ON REFLECTION OF STATIONARY SETS IN $\mathcal{P}_\kappa\lambda$

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Abstract. Let κ be an inaccessible cardinal, and let $E_0 = \{x \in \mathcal{P}_\kappa\kappa^+ : \text{cf } \lambda_x = \text{cf } \kappa_x\}$ and $E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is regular and } \lambda_x = \kappa_x^+\}$. It is consistent that the set E_1 is stationary and that every stationary subset of E_0 reflects at almost every $a \in E_1$.

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1. Introduction.

We study reflection properties of stationary sets in the space $\mathcal{P}_\kappa\lambda$ where κ is an inaccessible cardinal. Let κ be a regular uncountable cardinal, and let $A \supseteq \kappa$. The set $\mathcal{P}_\kappa A$ consists of all $x \subset A$ such that $|x| < \kappa$. Following [3], a set $C \subseteq \mathcal{P}_\kappa A$ is *closed unbounded* if it is $\subseteq$-cofinal and closed under unions of chains of length $< \kappa$; $S \subseteq \mathcal{P}_\kappa A$ is *stationary* if it has nonempty intersection with every closed unbounded set. Closed unbounded sets generate a normal κ-complete filter, and we use the phrase "almost all x" to mean all $x \in \mathcal{P}_\kappa A$ except for a nonstationary set.

Almost all $x \in \mathcal{P}_\kappa A$ have the property that $x \cap \kappa$ is an ordinal. Throughout this paper we consider only such x's, and denote $x \cap \kappa = \kappa_x$. If κ is inaccessible then for almost all x, κ_x is a limit cardinal (and we consider only such x's.) By [5], the closed unbounded filter on $\mathcal{P}_\kappa A$ is generated by the sets $C_F = \{x : x \cap \kappa \in \kappa$ and $F(x^{<\omega}) \subseteq x\}$ where F ranges over functions $F : A^{<\omega} \to A$. It follows that a set $S \subseteq \mathcal{P}_\kappa A$ is stationary if and only if every model M with universe $\supseteq A$ has a submodel N such that $|N| < \kappa$, $N \cap \kappa \in \kappa$ and $N \cap A \in S$. In most applications, A is identified with $|A|$, and so we consider $\mathcal{P}_\kappa\lambda$ where λ is a cardinal, $\lambda > \kappa$. For $x \in \mathcal{P}_\kappa\lambda$ we denote λ_x the order type of x.

We are concerned with *reflection* of stationary sets. Reflection properties of stationary sets of ordinals have been extensively studied, starting with [7]. So have been reflection principles for stationary sets in $\mathcal{P}_{\omega_1}\lambda$, following [2]. In this paper we concentrate on $\mathcal{P}_\kappa\lambda$ where κ is inaccessible.

Definition. Let κ be an inaccessible and let $a \in \mathcal{P}_\kappa\lambda$ be such that κ_a is a regular uncountable cardinal. A stationary set $S \subseteq \mathcal{P}_\kappa\lambda$ *reflects* at a if the set $S \cap \mathcal{P}_{\kappa_a} a$ is a stationary set in $\mathcal{P}_{\kappa_a} a$.

The question underlying our investigation is to what extent can stationary sets reflect. There are some limitations associated with cofinalities. For instance, let S and T be stationary subsets of λ such that every $\alpha \in S$ has cofinality ω, every

$\gamma \in T$ has cofinality ω_1, and for each $\gamma \in T$, $S \cap \gamma$ is a nonstationary subset of γ (cf. [4]). Let $\widehat{S} = \{x \in \mathcal{P}_\kappa\lambda : \sup x \in S\}$ and $\widehat{T} = \{a \in \mathcal{P}_\kappa\lambda : \sup a \in T\}$. Then $\widehat{S}$ does not reflect at any $a \in \widehat{T}$.

Let us consider the case when $\lambda = \kappa^+$. As the example presented above indicates, reflection will generally fail when dealing with the x's for which $\mathrm{cf}\,\lambda_x < \kappa_x$, and so we restrict ourselves to the (stationary) set

$$\{x \in \mathcal{P}_\kappa\lambda \ : \ \mathrm{cf}\,\kappa_x \le \mathrm{cf}\,\lambda_x\}$$

Since $\lambda = \kappa^+$, we have $\lambda_x \le \kappa_x^+$ for almost all x.

Let

$$E_0 = \{x \in \mathcal{P}_\kappa\kappa^+ \ : \ \kappa_x \text{ is a limit cardinal and } \mathrm{cf}\,\kappa_x = \mathrm{cf}\,\lambda_x\}\,,$$

$$E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ \ : \ \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\}\,.$$

The set E_0 is stationary, and if κ is a large cardinal (e.g. κ^+-supercompact) then E_1 is stationary; the statement "E_1 is stationary" is itself a large cardinal property (cf. [1]). Moreover, E_0 reflects at almost every $a \in E_1$ and consequently, reflection of stationary subsets of E_0 at elements of E_1 is a prototype of the phenomena we propose to investigate.

Below we prove the following theorem:

1.2. Theorem. *Let κ be a supercompact cardinal. There is a generic extension in which*

(a) *the set $E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x$ is inaccessible and $\lambda_x = \kappa_x^+\}$ is stationary, and*

(b) *for every stationary set $S \subseteq E_0$, the set $\{a \in E_1 : S \cap \mathcal{P}_{\kappa_a} a$ is nonstationary in $\mathcal{P}_{\kappa_a} a\}$ is nonstationary.*

A large cardinal assumption in Theorem 1.2 is necessary. As mentioned above, (a) itself has large cardinal consequences. Moreover, (b) implies reflection of stationary subsets of the set $\{\alpha < \kappa^+ : \mathrm{cf}\,\alpha < \kappa\}$, which is also known to be strong (consistency-wise).

2. Preliminaries.

We shall first state several results that we shall use in the proof of Theorem 1.2.

We begin with a theorem of Laver that shows that supercompact cardinals have a $\diamondsuit$-like property:

2.1. Theorem. [6] *If κ is supercompact then there is a function $f : \kappa \to V_\kappa$ such that for every x there exists an elementary embedding $j : V \to M$ with critical point κ such that j witnesses a prescribed degree of supercompactness and $(j(f))(\kappa) = x$.*

We say that the function f has *Laver's property.*

2.2. Definition. A forcing notion is $<\kappa$*-strategically closed* if for every condition p, player I has a winning strategy in the following game of length κ: Players I and II take turns to play a descending κ-sequence of conditions $p_0 > p_1 > \cdots > p_\xi > \cdots, \xi < \kappa$, with $p > p_0$, such that player I moves at limit stages. Player I wins if for each limit $\lambda < \kappa$, the sequence $\{p_\xi\}_{\xi<\lambda}$ has a lower bound.

It is well known that forcing with a $<\kappa$-strategically closed notion of forcing does not add new sequences of length $<\kappa$, and that every iteration, with $<\kappa$-support, of $<\kappa$-strategically closed forcing notions is $<\kappa$-strategically closed.

2.3. Definition. [8] A forcing notion satisfies the $<\kappa$*-strategic-κ^+-chain condition* if for every limit ordinal $\lambda < \kappa$, player I has a winning strategy in the following game of length λ:

Players I and II take turns to play, simultaneously for each $\alpha < \kappa^+$ of cofinality κ, descending λ-sequences of conditions $p_0^\alpha > p_1^\alpha > \cdots > p_\xi^\alpha > \cdots, \xi < \lambda$, with player II moving first and player I moving at limit stages. In addition, player I chooses, at stage ξ, a closed unbounded set $E_\xi \subset \kappa^+$ and a function f_ξ such that for each $\alpha < \kappa^+$ of cofinality κ, $f_\xi(\alpha) < \alpha$.

Player I wins if for each limit $\eta < \lambda$, each sequence $\langle p_\xi^\alpha : \xi < \eta\rangle$ has a lower bound, and if the following holds: for all $\alpha, \beta \in \bigcap_{\xi<\lambda} E_\xi$, if $f_\xi(\alpha) = f_\xi(\beta)$ for all

$\xi < \lambda$, then the sequences $\langle p^\alpha_\xi : \xi < \lambda\rangle$ and $\langle p^\beta_\xi : \xi < \lambda\rangle$ have a common lower bound.

It is clear that property (2.3) implies the κ^+-chain condition. It is proved in [8] that every iteration with $< \kappa$-support, of $< \kappa$-strategically κ^+-c.c. forcing notions satisfies the $< \kappa$-strategic κ^+-chain condition.

In Lemmas 2.4 and 2.5 below, $H(\lambda)$ denotes the set of all sets hereditarily of cardinality $< \lambda$.

2.4. Lemma. *Let S be a stationary subset of E_0. For every set u there exist a regular $\lambda > \kappa^+$, an elementary submodel N of $\langle H(\lambda), \in, \Delta, u\rangle$ (where Δ is a well ordering of $H(\lambda)$) such that $N \cap \kappa^+ \in S$, and a sequence $\langle N_\alpha : \alpha < \delta\rangle$ of submodels of N such that $|N_\alpha| < \kappa$ for every α, $N \cap \kappa^+ = \bigcup_{\alpha<\delta}(N_\alpha \cap \kappa^+)$ and for all $\beta < \delta$, $\langle N_\alpha : \alpha < \beta\rangle \in N$.*

Proof. Let $\mu > \kappa^+$ be such that $u \in H(\mu)$, and let $\lambda = (2^\mu)^+$; let Δ be a well ordering of $H(\lambda)$. There exists an elementary submodel N of $\langle H(\lambda), \in, \Delta\rangle$ containing u, S and $\langle H(\mu), \in, \Delta \restriction H(\mu)\rangle$ such that $N \cap \kappa^+ \in S$ and $N \cap \kappa$ is a strong limit cardinal; let $a = N \cap \kappa^+$.

Let $\delta = \text{cf}\ \kappa_a$. As $a \in S$, we have $\text{cf}\,(\sup a) = \delta$, and let γ_α, $\alpha < \delta$, be an increasing sequence of ordinals in $a - \kappa$, cofinal in $\sup a$. Let $\langle f_\alpha : \kappa \le \alpha < \kappa^+\rangle \in N$ be such that each f_α is a one-to-one function of α onto κ. (Thus for each $\alpha \in a$, f_α maps $a \cap \alpha$ onto κ_a.) There exists an increasing sequence β_α, $\alpha < \delta$, of ordinals cofinal in κ_a, such that for each $\xi < \alpha$, $f_{\gamma_\alpha}(\gamma_\xi) < \beta_\alpha$.

For each $\alpha < \delta$, let N_α be the Skolem hull of $\beta_\alpha \cup \{\gamma_\alpha\}$ in $\langle H(\mu), \in, \Delta \restriction H(\mu), \langle f_\alpha\rangle\rangle$. N_α is an elementary submodel of $H(\mu)$ of cardinality $< \kappa_a$, and $N_\alpha \in N$. Also, if $\xi < \alpha$ then $\gamma_\xi \in N_\alpha$ (because $f_{\gamma_\alpha}(\gamma_\xi) < \beta_\alpha$) and so $N_\xi \subseteq N_\alpha$.

As $N \cap \kappa$ is a strong limit cardinal, it follows that for all $\beta < \delta$, $\langle N_\alpha : \alpha < \beta\rangle \in N$. Also, $N_\alpha \subseteq N$ for all $\alpha < \delta$, and it remains to prove that $a \subseteq \bigcup_{\alpha<\delta} N_\alpha$.

As $\sup\{\beta_\alpha : \alpha < \delta\} = \kappa_a$, we have $\kappa_a \subseteq \bigcup_{\alpha<\delta} N_\alpha$. If $\gamma \in a$, there exists a $\xi < \alpha < \delta$ such that $\gamma < \gamma_\xi$ and $f_{\gamma_\xi}(\gamma) < \beta_\alpha$. Then $\gamma_\xi \in N_\alpha$ and so $\gamma \in N_\alpha$.

2.5. Lemma. *Let S be a stationary subset of E_0 and let P be a $<\kappa$-strategically closed notion of forcing. Then S remains stationary in V^P.*

Proof. Let $\dot{C}$ be a P-name for a club set in $\mathcal{P}_\kappa\kappa^+$, and let $p_0 \in P$. We look for a $p \le p_0$ that forces $S \cap \dot{C} \ne \emptyset$.

Let σ be a winning strategy for I in the game (2.2). By Lemma 2.4 there exist a regular $\lambda > \kappa^+$, an elementary submodel N of $\langle H(\lambda), \epsilon, \Delta, P, p_0, \sigma, S, \dot{C}\rangle$ (where Δ is a well-ordering) such that $|N| < \kappa$ and $N \cap \kappa^+ \in S$, and a sequence $\langle N_\alpha : \alpha < \delta\rangle$ of submodels of N such that $|N_\alpha| < \kappa$ for every α, $N \cap \kappa^+ = \bigcup_{\alpha<\delta}(N_\alpha \cap \kappa^+)$ and for all $\beta < \delta$, $\langle N_\alpha : \alpha < \beta\rangle \in N$.

We construct a descending sequence of conditions $\langle p_\alpha : \alpha < \delta\rangle$ below p_0 such that for all $\beta < \delta$, $\langle p_\alpha : \alpha < \beta\rangle \in N$: at each limit stage α we apply the strategy σ to get p_α; at each $\alpha + 1$ let $q \le p_\alpha$ be the Δ-least condition such that for some $M_\alpha \in \mathcal{P}_\kappa\kappa^+ \cap N$, $M_\alpha \supseteq N_\alpha \cap \kappa^+$, $M_\alpha \supseteq \bigcup_{\beta<\alpha} M_\beta$ and $q \Vdash M_\alpha \in \dot{C}$ (and let M_α be the Δ-least such M_α), and then apply σ to get $p_{\alpha+1}$. Since $M_\alpha \in N$, $N \models |M_\alpha| < \kappa$ and so $M_\alpha \subseteq N$; hence $M_\alpha \subseteq N \cap \kappa^+$. Since for all $\beta < \delta$, $\langle N_\alpha : \alpha < \beta\rangle \in N$, the construction can be carried out inside N so that for each $\beta < \delta$, $\langle p_\alpha : \alpha < \beta\rangle \in N$.

As I wins the game, let p be a lower bound for $\langle p_\alpha : \alpha < \delta\rangle$; p forces that $\dot{C} \cap (N \cap \kappa^+)$ is unbounded in $N \cap \kappa^+$ and hence $N \cap \kappa^+ \in \dot{C}$. Hence $p \Vdash S \cap \dot{C} \ne \emptyset$.

□

3. The forcing.

We shall now describe the forcing construction that yields Theorem 1.2. Let κ be a supercompact cardinal.

The forcing P has two parts, $P = P_\kappa * \dot{P}^\kappa$, where P_κ is the *preparation forcing* and P^κ is the *main iteration.* The preparation forcing is an iteration of length κ, with Easton support, defined as follows: Let $f : \kappa \to V_\kappa$ be a function with Laver's property. If $\gamma < \kappa$ and if $P_\kappa \restriction \gamma$ is the iteration up to γ, then the γ^{th} iterand $\dot{Q}_\gamma$ is trivial unless γ is inaccessible and $f(\gamma)$ is a $P_\kappa \restriction \gamma$-name for a $< \gamma$-strategically closed forcing notion, in which case $\dot{Q}_\gamma = f(\gamma)$ and $P_{\gamma+1} = P_\gamma * \dot{Q}_\gamma$. Standard forcing arguments show that κ remains inaccessible in V^{P_κ} and all cardinals and cofinalities above κ are preserved.

The main iteration $\dot{P}^\kappa$ is an iteration in V^{P_κ}, of length $2^{(\kappa^+)}$, with $< \kappa$-support. We will show that each iterand $\dot{Q}_\gamma$ is $< \kappa$-strategically closed and satisfies the $< \kappa$-strategic κ^+-chain condition. This guarantees that $\dot{P}^\kappa$ is (in V^{P_κ}) $< \kappa$-strategically closed and satisfies the κ^+-chain condition, therefore adds no bounded subsets of κ and preserves all cardinals and cofinalities.

Each iterand of $\dot{P}^\kappa$ is a forcing notion $\dot{Q}_\gamma = Q(\dot{S})$ associated with a stationary set $\dot{S} \subseteq \mathcal{P}_\kappa\kappa^+$ in $V^{P_\kappa * \dot{P}_\kappa \restriction \gamma}$, to be defined below. By the usual bookkeeping method we ensure that for every P-name $\dot{S}$ for a stationary set, some $\dot{Q}_\gamma$ is $Q(\dot{S})$.

Below we define the forcing notion $Q(S)$ for every stationary set $S \subseteq E_0$; if S is not a stationary subset of E_0 then $Q(S)$ is the trivial forcing. If S is a stationary subset of E_0 then a generic for $Q(S)$ produces a closed unbounded set $C \subseteq \mathcal{P}_\kappa\kappa^+$ such that for every $a \in E_1 \cap C$, $S \cap \mathcal{P}_{\kappa_a} a$ is stationary in $\mathcal{P}_{\kappa_a} a$. Since $\dot{P}^\kappa$ does not add bounded subsets of κ, the forcing $Q(\dot{S})$ guarantees that in V^P, $\dot{S}$ reflects at almost every $a \in E_1$. The crucial step in the proof will be to show that the set E_1 remains stationary in V^P.

To define the forcing notion $Q(S)$ we use certain models with universe in $\mathcal{P}_\kappa\kappa^+$. We first specify what models we use:

3.1. Definition. A *model* is a structure $\langle M, \pi, \rho\rangle$ such that

(i) $M \in \mathcal{P}_\kappa\kappa^+$; $M \cap \kappa = \kappa_M$ is an ordinal and $\lambda_M =$ the order type of M is at most $|\kappa_M|^+$

(ii) π is a two-place function; $\pi(\alpha, \beta)$ is defined for all $\alpha \in M - \kappa$ and $\beta \in M \cap \alpha$. For each $\alpha \in M - \kappa$, π_α is the function $\pi_\alpha(\beta) = \pi(\alpha, \beta)$ from $M \cap \alpha$ onto

$M \cap \alpha$, and moreover, π_α maps κ_M onto $M \cap \alpha$.

(iii) ρ is a two-place function; $\rho(\alpha,\beta)$ is defined for all $\alpha \in M - \kappa$ and $\beta < \kappa_M$. For each $\alpha \in M - \kappa$, ρ_α is the function $\rho_\alpha(\beta) = \rho(\alpha,\beta)$ from κ_M into κ_M, and $\beta \le \rho_\alpha(\beta) < \kappa_M$ for all $\beta < \kappa_M$.

Two models $\langle M, \pi^M, \rho^M\rangle$ and $\langle N, \pi^N, \rho^N\rangle$ are *coherent* if $\pi^M(\alpha,\beta) = \pi^N(\alpha,\beta)$ and $\rho^M(\alpha,\beta) = \rho^N(\alpha,\beta)$ for all $\alpha,\beta \in M \cap N$. M is a *submodel* of N if $M \subseteq N$, and $\pi^M \subseteq \pi^N$ and $\rho^M \subseteq \rho^N$.

3.2. Lemma. *Let M and N be coherent models with $\kappa_M \le \kappa_N$. If $M \cap N$ is cofinal in M (i.e. if for all $\alpha \in M$ there is a $\gamma \in M \cap N$ such that $\alpha < \gamma$), then $M \subseteq N$.*

Proof. Let $\alpha \in M$; let $\gamma \in M \cap N$ be such that $\alpha < \gamma$. As π^M_γ maps κ_M onto $M \cap \gamma$, there is a $\beta < \kappa_M$ such that $\pi^M_\gamma(\beta) = \alpha$. Since both β and γ are in N, we have $\alpha = \pi^M(\gamma,\beta) = \pi^N(\gamma,\beta) \in N$.

We shall now define the forcing notion $Q(S)$:

3.3 Definition. Let S be a stationary subset of the set $E_0 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x$ is a limit cardinal and $\operatorname{cf} \lambda_x = \operatorname{cf} \kappa_x\}$. A *forcing condition* in $Q(S)$ is a model $M = \langle M, \pi^M, \rho^M\rangle$ such that

(i) M is ω-closed, i.e. for every ordinal γ, if $\operatorname{cf}\gamma = \omega$ and $\sup(M \cap \gamma) = \gamma$, then $\gamma \in M$;

(ii) For every $\alpha \in M - \kappa$ and $\beta < \kappa_M$, if $\kappa_M \le \gamma < \alpha$, and if $\{\beta_n : n < \omega\}$ is a countable subset of β such that $\gamma = \sup\{\pi^M_\alpha(\beta_n) : n < \omega\}$, then there is some $\zeta < \rho^M_\alpha(\beta)$ such that $\gamma = \pi^M_\alpha(\zeta)$.

(iii) For every submodel $a \subseteq M$, if

$$a \in E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\},$$

then $S \cap \mathcal{P}_{\kappa_a} a$ is stationary in $\mathcal{P}_{\kappa_a} a$.

A forcing condition N is *stronger* than M if M is a submodel of N and $|M| < |\kappa_N|$.

The following lemma guarantees that the generic for Q_S is unbounded in $\mathcal{P}_\kappa\kappa^+$.

3.4. Lemma. *Let M be a condition and let $\delta < \kappa$ and $\kappa \le \varepsilon < \kappa^+$. Then there is a condition N stronger than M such that $\delta \in N$ and $\varepsilon \in N$.*

Proof. Let $\lambda < \kappa$ be an inaccessible cardinal, such that $\lambda \ge \delta$ and $\lambda > |M|$. We let $N = M \cup \lambda \cup \{\lambda\} \cup \{\varepsilon\}$; thus $\kappa_N = \lambda + 1$, and N is ω-closed. We extend π^M and ρ^M to π^N and ρ^N as follows:

If $\kappa \le \alpha < \varepsilon$ and $\alpha \in M$, we let $\pi^N_\alpha(\beta) = \beta$ for all $\beta \in N$ such that $\kappa_M \le \beta \le \lambda$. If $\alpha \in M$ and $\varepsilon < \alpha$, we define π^N_α so that π^N_α maps $\kappa_N - \kappa_M$ onto $(\kappa_N - \kappa_M) \cup \{\varepsilon\}$. For $\alpha = \varepsilon$, we define π^N_ε in such a way that π^N_ε maps λ onto $N \cap \varepsilon$.

Finally, if $\alpha, \beta \in N$, $\beta < \kappa \le \alpha$, and if either $\alpha = \varepsilon$ or $\beta \ge \kappa_M$ we let $\rho^N_\alpha(\beta) = \lambda$.

Clearly, N is a model, M is a submodel of N, and $|M| < |\kappa_N|$. Let us verify (3.3.ii). This holds if $\alpha \in M$, so let $\alpha = \varepsilon$. Let $\beta \le \lambda$, let $\{\beta_n : n < \omega\} \subseteq \beta$ and let $\gamma = \sup\{\pi^N_\varepsilon(\beta_n) : n < \omega\}$ be such that $\kappa \le \gamma < \varepsilon$. There is a $\zeta < \lambda = \rho^N_\varepsilon(\beta)$ such that $\pi^N_\varepsilon(\zeta) = \gamma$, and so (3.3.ii) holds.

To complete the proof that N is a forcing condition, we verify (3.3. iii). This we do by showing that if $a \in E_1$ is a submodel of N then $a \subseteq M$.

Assume that $a \in E_1$ is a submodel of N but $a \nsubseteq M$. Thus there are $\alpha, \beta \in a$, $\beta < \kappa \le \alpha$ such that either $\alpha = \varepsilon$ or $\beta \ge \kappa_M$. Then $\rho^a_\alpha(\beta) = \rho^N_\alpha(\beta) = \lambda$ and so $\lambda \in a$, and $\kappa_a = \lambda + 1$. This contradicts the assumption that κ_a is an inaccessible cardinal. $\square$

Thus if G is a generic for Q_S, let $\langle M_G, \pi_G, \rho_G\rangle$ be the union of all conditions in G. Then for every $a \in E_1$, that is a submodel of M_G, $S \cap \mathcal{P}_{\kappa_a} a$ is stationary in $\mathcal{P}_{\kappa_a} a$. Thus Q_S forces that S reflects at all but nonstationary many $a \in E_1$.

We will now prove that the forcing Q_S is $< \kappa$-strategically closed. The key technical devices are the two following lemmas.

Lemma 3.5. *Let $M_0 > M_1 > \cdots > M_n > \dots$ be an ω-sequence of conditions. There exists a condition M stronger than all the M_n, with the following property:*

(3.6)

If N is any model coherent with M such that there exists some $\gamma \in N \cap M$

$$\text{but } \gamma \notin \bigcup_{n=0}^{\infty} M_n, \text{ then } \kappa_N > \lim_n \kappa_{M_n}.$$

Proof. Let $A = \bigcup_{n=0}^{\infty} M_n$ and $\delta = A \cap \kappa = \lim_n \kappa_{M_n}$, and let $\pi^A = \bigcup_{n=0}^{\infty} \pi^{M_n}$ and $\rho^A = \bigcup_{n=0}^{\infty} \rho^{M_n}$. We let M be the ω-closure of $(\delta+\delta) \cup A$; hence $\kappa_M = \delta+\delta+1$. To define π^M, we first define $\pi^M_\alpha \supset \pi^A_\alpha$ for $\alpha \in A$ in such a way that π^M_α maps $\delta+\delta$ onto $M \cap \alpha$. When $\alpha \in M - A$ and $\alpha \geq \kappa$, we have $|M \cap \alpha| = |\delta|$ and so there exists a function π^M_α on $M \cap \alpha$ that maps $\delta+\delta$ onto $M \cap \alpha$; we let π^M_α be such, with the additional requirement that $\pi^M_\alpha(0) = \delta$. To define ρ^M, we let $\rho^M \supset \rho^A$ be such that $\rho^M(\alpha, \beta) = \delta+\delta$ whenever either $\alpha \notin A$ or $\beta \notin A$.

We shall now verify that M satisfies (3.3. ii). Let $\alpha, \beta \in M$ be such that $\alpha \geq \kappa$ and $\beta < \kappa$ and let $\gamma \in M$, $\kappa \leq \gamma < \alpha$, be an ω-limit point of the set $\{\pi^M_\alpha(\xi) : \xi < \beta\}$. We want to show that $\gamma = \pi^M_\alpha(\eta)$ for some $\eta < \rho^M_\alpha(\beta)$. If both α and β are in A then this is true, because $\alpha, \beta \in M_n$ for some n, and M_n satisfies (3.3 ii). If either $\alpha \notin A$ or $\beta \notin A$ then $\rho^M_\alpha(\beta) = \delta+\delta$, and since π^M_α maps $\delta+\delta$ onto $M \cap \alpha$, we are done.

Next we verify that M satisfies (3.6). Let N be any model coherent with M, and let $\gamma \in M \cap N$ be such that $\gamma \notin A$. If $\gamma < \kappa$ then $\gamma \geq \delta$ and so $\kappa_N > \delta$. If $\gamma \geq \kappa$ then $\pi^M_\gamma(0) = \delta$, and so $\delta = \pi^N_\gamma(0) \in N$, and again we have $\kappa_N > \delta$.

Finally, we show that for every $a \in E_1$, if $a \subseteq M$ then $S \cap \mathcal{P}_{\kappa_a} a$ is stationary. We do this by showing that for every $a \in E_1$, if $a \subseteq M$ then $a \subseteq M_n$ for some M_n.

Thus let $a \subseteq M$ be such that κ_a is regular and $\lambda_a = \kappa_a^+$. As $\kappa_a \leq \kappa_M = \delta+\delta+1$, it follows that $\kappa_a < \delta$ and so $\kappa_a < \kappa_{M_{n_0}}$ for some n_0. Now by (3.6) we have

$a \subseteq \bigcup_{n=0}^{\infty} M_n$, and since λ_a is regular uncountable, there exists some $n \geq n_0$ such that $M_n \cap a$ is cofinal in a. It follows from Lemma 3.2 that $a \subseteq M_n$.

Lemma 3.7. *Let $\lambda < \kappa$ be a regular uncountable cardinal and let $M_0 > M_1 > \cdots > M_\xi > \cdots$, $\xi < \lambda$, be a λ-sequence of conditions with the property that for every $\eta < \lambda$ of cofinality ω,*

(3.8)
If N is any model coherent with M_η such that there exists some $\gamma \in N \cap M_\eta$

$$\text{but } \gamma \notin \bigcup_{\xi<\eta} M_\xi, \text{ then } \kappa_N > \lim_{\xi\to\eta} \kappa_{M_\xi}.$$

Then $M = \bigcup_{\xi<\lambda} M_\xi$ is a condition.

Proof. It is clear that M satisfies all the requirements for a condition, except perhaps (3.3 iii). (M is ω-closed because λ is regular uncountable.) Note that because $|M_\xi| < \kappa_{M_{\xi+1}}$ for all $\xi < \lambda$, we have $|M| = |\kappa_M|$.

We shall prove (3.3 iii) by showing that for every $a \in E_1$, if $a \subseteq M$, then $a \subseteq M_\xi$ for some $\xi < \lambda$. Thus let $a \subseteq M$ be such that κ_a is regular and $\lambda_a = \kappa_a^+$.

As $\lambda_a = |a| \leq |M| = |\kappa_M|$, it follows that $\kappa_a < \kappa_M$ and so $\kappa_a < \kappa_{M_{\xi_0}}$ for some $\xi_0 < \lambda$. We shall prove that there exists some $\xi \geq \xi_0$ such that $M_\xi \cap a$ is cofinal in a; then by Lemma 3.2, $a \subseteq M_\xi$.

We prove this by contradiction. Assume that no $M_\xi \cap a$ is cofinal in a. We construct sequences $\xi_0 < \xi_1 < \cdots < \xi_n < \cdots$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_n < \cdots$ such that for each n,

$$\gamma_n \in a\,, \qquad \gamma_n > \sup(M_{\xi_n} \cap a)\,, \qquad \text{and} \qquad \gamma_n \in M_{\xi_{n+1}}$$

Let $\eta = \lim_n \xi_n$ and $\gamma = \lim_n \gamma_n$. We claim that $\gamma \in a$.

As λ_a is regular uncountable, there exists an $\alpha \in a$ such that $\alpha > \gamma$. Let β_n, $n \in \omega$, be such that $\pi^a_\alpha(\beta_n) = \gamma_n$, and let $\beta < \kappa_a$ be such that $\beta > \beta_n$ for all n. As M satisfies (3.3. ii), and $\gamma = \sup\{\pi^M_\alpha(\beta_n) : n < \omega\}$, there is some $\zeta < \rho^M_\alpha(\beta)$ such that $\gamma = \pi^M_\alpha(\zeta)$. Since $\zeta < \rho^M_\alpha(\beta) = \rho^a_\alpha(\beta) < \kappa_a$, we have $\zeta \in a$, and $\gamma = \pi^a_\alpha(\zeta) \in a$.

Now since $\gamma \in a$ and $\gamma > \sup(M_{\xi_n} \cap a)$ we have $\gamma \notin M_{\xi_n}$, for all n. As M_η is ω-closed, and $\gamma_n \in M_\eta$ for each n, we have $\gamma \in M_\eta$. Thus by (3.8) it follows that $\kappa_a > \lim_n \kappa_{M_{\xi_n}}$, a contradiction. □

Lemma 3.9. *Q_S is $< \kappa$-strategically closed.*

Proof. In the game, player I moves at limit stages. In order to win the game, it suffices to choose at every limit ordinal η of cofinality ω, a condition M_η that satisfies (3.8). This is possible by Lemma 3.5. □

We shall now prove that Q_S satisfies the $< \kappa$-strategic κ^+-chain condition. First a lemma:

Lemma 3.10. *Let $\langle M_1, \pi_1, \rho_1\rangle$ and $\langle M_2, \pi_2, \rho_2\rangle$ be forcing conditions such that $\kappa_{M_1} = \kappa_{M_2}$ and that the models M_1 and M_2 are coherent. Then the conditions are compatible.*

Proof. Let $\lambda < \kappa$ be an inaccessible cardinal such that $\lambda > |M_1 \cup M_2|$ and let $M = M_1 \cup M_2 \cup \lambda \cup \{\lambda\}$. We shall extend $\pi_1 \cup \pi_2$ and $\rho_1 \cup \rho_2$ to π^M and ρ^M so that $\langle M, \pi^M, \rho^M\rangle$ is a condition.

If $\alpha \in M_i - \kappa$, we define $\pi^M_\alpha \supset \pi_i$ so that π^M_α maps $\lambda - \kappa_{M_1}$ onto $M \cap \alpha$, and such that $\pi^M_\alpha(\beta) = \lambda$ whenever $\kappa \le \beta < \alpha$, $\alpha \in M_1 - M_2$ and $\beta \in M_2 - M_1$ (or vice versa). We define $\rho^M_\alpha \supset \rho_i$ by $\rho^M_\alpha(\beta) = \lambda$ for $\kappa_{M_1} \le \beta \le \lambda$. It is easy to see that M is an ω-closed model that satisfies (3.3 ii).

To verify (3.3 iii), we show that every $a \in E_1$ that is a submodel of M is either $a \subseteq M_1$ or $a \subseteq M_2$. Thus let a be a submodel of M, $a \in E_1$, such that neither $a \subseteq M_1$ nor $a \subseteq M_2$. First assume that $\kappa_a \le \kappa_{M_1}$. Then there are $\alpha, \beta \in a$ such that $\kappa \le \beta < \alpha$ and $\alpha \in M_1 - M_2$ while $\beta \in M_2 - M_1$ (or vice versa). But then $\pi^a(\alpha, \beta) = \pi^M(\alpha, \beta) = \lambda$ which implies $\lambda \in a$, or $\kappa_a = \lambda + 1$, contradicting the inaccessibility of κ_a.

Thus assume that $\kappa_a > \kappa_{M_1}$. Let $\alpha \in a$ be such that $\alpha \geq \kappa$, and then we have $\rho^a(\alpha, \kappa_{M_1}) = \rho^M(\alpha, \kappa_{M_1}) = \lambda$, giving again $\lambda \in a$, a contradiction. □

Lemma 3.11. *Q_S satisfies the $< \kappa$-strategic κ^+-chain condition.*

Proof. Let λ be a limit ordinal $< \kappa$ and consider the game (2.3) of length λ. We may assume that $\mathrm{cf}\,\lambda > \omega$. In the game, player I moves at limit stages, and the key to winning is again to make right moves at limit stages of cofinality ω. Thus let η be a limit ordinal $< \lambda$, and let $\{M_\xi^\alpha : \alpha < \kappa^+,\ \mathrm{cf}\,\alpha = \kappa\}$ be the set of conditions played at stage ξ.

By Lemma 3.5, player I can choose, for each α, a condition M_η^α stronger than each M_ξ^α, $\xi < \eta$, such that M_η^α satisfies (3.8). Then let E_η be the closed unbounded subset of κ^+

$$E_\eta = \{\gamma < \kappa^+ : M_\eta^\alpha \subset \gamma \quad \text{for all} \quad \alpha < \gamma\},$$

and let f_η be the function $f_\eta(\alpha) = M_\eta^\alpha \restriction \alpha$, this being the restriction of the model M_η^α to α.

We claim that player I wins following this strategy: By Lemma 3.7, player I can make a legal move at every limit ordinal $\xi < \lambda$, and for each α (of cofinality κ), $M^\alpha = \bigcup_{\xi<\lambda} M_\xi^\alpha$ is a condition. Let $\alpha < \beta$ be ordinals of cofinality κ in $\bigcap_{\xi<\lambda} E_\xi$ such that $f_\xi(\alpha) = f_\xi(\beta)$ for all $\xi < \lambda$. Then $M^\alpha \subset \beta$ and $M^\beta \restriction \beta = M^\alpha \restriction \alpha$, and because the functions π and ρ have the property that $\pi(\gamma, \delta) < \gamma$ and $\rho(\gamma, \delta) < \gamma$ for every γ and δ, it follows that M^α and M^β are coherent models with $\kappa_{M^\alpha} = \kappa_{M^\beta}$. By Lemma 3.10, M^α and M^β are compatible conditions. □

4. Preservation of the set E_1.

We shall complete the proof by showing that the set

$$E_1 = \{x \in \mathcal{P}_\kappa\kappa^+ : \kappa_x \text{ is inaccessible and } \lambda_x = \kappa_x^+\}$$

remains stationary after forcing with $P = P_\kappa * \dot{P}^\kappa$.

Let us reformulate the problem as follows: Let us show, working in V^{P_κ}, that for every condition $p \in \dot{P}^\kappa$ and every $\dot{P}^\kappa$-name $\dot{F}$ for an operation $\dot{F} : (\kappa^+)^{<\omega} \to \kappa^+$ there exists a condition $\overline{p} \leq p$ and a set $x \in E_1$ such that $\overline{p}$ forces that x is closed under $\dot{F}$.

As κ is supercompact, there exists by the construction of P_κ and by Laver's Theorem 2.1, an elementary embedding $j : V \to M$ with critical point κ that witnesses that κ is κ^+-supercompact and such that the κ^{th} iterand of the iteration $j(P_\kappa)$ in M is (the name for) the forcing $\dot{P}^\kappa$. The elementary embedding j can be extended, by a standard argument, to an elementary embedding $j : V^{P_\kappa} \to M^{j(P_\kappa)}$. Since j is elementary, we can achieve our stated goal by finding, in $M^{j(P_\kappa)}$, a condition $\overline{p} \leq j(p)$ and a set $x \in j(E_1)$ such that $\overline{p}$ forces that x is closed under $j(\dot{F})$.

The forcing $j(P_\kappa)$ decomposes into a three step iteration $j(P_\kappa) = P_\kappa * \dot{P}^\kappa * \dot{R}$ where $\dot{R}$ is, in $M^{P_\kappa * \dot{P}^\kappa}$, a $< j(\kappa)$-strategically closed forcing.

Let G be an M-generic filter on $j(P_\kappa)$, such that $p \in G$. The filter G decomposes into $G = G_\kappa * H * K$ where H and K are generics on $\dot{P}^\kappa$ and $\dot{R}$ respectively, and $p \in H$. We shall find $\overline{p}$ that extends not just $j(p)$ but each member of $j''H$ ($\overline{p}$ is a *master condition*). That will guarantee that when we let $x = j''\mathcal{P}_\kappa\kappa^+$ (which is in $j(E_1)$) then $\overline{p}$ forces that x is closed under $j(\dot{F})$: this is because $\overline{p} \Vdash j(\dot{F}) \restriction x = j''F_H$, where F_H is the H-interpretation of $\dot{F}$.

We construct $\overline{p}$, a sequence $\langle p_\xi : \xi < j(2^{\kappa^+})\rangle$, by induction. When ξ is not in the range of j, we let p_ξ be the trivial condition; that guarantees that the support of $\overline{p}$ has size $< j(\kappa)$. So let $\xi = j(\gamma)$ be such that $\overline{p} \restriction \xi$ has been constructed.

Let M the model $\bigcup\{j(N) : N \in H_\gamma\}$ where H_γ is the γ^{th} coordinate of H. The γ^{th} iterand of $\dot{P}^\kappa$ is the forcing $Q(S)$ where S is a stationary subset of E_0. In order for M to be a condition in $Q(j(S))$, we have to verify that for every submodel $a \subseteq M$, if $a \in j(E_1)$ then $j(S)$ reflects at a.

Let $a \in j(E_1)$ be a submodel of M. If $\kappa_a < \kappa_M = \kappa$, then $a = j''\overline{a} = j(\overline{a})$ for some $\overline{a} \in E_1$, and $\overline{a}$ is a submodel of some $N \in H_\gamma$. As S reflects at $\overline{a}$ it follows that $j(S)$ reflects at a.

If $\kappa_a = \kappa$, then $\lambda_a = \kappa^+$, and a is necessarily cofinal in the universe of M, which is $j''\kappa^+$. By Lemma 3.2, we have $a = M$, and we have to show that $j(S)$ reflects at $j''\kappa^+$. This means that $j''S$ is stationary in $\mathcal{P}_\kappa(j''\kappa^+)$, or equivalently, that S is stationary in $\mathcal{P}_\kappa\kappa^+$.

We need to verify that S is a stationary set, in the model $M^{j(P_\kappa)*j(\dot{P}_\kappa)\restriction j(\gamma)}$, while we know that S is stationary in the model $V^{P_\kappa * \dot{P}^\kappa\restriction\gamma}$. However, the former model is a forcing extension of the latter by a $<\kappa$-strategically closed forcing, and the result follows by Lemma 2.5.

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