \varphi)$ is a 1-commitment on the forcing notion $P \in \mathcal{H}(N_2)$ for $\tilde{M}$, an oracle with domain $S_{\tilde{M}}$

(b) $G_\varepsilon \subseteq P$ is generic over $V, \varphi_0 = (\nu_0 : \alpha \in S)$ where $\nu_0 = \nu_0[G_\varepsilon], \tilde{M}^1 = M[G_{\varphi_0}] = (M_{\delta_\varepsilon}[\nu_0[G_\varepsilon]] : \delta \in S_\varepsilon)$ for some one to one function $f$ from $P$ into $\omega_1$

(c) in $V[G_{\varphi_0}], Y_1 = (S^1, \Phi_1, \eta_1, \varphi_1)$ is a 0-commitment, $S \subseteq S^1$ mod $\mathcal{D}(M[G_{\varphi_0}], \Phi_1 \upharpoonright (S \cap S^1) = \Phi \upharpoonright (S \cap S^1), \eta_1 \upharpoonright (S \cap S^1), \varphi_1 \upharpoonright (S \cap S^1) = \varphi_0 \upharpoonright (S \cap S^1)$ and $(S^1, \Phi_1, \eta_1, \varphi_1) \in V$

(d) in $V[G_{\varphi_0}], Q$ is a forcing notion satisfying the 0-commitment $Y_1$ for $\tilde{M}^1$.

Then for some $P$-name $Q$ and 1-commitment $Y^2$ we have:

(a) $(P, Y, M) \leq^* (P * Q, Y^2, M)$

(b) $S^2 = S^1, \Phi^2 = \Phi_1, \eta^2 = \eta_1, \varphi^2 = \varphi_1, \nu_0[G_{\varphi_0}] = \nu^1$

(c) $Q[G_{\varphi_0}] = Q$

2) If for every $G_{\varepsilon} \subseteq P$ generic over $V$ there are $Q$ satisfying some $\psi_1$ and $(S^1, \Phi_1, \eta_1, \varphi_1) \in V[G_{\varphi_0}]$ as above satisfying some $\psi_2$, then we can demand

(d) $\models Q[G_{\varphi_0}], Y^2 as above satisfy $\psi_1, \psi_2$ respectively$.$

3) We may allow $((\Phi_\alpha, \eta_\alpha) : \alpha \in S^1)$ be a sequence of $P$-names and even $(P * Q)$-names.

Proof. Straight. □
Definition 2.8. For a pair \((\varphi, \eta)\) as in Definition 1.11, we say \(Z \subseteq \omega^2\) is positive for \((\varphi, \eta)\) when: for \(\chi\) large enough, the set \(\{N : N \prec (H(\chi), \in)\}\) is countable, \((\varphi, \eta) \in N\) and there is \(\nu \in Z\) which is \((N, Q^\varphi, \eta)\)-generic is stationary, equivalently not empty.

Claim 2.9. (iteration in successor case: increasing the commitment).

Assume
\[
\langle (P_i, Y_i, f_i) : i < \zeta \rangle \text{ is an } \bar{M}\text{-iteration and } \zeta = \xi + 1, S^{\bar{Y}_\xi} \subseteq S \subseteq S^*, S \subseteq \text{Dom}(\bar{M})
\]
(b) \((\bar{\varphi}_\alpha, \eta_\alpha) : \alpha \in S\setminus S^{\bar{Y}_\xi}\) is as required in Definition 1.11.

Lastly
\[
\text{Then we can find } (\bar{P}_\xi, Y_\xi, f_\xi) \text{ such that}
\]
(i) \((\bar{P}_\xi, Y_\xi, f_\xi) : i < \zeta + 1) \text{ is an } \bar{M}\text{-iteration}
(ii) \(P_\xi = \bar{P}_\xi, S^{Y_\xi} = S\),
(iii) \((\bar{\varphi}^{Y_\xi}, \bar{\eta}^{Y_\xi}) = (\bar{\varphi}_\alpha, \eta_\alpha)\) and \(\Vdash \bar{\varphi}^{Y_\xi} \in Z_\alpha\) when \(\alpha \in S\setminus S^{\bar{Y}_\xi}\).

Proof. Straight. \(\square\)

Claim 2.10. (iteration at successor: increasing the forcing)

Suppose
\[(a) \text{ assume in claim 2.9}
\]
(b) \(Q\) is a \(\mathbb{P}\)-name satisfying, for every \(G \subseteq \mathbb{P}\) generic over \(V\), the following:
(i) \(Q[G] \subseteq \mathbb{P}\) is a c.c.c. forcing notion of cardinality \(\aleph_1\)
(ii) \(\delta \in S_M : \mathbb{P} \upharpoonright \delta \in M_\delta\) and \(G \cap M_\delta\) is a generic subset of \(R^{\delta}_\varphi\) \(\bar{Q}_{\bar{\varphi}}/\bar{\eta}_\delta = \nu_\delta\) over \(M_\delta[\nu_\delta]\) which \(D_M\)-almost always occurs and \(Q[G] \upharpoonright \delta \in M_\delta\) and there is \(\mathbb{R}^* \in M_\delta\) such that \(R^{\delta}_\varphi \prec \mathbb{R}^*\) and \(R^{\delta}_\varphi\) \(\bar{Q}_{\bar{\varphi}}/\bar{\eta}_\delta\) \(\mathbb{P} \upharpoonright Q\) \(\in D_M[G]\).

Then we can find \((\bar{P}^+, Y^+)\) such that \((\mathbb{P}, Y) \leq^* (\bar{P}^+, Y^+) \in IS\) and the \(\mathbb{P}\)-name \(\mathbb{P}^+/\bar{G}_\varphi\) is equivalent to \(Q\).

Proof. Straight.

Now we draw an easy conclusion: consistently \(2^{\aleph_0} = \aleph_2\) and for each ideal defined naturally by \((\varphi, \eta)\) where \(\varphi\) absolutely defines a c.c.c. forcing \(Q_\varphi\) and \(\eta\) is a wide name for a read then for some set of \(\aleph_1\) reals is positive for this ideal and more. \(\square\)
§ 3. Conclusions

12.10 Conclusion 3.1. Assume \(((\hat{C}^*, \hat{X}^*))\) is as in 2.4. Let \(\Phi\) be a set of definitions of forcing notions with some real parameters, and \((S^*_i : i < \omega_2)\) is as in 1.8 for \(\mathcal{D}_M^i\) for some \(M\).

We can find \(\langle (P_i, Y_i, f_i, M^i) : i < \omega_2 \rangle\) such that

(a) it is an \((\hat{C}^*, \hat{X}^*)\)-iteration (for this we have to allow \(\alpha = \omega_2\) in Definition 2.4)

(b) \(P = \{P_i : i < \omega_2\}\) is a c.c.c. forcing notion of cardinality \(\aleph_2\) (so in \(V^P, 2^{\aleph_0} \leq \aleph_2\) and except in degenerated cases equality holds)

(c) \(S_{\leq i}^i = S^*_i\) from 1.8(3)

(d) if in \(V^P\) we have \((\tilde{\varphi}, \eta)\) is a case of \(\Phi\) as in 1.11, moreover \(\models_{P_i} \{\delta \in S_{\leq i+1}^i \setminus S^*_i : M^i_{\delta+1}[f^i_{\delta}(G^i_{\delta})]\} = \{\tilde{\varphi}, \eta\}\) as required in 1.11\(^{\ast}\}) \in \mathcal{D}_M^{i+1}\) (even less with more bookkeeping) and \(Z \subseteq (\langle 2 \rangle)^{\omega}\) is positive for \((\tilde{\varphi}, \eta), Y\), then

(α) \(\{\delta \in S_{\leq i+1}^i \setminus S^*_i : \langle \tilde{\varphi}_\delta, \eta_\delta \rangle/G^i_{\delta} = \langle \tilde{\varphi}, \eta \rangle\) and \(\nu_\delta[G^i_{\delta}] \in Z\) \(\in \mathcal{D}_M^{i+1}\): in fact the set is forced to include such old set (from \(V\)) by this we can get

(β) for some \(j > i, \delta \in S_{\leq j+1}^j \setminus S^*_j\) \(\Rightarrow \langle \tilde{\varphi}_\delta, \eta_\delta \rangle = \langle \tilde{\varphi}, \eta \rangle, \nu_\delta[G^i_{\delta}] \in Z\)

(e) if \(H\) is a pregiven function such that for every \(i < \omega_2\) and \((P, Y, M)\) satisfying the demands on \((P_i, Y_i, M): \) we have \((P, Y_M) \leq^* H(P, Y, M) \in IS\) such that \(H(P, Y, M)\) satisfies the demands from (a) + (c) on \((P_{j+1}, Y_{j+1}, M^{j+1})\), then we can demand \((\exists \delta > j)(\langle P_{j+1}, Y_{j+1}, M^{j+1} \rangle = H(P, Y, M))\); moreover, if \(S_H \subseteq \{\delta < \omega_2 : cf(\delta) = \aleph_1\} \) is stationary we can demand \(\{j < SH : \langle P_{j+1}, Y_{j+1}, M^{j+1} \rangle = H(P, Y, M)\}\) is stationary.

(Or course, we can promise this for \(\aleph_2\) such functions)

(f) similarly for \(S_H \subseteq \{\delta < \omega_2 : cf(\delta) = \aleph_1, \delta \notin \cup\{C_i^*: i\} \) but the domain is a sequence \((\langle P^n, Y^n, M^n \rangle : n < \omega), S_{\leq i}^n = S_{\leq i}^i, \langle i_n : n < \omega \rangle\) increases to \(i\).

Proof. Put together the previous claims. (Concerning clause (f) without loss of generality \(\{i < \omega_1 : otp(C^*_i) = 0\} \) is stationary) so in those stages we have no influence of clause (f) of 2.4: anyhow the influence of 2.4(f) is minor. \(\square\)

12.11 Discussion 3.2. We discuss here some possible extensions. We can add a version of the conclusion without the oracles, etc.

12.10a Claim 3.3. Assume \((S_i : i < \omega_2)\) is a sequence of pairwise almost disjoint stationary subsets of \(\omega_1\), each with diamond and \(i < j \Rightarrow S_i \subseteq S_j^+ \mod \mathcal{D}_{\omega_1}, \) so \(S^+_i \subseteq \omega_1\) and \(S_i \cap S^+_i = \emptyset\) and \((S^+_i/\mathcal{D}_{\omega_1} : i < \omega_2)\) is increasing

Then in the following game between the bookkeeper and the forcer, the bookkeeper has a winning strategy.

A play lasts \(\omega_2\) moves, before the \(\alpha\)-th move a sequence \(\langle (P_i, Q_i, M^i, Y_i) : i < \alpha \rangle\) is defined such that

(a) \(P_i\) a c.c.c. forcing notion of cardinality \(\aleph_1\), say \(\subseteq \mathcal{H}_{\leq i}(\aleph_2)\)

(b) \(Q_i\) is a \(P_i\)-name of a forcing notion of cardinality \(\leq \aleph_1\), say \(\subseteq \omega_1\)

(c) \(P_i\) is \(\leq^\ast\)-increasing

(d) \(P_{i+1}, P_i * Q_i\) are isomorphic over \(P_i\)

(e) \(M^i d\) is a \(P_i\)-name of an \(\aleph_1\)-oracle
(f) \( \mathcal{Y} \) is a \( \mathbb{P}_1 \)-name of an \( S_r \)-commitment.

In the \( i \)-th move:

(a) the bookkeeper chooses \( \mathbb{P}_i \) and a \( \mathbb{P}_i \)-name \( (\bar{N}^i, \mathcal{Y}^i) \) of an \( S^+_r \)-oracle and \( 0 \)-commitment

(b) the forcer chooses \( \mathcal{Q}_i \) and \( (\bar{M}^i, \mathcal{Y}_i) \), \( \mathbb{P}_i \)-names such that \( \mathcal{Q}_i \) satisfies \( (\bar{N}^i, \mathcal{Y}^i) \)
and \( (\bar{M}^i, \mathcal{Y}_i) \).

In the end the bookkeeper wins if

\[ i < j < \omega_2 \Rightarrow \mathbb{P}_j / \mathbb{P}_i \text{ satisfies } (\bar{M}^i, \mathcal{Y}_i). \]

Proof. Similar to earlier proofs. \( \square \)

We give an easy criterion for existence of forcing notion satisfying a given \( 0 \)-commitment and a (not complete) sub-forcing of given nicely definable one. The following uses more from [Sh:630].

Claim 3.4. Assume

(a) \( (\mathbb{P}, \leq, \leq_n)_{n<\omega} \) is a definition of a forcing notion satisfying condition \( A \) of Baumgartner with \( \leq_n \) as witness and \( ZFC^* \) says this, in a way preserved by suitable forcing

(b) \( \mathcal{Y} = (S, \Phi, \eta, \nu) \) is a \( 0 \)-commitment, so \( \tilde{\Phi} = \langle \tilde{\phi}_\alpha : \alpha \in S \rangle \)

(c) \( \mathbb{P} \) is absolutely nep such that for each \( \alpha \in S^\mathcal{Y} \) it is \( \leq_n \)-purely \( I_{Q^{\phi^\mathbb{P}}_\alpha} \)-preserving, i.e.

\[ (\ast) \text{ if } M \text{ is a } \mathbb{P} \text{-candidate and a } Q^{\phi^\mathbb{P}}_\alpha \text{-candidate, } p \in \mathbb{P}^M, n < \omega \text{ and } q \in (Q^{\phi^\mathbb{P}}_\alpha)^M \text{ then for some } p', \eta, \nu \text{ we have } p \leq_n p' \in \mathbb{P}, p' \text{ is } (M, \mathbb{P}) \text{-generic and } \nu \text{ is } (Q^{\phi^\mathbb{P}}_\alpha, \eta_\alpha) \text{-generic over } M \text{ satisfying } q \text{ (see [Sh:630])} \]
and \( p' \Vdash_P "\nu \text{ is } (Q^{\phi^\mathbb{P}}_\alpha, \eta_\alpha) \text{-generic over } M (G_\mathbb{P} \cap P^M)". \]

Then there is a c.c.c. forcing notion \( \mathbb{P}' \subseteq \mathbb{P} \) (not necessarily \( \mathbb{P}' \prec \mathbb{P} \)) satisfying the \( 0 \)-commitment \( \mathcal{Y} \) such that \( \Vdash_{\mathbb{P}'} "\text{for a club of } \delta < \omega_1, \phi^\mathbb{P} \text{, } \nu_{\tilde{\phi}_\delta}\". \)

Remark 3.5. 1) Why the \( \phi^\mathbb{P} \)'s? We hope it helps, for example in the following; suppose we are given \( f : \mathbb{R} \rightarrow \mathbb{R} \), we like to force \( A \subseteq \mathbb{R} \) which is not in \( I_{Q^{\phi^\mathbb{P}}_\alpha} \), see Definition in [Sh:630], and on which the function \( f \) is continuous; i.e. to force a continuous \( f^* \) such that \( \{ \eta \in "2 : f^*(\eta) = f(\eta)\} \in (I^{\phi^\mathbb{P}}_\delta)^+ \), see Definition in [Sh:630]. So not only do we like to find \( q \Vdash "\eta_{\tilde{\phi}} \text{ is } (Q_\delta, \eta_{\tilde{\phi}}) \text{-generic over } M_\delta(G_\mathbb{P})" \)
but also \( q \Vdash_P "\text{name } f^*(\eta_{\tilde{\phi}}) = f(\eta_{\tilde{\phi}})". \) This is what \( \phi \) says.

Proof. We choose by induction on \( \alpha < \omega_1 \), a pair \( (P_\alpha, \Gamma_\alpha) \) such that:

(a) \( P_\alpha \subseteq \mathbb{P} \) is countable

(b) \( \Gamma_\alpha \) is a countable family of predense subsets of \( P_\alpha \)

(c) if \( I \in \Gamma_\alpha \) and \( p \in P_\alpha \) and \( n < \omega \) then for some \( q \) we have \( p \leq_n q \in P_\alpha \) and \( I \) is predense above \( q \) in \( P \)

(d) \( P_\alpha \) is increasing continuous in \( \alpha \)

(e) \( \Gamma_\alpha \) is increasing continuous in \( \alpha \).
Case 1: $\alpha = 0$.
Trivial.

Case 2: $\alpha = \beta + 1$, $\beta$ non-limit or $(P_\beta, \Gamma_\beta) \notin M_\beta$.
Let $(P_\alpha, \Gamma_\alpha) = (P_\beta, \Gamma_\beta)$.

Case 3: $\alpha$ limit.
Let $(P_\alpha, \Gamma_\alpha) = (\bigcup_{\beta<\alpha} P_\beta, \bigcup_{\beta<\alpha} \Gamma_\beta)$.

Case 4: $\alpha = \delta + 1$ where $\delta$ is a limit ordinal and $(P_\delta, \Gamma_\delta) \in M_\delta$.
We can find $g \subseteq \text{Levy}(\aleph_0, |P|)^{M_\delta}$, generic over $M_\delta$ such that $\eta_\delta^*$ is still $Q_\delta$-generic over $M_\delta[g]$ (see [Sh:630], §6).

In $M_\delta[g]$ we define $P_\delta^+ = \{ p : M_\delta[g] \models p \in P \text{ and } I \in \Gamma_\alpha \Rightarrow I \text{ predense above } p' \}$
using the induction hypothesis, as in $M_\delta[g]$ the set $\Gamma_\delta$ is countable, so:

$(\ast)$ for every $p \in P_\delta$ and $n < \omega$ there is $p' \in P_\delta^+$ such that $P \models p \leq_n p'$.

Again by [Sh:630], §6 for every $n < \omega$ and $p \in P_\delta^+$, there is $q_{p,n} \in P_\delta$ such that $p \leq_n q_{p,n} \in P_\delta$, $q_{p,n} \in P$, $q_{p,n}$ is $(M_\delta[g], Q)$-generic and $q_{p,n} \Vdash \nu_\delta$ is a $(Q_\delta, \eta_\delta)$-generic real over $M_\delta[g][G_p]$.

Let $P_{\delta+1} = P_\delta \cup \{ q_{p,n} : p \in P_\delta^+ \text{ and } n < \omega \}$ and $\Gamma_{\delta+1} = \Gamma_\delta \cup \{ I_\delta \}$ where $I_\delta = \{ q_{p,n} : p \in P_\delta^+ \text{ and } n < \omega \}$. $\square_{3,4}$
§ 4. Relatives and closing remarks

The comments below try to connect [Sh:669], [Sh:895], [Sh:F699], [Sh:F1000]. A related work is [Sh:895], the main difference is that there the continuum is forced to be $\lambda^{+}$ and normally $\lambda > N_1$. Now if there is $\lambda = N_1$, still the construction is more general. Here, e.g.

(*): if the forcing notion $P$ satisfies the $\omega$-obligation $V$ then $\Vdash \langle \omega \rangle^V$ is non-meagre$^\dagger$.

[Why? Without loss of generality $|P| \leq N_1$ hence without loss of generality $P \subseteq \omega_1$.]

So if $B$ is a $P$-name of countable union of no-where-dense subsets of $\omega_2$ then for some $\delta \in S\gamma$ we have $B \in M_\delta$ and $P_\delta := P \cap \delta \in M_\delta$ and $\Vdash \langle G_\delta \cap \delta \rangle$ is a subset of $P_\delta$ which is generic over $M_\delta$.

So $\Vdash G_\delta \cap \delta$ is generic for $\langle P_\delta, M_\delta \rangle$ hence $\langle \omega \rangle^M_\delta$ is non-meagre in $M_\delta[G_\delta \cap \delta]$ hence has member $\rho \notin B^\dagger$.

By absoluteness $\Vdash \langle \omega \rangle^V \notin B^{V[P_m]}$, so we are done.

Now [Sh:895], for the case $\lambda = N_1$, this proof does not work. In fact we can avoid this but it requires some care.

We now describe one version of [Sh:895].

Definition 4.1. 1) A 2-commitment base is a sequence $p = \langle (P_\delta, <_\delta) : \delta \in S \rangle$ such that

- $S \subseteq \omega_1$ is stationary,
- $\leq_\delta$ is a well ordering of $\delta$
- for $\delta \in S$, $\bar{P}_\delta = \langle (P_\delta, >_\delta) : \gamma < \gamma_\delta \rangle$ is a $<_\delta$-increasing sequence of forcing notions
- $P_{\delta, \gamma}$ is $\subseteq_\delta$.

2) Above $p$ is positive for the normal filter $D$ on $\omega_1$ when:

- if $\langle P_\delta, >_\delta \rangle$ is a $<_\delta$-increasing sequence of c.c.c. forcing notion with a set of elements $\subseteq \alpha(\gamma), \alpha(\gamma) < \omega_1$ and $\bar{u} = \langle u_\alpha : \alpha < \omega_1 \rangle$ is $<_\delta$-increasing continuous sequence of countable sets of ordinals $< \alpha(\gamma)$ with union $\alpha(\gamma)$ the following set $\in D^+$

$$S_{p, u} = \{ \delta \in S_p : \text{otp}(u_\delta) \leq \text{otp}(\delta, <_{p, \delta}) \text{ and letting } h_\delta \text{ be the unique order preserving function from } u_\delta \text{ onto an initial segment of } (\delta, <_{p, \delta}) \text{ then } \gamma \in u_\delta \cap \gamma(\delta) \Rightarrow h''(P_\gamma \cap u_\delta) = P_{p, \delta, h(\gamma)} \rangle.$$  

3) A 2-commitment is a pair $(p, q)$ such that

- $p = \langle (P_\delta, <_\delta) : \delta \in S \rangle$ is a 2-commitment base
- $q = \langle P_{\delta}^+ : \delta \in S \rangle$
- $\bar{P}_{\delta}^+ = \langle P_{\delta, >_\delta}^+ : \gamma < \gamma(\delta) \rangle$
- $P_{\delta, \gamma} \leq P_{\delta, >_\delta}^+$ for $\gamma < \delta \in S$.

4) Let $\gamma(*) < \omega_2$. We say then $\langle P_\gamma : \gamma < \gamma(*) \rangle$ satisfies $(p, q, D)$ when
• $\mathcal{D}$ a normal filter on $\omega_1$ such that $S_p \in \mathcal{D}$

• if $\langle u_\alpha : \alpha < \omega_1 \rangle$ is as above then for some $W \in \mathcal{D}$ for every $\delta \in W \cap S_p$ letting $h_\delta$ be as there then for some function $h \supseteq h_\delta, \gamma \in \gamma(*) \cap u_\delta \Rightarrow \text{bb} P_{q,\delta,\gamma} \subseteq \text{Rang}(h) \cap h^{-1}(P_{q,\delta,\gamma}) = P_{\gamma}$.

**Remark 4.2.**
1) This definition fits [Sh:F699].
2) Replacing $\aleph_0$ by $\mu$-see [Sh:F1000].
3) We can translate to proof here is a winning strategy $\text{st}_\delta$ is [Sh:895] frame.
4) We may replace the sequence $P_\delta$ by a tree, then we can demand success on a club (or member of the filter). Necessary if we like to demand “no $\delta$ such that $G$ is generic for $(\mathbb{P} \cap \delta, M_\delta), M_\delta \models \text{"}|\delta| = \aleph_0"$”.
5) A natural question, continuing §3 is: for transparency assume $V \models \text{GCH}$ and for $\ell = 1, 2$ let $Q_\ell$ be a set of pairs $(\bar{\varphi}, \eta)$ so $\bar{\varphi}$ from $V$, we ask: is there a generic extension $V^P$ of $V$ such that:

- if $(\bar{\varphi}, \eta) \in Q_1$ then there is a $(\bar{\varphi}, \eta)$-positive set of cardinal $\aleph_0$
- if $(\bar{\varphi}, \eta) \in Q_2$ then there is no $(\bar{\varphi}, \eta)$-positive set of cardinal $\aleph_1$.

We surely can phrase sufficient conditions, but can we phrase sufficient and necessary conditions?
6) More complicated when we have $\bar{\varphi}$ above, so, i.e. allow real parameter in $\bar{\varphi}$, so we will have $2^{\aleph_0}$ such cases rather than $\aleph_1$. 
§ 5. Private Appendix

Question on $\oplus_5$ in the proof of Fact D, from pg.18:

Martin and Wolfgang would like to know if $Q^*$ is necessary. They think that every filter $G \subseteq Q$ which is $Q$-generic over $M_\delta$ will satisfy the following (and therefore belong to $\text{gen}^+(Q)$):

- $G \cap P^{+}_{t,\bar{\psi}}$ is generic over $M_\delta(t)$ (the same holds for $G \cap P^t$)
- $G \cap P_{s,n}$ is generic over $M_\delta$ (for each $n < \omega$)
- $\nu_\delta(G \cap P\{x,n(\ast)\} = \eta(G \cap P^r_{t,\bar{\psi}})$.

Discussion 5.1. (based on 2010.3.18 remark) after coments of Hecke

1) In Definition 1.11(2) in $(\ast)_2(d)$ should we separate for obligations of kind 1.
2) In the proof of $(\ast)_8$ in Main Fact D inside the proof of 2.1, the paragraph on “why clause (b)?” Heike thought $n$ not $n(\ast)$.

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