

On incomparability and related cardinal functions on ultraproducts of Boolean algebras

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ABSTRACT: Let C denote any of the following cardinal characteristics of Boolean algebras: incomparability, spread, character, π -character, hereditary Lindelöf number, hereditary density. It is shown to be consistent that there exists a sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras and an ultrafilter D on κ such that

$$C(\prod_{i < \kappa} B_i/D) < |\prod_{i < \kappa} C(B_i)/D|.$$

This answers a number of problems posed in [M].

Introduction

For a number of cardinal characteristics C of Boolean algebras it makes sense to ask whether it is consistent to have a sequence $\langle B_i : i < \kappa \rangle$ of Boolean algebras and an ultrafilter D on κ such that

$$C(\prod_{i < \kappa} B_i/D) < |\prod_{i < \kappa} C(B_i)/D|.$$

For C being the length this was proved in [MSh]. The same method of proof can be used to get the analogous thing for C being any one of the following: incomparability (Inc), spread (s), character (χ), π -character ($\pi\chi$), hereditary Lindelöf number (hL), hereditary density (hd). This answers problems 47, 48, 52, 56, 60 of [M]. For irredundancy (Monk's problem 25) this will be done in a subsequent paper of the first author. We won't define these notions here, as they are very clearly defined on pp. 2,3 in [M]. We assume that the reader has a good knowledge of [MSh] and [Mg].

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For C a cardinal function of Boolean algebras which is defined as the supremum of all cardinals which have a certain property, we define C^+ as the least cardinal κ such that this property fails for every cardinal $\lambda \geq \kappa$. Note that all cardinal functions mentioned above are of this form. For χ , note that in [M] $\chi(B)$ has been defined as the minimal κ such that every ultrafilter on B can be generated by κ elements. Clearly, $\chi(B)$ can be equivalently defined as

$$\sup\{\chi(U) : U \text{ is an ultrafilter on } B\},$$

where $\chi(U)$ is the minimal size of a generating subset of U .

In §1 below we deal with incomparability. In §2 we shall show that the results for all the other characteristics can be deduced from this relatively easily.

A key notion for the proofs is that of μ -entangled linear order, μ being a cardinal (see definition 1.1 below). The reason for this is the following observation of Shelah (see [M, p.225]), where $Int(I)$ denotes the interval algebra of some linear order I , i.e. the subalgebra of $\mathcal{P}(I)$ generated by the half-open intervals of the form $[a, b)$, $a, b \in I$.

Fact. *Let μ be a regular uncountable cardinal and let I be a linear order. The following are equivalent:*

- (1) I is μ -entangled.
- (2) There is no incomparable subset of $Int(I)$ of size μ .

For b a member of some Boolean algebra B , b^1 denotes b and b^0 denotes the complement of b . By $Ult(B)$ we denote the Stone space of B .

1. Incomparability

Definition 1.1. Let $(I, <)$ be a linear order and let $D \subseteq \mathcal{P}(\kappa)$ for some infinite cardinal κ .

(1) $(I, <)$ is called (δ, γ) -entangled, where $\gamma < \delta$ are cardinals, if for every family $\langle t_{\alpha, \varepsilon} : \alpha < \delta, \varepsilon < \varepsilon(*) < \gamma \rangle$ of pairwise distinct members of I and for every $u \subseteq \varepsilon(*)$ there exist $\alpha < \beta < \delta$ such that

$$\forall \varepsilon < \varepsilon(*) \quad t_{\alpha, \varepsilon} < t_{\beta, \varepsilon} \Leftrightarrow \varepsilon \in u.$$

If $\gamma = \omega$ we say that $(I, <)$ is δ -entangled.

(2) $(I, <)$ is called (δ, D) -entangled if for every sequence $\langle t_{\alpha, \varepsilon, l} : \alpha < \delta, \varepsilon \in A, l < n \rangle$ of pairwise distinct members of I , where $A \in D$ and $n < \omega$, and for every $u \subseteq n$ there exist $\alpha < \beta < \delta$ and $B \subseteq A$, $B \in D$, such that

$$\forall \varepsilon \in B \forall l < n \quad t_{\alpha, \varepsilon, l} < t_{\beta, \varepsilon, l} \Leftrightarrow l \in u.$$

Note that if $\kappa < \gamma$ and $(I, <)$ is (δ, γ) -entangled then $(I, <)$ is (δ, D) -entangled for every $D \subseteq \mathcal{P}(\kappa)$.

In the sequel, if $\gamma < \delta$ are regular cardinals, by $C(\gamma, \delta)$ we denote the partial order to add δ Cohen subsets to γ . More precisely,

$$C(\gamma, \delta) = \prod_{i < \delta} Q_i \quad (< \gamma\text{-support product})$$

where $Q_i = (\mathcal{C}^{\gamma} 2, \supseteq)$. Clearly Q is γ -directedly-closed.

Lemma 1.2. *Let $\gamma < \delta$ be regular cardinals such that $\forall \alpha < \delta \quad \alpha^{<\gamma} < \delta$. Let $L = \{\eta_i : i < \delta\}$ be $C(\gamma, \delta)$ -generic. Letting $<_{lex}$ denote the lexicographic order on ${}^\gamma 2$, we have that $(L, <_{lex})$ is a (δ, γ) -entangled linear order.*

Proof: Suppose $p \Vdash_{C(\gamma, \delta)} \langle \dot{\tau}(\alpha, \varepsilon) : \alpha < \delta, \varepsilon < \varepsilon(*) < \gamma \rangle$ is a sequence of pairwise distinct ordinals below δ such that the family $\langle \dot{\eta}_{\dot{\tau}(\alpha, \varepsilon)} : \alpha < \delta, \varepsilon < \varepsilon(*) < \gamma \rangle$ contradicts (δ, γ) -entangledness of L , witnessed by set $u \subseteq \varepsilon(*)$ ". Here η_i is the canonical name for the i th Cohen subset of γ .

As $C(\gamma, \delta)$ does not add new ordinal sequences of length $< \gamma$ we may assume that $u \in V$. For the same reason, for each $\alpha < \delta$ we may pick $p_\alpha \leq p$ such that p_α decides the value of $\langle \dot{\tau}(\alpha, \varepsilon) : \varepsilon < \varepsilon(*) \rangle$, say as $\langle \tau(\alpha, \varepsilon) : \varepsilon < \varepsilon(*) \rangle$. By the Δ -system Lemma and some thinning out there exists $Y \in [\delta]^\delta$ and $r \in [\delta]^{<\gamma}$ such that for all $\alpha, \beta \in Y$, $\alpha \neq \beta$, we have:

- (i) $\text{dom}(p_\alpha) \cap \text{dom}(p_\beta) = r$,
- (ii) $\{\tau(\alpha, \varepsilon) : \varepsilon < \varepsilon(*)\} \subseteq \text{dom}(p_\alpha)$,
- (iii) $\langle p_\alpha(i) : \alpha \in Y \rangle$ is constant for every $i \in r$,
- (iv) $\langle p(\tau(\alpha, \varepsilon)) : \alpha \in Y \rangle$ is constant for every $\varepsilon < \varepsilon(*)$.

Pick $\alpha, \beta \in Y$ with $\alpha < \beta$. Note that p_α, p_β are compatible, and hence $\{\tau(\alpha, \varepsilon) : \varepsilon < \varepsilon(*)\} \cap \{\tau(\beta, \varepsilon) : \varepsilon < \varepsilon(*)\} = \emptyset$.

Define $q \leq p_\alpha, p_\beta$ by $\text{dom}(q) = \text{dom}(p_\alpha) \cup \text{dom}(p_\beta)$ and

$$q(i) = \begin{cases} p_\alpha(i) \wedge 0 & i \in \{\tau(\alpha, \varepsilon) : \varepsilon \in u\} \\ p_\alpha(i) \wedge 1 & i \in \{\tau(\alpha, \varepsilon) : \varepsilon \in \varepsilon(*) \setminus u\} \\ p_\beta(i) \wedge 0 & i \in \{\tau(\beta, \varepsilon) : \varepsilon \in \varepsilon(*) \setminus u\} \\ p_\beta(i) \wedge 1 & i \in \{\tau(\beta, \varepsilon) : \varepsilon \in u\} \\ p_\alpha(i) & i \in \text{dom}(p) \setminus \{\tau(\alpha, \varepsilon) : \varepsilon < \varepsilon(*)\} \\ p_\beta(i) & \text{otherwise.} \end{cases}$$

Then clearly

$$q \Vdash (\forall \varepsilon < \varepsilon(*)) \eta_{\dot{\tau}(\alpha, \varepsilon)} <_{lex} \eta_{\dot{\tau}(\beta, \varepsilon)} \Leftrightarrow \varepsilon \in u,$$

a contradiction. □

Assume GCH. Let μ be a supercompact cardinal and let $\kappa < \mu$ be a measurable cardinal. Fix D a normal measure on κ . By [L] we may assume that the supercompactness of μ cannot be destroyed by any μ -directedly-closed forcing.

For any ordinal α let $F(\alpha)$ denote the least inaccessible cardinal above α , if it exists. We assume that $F(\mu)$ exists and denote it with λ .

Let $Q = C(\mu, \lambda)$. Clearly Q is μ -directedly-closed and $V^Q \models 2^\mu = \lambda$.

Work in V^Q . Let U be a normal fine measure on $[H((2^\lambda)^+)]^{<\mu}$. By Lemma 1.2, $H((2^\lambda)^+) \models$ “there exists a (λ, μ) -entangled linear order on λ ”. Therefore the set A of all $a \in [H((2^\lambda)^+)]^{<\mu}$ such that

- (1) $(a, \in) \prec (H((2^\lambda)^+), \in)$,
- (2) $a \cap \mu$ is measurable,
- (3) the Mostowski collapse of a is $H((2^{F(a \cap \mu)})^+)$,
- (4) $H((2^{F(a \cap \mu)})^+) \models$ there is a $(F(a \cap \mu), a \cap \mu)$ -entangled linear order on $F(a \cap \mu)$, call it $J_{a \cap \mu}^*$,
- (5) $H((2^{F(a \cap \mu)})^+) \models 2^{a \cap \mu} = F(a \cap \mu)$

belongs to U .

By well-known arguments on large cardinals and elementary embeddings we can build a sequence $\bar{U} = \langle U_\alpha : \alpha < \kappa \rangle$ of normal measures on μ such that

- (a) $\alpha < \beta \Rightarrow U_\alpha < U_\beta$ (i.e. $U_\alpha \in \text{Ult}(V, U_\beta)$),
- (b) $\{a \cap \mu : a \in A\} \in U_\alpha$ for all $\alpha < \kappa$.

The main fact which is used for this is the following lemma which goes back to [SRK]. We thank James Cummings for reconstructing the proof for us.

Lemma 1.3 For all $a \in V_{\mu+2}$ there exists a normal measure U on μ such that $a \in \text{Ult}(V, U)$ and $\{a \cap \mu : a \in A\} \in U$.

Sketch of proof: Let $j : V \rightarrow M$ be the elementary embedding defined by the normal fine measure U above. Fix $<$ a wellordering of V_μ and let

$$<^* = j(<) \upharpoonright V_{\lambda^+}.$$

Assuming that the Lemma is false, let $b \in V_{\mu+2}$ be the $<^*$ -minimal counterexample. Let $\bar{U} = \{B \subseteq \mu : \mu \in j(B)\}$ be the normal measure on μ induced by j and let $i : V \rightarrow N$ be the corresponding elementary embedding. As usual we have another elementary embedding $k : N \rightarrow M$, defined by $k([f]_{\bar{U}}) = j(f)(\mu)$, such that $j = k \circ i$ (see [J, p.312]). As $V_{\mu+2} \subseteq M$ we have that

$$M \models b \text{ is the } j(<)\text{-minimal counterexample.}$$

By elementarity there must exist $\bar{b} \in N$ such that $k(\bar{b}) = b$ and

$$N \models \bar{b} \text{ is the } i(<)\text{-minimal counterexample.}$$

Note that $b \notin N$, as b is a counterexample. Also note that $V_{\mu+1} \subseteq \text{ran}(k)$. As k is simply the inverse of the transitive collapse map on $\text{ran}(k)$, we conclude $\bar{b} = b$ and hence $b \in N$, a contradiction. \square

Let $Q(\bar{U})$ denote Magidor's forcing to change the cofinality of μ to κ by adding a normal sequence $\langle \mu_i : i < \kappa \rangle$ cofinal in μ . Fix such a $Q(\bar{U})$ -generic sequence with $\mu_0 > 2^\kappa$. We let

$$\mu'_i = \mu_{\omega_i}, \theta_i = F(\mu_{i+1}), \lambda_i = F(\mu_{\omega_i}), J_i = J_{\mu_{i+1}}^*.$$

Lemma 1.4 For every $i < \kappa$, $V^{Q^*Q(\bar{U})} \models \text{“}F(\mu_{i+1}) = \theta_i \text{ and } J_i \text{ is } (\theta_i, D)\text{-entangled”}$.

Proof: Work in V^Q . Let $\langle \dot{\mu}_i : i < \kappa \rangle$ be a $Q(\bar{U})$ -name for the generic sequence. Fix $i < \kappa$. Let $p \in Q(\bar{U})$ such that p decides $\dot{\mu}_j$ as μ_j for $j \in \{i, i+1, i+2\}$. We may assume that the domain of the first coordinate of p is $\{i, i+1, i+2\}$. By the main arguments of [Mg], especially [Mg, Lemma 5.3], it follows that forcing $Q(\bar{U})$ below p factors as $P_{\mu_i}^i * Q^i$, where $P_{\mu_i}^i$ is the union of μ_i many μ_i -directed suborders each of them of size $\leq 2^{\mu_i}$, and Q^i does not add new subsets to μ_{i+2} . Hence clearly $V^{Q^*Q(\bar{U})} \models F(\mu_{i+1}) = \theta_i$.

Now suppose $p \Vdash \text{“}\langle \dot{t}_{\alpha, \varepsilon, l} : \alpha < \theta_i, \varepsilon \in A, l < n \rangle \text{ is a one-to-one family of elements of } J_i\text{”}$. By [Mg, Lemma 4.6], for each $\alpha < \theta_i$ we can find $p_\alpha \leq p$ such that p_α and p have the

same first coordinate and for all $\varepsilon \in A$ and $l < n$ there exists $w_{\alpha,\varepsilon,l} \in [i]^{<\omega}$ such that below p_α , the value of $\dot{t}_{\alpha,\varepsilon,l}$ depends only on the value of $\langle \dot{\mu}_j : j \in w_{\alpha,\varepsilon,l} \rangle$. As D is κ -complete and $i < \kappa$, there exists $B_{\alpha,l} \in D$ and $w_{\alpha,l}$ such that $w_{\alpha,\varepsilon,l} = w_{\alpha,l}$ for all $\varepsilon \in B_{\alpha,l}$ and $l < n$. Let $w_\alpha = \bigcup_{l < n} w_{\alpha,l}$, $B_\alpha = \bigcap_{l < n} B_{\alpha,l}$. As $2^\kappa < \theta_i$ and $i < \theta_i$ we can find $Y \subseteq \theta_i$ of size θ_i , $w^* \in [i]^{<\omega}$ and $B \in D$ such that $B_\alpha = B$ and $w^* = w_\alpha$ for all $\alpha \in Y$. Let v be the domain of the first coordinate of any p_α . By [Mg, Lemma 3.3], for each $\alpha \in Y$ we can find $p'_\alpha \leq p_\alpha$ such that the domain of the first coordinate of p'_α is $w^* \cup v$. Then p'_α decides $\langle \dot{\mu}_j : j \in w^* \rangle$, say as $\langle \mu_j^\alpha : j \in w^* \rangle$, and hence p'_α decides $\langle \dot{t}_{\alpha,\varepsilon,l} : \varepsilon \in B, l < n \rangle$, say as $\langle t_{\alpha,\varepsilon,l} : \varepsilon \in B, l < n \rangle$. Note that this sequence is one-to-one. As $\mu_i < \theta_i$ we can find $Y' \subseteq Y$ of size θ_i and $\langle \mu_j : j \in w^* \rangle$ such that $\langle \mu_j^\alpha : j \in w^* \rangle = \langle \mu_j : j \in w^* \rangle$ and $(p_\alpha)_{i+2} = (p_\beta)_{i+2}$ (we use the notation of [Mg, p.67]), for all $\alpha, \beta \in Y'$. By [Mg, Lemma 4.1] it follows that p_α and p_β are compatible for all $\alpha, \beta \in Y'$. It follows that $\langle t_{\alpha,\varepsilon,l} : \alpha \in Y', \varepsilon \in B, l < n \rangle$ is a one-to-one family. Applying (θ_i, D) -entangledness of J_i in V^Q , for any $u \subseteq n$ we obtain $B' \in D$, $B' \subseteq B$ and $\alpha < \beta$, $\alpha, \beta \in Y'$, such that for all $\varepsilon \in B'$, for all $l < n$, $t_{\alpha,\varepsilon,l} < t_{\beta,\varepsilon,l} \Leftrightarrow l \in u$. What we have shown suffices to prove the Lemma. \square

For every $i < \kappa$ we define a linear order $I_i \subseteq \prod_{j < \omega_i} \theta_j$ as follows: For every $i' < \omega_i$ fix a family $\langle A^\rho : \rho \in \prod_{j < i'} \theta_j \rangle$ of pairwise disjoint subsets of $\theta_{i'} \cap \text{Card}$, each of them of cardinality $\theta_{i'}$. This is possible as $|\prod_{j < i'} \theta_j| < \theta_{i'}$ and $\theta_{i'}$ is a regular limit cardinal. Let I_i be the set of all $\eta \in \prod_{j < \omega_i} \theta_j$ such that for all $j < \omega_i$, $\eta(j) \in A^{\eta \upharpoonright j}$. Define a linear order $<_i$ on I_i as follows: For distinct $\eta, \nu \in I_i$ let $\varepsilon = \min\{j < \omega_i : \eta(j) \neq \nu(j)\}$. Now let

$$\eta <_i \nu \Leftrightarrow \begin{cases} \varepsilon \text{ is even and } \eta(\varepsilon) <_{J_\varepsilon} \nu(\varepsilon), \text{ or} \\ \varepsilon \text{ is odd and } \eta(\varepsilon) < \nu(\varepsilon). \end{cases}$$

We claim that in I_i we can choose a one-to-one family $\langle \eta_{\zeta,\varepsilon}^i : \varepsilon \leq \zeta < \lambda_i \rangle$ such that the following hold:

- (a) $\forall \zeta_1 < \zeta_2 \forall \varepsilon_1 \leq \zeta_1 \forall \varepsilon_2 \leq \zeta_2 \quad \eta_{\zeta_1,\varepsilon_1}^i <_{J_{\omega_i}^{bd}} \eta_{\zeta_2,\varepsilon_2}^i$,
- (b) $\langle \eta_{\zeta,0}^i : \zeta < \lambda_i \rangle$ is cofinal in $\prod_{j < \omega_i} \theta_j / J_{\omega_i}^{bd}$,
- (c) the mapping $\langle (\eta_{\zeta,2\varepsilon}^i, \eta_{\zeta,2\varepsilon+1}^i) : \varepsilon < \zeta \rangle$ is $<_i$ -preserving.

Here $J_{\omega_i}^{bd}$ denotes the ideal of bounded subsets of ω_i . For the construction of such a family remember from [MSh] that $\prod_{j < \omega_i} \theta_j / J_{\omega_i}^{bd}$ has true cofinality λ_i . Clearly, in I_i we can find a family $\langle \eta_\zeta^i : \zeta < \lambda_i \rangle$ which is increasing and cofinal in $\prod_{j < \omega_i} \theta_j / J_{\omega_i}^{bd}$ and satisfies

$\eta_\zeta^i(j) \cdot 3 < \eta_{\zeta+1}^i(j)$ for almost all $j < \omega i$. Now let $\zeta < \lambda_i$ and $2\varepsilon < \zeta$. Define $\eta_{\zeta, 2\varepsilon}^i$ and $\eta_{\zeta, 2\varepsilon+1}^i$ by letting

$$\eta_{\zeta, 2\varepsilon+1}^i(j) = \begin{cases} \eta_\zeta(j) & \text{if } j \text{ is even,} \\ \eta_\zeta(j) + \eta_\varepsilon(j) & \text{if } j \text{ is odd and } l = 0, \\ \eta_\zeta(j) + \eta_\varepsilon(j) \cdot 2 & \text{if } j \text{ is odd and } l = 1. \end{cases}$$

It is easy to see that this definition works.

Lemma 1.5. *In $V^{Q*Q}(\bar{U})$ the following holds: Whenever $\langle J_i : i < \kappa \rangle$ is a family such that for every i , J_i is a (θ_i, D) -entangled linear order on θ_i and I_i is defined as above, then $\langle I_i, <_i \rangle$ is (λ_i, D) -entangled but not (λ', \aleph_0) -entangled for any $\lambda' < \lambda_i$.*

Proof: The last statement easily follows from the existence of the family $\langle \eta_{\zeta, \varepsilon}^i : \varepsilon \leq \zeta < \lambda_i \rangle$. Let $\langle t_{\alpha, \varepsilon, l} : \alpha < \lambda_i, \varepsilon \in A, l < n \rangle$ be a family of pairwise distinct members of I_i , where $A \in D$ and $n < \omega$. Hence

$$t_{\alpha, \varepsilon, l} = \eta_{\zeta(\alpha, \varepsilon, l), \nu(\alpha, \varepsilon, l)}^i$$

for some $\nu(\alpha, \varepsilon, l) \leq \zeta(\alpha, \varepsilon, l) < \lambda_i$. Fix $\alpha < \lambda_i$. As $i < \kappa$ there is $A'_\alpha \in D$ and $i_\alpha^* < \omega i$ such that for all distinct $\varepsilon, \varepsilon' \in A'_\alpha$ and $l, m < n$ do we have $t_{\alpha, \varepsilon, l} \upharpoonright i_\alpha^* \neq t_{\alpha, \varepsilon', l} \upharpoonright i_\alpha^*$, $t_{\alpha, \varepsilon, l} \upharpoonright i_\alpha^* \neq t_{\alpha, \varepsilon, m} \upharpoonright i_\alpha^*$ and $t_{\alpha, \varepsilon, l} \upharpoonright i_\alpha^* \neq t_{\alpha, \varepsilon', m} \upharpoonright i_\alpha^*$. As $2^\kappa < \lambda_i$ we may assume that $\langle A'_\alpha : \alpha < \lambda_i \rangle$ and $\langle i_\alpha^* : \alpha < \lambda_i \rangle$ are constant, say with values A^*, i^* . As in [Sh462, Claim 3.1.1] one shows that there must exist cofinally many even $j \in (i^*, \omega i)$ such that for every $\xi < \theta_j$ there is $\alpha < \lambda_i$ with the property $\forall \varepsilon \in A^* \forall l < n \quad t_{\alpha, \varepsilon, l}(j) > \xi$. Fix such j . Construct an increasing sequence $\langle \alpha(\nu) : \nu < \theta_j \rangle$ such that

$$\forall \nu < \rho < \theta_j \forall \varepsilon, \varepsilon' \in A^* \forall l, m < n \quad t_{\alpha(\nu), \varepsilon, l}(j) < t_{\alpha(\rho), \varepsilon', m}(j).$$

As $(\prod_{l < j} \theta_l)^\kappa < \theta_j$, we may assume that the sequence $\langle \langle t_{\alpha(\nu), \varepsilon, l} \upharpoonright j : \varepsilon \in A^*, l < n \rangle : \nu < \theta_j \rangle$ is constant. Note that by construction,

$$\langle t_{\alpha(\nu), \varepsilon, l}(j) : \nu < \theta_j, \varepsilon \in A^*, l < n \rangle$$

is a sequence of pairwise distinct members. We can apply (θ_j, D) -entangledness of J_j and, for given $u \subseteq n$, we get $B \in D$, $B \subseteq A^*$, and $\nu < \rho < \theta_j$ such that

$$\forall \varepsilon \in B \forall l < n \quad t_{\alpha(\nu), \varepsilon, l}(j) <_{J_i} t_{\alpha(\rho), \varepsilon, l}(j) \Leftrightarrow l \in u.$$

By construction we conclude that

$$\forall \varepsilon \in B \forall l < n \quad t_{\alpha(\nu), \varepsilon, l} <_i t_{\alpha(\rho), \varepsilon, l} \Leftrightarrow l \in u.$$

□

Lemma 1.6. *Letting $I = \prod_{i < \kappa} I_i / D$, I is (λ, \aleph_0) -entangled in $V^{Q*Q}(\bar{U}) * \text{Coll}(\mu^+, < \lambda)$.*

Proof: By Lemmas 1.4 and 1.5 and as $\text{Coll}(\mu^+, < \lambda)$ does not add new subsets to μ , in $V^{Q*Q}(\bar{U}) * \text{Coll}(\mu^+, < \lambda)$ it is true that I_i is (λ_i, D) -entangled and D is a normal fine measure on κ . Moreover, note that $\prod_{i < \kappa} \lambda_i / D$ has order-type λ in $V^{Q*Q}(\bar{U}) * \text{Coll}(\mu^+, < \lambda)$. This is true because it holds in $V^{Q*Q}(\bar{U})$ by [MSh] and because $\text{Coll}(\mu^+, < \lambda)$ does not add new functions to $\prod_{i < \kappa} \lambda_i$. As $V^{Q*Q}(\bar{U}) * \text{Coll}(\mu^+, < \lambda) \models \lambda = \mu^{++}$ we have that the cofinality of $\prod_{i < \kappa} \lambda_i / D$ is λ .

Let $\langle t_\alpha^l : \alpha < \lambda, l < n \rangle$, $n < \omega$, be a family of pairwise distinct elements of I . So t_α^l is of the form

$$t_\alpha^l = \langle \eta_{\zeta_i(\alpha, l), \varepsilon_i(\alpha, l)}^i : i < \kappa \rangle / D,$$

where $\eta_{\zeta_i(\alpha, l), \varepsilon_i(\alpha, l)}^i \in I_i$. By the above observations, wlog we may assume that

(*1) $\langle \langle \zeta_i(\alpha, l) : i < \kappa \rangle / D : \alpha < \lambda \rangle$ is increasing and cofinal in $\prod_{i < \kappa} \lambda_i / D$, for every $l < n$.

For every $\alpha < \lambda$ and $i < \kappa$ there is $j < \omega$ such that for every $l < m < n$, if $\langle \zeta_i(\alpha, l), \varepsilon_i(\alpha, l) \rangle \neq \langle \zeta_i(\alpha, m), \varepsilon_i(\alpha, m) \rangle$ then

$$\eta_{\zeta_i(\alpha, l), \varepsilon_i(\alpha, l)}^i \upharpoonright j \neq \eta_{\zeta_i(\alpha, m), \varepsilon_i(\alpha, m)}^i \upharpoonright j.$$

By Los' Theorem and since D is normal, there exist $B'_\alpha \in D$ and $j'_\alpha < \kappa$ such that for all $i \in B'_\alpha$ and $l < m < n$ we have

$$\eta_{\zeta_i(\alpha, l), \varepsilon_i(\alpha, l)}^i \upharpoonright j'_\alpha \neq \eta_{\zeta_i(\alpha, m), \varepsilon_i(\alpha, m)}^i \upharpoonright j'_\alpha.$$

As $2^\kappa < \lambda$, wlog we may assume that

(*2) there are $B^1 \in D$ and $j' < \kappa$ such that for all $\alpha < \lambda$, $i \in B^1$ and $l < m < n$

$$\eta_{\zeta_i(\alpha, l), \varepsilon_i(\alpha, l)}^i \upharpoonright j' \neq \eta_{\zeta_i(\alpha, m), \varepsilon_i(\alpha, m)}^i \upharpoonright j'.$$

Moreover we have

($*_3$) there exist $B^2 \in D$, $B^2 \subseteq B^1$, and $\langle j_i^2 : i \in B^2 \rangle$ such that $j_i^2 < \omega i$ and for every $g \in \prod_{i \in B^2} \lambda_i$, $f_i \in \prod \{\theta_j : j_i^2 \leq j < \omega i\}$ and $\alpha < \lambda$ we can find $\beta \in (\alpha, \lambda)$ such that for every $i \in B^2$, $j_i^2 \leq j < \omega i$ and $l < n$ we have $g(i) < \zeta_i(\beta, l)$ and

$$f_i(j) < \eta_{\zeta_i(\beta, l), \varepsilon_i(\beta, l)}^i(j).$$

If ($*_3$) failed, for every candidate $y = \langle B^y, \langle j_i^y : i \in B^y \rangle \rangle$ to satisfy ($*_3$) we had g^y , $\langle f_i^y : i \in B^y \rangle$, α^y which witness that y does not satisfy ($*_3$). Note that there are only 2^κ candidates. Let

$$\alpha = \sup\{\alpha^y : y \text{ is a candidate}\}$$

and

$$f_i(j) = \sup\{f_i^y(j) : y \text{ is a candidate and } j \in \text{dom}(f_i^y)\}.$$

As there are only 2^κ candidates we have $\alpha < \lambda$ and $f_i(j) < \theta_j$. We can choose $\beta_i < \lambda_i$ such that $\beta_i > \zeta_i(\alpha, l)$ for every $l < n$ and

$$f_i <_{J_{\omega_i}^{bd}} \eta_{\beta_i, \varepsilon}^i$$

for every $\varepsilon \leq \beta_i$. Finally we define $g \in \prod_{i < \kappa} \lambda_i$ by letting

$$g(i) = \sup\{g^y(i) : y \text{ is a candidate and } i \in B\} \cup \{\beta_i + 1\}.$$

By ($*_1$) we can find $\gamma \in (\alpha, \lambda)$ and $B \in D$ such that $g(i) < \langle \zeta_i(\gamma, l) : i < \kappa \rangle$ for all $i \in B$ and $l < n$. By construction, for every $i \in B$ there is $j_i < \omega i$ such that for all $j_i \leq j < \omega i$ and $l < n$

$$f_j(j) < \eta_{\zeta_i(\gamma, l), \varepsilon_i(\gamma, l)}^i(j).$$

Then $y = \langle B, \langle j_i : i \in B \rangle \rangle$ is a candidate which contradicts the definition of α , $\langle f_i : i < \kappa \rangle$, g . This finishes the proof of ($*_3$).

As D is normal, wlog we may assume that in ($*_3$), $\langle j_i^2 : i \in B^3 \rangle$ is constant with value $j^2 < \kappa$. Now choose $i^* \in B^3$ even with $\max\{j^1, j^2\} < i^*$. Using ($*_3$) it is straightforward to find an increasing sequence $\langle \alpha(\nu) : \nu < \theta_{i^*} \rangle$ in λ such that for all $i \in B^3 \setminus i^* + 1$ and $l, m < n$ we have

$$\eta_{\zeta_i(\alpha(\nu), l), \varepsilon_i(\alpha(\nu), l)}^i(i^*) < \eta_{\zeta_i(\alpha(\nu+1), l), \varepsilon_i(\alpha(\nu+1), l)}^i.$$

As $(\prod_{j < i^*} \theta_j)^\kappa < \theta_{i^*}$, wlog we may assume that

$$\langle \langle \eta_{\zeta_i(\alpha(\nu), l)\varepsilon_i(\alpha(\nu), l)}^i \upharpoonright i^* : i \in B^3 \setminus (i^* + 1), l < n \rangle : \nu < \theta_{i^*} \rangle$$

is constant. By construction we have that, letting

$$s_{\nu, i, l} = \eta_{\zeta_i(\alpha(\nu), l)\varepsilon_i(\alpha(\nu), l)}^i(i^*),$$

$$\langle s_{\nu, i, l} : \nu < \theta_{i^*}, i \in B^3 \setminus (i^* + 1), l < n \rangle$$

is a sequence of pairwise distinct members of I_{i^*} . Hence by Lemma 1.5, for every $u \subseteq n$ we can find $\nu < \xi < \theta_{i^*}$ and $A \in D$, $A \subseteq B^3 \setminus (i^* + 1)$ such that for all $i \in A$ and $l < n$ we have

$$s_{\nu, i, l} < s_{\xi, i, l} \Leftrightarrow l \in u.$$

This implies

$$t_{\alpha(\nu)}^l < t_{\alpha(\xi)}^l \Leftrightarrow l \in u,$$

which finishes the proof. \square

As a corollary we obtain the following:

Theorem 1.7 *For $i < \kappa$ let I_i be the linear order defined above and let $B_i = \text{Int}(I_i)$.*

*In the model $V^{Q*Q(\bar{U})} * \text{Coll}(\mu^+, < \lambda)$ the following hold:*

- (i) $\text{Inc}(B_i) = \text{Inc}^+(B_i) = \lambda_i$ for all $i < \kappa$, and hence $\prod_{i < \kappa} \text{Inc}(B_i)/D = \lambda = \mu^{++}$,
- (ii) $\text{Inc}^+(\prod_{i < \kappa} B_i/D) \leq \lambda$ and hence $\text{Inc}(\prod_{i < \kappa} B_i/D) \leq \mu^+$.

Proof: (i) follows from the fact mentioned in the introduction and Lemma 1.5. Note that Lemma 1.5 holds also in $V^{Q*Q(\bar{U})} * \text{Coll}(\mu^+, < \lambda)$ as $\text{Coll}(\mu^+, < \lambda)$ does not add new subset of μ .

(ii) follows from the same fact, by Lemma 1.6 and by the fact that $\prod_{i < \kappa} B_i/D$ is isomorphic to $\text{Int} \prod_{i < \kappa} I_i/D$. This last fact holds by Los' Theorem and as D is \aleph_1 -complete.

\square

2. Other characteristics

Definition 2.1. If $(I, <)$ is a linear order, by $Sq(I)$ we denote the Boolean subalgebra of $(\mathcal{P}(I^2), \subseteq)$ generated by sets of the form

$$X_{a,b} = \{(a', b') \in I^2 : a' < a \text{ and } b' < b\},$$

for $a, b \in I$.

Recall that a sequence $\langle y_\alpha : \alpha < \lambda \rangle$ of elements of some Boolean algebra is *left-separated* iff for every $\alpha < \lambda$, y_α does not belong to $Id\langle y_\beta : \beta > \alpha \rangle$, the ideal generated by $\langle y_\beta : \beta > \alpha \rangle$. Similarly, $\langle y_\alpha : \alpha < \lambda \rangle$ is *right-separated* if for every $\alpha < \lambda$, y_α does not belong to $Id\langle y_\beta : \beta < \alpha \rangle$, the ideal generated by $\langle y_\beta : \beta < \alpha \rangle$.

Lemma 2.2. *Suppose $(I, <)$ is a λ -entangled linear order, where $\lambda > \omega$ is regular. Then $Sq(I)$ has neither a left-separated nor a right-separated sequence of length λ .*

Proof: We prove the Lemma only for right-separated sequences. The proof for left-separated sequences is similar. Suppose $\langle y_\alpha : \alpha < \lambda \rangle$ is a right-separated sequence in $Sq(I)$. We shall obtain a contradiction. Each y_α is a finite union of finite intersections of sets of the form $X_{a,b}$ or $-X_{a,b}$. One of these finite intersections does not belong to $Id\langle y_\beta : \beta < \alpha \rangle$. Hence wlog we may assume that each y_α is such a finite intersection. As $cf(\lambda) > \omega$, wlog there exist $n < \omega$ and $\eta : n \rightarrow 2$ such that

$$y_\alpha = \bigcap_{l < n} X_{a(\alpha,l), b(\alpha,l)}^{\eta(l)}$$

for some $a(\alpha, l), b(\alpha, l) \in I$, for all $\alpha < \lambda$.

Case I: $\exists l < n \quad \eta(l) = 1$.

As the intersection of any two sets of the form $X_{a,b}$ has the same form, wlog we may assume that $\eta(0) = 1$ and $\eta(l) = 0$ for all $0 < l < n$. We may also assume that $0 < l < l' < n$ implies $a(\alpha, l) \neq a(\alpha, l')$, $b(\alpha, l) \neq b(\alpha, l')$ and $a(\alpha, l) < a(\alpha, l') \Leftrightarrow b(\alpha, l) > b(\alpha, l')$, for all $\alpha < \lambda$. Otherwise we could choose a smaller n . Hence we have two subcases according to whether $a(\alpha, 1) < \dots < a(\alpha, n-1)$ and $b(\alpha, 1) > \dots > b(\alpha, n-1)$ or $a(\alpha, 1) > \dots > a(\alpha, n-1)$ and $b(\alpha, 1) < \dots < b(\alpha, n-1)$ holds. We assume the first alternative holds. The second one is symmetric.

For fixed $\alpha < \lambda$ define the following sets:

$$z_0 = X_{a(\alpha,0), b(\alpha,0)} - X_{a(\alpha, n-1), b(\alpha,0)},$$

$$z_1 = X_{a(\alpha, n-1), b(\alpha,0)} - X_{a(\alpha, n-1), b(\alpha, n-1)} - X_{a(\alpha, n-2), b(\alpha,0)},$$

...

$$z_{n-2} = X_{a(\alpha,2), b(\alpha,0)} - X_{a(\alpha,2), b(\alpha,2)} - X_{a(\alpha,1), b(\alpha,0)},$$

$$z_{n-1} = X_{a(\alpha,1), b(\alpha,0)} - X_{a(\alpha,1), b(\alpha,1)}.$$

Note that $y_\alpha = \bigcup_{j < n} z_j$. Hence there exists $j < n$ such that $z_j \notin Id\langle y_\beta : \beta < \alpha \rangle$. Wlog we may assume that j is the same for all $\alpha < \lambda$ and that $y_\alpha = z_j$ for all $\alpha < \lambda$. Then y_α has the form $X_{a,b} - X_{a',b} - X_{a,b'}$ or $X_{a,b} - X_{a',b}$ or $X_{a,b} - X_{a,b'}$ for some $a' < a$ and $b' < b$. Let us assume y_α is of the first form. The others are even easier to handle. Hence we have

$$y_\alpha = X_{c(\alpha,0),d(\alpha,0)} - X_{c(\alpha,0),d(\alpha,1)} - X_{c(\alpha,1),d(\alpha,0)},$$

where $c(\alpha, 1) < c(\alpha, 0)$ and $d(\alpha, 1) < d(\alpha, 0)$.

Choose $F \subseteq 2 \times 2$ maximal such that there exist $\sigma : F \rightarrow I$ and cofinally many $\alpha \in \lambda$ with the property that $(0, j) \in F$ implies $c(\alpha, j) = \sigma(0, j)$ and $(1, j) \in F$ implies $d(\alpha, j) = \sigma(1, j)$ for all $j < 2$. Wlog we may assume that the above holds for all $\alpha < \lambda$ and that for all $\alpha < \beta < \lambda$ and $(i, j) \in 2 \times 2 \setminus F$, if $i = 0$ then $c(\alpha, j) \neq c(\beta, j)$ and if $i = 1$ then $d(\alpha, j) \neq d(\beta, j)$. Depending on F we have 16 cases to consider. However we consider only the case $F = \emptyset$, as the others are similar.

We have more subcases to consider according to the order-type of the sequence $\langle c(\alpha, 0), c(\alpha, 1), d(\alpha, 0), d(\alpha, 1) \rangle$. Wlog we may assume that it does not depend on α . We only work through two typical examples. Let us first assume that this sequence consists of pairwise distinct elements. As we assumed $F = \emptyset$ we conclude that $\langle c(\alpha, j), d(\alpha, j) : \alpha < \lambda, j < 2 \rangle$ is a family of pairwise distinct elements. By λ -entangledness of I we get $\alpha > \beta$ such that $c(\alpha, 0) < c(\beta, 0)$, $d(\alpha, 0) < d(\beta, 0)$, $c(\alpha, 1) > c(\beta, 1)$ and $d(\alpha, 1) > d(\beta, 1)$. We conclude $y_\alpha \leq y_\beta$, a contradiction. Now suppose $c(\alpha, 0) = d(\alpha, 0) < c(\alpha, 1) < d(\alpha, 1)$. In this case the family $\langle c(\alpha, j), d(\alpha, 1) : \alpha < \lambda, j < 2 \rangle$ consists of pairwise distinct elements. By λ -entangledness we obtain $\alpha > \beta$ such that $c(\alpha, 0) < c(\beta, 0)$, $c(\alpha, 1) > c(\beta, 1)$ and $d(\alpha, 1) > d(\beta, 1)$. Again we conclude $y_\alpha \leq y_\beta$, a contradiction. The other cases are similar.

Case II: $\forall l < n \quad \eta(l) = 0$.

Again we may assume that $a(\alpha, 0) < a(\alpha, 1) < \dots < a(\alpha, n-1)$ and $b(\alpha, 0) > b(\alpha, 1) > \dots > b(\alpha, n-1)$ for all $\alpha < \lambda$. Notice that wlog we may assume that

$$X_{a(\alpha, n-1), b(\alpha, 0)}^0 \notin Id\langle y_\beta : \beta < \alpha \rangle$$

for all $\alpha < \lambda$, as otherwise we may replace y_α by $y_\alpha \cap X_{a(\alpha, n-1), b(\alpha, 0)}$ and proceed as in Case I. Hence wlog

$$y_\alpha = X_{a(\alpha, n-1), b(\alpha, 0)}^0$$

for all $\alpha < \lambda$. Let $a_\alpha = a(\alpha, n - 1)$, $b_\alpha = b(\alpha, 0)$. Clearly, if $a_\alpha = a_\beta$ for some $\alpha < \beta$ then $b_\alpha < b_\beta$, as otherwise $y_\alpha \leq y_\beta$. Similarly, $b_\alpha = b_\beta$ implies $a_\alpha < a_\beta$. As a λ -entangled linear order does not have any increasing or decreasing sequences of length λ , wlog we may assume that both families $\langle a_\alpha : \alpha < \lambda \rangle$ and $\langle b_\alpha : \alpha < \lambda \rangle$ are one-to-one. By a similar argument we may assume that $a_\alpha \neq b_\alpha$ for all $\alpha < \lambda$ and also that $a_\alpha \neq b_\beta$ for all $\alpha \neq \beta$. We can apply λ -entangledness of I to the family $\langle a_\alpha, b_\alpha : \alpha < \lambda \rangle$ and get some $\alpha > \beta$ such that $a_\alpha > a_\beta$ and $b_\alpha > b_\beta$. Hence $y_\alpha \leq y_\beta$, a contradiction. \square

Lemma 2.3. *Let $(I, <)$ be a linear order and μ a cardinal such that there exist $\{(a_\alpha, b_\alpha) : \alpha < \mu\} \subseteq I^2$ and $c \in I$ with the property that $a_\alpha \neq a_\beta$, $b_\alpha < c$ and that $a_\alpha < a_\beta$ implies $b_\alpha < b_\beta$ for all $\alpha, \beta < \mu$, $\alpha \neq \beta$. Then $s^+(Sq(I)) > \mu$ holds.*

Proof: Let $y_\alpha = X_{a_\alpha, c} - X_{a_\alpha, b_\alpha}$, for $\alpha < \mu$. Note that

$$y_\alpha \not\leq \bigcup_{\beta \in F} y_\beta$$

for all $\alpha < \mu$ and finite $F \subseteq \mu$ with $\alpha \notin F$. Indeed, let $F_0 = \{\beta \in F : a_\alpha < a_\beta\}$, $F_1 = F \setminus F_0$, let β_0 be the subscript of the smallest a_β , $\beta \in F_0$ and let β_1 be the subscript of the largest a_β , $\beta \in F_1$. Then $y_\alpha \setminus \bigcup_{\beta \in F} y_\beta = y_\alpha \setminus (y_{\beta_0} \cup y_{\beta_1})$. As $(a_\alpha, b_\alpha) \in y_\alpha \setminus (y_{\beta_0} \cup y_{\beta_1})$ we are done. Hence there exists a family of ultrafilters $\langle U_\alpha : \alpha < \mu \rangle$ with $y_\alpha \in U_\alpha$ and $-y_\beta \in U_\alpha$ for all $\alpha \neq \beta$. Then $\langle U_\alpha : \alpha < \mu \rangle$ is a discrete set of cardinality μ in the Stone space of $Sq(I)$. \square

Corollary 2.4. *Using the notation of §1, letting $B_i = Sq(I_i)$ for $i < \kappa$, in the model $V^{Q*Q(\bar{V})} * Coll(\mu^+, < \lambda)$ the following hold:*

(i) $s(B_i) = s^+(B_i) = hL(B_i) = hL^+(B_i) = hd(B_i) = hd^+(B_i) = \lambda_i$ for all $i < \kappa$, and hence $|\prod_{i < \kappa} s(B_i)/D| = |\prod_{i < \kappa} hL(B_i)/D| = |\prod_{i < \kappa} hd(B_i)/D| = \lambda = \mu^{++}$,

(ii) $hL^+(\prod_{i < \kappa} B_i/D) = hd^+(\prod_{i < \kappa} B_i/D) \leq \lambda$ and hence $s(\prod_{i < \kappa} B_i/D)$, $hL(\prod_{i < \kappa} B_i/D)$ and $hd(\prod_{i < \kappa} B_i/D)$ are all at most μ^+ . \square

Proof: We first prove (i). The proofs of Theorem 6.7 and Lemma 6.8 in [M] show that for every Boolean algebra B , if $hd(B) = \kappa$, κ being regular and infinite, then $hd(B)$ is attained (i.e. there exists a subspace $X \subseteq Ult(B)$ with $d(X) = \kappa$) iff B has a left-separated sequence of length κ . Similarly, the proof of Theorem 15.1 in [M] shows that if $hL(B)$ is regular and infinite, then $hL(B) = \kappa$ is attained iff B has a right-separated sequence of length κ . As trivially $s^+(B) \leq \min\{hL^+(B), hd^+(B)\}$ and hence $s(B) \leq$

$\min\{hL(B), hd(B)\}$ holds, we conclude that all cardinal coefficients of B_i mentioned in (i) are at most λ_i . That they are at least λ_i follows from Lemma 2.3, the construction of I_i and the trivial fact that every linear order of cardinality μ^+ , for some cardinal μ , has a subset of size μ which has an upper bound.

In order to prove (ii) note that by Los' Theorem and \aleph_1 -completeness of D we have that $\prod_{i<\kappa} B_i/D$ is isomorphic to $Sq(\prod_{i<\kappa} I_i/D)$. By Lemmas 1.6 and 2.2 and the previous argument we get (ii). \square

Definition 2.5. Let $\langle y_\alpha : \alpha < \lambda \rangle$ be a one-to-one enumeration of some infinite linear order $(J, <_J)$. Define a linear order $(L(J), <)$ by letting $L(J) = \{(y_\alpha, \beta) : \alpha < \lambda, \beta < \alpha\}$ and

$$(y_\alpha, \beta) < (y_{\alpha'}, \beta') \Leftrightarrow (y_\alpha <_J y_{\alpha'}) \vee (y_\alpha = y_{\alpha'} \wedge \beta < \beta').$$

Lemma 2.6. *Let σ be an infinite, regular cardinal which is not the successor of a singular cardinal. Let $(J, <_J)$ be a linear order of size λ which does not have any increasing or decreasing chain of length λ . Then*

$$\chi^+(Int(L(J))) = \pi\chi^+(Int(L(J))) = \lambda$$

holds.

Proof: As trivially $\pi\chi^+(B) \leq \chi^+(B)$ holds for every Boolean algebra B , it suffices to show $\chi^+(Int(L(J))) \leq \lambda$ and $\pi\chi^+(Int(L(J))) \geq \lambda$. Let U be an ultrafilter on $Int(L(J))$. Let

$$L_U = \{z \in L(J) : (-\infty, z) \in U\}.$$

Clearly L_U is a (possibly empty) end-segment of $L(J)$. It is straightforward to see that

$$\chi(U) \leq cf(L \setminus L_U) + cf(L_U^*),$$

where the cofinality of a linear order is the minimal length of a well-ordered cofinal subset, and L_U^* is the inverse order of L_U . We claim that $cf(L \setminus L_U) + cf(L_U^*) < \lambda$. Let us first consider $cf(L \setminus L_U)$. If $(L \setminus L_U) \cap J \times \{0\}$ is unbounded in $L \setminus L_U$ then $cf(L \setminus L_U)$ equals the cofinality of some well-ordered increasing chain in J , which is assumed to be $< \lambda$. Otherwise $L \setminus L_U \subseteq \{(y_\beta, \gamma) : \beta \leq \alpha, \gamma < \beta\}$ for some $\alpha < \lambda$. Then $cf(L \setminus L_U) \leq |\alpha| < \lambda$. We conclude $\chi^+(Int(L(J))) \leq \lambda$.

In order to prove $\pi\chi^+(Int(L(J))) \geq \lambda$ let $\sigma < \lambda$ be regular. Let U be the ultrafilter on $Int(L(J))$ generated by the intervals

$$[(y_{\sigma+1}, \alpha), (y_{\sigma+1}, \sigma)), \quad \alpha < \sigma.$$

Now let $Y \subseteq Int(L(J)) \setminus \{0\}$ be dense in U . If $|U| < \sigma$ there exists $y \in Y$ such that $y \subseteq [(y_{\sigma+1}, \alpha), (y_{\sigma+1}, \sigma))$ holds for cofinally many $\alpha < \sigma$. This is clearly impossible. \square

Corollary 2.7. *Using the notation of §1 and definition 2.5, letting $B_i = Int(L(I_i))$ for $i < \kappa$, in the model $V^{Q*Q(\bar{U})*Coll(\mu^+, < \lambda)}$ the following hold:*

(i) $\pi(B_i) = \pi^+(B_i) = \pi\chi(B_i) = \pi\chi^+(B_i) = \lambda_i$ for all $i < \kappa$, and hence $|\prod_{i < \kappa} \chi(B_i)/D| = |\prod_{i < \kappa} \chi^+(B_i)/D| = \lambda = \mu^{++}$,

(ii) $\chi^+(\prod_{i < \kappa} B_i/D) = \pi\chi^+(\prod_{i < \kappa} B_i/D) = \lambda$ and hence $\chi(\prod_{i < \kappa} B_i/D) = \pi\chi(\prod_{i < \kappa} B_i/D) = \mu^+$. \square

Proof: First note that I_i , $i < \kappa$, has a dense subset of size $\mu_{\omega i}$. Indeed, for each $s \in \bigcup_{j' < \omega i} \prod_{j < j'} \theta_j$ choose $\eta_s \in I_i$ with $s \subseteq \eta_s$ if this is possible. It is easy to see that the collection of all these η_s is dense in I_i . As there are only $\mu_{\omega i}$ many s we are done. Hence clearly I_i does not have a well-ordered increasing or decreasing chain of length λ_i . Hence by Lemma 2.6 we have (i). By Los' Theorem and \aleph_1 -completeness of D we have that $\prod_{i < \kappa} B_i/D$ is isomorphic to $Int(L(\prod_{i < \kappa} I_i/D))$. By Los' Theorem again and as $\prod_{i < \kappa} \lambda_i = \lambda$, it follows that $\prod_{i < \kappa} I_i/D$ does not have a well-ordered increasing or decreasing chain of length λ . By Lemma 2.6 we conclude (ii). \square

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