A PARTITION THEOREM
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Abstract. We deal with some relatives of the Hales Jewett theorem with primitive recursive bounds.

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Content

§0 Introduction

We prove the following: there is a primitive recursive function $f^*_-(-,-)$, in three variables in the integers, such that: for all natural numbers t, n, c and k, if $n > 0, t > 0$ and $k \geq f^*_t(n,c)$, then the following holds. Assume that Λ is an alphabet with $n > 0$ letters, M is the family of non empty subsets of $\{1,\ldots,k\}$ with at most t members and V is the set of functions from M to Λ and d is a c-colouring of V (i.e. a function with domain V and range with at most c members). Then there is a d-monochromatic V-line, which means that there are $w \subseteq \{1,\ldots,k\}$, with at least t members and function ρ from $\{u \in M : u$ not a subset of $w\}$ to Λ such that

(∗) $d \restriction L_\rho$ is constant which means $\rho, \nu \in L_\rho \Rightarrow d(\rho) = d(\nu)$ where $L_\rho =:$ $\{\eta \in V :$ the function η extend ρ and for each $s = 1,\ldots,t$ it is constant on $\{u \in M : u \subseteq w$ has exactly s members $\}\}$,
(for $t = 1$ those are the Hales-Jewett numbers).

A second theorem relates to the first just as the affine Ramsey theorem of Graham, Leeb and Rothschild (which continues the n-parameter Ramsey theorem of Graham and Rothschild), relates to the Hales Jewett theorem. We also note an infinitary related theorem parallel to the Galvin-Prikry theorem and the Carlson-Simpson theorem.

Let us review history and background, not repeating what is said in [GRS80]. In the late seventies, Furstenberg and Sarkozy independently proved that if $(p(x)$ is a polynomial in $\mathbb{Z}[\mathbf{x}]$ satisfying $p(0) = 0$ and $A \subseteq \mathbb{N}$ is a set of positive density then for some $a, b \in A$ and $n \in \mathbb{N}$ we have $a - b = \mathbf{p}(n)$. Bergelson and Leibman [BeLe96] continuing Furstenberg [Fu] prove (they also prove a density theorem like Szemerédi): if r, k, t, m are natural numbers, $\mathbf{p}_{\ell,s}(x)$ for $\ell = 1,\ldots,k$ and $s = 1,\ldots,t$ are polynomials with rational coefficients, taking integer values at integers, and vectors $\bar{v}_1,\ldots,\bar{v}_t \in {}^m\mathbb{Z}$ and any $r-$colouring of ${}^m\mathbb{Z}$ there are $\bar{a} \in {}^m\mathbb{Z}$ and $n \in \mathbb{Z}(n \neq 0)$ such that the set $S(\bar{a},n) = \{\bar{a} + \S_{j=1,t}\mathbf{p}_{i,j}(n)\bar{v}_j : i = 1,\ldots,k\}$ is monochromatic.

Bergelson and Leibman [BeLe99] prove a theorem, "set polynomial extension", which is, in a different formulation, like the first theorem described above but without a bound (i.e. the primitive recursiveness). Their method is infinitary so does not seem to give even the weak bound in 2.5 (one with triple induction), and certainly does not give primitive recursive bounds.

Naturally our proofs continue [Sh 329]. We thank friends of Bruce Rothchild for telling us about [BeLe99] of which we were not aware and other helpful comments. See a discussion of related problems in [Sh 702].

No previous knowledge is assumed (if the reader does not like the notions of "structure" and "vocabulary" which are fully explained in Definition 1.1, he can use the alternative in 1.2).

The connection between the theorems stated in the beginning and our framework is as in 1.16.

0.1 Notation.

(a) We use Λ for a finite alphabet, always non empty, members of which are denoted by α, β, γ

(b) We use M, N to denote structures (see Definition 1.1) which serve as index sets, so we call them index models. We use τ to denote vocabularies, (see Definition 1.1), F to denote function symbols and P to denote predicate symbols. Let fim mean full index model.

(c) We use $n, m, k, \ell, i, j, c, r, s, t$ to denote natural numbers, but usually n is the number of letters, i.e. the number of members in an alphabet; k the dimension of the index models and $c \geq 1$ the number of colours

(d) $|X|$ and also $\text{card}(X)$ denote the number of elements of the set X

(e) We use η, ν, ρ to denote members of (combinatorial) spaces, we use V, U to denote (combinatorial) spaces and a, b to denote elements of M, N and d to denote colourings, p to denote the "type" of a point in a line and $\mathfrak{p}$ to denote the type of a line or a space (see Definition 1.9(3)). We use L to denote (combinatorial) lines, S to denote (combinatorial) subspaces

(f) *A* bar on a symbol, say $\bar{x}$ denote finite sequences of such objects with the length of $\bar{x}$ being $\ell g(\bar{x})$ and the i-th element in the sequence being x_i (and for $\bar{x}_m$ or $\bar{x}^m$ it is x_i^m)

(g) let ${}^m X = \{(x_0, \ldots, x_{m-1}) : x_0, \ldots, x_{m-1} \in X\}$.

0.2 Definition. 1) For $m \geq 1$, let $\mathbb{E}_m$ be the minimal class of functions from natural numbers to natural numbers (with any number of places) that is closed under composition, which for $m = 1$ contains $0, 1, x+1$ and the projection functions, and for $m > 1$ contains any function which we get by inductive definition on functions from $\mathbb{E}_{m-1}$. See [Ro84], so $\mathbb{E}_3$ is the family of polynomials, $\mathbb{E}_4$ contains the tower function (and $\mathbb{E}_5$ contains the so-called waw function) and $\bigcup\limits_{m \geq 1} \mathbb{E}_m$ is the family of primitive recursive functions, and the "simplest" function not there is the Ackerman function.
2) We allow an object like $\Lambda, \bar{\Lambda}$ to be one of the arguments meaning, if $\bar{\Lambda} = \langle \Lambda_F :$

$F \in I\rangle$ then we replace it by $\prod\limits_{F \in \tau}(|\Lambda_F| + |\text{arity } \tau(F)|)$ and Λ we replace by $|\Lambda|$. Abusing notation, we may say "f is in $\mathbb{E}_n$" instead of "f is bounded by a function from $\mathbb{E}_n$", also writing $f_{\bar{\Lambda}}(-,\dots)$ we count $\bar{\Lambda}$ as one of the arguments.

2) We can define the Ackerman function $A_n(m)$ by double induction:

$A_0(m) = m+1$,

$A_n(0) = n$,

$A_{n+1}(m+1) = A_n(A_{n+1}(m))$.

(In a sense it is the simplest, smallest function which is not primitive recursive).

0.3 Definition. 1) Let $\mathrm{RAM}(t,\ell,c)$ be the Ramsey number, i.e. the first k such that $k \rightarrow (t)^\ell_c$ see below.

2) Let $k \rightarrow (t)^\ell_c$ mean that if A is a set with k elements, and d is a c-colouring of $[A]^\ell =: \{B : B \text{ is a subset of } A \text{ with } \ell \text{ elements}\}$, that is a function with this domain and range of cardinality $\le c$, then for some $A_1 \in [A]^t$ we have $d \restriction [A_1]^\ell$ is constant.

3) Let $\mathrm{HJ}(n,m,c)$ be the Hales-Jewett number for getting a monochromatic subspace of dimension m, when the colouring has c colours and for an alphabet with n members. (This is, by our subsequent definitions, $f^1(\bar{\Lambda},m,c)$ when $\tau(\bar{\Lambda}) = \{\mathrm{id}\}$, and Λ_{id} has n members, see Definition 1.11).

§1 Basic definitions

We can look at the Hales-Jewett theorem in geometric terms: $\mathbb{R}$ is replaced by Λ, a finite alphabet; the k-dimensional euclidean space is replaced by ${}^{[1,k]}\Lambda$; (or ${}^{[0,k)}\Lambda$), essentially the set of sequences of length k of members of the alphabet Λ; a subspace is replaced by the set of solutions $(x_1,\dots,x_k) \in {}^{[1,k]}\Lambda$ of a family of linear equations, which here means just $x_i = \alpha$ (where $\alpha \in \Lambda, 1 \le i \le k$) or $x_i = x_j$. Here the basic set $[1,k]$ is replaced by a structure M, a τ-fim (= full index model, see below). Such basic definitions are given in this section.

We define a "space over an index model of dimension k, over an alphabet Λ of size n", we define for such a space lines and more. We then define the function f^1, such that for every n, if k is $f^1_\tau(n,c)$ then for every colouring of the space by $\le c$ colours, there is a monochromatic line (in the appropriate interpretation.) Of course the use of id as a special function symbol is not really needed, also we can waive the linear order on P^M, and the set of automorphisms of the resulting structure are natural for our purpose, but not for the structures from 1.13(3); but at present those decisions do not matter.

If Definition 1.1(1) is not clear, try Definition 1.2. The point is that though we like, for a finite linear order J, to colour functions from $\{u : u \subseteq J, 1 \le |u| \le t\}$ to an alphabet Λ, we may need to use several copies of the same u, and we find it convenient to consider nondecreasing sequences of length r instead of subsets of size r (you can say that we consider subsets of J with possibly multiple membership).

1.1 Definition. 1) We call M a full index model [fim or τ-fim or fim for τ] if M consists of a set τ of symbols, the universe M, and appropriate interpretation for each symbol of τ; in full detail if:

- (a) the vocabulary $\tau = \tau_M = \tau(M)$ of M includes a unary predicate[1] P, a binary predicate $<$, and finitely many function symbols F, F being arity(F)-place and no other symbols (so F varies over such function symbols). We may write arity$^\tau(F)$ for arity(F). We usually treat τ as the set of function symbols in τ
- (b) univ(M), the universe of M which is a finite, nonempty set; we may write $a \in M$ instead of $a \in$ univ(M) and $B \subseteq M$ instead of $B \subseteq$ univ(M)
- (c) P^M is a non empty subset of the universe of M and we call its cardinality dim(M), the dimension of M
- (d) $<^M$ is a linear order of P^M, so $x <^M y$ implies $x, y \in P^M$

[1] a predicate P is just a symbol such that in τ-structures M it is interpreted as P^M, similarly for function symbols.

(e) for $F \in \tau, F^M$ is a function with arity ${}^\tau(F)$-places from P^M to univ(M) such that

(f) the function F^M is symmetric, i.e. does not depend on the order of the arguments, and so writing $F^M(a_1,\dots,a_r)$ we assume $a_1,\dots,a_r \in P^M$ and, of course, $r =$ arity$^\tau(F)$, and if not said otherwise we assume $a_1 \le^M a_2 \le^M \dots \le^M a_r$

(g) if $F_1^M(a_1,\dots,a_r) = F_2^M(b_1,\dots,b_t)$ then $F_1 = F_2$ (hence $r = t$) and $a_\ell = b_\ell$ for $\ell = 1,\dots,t$ (under the convention from clause (f)) and every $b \in M \setminus P^M$ has this form. So we let base$_M(b) =: \{a_1,\dots,a_r\}$ and let base$_\ell(b) =$ base$_{M,\ell}(b) =: a_\ell$ where $b = F^M(a_1,\dots,a_r)$ (and $a_1 \le^M a_2 \le^M \dots \le^M a_r$ of course) and $F_{M,b} =: F$; those are well defined by the previous sentence.

(h) id^M is the identity function on P^M, so id is always considered a unary function symbol of τ.

2) For τ as in part 1), let arity(τ) be Max$\{$arity$(F) : F \in \tau\}$, so it is at least 1 by clause (h) of part 1) and let $\bar m[\tau] =: \langle m_t^\tau : t = 1,\dots,\text{arity}(\tau)\rangle$ where m_t^τ is the number of $F \in \tau$ with arity t; and we call $\bar m^\tau$ the signature of τ, of course when saying "the signature of M" we mean "of $\tau(M)$".
3) For M a fim we call $B \subseteq M$ closed in M (or M-closed) if for $b = F^M(a_1,\dots,a_s)$ we have $b \in B$ iff $a_1,\dots,a_s \in B$. Let the closure of A in M or cl$_M(A)$ for $A \subseteq M$, be the minimal M-closed set $B \subseteq M$ which includes A. We do not strictly distinguish between a closed subset B of M and the model $M \restriction B$ (which are fims with the same vocabulary, see part 6), 7)).
4) For τ-index models M, N with $\tau_M = \tau_N$ let PHom(M,N) be the set of functions f from P^M into P^N such that $x \le^M y \Leftarrow f(x) \le^N f(y)$. Let Hom$(M,N)$ be the set of functions f from M into N such that $f \restriction P^M \in$ PHom(M,N) and $b = F^M(a_1,\dots,a_t)$ implies $f(b) = F^N(f(a_1),\dots,f(a_t))$. Let PHm$(M,N)$ be the set of functions f from P^M into P^N, and let Hm(M,N) be the set of functions f from M into N such that $f \restriction P^M \in PHm(M,N)$ and $b = F^M(a_1,\dots,a_t)$ implies $f(b) = F^N(f(a_1),\dots,f(a_t))$ recalling that F^M is symmetric.
5) Let Sort$^M(F)$ be the range of F^M.
6) For M a τ-form and non empty subset u of P^M, we say that N is the submodel of M with universe $B = cl_M(u), N = M \restriction B$ in symbols, if N is the unique τ-fims such that

univ$(N) = c\ell_M(u)$

$P^N = u$

$<^N = <^M \restriction u$, i.e. $a <^N b \Leftrightarrow a \in u \;\&\; b \in u \;\&\; a <^M b$

$F^N = F^M \restriction u$, (i.e. $F^N(a_1,\dots,a_r) = b$ iff $r =$ arity$(F), a_1,\dots,a_r \in u$ and $b = F^M(a_1,\dots,a_r)$).

So N is uniquely determined by M and univ(N) when N is a submodel of N; and there is $N \subseteq M$, univ$(N) = B$ iff B is a closed subset of M.

7) We say that N is a submodel of M if for some non empty $u \subseteq P^M, N$ is the submodel of M with universe $c\ell_M(u)$. We also say M extends N; if P^N is an initial segment of P^M we say M end-extends N.

An alternative presentation of 1.1(1) is:

1.2 Definition. We say that τ is a t-vocabulary for

- (A) τ is a function with finite domain, but allow to write $F \in \tau$ instead $F \in \mathrm{Dom}(\tau)$ and write $\mathrm{arity}^\tau(F)$ instead of $\tau(F)$. We may write $F \in \tau$ instead $F \in \mathrm{Dom}(\tau)$, always id $\in \tau$ so we may not say it.
- (B) t is a natural number. Omitting t means for some t; we let $t = t(\tau)$
- (C) if $F \in \tau$ then $\mathrm{arity}^\tau(F) \in \{1, 2, \ldots, t\}$
- (D) $t = \max\{\mathrm{arity}^\tau(F), \mathrm{arity}^\tau(\mathrm{id}) : F \in \tau\}$
- (E) $\mathrm{id} \in \tau, \mathrm{arity}^\tau(F), \mathrm{arity}^\tau(\mathrm{id}) = 1$.

2) We say that M is a τ-fim (or fim for τ) if for some linear order $J, M = M_\tau(J)$, see below.
3) For a linear order J and vocabulary τ let $M = M_\tau(J)$ be the following object: it consists of

- (a) $\tau_M = \tau(M) = \tau$
- (b) $\mathrm{univ}(M) = \{(F, \bar{a}) : \bar{a}$ is a $\leq_J$-increasing sequence of members of $J, F \in \tau$ and $\ell g(\bar{a}) = \mathrm{arity}^\tau(F)\}$ but if $\bar{a} = \langle a \rangle$ we may write a instead of $\bar{a}$
- (c) $P^M = \{(\mathrm{id}, \{a\}) : a \in J\}$, we tend to identify $a \in J$ with (id,$\{a\}$)
- (d) $<^M$ is the following linear order on P^M

$$(\mathrm{id}, \{a\}) <^M (\mathrm{id}, \{b\}) \text{ iff } a <_J b$$

- (e) for $F \in \tau$ with $r = \mathrm{arity}^\tau(F)$ let F^M be an r-place function from P^M to $\mathrm{univ}(M)$ defined by

$$F(a_1, \ldots, a_r) = (F, (a_{\pi(1)}, \ldots, a_{\pi(r)}))$$

  where π is a permutation of $\{1, \ldots, r\}$ such that $a_{\pi(1)} \leq_J a_{\pi(2)} \leq_J \ldots \leq_J a_{\pi(r)}$.

4) The signature of τ is $\langle m_s^\tau : s = 1, \ldots, s\rangle$ where $m_s^\tau = |\{F \in \tau : \mathrm{arity}^\tau(F) = m\}|$.
5) In 2), if $x = (F, \bar{a})$ then we let $\mathrm{base}_M(x) = \{a_\ell : \ell = 1, \ldots, \ell g(\bar{a})\}$, $\dim(M) = |P^M|$.

1.3 Fact. 1) For any finite linear order J and vocabulary $\tau, M_\tau(J)$ as defined in 1.2(3) is really a τ-fim.

1.4 Fact. 1) For any $f \in \mathrm{PHom}(M,N)$ there is a unique $\hat{f} \in \mathrm{Hom}(M,N)$ which extends f.
2) For any $f \in \mathrm{PHm}(M,N)$ there is a unique $\hat{f} \in \mathrm{Hm}(M,N)$ which extends f.

1.5 Claim/Definition. 1) For any fim M there is a polynomial $\mathbf{p}(x)$, with rational coefficients but positive integers as values for x a positive integer such that for $u \subseteq P^M$, the set $\mathrm{cl}_M(u)$ has exactly $\mathbf{p}(|u|)$ members.
2) Fixing a vocabulary τ, let $\mathbf{p}(x) = \Sigma\{\mathbf{p}_F(x) : F \in \tau\}$ where $\mathbf{p}_F(x) = \Sigma\{\mathrm{binom}(x, \langle m_1,\dots,m_n\rangle) : 1 \le n \le \mathrm{arity}^\tau(F)$ and $m_1 + m_2 + \ldots + m_r = \mathrm{arity}^\tau(F)\}$. Then for any τ-fim M we have $\mathrm{card}(\mathrm{univ}(M)) = \mathbf{p}(|P^M|)$ and $|\mathrm{Sort}^F(M)| = \mathbf{p}_F(|P^M|)$, recall Definition 1.1(5).
3) Now $\mathbf{p}(x)$ depends on the signature of τ only and so we shall denote it by $\mathbf{p}_\tau(x)$ or $\mathbf{p}_M(x)$. Note that $\mathbf{p}_\tau(0) = 0$.

1.6 Definition. 1) We say that τ is a canonical vocabulary for t (or t-canonical) and write $\tau = \tau_t$ if $\tau = \{F_1,\dots,F_t,P,<\}$ where $\mathrm{arity}(F_s)$ is s, F_1 is also called id; (in Definition 1.2 we have $\mathrm{Dom}(\tau) = \{F_1,\dots,F_t\}$ and $\tau(F_r) = r$).
2) We say that M is a (J,t)-canonical fim if:

- (a) J is a finite linear order
- (b) M is a fim with the t-canonical vocabulary
- (c) $(P^M, <^M)$ is J
- (d) F_1^M is the identity on P^M
- (e) for $r = 2,\dots,t$ the function F_r^M is $F_r^M(a_1,\dots,a_r) =: \langle a_{\pi(1)},\dots,a_{\pi(r)}\rangle$ whenever π is a permutation of $\{1,\dots,r\}$ such that $a_{\pi(1)} \le_J \dots \le_J a_{\pi(r)}$.

1.7 Definition. 1) Let M be a fim with vocabulary $\tau = \tau_M$ and let $\{A_1,A_2\}$ be a partition of P^M into convex sets such that $A_1 <^M A_2$ which means that $(\forall a_1 \in A_1)(\forall a_2 \in A_2)[a_1 <^M a_2]$. We define a vocabulary $\tau_{M,A_1,A_2} = \tau(M,A_1,A_2)$. It contains, in addition to the symbols $P,<$, for each function symbol F of τ and a $\le^M$ $-$increasing sequence $\bar{a}_1$ from A_1 and a $\le^M$-increasing sequence $\bar{a}_2$ from A_2 such that $\ell g(\bar{a}_1) + \ell g(\bar{a}_2) < \mathrm{arity}^\tau(F)$ a function symbol called $F_{\bar{a}_1,\bar{a}_2}$ with $\mathrm{arity}^{\tau(M,A_1,A_2)}(F)$ being $\mathrm{arity}^\tau(F) - \ell g(\bar{a}_1) - \ell g(\bar{a}_2)$.
We identify $F \in \tau$ with $F_{\langle\rangle,\langle\rangle}$ and so consider $\tau_{M,\bar{a}_1,\bar{a}_2}$ an extension of τ.
2) Let $\bar{m} = \bar{m}[\tau,k_0,k_1]$ be $\bar{m}[\tau_{M,A_0,A_1}]$, the signature of τ_{M,A_0,A_2} whenever M is a

τ-fim of dimension $k_0 + k_1$ and A_0 is the set of k_0 first members of P^M and A_1 is the set of k_1 last members of P^M.
3) Let M^τ_k be a fim of vocabulary τ and dimension k, say $P^M = \{1, \dots, k\}$, i.e. as in Definition 1.2. Let $\tau^{[k,\ell]}$ be $\tau_{M^\tau_{k+\ell}, A_0, A_1}$ where A_0 is the set of the first k members of P^M and A_1 is the set of the last ℓ members of P^M.

1.8 Definition. 1) Let $\bar\Lambda$ denote a sequence $\langle \Lambda_F : F \in \tau\rangle$ where Λ_F is a finite (non empty) alphabet, and we let $\tau[\bar\Lambda] = \tau$, as $\bar\Lambda$ determines τ. We call $\bar\Lambda$ an alphabet sequence (for τ) or a τ-alphabet sequence.
We may write (τ, Λ) instead $\bar\Lambda$ if $\tau = \tau[\bar\Lambda]$ and $\Lambda_F = \Lambda$ for every $F \in \tau$.
2) We say p is a $\bar\Lambda$-type if p is a function with domain $\tau = \tau[\bar\Lambda]$ such that $p(F) \in \Lambda_F$ for $F \in \tau$; let $\mathfrak{p}, \mathfrak{q}$ denote non empty sets of $\bar\Lambda$-types;. We identify a set of $\bar\Lambda$-types with the characteristic function that it defines, i.e. $\mathfrak{p}$ is a function from the set of $\bar\Lambda$-types to $\{0, 1\}$, so we assume that from $\mathfrak{p}$ we can reconstruct $\bar\Lambda$ hence $\tau[\bar\Lambda]$. Let $\mathfrak{p}_{\bar\Lambda}$ be the set of all $\bar\Lambda$-types. We may write Λ instead of $\bar\Lambda$ if $\Lambda_F = \Lambda$ for every $F \in \tau$ and then let $\mathfrak{p}_{\tau,\Lambda}$ be the set of constant (τ, Λ)-types.

1.9 Definition. 1) For $\bar\Lambda$ a τ-alphabet sequence and M a τ-fim, let $V = \mathrm{Space}_{\bar\Lambda}(M)$ be defined as follows:

> its set of elements is the set of functions η with domain M, such that $b \in \mathrm{Sort}^M(F) \Rightarrow \eta(b) \in \Lambda_F$; we assume that from V we can reconstruct M and Λ.

2) We say d is a C-colouring of V, if d is a function from V into C; we say that d is a c-colouring if C has c members and the default value of C is $[0, c) = \{0, 1, \dots, c-1\}$.
3) We say L is a V-line or a line of V if for $\mathfrak{q} = \mathfrak{p}_{\bar\Lambda}$ we have: L is a $(V, \mathfrak{q})$-line or a $\mathfrak{q}$-line of V; (see below).
4) For $\mathfrak{q}$ a (non empty) subset of $\mathfrak{p}_{\bar\Lambda}$ let "L is a $(V, \mathfrak{q})$-line" or a "$\mathfrak{q}$-line of V" mean: L is a subset of V such that for some subset $\mathrm{supp}(L) = \mathrm{supp}_M(L)$ of M we have:

- (a) $\mathrm{supp}(L) \cap P^M$ is non empty and we call it $\mathrm{supp}^P(L)$
- (b) $\mathrm{supp}(L)$ is the M-th closure of $\mathrm{supp}^P(L)$
- (c) for any $\eta, \nu \in L$ we have $\eta \restriction (M \backslash \mathrm{supp}_M(L)) = \nu \restriction (M \backslash \mathrm{supp}_M(L))$
- (d) for any $\eta \in L$ there is $p \in \mathfrak{q}$ such that: if $b \in \mathrm{supp}_M((L)$ then $\eta(b) = p(F_{M,b})$
- (e) For any $p \in \mathfrak{q}$ there is $\eta \in L$ as in clause (d).

5) For L as above and $p \in \mathfrak{q}$ let $\mathrm{pt}_L(p)$ be the unique $\nu \in L$ such that for every $a \in \mathrm{supp}_M(L)$ we have $\nu(a) = p(F_{M,b})$. For $\mathfrak{q}^* \subseteq \mathfrak{q}$, the $\mathfrak{q}^*$-subline of a $\mathfrak{q}$-line L is

$\{\mathrm{pt}_L(p) : p \in \mathfrak{q}^*\}$.
6) For a colouring d of V, we say a V-line (or a $(V, \mathfrak{q})$-line) L is $d-$monochromatic if d is constant on L.
7) When we are given $M, \tau, \bar\Lambda, V$ as in part (1) and in addition we are given m, we define when S is an m-dimensional V-subspace, or m-dimensional subspace of V. It means that for some sequence $\langle M_\ell : \ell < m\rangle$ we have

- (a) each M_ℓ is a submodel of M, (see Definition 1.1(7))
- (b) if $\ell_1 < \ell_2 < m$ then M_{ℓ_1}, M_{ℓ_2} are disjoint,
- (c) there is $\rho = \rho^S$, a function with domain $(M \setminus c\ell(\cup\{M_\ell : \ell < m\}))$ such that $\rho(b) \in \Lambda_{F_{M,b}}$ for every $b \in \mathrm{Dom}(\rho)$, and there is an $m-$dimensional $\tau-$fim K say $K = M^\tau_{[0,m)}$ such that the following holds. Let N be the submodel of M with universe $c\ell_M(\bigcup\limits_{\ell<m} M_\ell)$.

<u>There is</u> $f \in \mathrm{Hm}(N, K)$ which is onto K such that $f \restriction P^{M_\ell}$ is constant for each $\ell, \langle \mathrm{Rang}(f \restriction P^{M_\ell}) : \ell < \omega\rangle$ are pairwise disjoint and:

- $(*)$ $\nu \in S$ iff ν extends ρ and for some $\varrho \in \mathrm{Space}_{\bar\Lambda}(K)$ we have $b \in N \Rightarrow \nu(b) = \varrho(\hat f(b))$.

8) We call S convex if

- (a) for $\ell_1 < \ell_2 < m$ and $a_1 \in M_{\ell_1}$ and $a_2 \in M_{\ell_2}$ we have $a_1 <^M a_2$ and
- (b) $f \in \mathrm{Hom}(N, K)$.

9) For S as above and (see Definition 1.7(3)) $\varrho \in \mathrm{Space}_\Lambda(M^\tau_{[0,m)})$ we define $\mathrm{pt}_S(\varrho)$ as the unique $\nu \in S$ as above in $(*)$ of part (7).
10) We call S weakly converse if $\ell_1 < \ell_2 \Rightarrow (\exists a_1 \in M_{\ell_1})(\forall a_2 \in M_{\ell_2})(a_1 <^M a_2)$.

We may define now a natural function, which is our main concern here:

1.10 Definition. 1) Let $f^1(\mathfrak{p}, c)$ where $\mathfrak{p} \subseteq \mathfrak{p}_{\bar\Lambda}$ (and $\bar\Lambda = \langle \Lambda_F : F \in \tau\rangle$ an alphabet sequence) be the minimal k such that for any $\tau_{\bar\Lambda}-$fim M of dimension k, we have:

- $(*)_k$ for any c-colouring d of $V = \mathrm{Space}_{\bar\Lambda}(M)$ there is a $\mathfrak{p}$-line L of V which is d-monochromatic, i.e. such that $p, q \in \mathfrak{p}$ implies that $\mathrm{pt}_L(p)$, $\mathrm{pt}_L(q)$ have the same colour (by d).

If k does not exist we may say it is ω or is ∞. We may write $f^1_\tau(\mathfrak{p}, c)$ or $f^1(\mathfrak{p}, c; \tau)$ to stress the role of τ.
2) If $\mathfrak{p} = \mathfrak{p}_{\bar\Lambda}$ we may write $f^1(\bar\Lambda, c)$. If $\Lambda_F = \Lambda$ for every $F \in \tau = \tau[\bar\Lambda]$ then we may

write $f^1_\tau(\Lambda, c)$; in this case we can replace Λ by $|\Lambda|$. Clearly only $\bar{m}^\tau$ is important so we may write it instead of τ. Also we may write $f^1_\tau(\bar{n}, c)$ for $f^1_\tau(\bar{\Lambda}, c)$ whenever $\bar{n} = \langle n_F : F \in \tau \rangle$ and $|\Lambda_F| = n_F$.

We can, of course, use the multidimensional versions of those definitions.

1.11 Definition. Let $f^1(\bar{\Lambda}, m, c)$ where $\bar{\Lambda} = \langle \Lambda_F : F \in \tau \rangle$ be the minimal k such that

$(**)_k$ for any τ-fim M of dimension k we have: for any $c-$colouring d of $\mathrm{Space}_{\bar{\Lambda}}(M)$ there is a convex subspace S of V of dimension m which is d-monochromatic, i.e. such that all the points in S have the same colour (by d), if k does not exist we say it is ω or is ∞.

We may write $f^1_\tau(\bar{\Lambda}, m, c)$ or $f^1_\tau(\Lambda, m, c)$ etc. as before. Clearly only $\bar{m}^\tau$ is important (rather than τ), so we may write only it. We may replace Λ by $|\Lambda|$. We may replace $\bar{\Lambda}$ by $\langle n_F : F \in \tau \rangle$ when $n_F = |\Lambda_F|$.

1.12 Remark. In the present paper, it does not really matter if in Definition 1.11 we omit the demand that S is convex.

The function has some obvious monotonicity properties, we mention those we shall actually use.

1.13 Claim. *1) For $\ell = 1, 2$ assume $\bar{\Lambda}^\ell$ is an alphabet sequence for the vocabulary τ^ℓ and arity$(\tau^1) \leq$ arity(τ^2) and for each $m = 1, \ldots,$ arity(τ^1) we have*

$$\Pi\{|\Lambda^1_F| : F \in \tau^1 \text{ has arity } m\} \leq \Pi\{|\Lambda^2_F| : F \in \tau^2 \text{ has arity } m\}.$$

Then $f^1(\bar{\Lambda}^{1,c}) \leq f^1(\bar{\Lambda}^{2,c})$.
2) For $\ell = 1, 2$ assume $\bar{\Lambda}^\ell$ is an alphabet sequence for the vocabulary τ^ℓ and $\tau^1 \subseteq \tau^2$ and $\bar{\Lambda}^1 = \bar{\Lambda}^2 \restriction \tau^1$ and $F \in \tau^2 \setminus \tau^1 \Rightarrow |\Lambda^2_F| = 1$.
Then $f^1(\bar{\Lambda}^1, c) = f^1(\bar{\Lambda}^2, c)$.

Proof. Straightforward. $\square_{1.13}$

1.14 Definition. 1) We define, for $\ell = 1, 2, 3$ what is a fim$^\ell$; we just replace in Definition 1.1, clauses $(f), (g)$ by

$(f)_\ell$ F^M is an arity ${}^\tau(F)$-place function from P^M to M such that if $F^M(a_1, \dots, a_r)$ is well defined (so $r = \text{arity}(F)$) then $a_1, \dots, a_m \in P^M$ and $\ell = 1$ implies the function is symmetric, i.e. does not depend on the order of the variables, so if not said otherwise we assume $a_1 \le^M a_2 \le^M \dots \le^M a_r$

$(g)_\ell$ if $F_1^M(a_1, \dots, a_r) = F_2^M(b_1, \dots, b_t)$ and $\ell \in \{1, 2\}$ then $F_1 = F_2$ (hence $r = t$) and $\ell = 2 \Rightarrow \bigwedge\limits_{s=1,\dots,r} a_s = b_s$ and

$$\ell = 1 \wedge \bigwedge_{s=1,\dots,r-1} a_s \le^M a_{s+1} \wedge \bigwedge_{s=1,\dots,r-1} b_s \le^M b_{s+1} \Rightarrow \bigwedge_{s=1,\dots,r} a_s = b_s.$$

So we let $\text{base}_M(b) =: \{a_1, \dots, a_r\}$ and when $\ell = 1, 2$ let $\text{base}_s(b) = \text{base}_{M,s}(b) =^{df} a_s$ where $b = F^M(a_1, \dots, a_r)$ (and if $\ell = 1$ then $a_1 \le^M a_2 \le^M \dots \le^M a_r$, of course) and $F_{M,b} =: F$; those are well defined by the demand above.

$(g)'_\ell$ if $\ell \in \{1, 2, 3\}$ and $b \in M \backslash P^M$ then for some $F \in \tau$ and $a_1, ..., a_{\text{arity}(F)} \in P^M$ we have $b = F^M(a_1, ..., a_{\text{arity}(F)})$.

So $\ell = 1$ is the old notion and for $\ell = 3$ we require very little.
2) We define $f^\ell_\lambda(\bar{\Lambda}, c)$ as in Definition 1.11 for fim$^\ell$ (so again $\ell = 1$ is our standard case).

1.15 Claim. *Let τ be a vocabulary and $\tau_\circ = \{G_{F,\pi} : F \in \tau$ and π is a permutation of $\{1, \dots, arity(F)\}\}$ with arity $(G_{F,\pi}) = arity(F)$.*
<u>*Then*</u>

(α) *If $\bar{\Lambda}$ is a $\tau-$alphabet sequence and $\bar{\Lambda}^\circ = \langle \Lambda^\circ_G : G \in \tau_\circ \rangle$ where $\Lambda^\circ_{G_{F,\pi}} = \Lambda_F$ then $f^2{}_\tau(\bar{\Lambda}, c) \le f^1_{\tau_\circ}(\bar{\Lambda}^\circ, c)$*

(β) *for $\bar{\Lambda}$ a $\tau-$alphabet sequence we have:*
$f^3_\tau(\bar{\Lambda}, c)$ is at most $RAM(f^2_\tau(\bar{\Lambda}^\circ, c)$, $arity(\tau), c^*)$ where e.g. c^* depends on τ only (and RAM stand for Ramsey number (see 0.3(1))

(γ) $f^1_\tau(\bar{\Lambda}, c) \le f^2_\tau(\bar{\Lambda}, c)$

(δ) $f^2_\tau(\bar{\Lambda}, c) \le f^3_\tau(\bar{\Lambda}, c)$.

Proof. Straightforward. $\square_{1.15}$

1.16 Claim. *If $n > 0, t > 0, c > 0$, then letting $\tau = \tau_t$ (see Definition 1.6(1)), $\Lambda = \{1, \dots, n\}$, then $f^1_\tau(\Lambda, c)$ from Definition 1.10(2) can serve as $f^*_t(n, c)$ from the beginning of the introduction.*

Proof. Recall that M there is $\{u : u \subseteq \{1, \dots, k\}, 1 \leq |u| \leq t\}$. Let $J = ([1,k], <)$ and $M^* = M_\tau(J)$ and $f : M \to \text{ univ}(M^*)$ is defined by: for if $1 \leq j_1 < j_2 < \dots < j_r \leq k, r \in \{1, \dots, t\}$ then $f(\{j_1, \dots, j_r\}) = (F_r, \langle j_1, \dots, j_r \rangle)$. So f is 1-to-1 and for $c : {}^M\Lambda \to C$ let $c^* : \text{Space}_\Lambda(M) \to C$ be $c^*(\rho) = c(\nu)$ if $\nu \in {}^M C$, and $(\forall x \in M)(\nu(x) = \rho(f(x)))]$.
The rest should be clear. $\square_{1.16}$

§2 Proof of the partition Theorem with a bound

Except Definitions 2.1, 2.2 this section is for the reader convenience only, as it gives a proof of a weaker version of the first theorem (with a bound which we get by triple induction). Later in 4.1 - 4.10 we give a complete proof with the primitive recursive bound, formally not depending on the proofs here. The strategy is to make the $b \in M$ with $|\mathrm{base}_M(b)|$ maximal immaterial. We first define some help functions.

2.1 Definition. 1) We call a vocabulary τ monic if there is a unique function symbol of maximal arity, we then denote it by $F^{\max}_\tau$.
2) For $a \in P^M$ let M_a be $\mathrm{cl}_M(\{P^M \setminus \{a\}\})$.
3) For $V = \mathrm{Space}_{\bar\Lambda}(M)$ and N a closed subset of M and $H \in \tau$, we say that a colouring d of V is $(N, \alpha, H)-$invariant <u>if</u>: $\alpha \in \Lambda_H$, and the following holds, for any $a \in P^N$:

($*$) if $\nu, \eta \in V$ and $\nu \restriction M_a = \eta \restriction M_a$ and $[b \in M \wedge\ \mathrm{base}(b) = \{a\} \wedge F_{M,b} = H \Rightarrow \nu(b) = \alpha = \eta(b)]$ then $d(\nu) = d(\eta)$.

4) In part (3) we write $(\ell, \alpha, H)-$monochromatic if the above N is such that P^N is the set of the last ℓ members of P^M. We write $(M, \alpha, H)-$monochromatic if in part (3) we have $M = N$.
5) In parts (3) and (4) we may omit H when τ is monic and $H = F^{\max}_\tau$. Replacing α by Λ^* means that Λ^* is a subset of Λ_H and the demand holds for every $\alpha \in \Lambda^*$. Replacing Λ^* by 0 means Λ^*.

2.2 Definition. Let f^0 be defined as follows. First, $f^0_{\bar\Lambda}(n, \ell, c) = f^0_{\tau,\bar\Lambda}(n, \ell, c)$ is defined iff $\tau = \tau[\bar\Lambda]$ is monic with $H = F^{\max}_\tau$ and $\bar\Lambda$ is an alphabet sequence for τ and $n \leq |\Lambda_H|$ and $n < |\Lambda_H| \vee (n = |\Lambda_H| \wedge \ell = 0)$.
Second, $f^0_{\bar\Lambda}(n, \ell, c)$ is the first k (natural number, if not defined we can understand it as ∞ or ω or "does not exist") such that $(*)_k$ below holds, where:

$(*)_k$ If clauses (a)-(f) below hold <u>then</u> there is a d-monochromatic line of V, where :

- (a) M is a fim of vocabulary τ
- (b) the dimension of M is k
- (c) $V = \mathrm{Space}_{\bar\Lambda}(M)$
- (d) Λ° is a subset of Λ_H with exactly n members
- (e) d is an $(M, \Lambda^\circ, H)-$invariant colouring of V
- (f) if $\ell \neq 0$, then there is an α such that $\alpha \in \Lambda_H \setminus \Lambda^\circ$ and d is (ℓ, α, H)-invariant.

Immediate connections are:

2.3 Observation. 1) The function $f^0_{\tau,\bar{\Lambda}}(n,\ell,c)$ increases with c and decreases with ℓ and n.
2) The function $f^0_{\tau,\bar{\Lambda}}(n,\ell,c)$ depends just on n,ℓ,c and the set $\{(\text{arity}(F),|\Lambda_F|) : F\in\tau\}$ (possibly with multiple membership), so if $\Lambda_F=\Lambda$ we may replace τ by its $\bar{m}^\tau$ (similarly for other such functions).
3) In definitions 1.10, 1.11, 2.2 the demand $(*)_{k'}$ holds for any k' larger then k.
4) $f^0_{\bar{\Lambda}}(0,0,c)=f^1(\bar{\Lambda},c)$.
5) If τ is monic and $H=F^{\max}_\tau$ and $\tau^-=\tau\backslash\{H\}$ then $f^0_{\bar{\Lambda}}(|\Lambda_H|,0,c)=f^1(\bar{\Lambda}\restriction\tau^-,c)$.
6) If $\ell^*=f^0_{\bar{\Lambda}}(n+1,0,c)$ then $f^0_{\bar{\Lambda}}(n,\ell^*,c)=\ell^*$.

Proof. Trivial.

2.4 Main Claim. *Assume*

(a) *$\bar{\Lambda}$ is an alphabet sequence for a vocabulary $\tau=\tau[\bar{\Lambda}]$, and $n<|\Lambda_H|$*
(b) *τ is a monic vocabulary with $H=F^{\max}_\tau$*
(c) *$k_0\geq f^0_{\bar{\Lambda}}(n,\ell+1,c)$ and $k_0>\ell$*
(d) *K is a τ-fim of dimension k_0-1 and A_2 is the set of last ℓ elements of P^K and A_1 is the set of the first $(k_0-\ell-1)$-elements of P^K (this K serves just for notation)*
(e) *τ^* is the vocabulary $(\tau_{K,A_1,A_2})\setminus\{H\}$; see Definition 1.7(3); so*
  (i) *arity$(\tau^*)<$ arity(τ),*
  (ii) *proj is the following function from τ^* to τ: it maps $F_{K,\bar{a}_1,\bar{a}_2}$ to F so proj$\restriction\tau$ is the identity,*
  (iii) *$\bar{\Lambda}^*=:\langle\Lambda^*_F:F\in\tau^*\rangle$ where $\Lambda^*_F=\Lambda_{\text{proj}(F)}$.*

(f) *$c^*=:c^{\text{card}(\text{Space}_{\bar{\Lambda}}(K))}$.*

<u>Then</u>

$$f^0_{\bar{\Lambda}}(n,\ell,c)\leq k_0+f^1(\bar{\Lambda}^*,c^*)-1.$$

Proof. Let $k_1=f^1(\bar{\Lambda}^*,c^*)$ and let $k=k_0+k_1-1$, so it suffices to prove that $k\geq f^0_{\bar{\Lambda}}(n,\ell,c)$. For this it is enough to check $(*)_k$ from Definition 2.2(1), so let

Λ° be a subset of Λ_H with n elements and $\alpha^* \in \Lambda_H \setminus \Lambda^*$, also let M be a fim of vocabulary τ and dimension k (i.e. P^M is with k members), $V = \mathrm{Space}_{\bar{\Lambda}}(M)$, and d an (ℓ, α^*, H)-invariant and (M, Λ°, H)-invariant C-colouring of V such that C has $\leq c$ members. So we just have to prove that the conclusion of Definition 2.2(1) holds, which means that: there is a d-monochromatic line of V.

Let $w_1 =: \{a : a \in P^M$ and the number of $b <^M a$ is $\geq k_0 - \ell - 1$ but is $< k_0 - \ell - 1 + k_1\}$, hence in w_1 there are k_1 members, and let w_0 be the set of first $k_0 - \ell - 1$ members of P^M by $<^M$, and lastly let w_2 be the set of the ℓ last members of M by $<^M$. So w_0, w_1, w_2 form a convex partition of P^M.

Now we let K be M restricted to $\mathrm{cl}_M(w_0 \cup w_2)$, (note that this gives no contradiction to the assumption on K i.e. clause (d) of the assumptions, as concerning K there, only its vocabulary and dimension are important and they fit). Let K^+ be a fim with vocabulary τ and dimension k_0, let $g_0 \in \mathrm{PHom}(M, K^+)$ be the following function from P^M onto P^{K^+}: it maps all the members of w_1 to one member of P^{K^+} which we call b^*, it is a one to one order preserving function from w_2 onto $\{b \in P^{K^+} : b^* <^{K^+} b\}$ and it is a one to one order preserving function from w_0 onto $\{b \in P^{K^+} : b >^{K^+} b^*\}$. Let $g \in \mathrm{Hom}(M, K^+)$ be the unique extension of g_0; without loss of generality g_0 is the identity on w_0 and on w_2 hence without loss of generality g is the identity on K, it exists by 1.4. Hence clearly $w_0 = A_1$ and $w_2 = A_2$ so $\tau^* = \tau_{K,w_0,w_2} \backslash \{H\}$.

Next recall that the vocabulary $\tau^* = \tau_{K,w_o,w_2} \setminus \{H\}$ is a well defined vocabulary (see Definition 1.7(1) and remember that $\tau \subseteq \tau_{K,w_0,w_2}$ so $H \in \tau_{K,w_0,w_2}$). Next we shall define a τ^*-model N. Its universe is $(M \setminus K) \setminus A^*$ where $A^* =: \{b \in M$: $\mathrm{base}_M(b) \subseteq w_1$ and $F_{M,b} = H\}$, we let P^N be w_1 and $<^N$ be $<^M \restriction P^N$. Now we have to define each function $F^N_{K,\bar{a}_1,\bar{a}_2}$, say of arity r, where $F \in \tau, \bar{a}_1$ a non decreasing sequence from w_0 and $\bar{a}_2$ a non decreasing sequence from w_2, and $\ell g(\bar{a}_1) + \ell g(\bar{a}_2) <$ $\mathrm{arity}^\tau(F)$ and $\mathrm{arity}^{\tau^*}(F_{K,\bar{a}_1,\bar{a}_2}) < \mathrm{arity}(\tau)$. Note that the last condition is equivalent to: if $F = H$ then at least one of the sequences $\bar{a}_1, \bar{a}_2$ is not empty.

For $b_1 \leq^N \ldots \leq^N b_t \in P^N$ we let $F^N_{\bar{a}_1,\bar{a}_2}(b_1, \ldots, b_t)$ be equal to $b = F^M(\bar{a}_1, b_1, \ldots, b_t, \bar{a}_2) = F^M(a^1_1, a^1_2, \ldots, a^1_{\ell g(\bar{a}_1)}, b_1, \ldots, b_t, a^2_1, \ldots, a^2_{\ell g(\bar{a}_2)})$.

It is easy to check that the number of arguments is right and also the sequence they form is $\leq^M$ $-$increasing, so this is well defined and b belongs to M, but still we have to check that it belongs to N. First note that it does not belong to K, as if $b \in K$ then $\mathrm{base}_{\ell g(\bar{a}_1)+1}(b) \in K$ and it is just b_1 which belongs to w_1, contradiction. Second note that it does not belong to A^*; this holds as we have substructed $H = F_{<>,<>}$ when we have defined τ^*.

Lastly it is also trivial to note that every member of N has this form. It is easy to check that N is really a τ^*-fim. We next let $V^* = \mathrm{Space}_{\bar{\Lambda}^*}(N)$ and let $C^* = \{g : g$ is a function from $\mathrm{Space}_{\bar{\Lambda}}(K)$ to $C\}$ and we shall define a C^*-colouring d^* of V^*. For $\eta \in V^*$ let $d^*(\eta)$ be the following function from $\mathrm{Space}_{\bar{\Lambda}}(K)$ to C, letting ϱ be

the function with domain A^* which is constantly α^*: for $\nu \in$ $\mathrm{Space}_{\bar\Lambda}(K)$ we let $(d^*(\eta))(\nu) = d(\eta \cup \nu \cup \varrho)$.
Clearly the function $d^*(\eta)$ is a C^*-colouring of $\mathrm{Space}_{\bar\Lambda}(K)$. How many such functions are there? The domain has clearly $\mathrm{card}(\mathrm{Space}_{\bar\Lambda}(K))$ members, (we can get slightly less if $\ell > 0$, but with no real influence). The range has at most c members, so the number of such functions is at most $c^{\mathrm{card}(\mathrm{Space}_{\bar\Lambda}(K))}$, a number which we have called c^*. So d^* is a c^*-colouring of V^*.

Now as we have chosen $k_1 = f^1(\bar\Lambda^*, c^*)$ we can apply Definition 2.2 to $V^* = \mathrm{Space}_{\bar\Lambda^*}(N)$ and d^*; so we can find a d^*-monochromatic V^*-line and we call it L^*. Let h be the function from $U =: \mathrm{Space}_{\bar\Lambda}(K^+)$ to V defined as follows:

$(*)$ $h(\rho) = \nu$ iff:
- (a) $\nu \in V, \rho \in U$,
- (b) $\nu \restriction K = \rho \restriction K$
- (c) if $b \in N \setminus \mathrm{supp}_N(L^*)$ (see Definition 1.9(3)) then $\nu(b) = \eta(b)$ for every $\eta \in L^*$
- (d) if $a \in A^* \setminus c\ell_M(\mathrm{supp}_N(L^*))$ then $\rho(a) = \alpha^*$
- (e) if $a \in \mathrm{supp}_N(L^*)$, (so $a \in N, F \in \tau^-, F_{N,a} = F_{K,\bar a_1,\bar a_2}$, $\mathrm{base}_N(a) \subseteq \mathrm{supp}^P_N(L^*)$), and $b \in K^+, F_{K^+,b} = F, b = F(\bar a_1, b^*, \dots, b^*, \bar a_2)$ (with the number of cases of b^* being $\mathrm{arity}^{\tau^*}(F_{K,\bar a_1,\bar a_2})$) then $\rho(b) = \nu(a)$
- (f) if $a \in A^* \cap c\ell_M(\mathrm{sup}_N(L^*))$ and $b \in K^+$ is $H(b^*, \dots, b^*)$ then $\rho(b) = \nu(a)$.

Let the range of h be called S. Now clearly

$\otimes_1(\alpha)$ h is a one to one function from U onto $S \subseteq V$
(β) S has $|\mathrm{Space}_{\bar\Lambda}(K^+)|$ members
(γ) S is a subspace of V of dimension k_0, such that $h(\rho) =$ $\mathrm{pt}_S(\rho)$, see 1.9(7).

Now clearly

$\otimes_2$ there is a C-colouring d° of U such that:

$$d^\circ(\nu) = d(h(\nu)) \text{ for } \nu \in U.$$

and

$\otimes_3(a)$ d° is (K^+, Λ^*)-invariant
(b) d° is $(\ell + 1, \alpha^*, H))$-invariant.
[Why? Reflect].

Applying the definition of $k_0 \geq f^0_{\tau,\bar\Lambda}(n,\ell+1,c)$, that is Definition 2.2 to $\bar\Lambda, \alpha^*, U, d^\circ$ we can conclude that there is a d°-monochromatic U-line L°. Let $L =: \{h(\rho) : \rho \in L^\circ\}$. It is easy to check that L is as required. $\square_{2.4}$

As a warm up for the later bounds we prove:

2.5 Theorem. *1) The function $f^1_\tau(\bar\Lambda, c)$ is well defined, i.e. always get value which is, a natural number.*
2) Moreover f^1 has a bound which we have got by triple induction.
3) Similarly the function f^0.

Proof. 1) The proof follows by induction, the main induction is on $t = \text{arity}(\tau_{\bar\Lambda})$. Now by observation 1.13(1) without loss of generality τ is monic, i.e. has a unique function symbol of arity t, called $H =: F^{\max}_\tau$. Fixing t, we prove by induction on $s = |\Lambda_H|$.

Case 0: $t = 1$.
This is Hales-Jewett theorem (on a bound see [Sh 329] and [GRS80]).

Case 1: $t > 1, s = 1$.
By Claim 1.13(2) we can decrease t.

Case 2: $t > 1, s \geq 2$.
We note that $f^1(\bar\Lambda, c) = f^0_{\bar\Lambda}(0,0,c)$ by 2.3(4) so it is enough to bound the later one. But by 2.3(5) we know $f^0_{\bar\Lambda}(|\Lambda_H|, 0, c) = f^1(\bar\Lambda \restriction \tau^-, c)$ where $\tau^- =: \tau \setminus \{H\}$, but for the later one we have a bound by the induction hypothesis on t as $\text{arity}(\tau^-) < t$, so we have a bound on $f^0_{\bar\Lambda}(|\Lambda_H|, 0, c)$. By the last two sentences together, it is enough to find a bound to $f^0_{\bar\Lambda}(n, 0, c)$ by downward induction on $n \leq |\Lambda_H|$, and on the one hand we have the starting case $n = |\Lambda_H|$ and on the other hand the case $n = 0$ gives the desired conclusion. So assume we know for $n+1$ and we shall do it for n. Let $\ell^* =: f^0_{\bar\Lambda}(n+1, 0, c)$, so we know that $\ell^* = f^0_{\bar\Lambda}(n, \ell^*, c)$ by 2.3(6), so we by downward induction on $\ell \leq \ell^*$ give a bound to $f^0_{\bar\Lambda}(n, \ell, c)$. So we are left with bounding $f^0_{\bar\Lambda}(n, \ell, c)$ given bound for $f^0_{\bar\Lambda}(n, \ell+1, c)$ (and also $f^1_{\tau_\circ}(\bar\Lambda^\circ, c^\circ)$ whenever $\text{arity}(\tau_\circ) < t$). For this 2.4 was designed, it says $f^0_{\bar\Lambda}(n, \ell, c) \leq f^0_{\bar\Lambda}(n, \ell+1, c) + f^1_{\tau^*}(\bar\Lambda^*, c) + 1$ where $\tau^*, \bar\Lambda^*$ were defined there and $\text{arity}(\tau^*) < \text{arity}(\tau)$. (Well, we have to assume that $\ell < f^0_{\bar\Lambda}(n, \ell+1, c)$, but otherwise use $\ell + 1 + f^1_{\tau^*}(\bar\Lambda^*, c) + 1$).
2), 3) Should be clear. $\square_{2.5}$

§3 Higher Dimension Theorems

Concerning the multidimensional case (see Definition 1.11):

3.1 Conclusion. 1) For any $\bar{\Lambda}, m$ and c, we have $f^1(\bar{\Lambda}, m, c)$ is well defined (with bound as in the proof, actually using one further induction using only $f^1_{\tau_i}(\bar{\Lambda}, c)$ for suitable τ_i-s in the i-step).
2) We can naturally define τ-fim of dimension $\aleph_0$ and convex subspaces, and prove that for any τ-fim M of dimension $\aleph_0$ and alphabet sequence $\bar{\Lambda}$, (with each Λ_F and τ finite (of course)) the following holds: if $\text{Space}_M(\bar{\Lambda})$ is the union of finitely many Borel subsets, then some convex subspace S of dimension $\aleph_0$ is included in one of those Borel subsets.

Proof. 1) For simplicity (and without loss of generality by Definition 1.13(1)) we have $\bar{\Lambda}$ is constantly Λ, so each Λ_F is Λ, a fixed alphabet. We choose by induction on $i = 0, \dots, m$ the objects M_i, τ_i, k_i and c_i such that:

- (a) $k_0 = 0$ and $k_i < k_{i+1}$
- (b) M_i is a fim for τ of dimension k_i; (we allow empty fim, if you do not like it start with $k_0 = 1$)
- (c) M_{i+1} is an end extension of M_i, (see Definition 1.1(7))
- (d) $\tau_i = \tau_{M_i, P^{M_i}, \emptyset}$ (see Definition 1.7(1))
- (e) c_0 is c and c_{i+1} is $c^{|\text{Space}_\Lambda(k_i+m-i)|}$
- (f) $k_{i+1} = k_i + f^1_{\tau_i}(\Lambda, c_i)$.

There is no problem to carry over the definition and we can prove that $k_m \geq f^1_\tau(\Lambda, m, c)$.
The proof is straight.
2) Such theorems are closed relatives of theorems on appropriate forcing notions, so as it is a set theoretical theorem in the proof we use forcing. Specifically we use the general treatment of creature forcing of [RoSh 470]. For any finite non empty $u \subseteq \omega$ let $M^\tau_u = M^\tau[u]$ be a τ-model with $(P^{M^\tau_u}, \leq^{M^\tau_u}) = (u, \leq)$, and without loss of generality $u_1 \subseteq u_2 \Rightarrow M^\tau_{u_1} \subseteq M^\tau_{u_2}$. So for infinite $u \subseteq \omega$ we have $M^\tau_u = \cup\{M^\tau_{u_1} : u_1 \subseteq u \text{ finite}\}$ is a well defined τ-fim of dimension $\aleph_0$.
A $\bar{\Lambda}$-creature $\mathfrak{c}$ consists of a convex subspace $S^{\mathfrak{c}} = S[\mathfrak{c}]$ of some $\text{Space}_{\bar{\Lambda}}(M_{\mathfrak{c}}), M_{\mathfrak{c}} = M^\tau_u$ for some finite non empty $u = u[\mathfrak{c}]$ of the form $[n, m] = [n_{\mathfrak{c}}, m_{\mathfrak{c}}]$.
For creatures $\mathfrak{c}_1, \dots, \mathfrak{c}_k$ we let $\Sigma(\mathfrak{c}_1, \dots, \mathfrak{c}_k)$ be well defined iff $m_{\mathfrak{c}_\ell} = n_{\mathfrak{c}_{\ell+1}}$ for $\ell \in [1, k)$ and it is the set of $\bar{\Lambda}-$creatures $\mathfrak{c}$ such that $n_{\mathfrak{c}} = n_{\mathfrak{c}_1}, m_{\mathfrak{c}} = m_{\mathfrak{c}_k}$ and $\eta \in S^{\mathfrak{c}} \wedge \ell \in [1, k) \Rightarrow \eta \restriction M_{\mathfrak{c}} \in S^{\mathfrak{c}_\ell}$.

So the forcing notion $\mathbb{Q}$ is well defined by [RoSh 470] for the case "the lim-sup of the norms is infinity". So a condition p has the form $\langle \eta, \mathfrak{c}_1, \mathfrak{c}_2, \dots \rangle = \langle \eta^p, \mathfrak{c}_1^p, \mathfrak{c}_2^p, \dots \rangle$ where for $t = 1, 2, \dots, \mathfrak{c}_t^p$ is a $\bar{\Lambda}$-creature satisfying $m_{\mathfrak{c}^p_{t+1}} = n_{\mathfrak{c}^p_t}$ and for some $\eta^* \in \text{Space}_{\bar{\Lambda}}(M^\tau_\omega)$, see below, we have

$$\begin{aligned} \eta = \eta^* \restriction \{a \in M^\tau_\omega : \ & \text{for no } \ell, k, \mathfrak{c} \text{ do we have} \\ & 1 \le \ell < k, \mathfrak{c} \in \Sigma(\mathfrak{c}_\ell, \mathfrak{c}_{\ell+1}, \dots, \mathfrak{c}_k) \text{ and} \\ & a \in M^\tau_{[m_{\mathfrak{c}_\ell}, n_{\mathfrak{c}_k})} \setminus \text{Dom}(\rho^{S_\mathfrak{c}})\}. \end{aligned}$$

Let $\mathbb{B} =: \text{Space}_{\bar{\Lambda}}(M^\tau_\omega) = \{\rho : \rho \text{ is a function with domain } M^\tau_\omega \text{ satisfying } f(b) \in \Lambda_{F(b)}\}$ where $F(b) = F_{M^\tau_\omega, b}$. We say that $\rho \in \mathbb{B}$ obeys $p \in \mathbb{Q}$ if $\eta^p \subseteq \rho$ and for $t = 1, 2, \dots$ we have for some $\mathfrak{c} \in \Sigma(\mathfrak{c}_1, \dots, \mathfrak{c}_t), \rho \restriction \text{univ}(M_\mathfrak{c}) \in S^\mathfrak{c}$. It is proved there that such forcing notions have many good properties. In particular let $\text{cont}(p) = \{\rho : \rho \in \mathbb{B} \text{ obeys } p\}$ and defining the $\mathbb{Q}$-name $\hat{f} = \cup\{\eta^p : p \in \underset{\sim}{G}_\mathbb{Q}\}$. Now note that:

- (a) $p \Vdash_\mathbb{Q}$ "$\hat{f} \in \text{cont}(p)$"
- (b) if $N \prec (\mathscr{H}(\chi), \in)$ is countable, the definition of the given finitely many Borel sets belongs to N, and $p \in \mathbb{Q} \cap N$, then we can find q such that
  - (i) $p \le q$
  - (ii) every $f \in \text{cont}(q)$ is a generic for $\mathbb{Q}$ over N
  - (iii) for some $p' \le n'$ we have $p \le p' \in N \cap \mathbb{Q}, p' \le q$ and $p' \Vdash_\mathbb{Q}$ "$\hat{f} \in A_{n'}$".

Together we conclude that $\text{cont}(q) \subseteq A_{n'}$ and we are done. $\square_{3.1}$

We turn to relating the old results from Bergelson Leibman [BeLe96].

3.2 Conclusion. 1) Assume that

- (a) τ is a t-canonical vocabulary (see 1.6)
- (b) $k = f^1_\tau(\Lambda, c)$, Ł a (finite) alphabet
- (c) R is a ring, and $r_1, \dots, r_k \in R$
- (d) for $\alpha \in \Lambda, \mathbf{p}_\alpha(x)$ is a polynomial over R (i.e. with parameters in R)
- (e) d is a c-colouring of R (actually enough to consider a finite subset, the range of g in the proof below).

Then we can find y, z and $w \subseteq \{1, \dots, k\}$ such that

(α) $y \in R$ and $z = \Sigma_{\ell \in w} r_\ell \in R$

(β) the set $\{y + \mathbf{p}_\alpha(z) : \alpha \in \Lambda\}$ is d-monochromatic.

(2) Assume that:

(a) τ is a vocabulary of arity t, such that for each $s = 1, \dots, t$ in τ there are exactly m^* function symbols of arity s

(b) $k = f^1_\tau(\Lambda, c)$, Λ a (finite) alphabet; see Definition 1.10(2)

(c) R is a ring, and $r_1, \dots, r_k \in R$

(d) for $\alpha \in \Lambda$ and $m < m^*$, $\mathbf{p}_{\alpha,m}(x)$ is a polynomial over R (i.e. with coefficients in R)

(e) d is a c-colouring of ${}^{m^*}R = \{\langle y_m : m < m^* \rangle : y_0, \dots, y_{m^*-1} \in R\}$ (actually enough to consider a finite subset, the range of g in the proof below).

Then we can find y, z and $w \subseteq \{1, \dots, k\}$ such that

(α) $y \in R$ and $z = \Sigma_{\ell \in w} r_\ell \in R$

(β) the set $\{\langle y + \mathbf{p}_{\alpha,m}(z) : m < m^* \rangle : \alpha \in \Lambda\}$ is d-monochromatic.

Proof. 1) Let M be a fim for τ of dimension k and let h be a one to one order preserving function from P^M onto $\{1, \dots, k\}$. We define a function g from $V = \mathrm{Space}_\Lambda(M)$ to R. For $\eta \in V$ we let $g(\eta) = \sum\limits_{b \in M} g_b(\eta(b))$ where g_b is the following function from Λ to R. For $b = F(b_1, \dots, b_t) \in M$ and $\alpha \in \Lambda$ we let $g_b(\alpha)$ be zero if $\langle b_1, b_2, \dots, b_t \rangle$ is with repetitions and otherwise we consider $\mathbf{p}_\alpha(\sum\limits_{i=1,t} r_{h(b_i)})$, expand it as sum of monoms in $r_1, \dots, r_k$, and let $g_b(\alpha)$ be the sum of those monoms for which $\{r_j : j \in \{1, \dots, k\}$ and r_j appear in the monom$\} = \{h(b_1), \dots, h(b_t)\}$. Now we define a $c-$colouring d^* of V by $d^*(\eta) = d(g(\eta))$. Let L be a d^*-monochromatic line of V , let $\mathrm{supp}_M(L) = N$. Now let $y =: \sum\limits_{b \in M \setminus N} g_b(\mathrm{pt}_L(\alpha))$ for any $\alpha \in \Lambda$, note that all the $\alpha \in \Lambda$ give the same value. Let $w =: \{h(b) : b \in \mathrm{supp}^P_M(L)\}$, recalling Definition 1.9(5) and so $z = \sum\limits_{\ell \in w} r_\ell$, now check.

Note that algebraically it is more natural to define g differently, working with the rank of the monom rather than with the set of variables appearing.

2) Similarly, left to the reader. $\square_{3.2}$

3.3 Discussion. It is natural to ask:

(1) Can we generalize the Graham Rothschild theorem? (see [GR71], [GRS80])
(2) Can we get here primitive recursive bounds?
(3) Can we prove the density version of the theorem 2.5?

Below we answer positively questions (1),(2), we believe that the answer to question (3) is positive too but probably it requires methods of dynamical systems, see the book Furstenberg [Fu81].

3.4 Definition. We define $f^4(\bar{\Lambda}, t, \ell, c) = f^4_\tau(\bar{\Lambda}, t, \ell, c)$ where $0 \le \ell < t$ as follows. It is the minimal k such that: if M is fim for τ with $k = \dim(M), V = \mathrm{Space}_{\bar{\Lambda}}(M)$ and d is a c-colouring of $\{S : S \text{ is an } \ell\text{-subspace of } V\}$ <u>then</u> for some subspace U of V of dimension t, all the ℓ-subspaces of U (equivalently, ℓ-subspaces of V which are contained in U) have the same colour by d.

3.5 Theorem. *1) For any $\bar{\Lambda}$,t, ℓ, c as in Definition 3.3, the function $f^4(\bar{\Lambda}, t, \ell, c)$ is well defined, i.e. is finite.*
2) Let $m = RAM(t, \ell, c)$, see Definition 0.3(1), where τ is a vocabulary and $\bar{\Lambda}$ is a τ-alphabet sequence, and define k_i for $i = 0, \dots, m$ by induction on i as follows (on $\tau^{[k,r]}$ see 1.7(3)): $k_0 = 0, \bar{\Lambda}^0 = \bar{\Lambda}$ and $k_{i+1} = k_i + f^1_{\tau_i}(\bar{\Lambda}^i, c_i)$ where $\tau_i =: \tau^{[k_i,m-i]}$ and $\bar{\Lambda}^i$ is a $\tau^{[k,m-i]}$-alphabet sequence, and $\Lambda^i_{F_{M^\tau_{k_i+m-i},\bar{a}_1,\bar{a}_2}}$ has $|\Lambda^i_F|+\ell+|M^\tau_{k_i+m-i}|$ members and $c_i = c^{\mathrm{card}(\mathrm{Space}_{\bar{\Lambda}^i}(M^\tau_{k_i+m-i}))}$.
<u>*Then*</u> $f^4_\tau(\Lambda, t, \ell, c) \le k_m$.

Proof. 1) Follows from (2).
2) Without loss of generality $\bar{\Lambda} = \langle \Lambda_F : F \in \tau \rangle$ is a sequence of pairwise disjoint sets and $\Lambda = \cup\{\Lambda_F : F \in \tau\}$.
Let $N = M^\tau_\ell$ (see notation in 1.7(3), recall that ℓ is the dimension of the subspaces we are colouring) and let $\{\gamma_a : a \in N\}$ list a set disjoint to Λ without repetitions. We choose for $i = 0, \dots, m$ the objects $k_i, \tau_i, \bar{\Lambda}^i$ (consistently with what is said in the statement of the theorem) and M_i, M^+_i, by induction on i as follows:

$\otimes_1(a)$ $k_0 = 0$ and $k_i < k_{i+1}$
(b) M_i is a fim for τ of dimension k_i (we allow empty fim, the space is a singleton, if you do not like it start with $k_0 = 1$)
(c) M_{i+1} an end extension of M_i and M^+_i is an end extension of M_i (so both have vocabulary τ) and M^+_i has dimension $k_i + m - i$

(d) $\tau_i = \tau_{M_i^+, P^{M_i}, P^{M_i^+} \setminus P^{M_i}}$ (see Definition 1.7(3))

(e) $\bar\Lambda^0 = \bar\Lambda$ and for $F \in \tau$ and $\Lambda^i_{F_{\bar a_1, \bar a_2}}$ is the disjoint union of Λ_F, Λ^*_F and Λ^{**}_F where $\Lambda^*_F =: \{\gamma_b : b \in N$ and $F_{N,b} = F\}$ and $\Lambda^{**}_F =: \{\beta_b : b \in M_i^+$ such that $F_{M_i^+, b} = F\}$ (and no two letters are incidentally equal, of course)

(f) c_0 is c and c_{i+1} is $c^{\mathrm{card}(\mathrm{Space}_{\bar\Lambda_i}(M^\tau_{k_i+m-i}))}$

(g) $k_{i+1} = k_i + f^1_{\tau_i}(\bar\Lambda^i, c_i)$.

Let $k = k_m, M = M_m$ and let $V_i = \mathrm{Space}_{\bar\Lambda}(M_i^\tau)$ and $V = V_m$. We shall regard an ℓ-subspace Φ of V as a function from M to $\Lambda^* =: \{\gamma_b : b \in N\} \cup \Lambda$, such that (recall $\Lambda = \cup\{\Lambda_F : F \in \tau\}$):

$\otimes_2(a)$ $\Phi(b) \in \Lambda^*_{F_{M,b}} \cup \Lambda_{F_{M,b}}$, see clause (e) of $\otimes_1$

(b) if $b \in M, \alpha \in \Lambda$ and $(\forall \nu)[\nu \in \Phi \Rightarrow \nu(b) = \alpha]$ then $\Phi(b) = \alpha$

(c) if $b \in M, a \in N$ and for every $\rho \in \mathrm{Space}_{\bar\Lambda}(N)$ we have $(\mathrm{pt}_\Phi(\rho))(b) = \rho(a)$ then $\Phi(b) = \gamma_a$.
(Reflect on the meaning of ℓ-subspace of M, i.e. Definition 1.9(7) and it should be clear.)

Let d be a c-colouring of the set of ℓ-subspaces of V. We shall define by downward induction on $i < m$ a pair (A_i, ρ_i) such that

$(*)_0(i)$ A_i is a (non empty) subset of $P^{M_{i+1}}$ disjoint to M_i and

(ii) ϱ_i is a function from $B_i =: M \setminus \mathrm{cl}_M(M_i \cup \bigcup\limits_{j=i,\dots,m-1} A_j) \setminus \bigcup\limits_{j=i+1,\dots,m-1} B_j$ into $\Lambda \cup \{\beta_a : a \in M_i^+\}$ such that $b \in B_i \Rightarrow \varrho_i(b) \in \Lambda_{F_{M,b}} \cup \Lambda^{**}_{F_{M,b}}$.

We let $\mathbf{R}_i$ denote the family of ℓ-subspaces Φ of V which satisfies:

$(*)_1(a)$ if j satisfies $i \le j < m$ and $b \in B_j$ and $\varrho_j(b) \in \Lambda$ then $\Phi(b) = \varrho_j(b)$

(b) if j satisfies $i \le j < m$ and $b \in B_j$ and $\varrho_j(b) = \beta_a$ where $a \in M_j$ then $\Phi(b) = \Phi(a)$

(c) if b_1, b_2 satisfies the following then $\Phi(b_1) = \Phi(b_2)$ where the demand is:

(i) $b_1, b_2 \in \mathrm{cl}_M(M_i \cup \bigcup\limits_{j=i,\dots,m-1} A_j)$ and

(ii) $F_{M,b_1} = F_{M,b_2}$ and for every $r \in \{1, \dots, \mathrm{arity}(F_{M,b_1})\}$ we have: $\mathrm{base}_{M,r}(b_1) = \mathrm{base}_{M,r}(b_2)$ or they both belong to the same A_j for some $j \in \{i, \dots, m-1\}$.

Now A_i, B_i, ϱ_i will be chosen such that the following condition holds:

$(*)_2$ If $\Phi, \Psi \in \mathbf{R}_i$ satisfy the clauses (a),(b) below then $d(\Phi) = d(\Psi)$ where

(a) $\Phi \restriction \mathrm{cl}_M(M_i \cup \bigcup\limits_{j=i+1,\dots,m-1} A_j) = \Psi \restriction \mathrm{cl}_M(M_i \cup \bigcup\limits_{j=i+1,\dots,m-1} A_j)$

(b) if $b \in N$ and $\gamma_b \in \mathrm{Rang}(\Phi \restriction M_{i+1})$ then $\gamma_b \in \mathrm{Rang}(\Phi \restriction M_i)$.

Suppose now that we have carried this induction, and we shall show that this suffice. Let S be the following subset of V:

$(*)_3$ $\eta \in S$ iff

(a) if $i < m$ and $b \in B_i$ and $\varrho_i(b) \in \Lambda$ then $\eta(b) = \varrho_i(b)$

(b) if $i < m$ and $b \in B_i$ and $\varrho_i(b) = \beta_a$ and $a \in M_i$ then $\eta(b) = \eta(a)$.

(c) if $b^\ell = F(b_1^\ell, \dots, b_t^\ell)$ for $\ell = 1, 2$ and $[r \in \{1, \dots, t\} \wedge b_r^1 \neq b_r^2 \Rightarrow (\exists i)(b_r^1 \in A_i \ \& \ b_i^2 \in A_i)]$ <u>then</u> $\eta(b^1) = \eta(b^2)$.

Clearly S is an $m-$subspace of V, and we may by $(*)_2$ above show that:

$(*)_4$ if Φ is an ℓ-subspace of S, then $d(\Phi)$ can be computed from $J[\Phi] =: \{$ Min $\{i : \Phi \restriction A_i$ is constantly the r-th member of $P^N\} : r < \ell\}$.

Clearly $J[\Phi]$ is a subset of $\{0, \dots, m-1\}$ with exactly ℓ elements.
So for some function e, with domain the family of subsets of $\{0, \dots, m-1\}$ with ℓ elements, we have: if Φ is an ℓ-subspace of S then $d(\Phi) = e(J[\Phi])$. Clearly the set $\mathrm{Rang}(e)$ has $\le |\mathrm{Rang}(d)|$ elements.
By Ramsey's theorem and the choice of m, there is a subset w of $\{0, \dots, m-1\}$ with t members such that the function e is constant on the family of subsets of w with ℓ elements. Let U be a subspace of S of dimension t such that if $b \in M$, base(b) not a subset of $\bigcup\limits_{i \in w} A_i$ then $\langle \nu(b) : b \in U\rangle$ is constant (and, of course, the constant value belongs to $\Lambda_{F_{M,b}}$).
Clearly U is as required. So it is enough to prove that we can carry the induction, that is, assume that $i \in \{0, \dots, m-1\}$ and $\langle A_j, \varrho_j : \varrho = i+1, i+2, \dots, m-1\rangle$ are as required (in $(*)_0 + (*)_2$); and we have to choose (A_i, ϱ_i) as required. But this is obvious by the choice of k_{i+1} see clause (g) of $\otimes_1$ above. $\square_{3.5}$

§4 The main Theorem

Now we turn to the obtainment of primitive recursive bounds. The idea is that we decrease the dependency from below, dealing with the unary functions each time (rather than dealing with $H \in \tau$ of maximal arity).

In the definition below, we shall use the case $r = 1$.

4.1 Definition. 1) Recall that for $a \in P^M$ we let M_a be $\mathrm{cl}_M(P^M \setminus \{a\})$, that is M restricted to this set.
2) For $V = \mathrm{Space}_{\bar\Lambda}(M)$ and $u \subseteq P^M$ we say that a colouring d of V is (u,r)-base-invariant if the following holds, for any $a \in u$:

- $(*)$ if $\nu, \eta \in V$ and $\nu \restriction M_a = \eta \restriction M_a$ and $[b \in M \wedge r < |\{i : i = 1, \dots, \mathrm{arity}(F_{M,b})$ and $\mathrm{base}_{M,i}(b) = a\}| \Rightarrow \nu(b) = \eta(b)]$ then $d(\nu) = d(\eta)$.

3) We write (ℓ, r)-base-invariant if above u is the set of the last ℓ members of P^M.
4) We may replace u by a submodel N of M, meaning $u = P^N$.

4.2 Definition. Let f^6 be defined as follows. First, $f^6_{\bar\Lambda}(\ell, c) = f^6(\bar\Lambda, \ell, c) = f^6_\tau(\bar\Lambda, \ell, c)$ is defined <u>iff</u> $\bar\Lambda$ is an alphabet sequence for a vocabulary τ. Second, let $f^6_{\bar\Lambda}(\ell, c)$ be the first k (natural number, if not defined we can understand it as ∞ or ω or "does not exist") such that $(*)_k$ below holds, where:

- $(*)_k$ If clauses (a)-(d) below hold <u>then</u> there is a d-monochromatic line of V, where:
  - (a) M is a fim of vocabulary τ
  - (b) the dimension of M is k
  - (c) $V = \mathrm{Space}_{\bar\Lambda}(M)$
  - (d) d is an $(\ell, 1)$-base-invariant colouring of V.

Immediate connections are:

4.3 Observation. 1) The function $f^6_{\bar\Lambda}(\ell, c)$ increases with c and decreases with ℓ.
2) We have $f^6_{\tau_1}(\bar\Lambda^1, \ell_1, c_1) \le f^6_{\tau_2}(\bar\Lambda^2, \ell_2, c_2)$ if:

- (a) $c_1 \le c_2, \ell_1 \ge \ell_2$, and
- (b) $s \le \mathrm{arity}(\tau_1) \Rightarrow \Pi\{|\Lambda^1_F| : F \in \tau_1,\ \mathrm{arity}^\tau(F) = s\} \le \Pi\{|\Lambda^2_F| : F \in \tau_2,\ \mathrm{arity}^\tau(F) = s\}$ and
- (c) $\mathrm{arity}(\tau_1) < s \le \mathrm{arity}\ (\tau_2) \wedge F \in \tau_2 \wedge \ \mathrm{arity}(F) = s \Rightarrow |\Lambda^2_F| = 1$.

3) In definition 4.2 the demand $(*)_{k'}$ holds for any k' larger than k.
4) $f^6_{\bar\Lambda}(0,c) = f^1(\bar\Lambda, c)$.

Proof. Trivial, e.g.
4) The point is that in $(*)$ of 4.1, the implication $[b \in M \wedge 0 < |\{i : i = 1,\dots,\text{arity}(F_{M,b}) \text{ and } \text{base}_{M,i}(b) = a\}| \Rightarrow \nu(b) = \eta(b)]$ the antecedent means just $b \in M \backslash M_a$ so the implication means $\eta \restriction (M\backslash M_a) = \nu \restriction (M\backslash M_b)$, so the assumption in $(*)$ says $\eta = \nu$ hence $(*)$ is an empty demand. $\square_{4.3}$

4.4 Claim. *Assume*

- (a) *τ is a vocabulary of arity > 1 and $\bar\Lambda$ is a $\tau-$alphabet sequence*
- (b) *τ^* is the following vocabulary:*
  $\{G_{F,e} : F \in \tau$ and e is a convex equivalence relation on $\{1,\dots,\, arity^\tau(F)\}$ such that if $arity^\tau(F) > 1$ then each e-equivalence class has at least two elements$\}$ with $arity^{\tau^*}(G_{F,e}) =$ the number of e-equivalence classes and for some $H \in \tau$ of maximal arity, letting $e =: \{(i,j) : i,j \in [1, arity^{\tau^*}(H)]\}$ we identify $G_{H,e}$ with $id \in \tau^*$
- (c) *$\bar\Lambda^*$ is the following τ^*-alphabet sequence: $\Lambda^*_{G_{F,e}} = \Lambda_F$*
- (d) *$\ell^* = f^1_{\tau^*}(\bar\Lambda^*, c)$.*

<u>Then</u> *$f^6_{\bar\Lambda}(\ell^*, c) \le \ell^*$.*

Proof. Let M be a fim of vocabulary τ and dimension ℓ^* and $V = \text{Space}_{\bar\Lambda}(M)$ and d is a c-colouring of V which is $(\ell^*, 1)$-base-invariant; it suffices to find a monochromatic V-line L.
Let M^* be a fim of vocabulary τ^* and dimension ℓ^* and $V^* = \text{Space}_{\bar\Lambda^*}(M^*)$. Let g_0 be an isomorphism from $(P^M, <^M)$ onto $(P^{M^*}, <^{M^*})$. We define a partial function g from M into M^* as follows; if $b = F^M(b_1,\dots,b_t)$ so $t = \text{arity}^\tau(F)$ and $b_1 \le^M b_2 \le^M \dots \le^M b_t$ and $e = \{(i,j) : b_i = b_j\}$ and $G_{F,e} \in \tau^*$ is well defined (i.e. $t = 1$ or every e-equivalence class has at least two elements) and the e-equivalence classes are $[s_i, s_{i+1})$ for $i = 1,\dots,\text{arity}^{\tau^*}(G_{F,e}) - 1$ and $1 = s_1 < s_2 < \dots < s_{\text{arity}(G_{F,e})} = t+1$ then $g(b) = G^{M^*}_{F,e}(g_0(b_{s_1}),\dots,g_0(b_{s_{\text{arity}(G_{F,e})-1}}))$.
Note:

- $(*)_1$ g is really a partial function from M to M^*
- $(*)_2$ if $\eta,\nu \in V$ and $\eta \restriction \text{Dom}(g) = \nu \restriction \text{Dom}(g)$ then $d(\eta) = d(\nu)$.

[Why? By the transitivity of equality, it is enough to consider the case that for some $a^* \in M \setminus \mathrm{Dom}(g)$ we have $\{a^*\} = \{a \in M : \eta(a) \neq \nu(a)\}$. Clearly by the definition of g for some $a \in P^M$ we have $(\exists! i)[\mathrm{base}_{M,i}(a^*) = a]$. Now we know that d is $(\ell^*, 1)$-base invariant (see clause (d) of $(*)_k$ from Definition 4.2). Hence by Definition 4.1(3), d is $(P^M, 1)$-base invariant. Hence by Definition 4.1(2), our η, ν satisfies $(*)$ of Definition 4.1(2).
Now the first assumption in $(*)$, i.e. $\eta \restriction M_a = \nu \restriction M_a$ holds as by the choice of a we have $a^* \notin M_a$ as we have $(\forall b \in M)(b \neq a^* \to \eta(a) = \nu(a))$. Also the second assumption of $(*)$ holds as the antecedent fails for $b = a^*$ (as $1 = |\{i : \mathrm{base}_{M,i}(a^*) = a\}|$ and its conclusion, $\eta(b) = \nu(b)$ holds for $b \in M \setminus \{a^*\}$. We can conclude that $d(\eta) = d(\nu)$ as required.]

$(*)_3$ we can define a c-colouring d^* of V^* such that: if $\eta \in V, \nu \in V^*$, and $[b \in \mathrm{Dom}(g) \Rightarrow \eta(b) = \nu(g(b))]$ then $d(\eta) = d^*(\nu)$.

[Why? By $(*)_2$.]

$(*)_4$ for any V^*-line L^* there is a $V-$line L such that for every $\eta \in L$ for some $\nu \in L^*$ we have $d(\eta) = d^*(\nu)$.

[Why? Reflect. In details, let $w^* = \mathrm{supp}^P(L^*)$ and $N^* = \mathrm{supp}(L^*)$ and ν^* is the function with domain $M^* \setminus N^*$ such that for every b from this set and $\nu \in L^*$ we have $\nu(b) = \nu^*(b)$. Let $w =: \{b \in P^M : g_0(b) \in w^*\}$ and let $N =: \mathrm{cl}_M(w)$ and choose a function η^* with domain $M \setminus N$ such that for every $b \in M \setminus N$ we have $\eta^*(b) = \nu^*(g(b))$ if $b \in \mathrm{Dom}\,(g)$ and is any member of $\Lambda_{F_{M,b}}$ otherwise. Let L be the V-line such that $\mathrm{supp}(L) = N$ and for every $\eta \in L$ the function η extends η^*. Clearly, L is a V-line and let $\eta \in L$ and we should check the desired conclusion. So there is $p \in \mathfrak{p}_{\bar\Lambda}$ such that $\eta = \mathrm{pt}_L(p)$; now we define $q \in \mathfrak{p}_{\bar\Lambda^*}$ as follows: $q(G_{F,e}) = p(F)$, the later belongs to Λ_F which is equal to $\Lambda^*_{G_{F,e}}$. Let $\nu = \mathrm{pt}_{L^*}(q)$ and we should just check that η, ν are as in $(*)_3$ above so we are done.]

By the assumption $\ell^* = f^1(\bar\Lambda^*, c)$ (see clause (d) in the assumption), hence there is a d^*-monochromatic V^*-line L^*. Apply $(*)_4$ to it, so there is a d-monochromatic V-line and so we are done. $\square_{4.4}$

4.5 Definition. 1) Assume the following:

- (i) $\bar\Lambda$ is an alphabet sequence for the vocabulary τ
- (ii) $\mathbb{P} \subseteq \{(p,q) : p, q$ are $\bar\Lambda$-types$\}$, see Definition 1.8
- (iii) $m, c > 0$.

We define $f^7_{\bar\Lambda}(\mathbb{P}, m, c)$ as the first k (if there is no such k it is ω or ∞ or undefined) such that $(*)_k$ stated below holds, where

$(*)_k$ if clauses (a)-(e) below hold, <u>then</u> there is a subspace S of V of dimension m, satisfying:
if L is a V-line $\subseteq S$, then L is $(\mathbb{P}, d)$-monochromatic which means:
if $(p, q) \in \mathbb{P}$ then $d(\mathrm{pt}_L(p)) = d(\mathrm{pt}_L(q))$ where

- (a) M is a fim of vocabulary τ
- (b) M has dimension k
- (c) $V = \mathrm{Space}_{\bar\Lambda}(M)$
- (d) $\mathbb{P}$ is a subset of $\{(p,q) : p, q \in \mathfrak{p}_{\bar\Lambda}$ and $[F \in \tau \wedge$ arity$(F) > 1 \Rightarrow p(F) = q(F)]\}$
- (e) d is a c-colouring of V.

2) Let $\mathbb{P}_{\bar\Lambda} = \{(p,q) : p, q \in \mathfrak{p}_{\bar\Lambda}$ and $[F \in \tau \wedge$ arity$(F) > 1 \Rightarrow p(F) = q(F)]\}$.

4.6 Main Claim. *Assume*

- (a) $\bar\Lambda$ *is an alphabet sequence for a vocabulary* $\tau = \tau[\bar\Lambda]$
- (b) $k_0 \geq f^6_{\bar\Lambda}(\ell + 1, c)$ *and* $k_0 > \ell$
- (c) K *is a* $\tau-$*fim of dimension* $k_0 - 1$ *and* A_2 *is the set of the last* ℓ *elements and* A_1 *is the set of the first* $(k_0 - \ell - 1)$*-elements (this* K *serves just for notation)*
- (d) τ^* *is the vocabulary* τ_{K,A_1,A_2}*, see Definition 1.7(1) and proj is the following function from* τ^* *to* τ*: it maps* $F_{K,\bar a_1,\bar a_2}$ *to* F *and* $\bar\Lambda^* =: \langle \Lambda^*_F : F \in \tau^* \rangle$ *where* $\Lambda^*_F = \Lambda_{\mathrm{proj}(F)}$*, so proj* $\restriction \tau$ *is the identity*
- (e) $c^* =: c^{\mathrm{card}(\mathrm{Space}_{\bar\Lambda}(K))}$.

<u>Then</u> $f^6_{\bar\Lambda}(\ell, c) \leq k_0 + f^7_{\bar\Lambda^*}(\mathbb{P}_{\bar\Lambda^*}, 1, c^*) - 1$.

Remark. This is similar to the proof of 2.4, but for completeness we do it in full.

Proof. Let $k_1 = f^7_{\bar\Lambda^*}(\mathbb{P}_{\bar\Lambda^*}, 1, c^*)$ and let $k = k_0 + k_1 - 1$, so it suffices to prove that $k \geq f^6_{\bar\Lambda}(n, \ell, c)$. For this it is enough to check $(*)_k$ from Definition 4.2. So let M be a fim of vocabulary τ and dimension k (that is P^M is with k members), $V = \mathrm{Space}_{\bar\Lambda}(M)$, and d an $(\ell, 1)$-base-invariant C-colouring of V such that C has $\leq c$ members. So we just have to prove that the conclusion of Definition 4.2 holds, which means that there is a monochromatic V-line.
Let $w_1 =: \{a : a \in P^M$ and the number of $b <^M a$ is $\geq k_0 - \ell - 1$ but is

$< k_0 - \ell - 1 + k_1\}$ hence in w_1 there are k_1 members, and let w_0 be the set of first $k_0 - \ell - 1$ members of P^M by $<^M$, lastly let w_2 be the set of the ℓ last members of M by $<^M$. So w_0, w_1, w_2 form a convex partition of P^M.

Now we let K be M restricted to $\mathrm{cl}_M(w_0 \cup w_2)$, (note that this gives no contradiction to the assumption on K, as concerning K there, only its vocabulary and dimension are important and they fit) hence $w_0 = A_1, w_2 = A_2$. Let K^+ be a fim with vocabulary τ and dimension k_0, let $g_0 \in \mathrm{PHom}(M, K^+)$ be the following function from P^M onto P^{K^+}: it maps all the members of w_1 to one member of P^{K^+} which we call b^*, it is a one to one order preserving function from w_2 onto $\{b \in P^{K^+} : b^* <^{K^+} b\}$ and it is a one to one order preserving function from w_0 onto $\{b \in P^{K^+} : b <^{K^+} b^*\}$. Let $g \in \mathrm{Hom}(M, K^+)$ be the unique extension of g_0; without loss of generality g_0 is the identity on w_0 and on w_2 hence without loss of generality g is the identity on K; it exists by 1.4.

Next recall that the vocabulary $\tau^* = \tau_{K,w_o,w_2}$ is a well defined vocabulary (see Definition 1.7(1)). Next we shall define a τ^*-fim N. Its universe is $M \setminus K$; we let P^N be w_1 and $<^N$ be $<^M \restriction P^N$. Now we have to define the function $F^N_{\bar a_1, \bar a_2}$, say of arity r, where $F \in \tau, \bar a_1$ a non decreasing sequence from w_0 and $\bar a_2$ a non decreasing sequence from w_2, and $\ell g(\bar a_1) + \ell g(\bar a_2) < \mathrm{arity}(F)$. So $r = \mathrm{arity}(F) - \ell g(\bar a_1) - \ell g(\bar a_2)$. For $b_1 \le^N \ldots \le^N b_r \in P^N$ we let $F^N_{\bar a_1, \bar a_2}(b_1, \ldots, b_r)$ be equal to $b = F^M(\bar a_1, b_1, \ldots, b_r, \bar a_2) = F^M(a^1_1, a^1_2, \ldots, a^1_{\ell g}(\bar a_1), b_1, \ldots, b_r, a^2_1, \ldots, a^2_{\ell g}(\bar a_2))$.

It is easy to check that the number of arguments is right and also the sequence they form is $\le^M$ $-$increasing, so this is well defined and b belongs to M, but still we have to check that it belongs to N. But $N = M \setminus K$ and if $b \in K$ then $\mathrm{base}_{\ell g(\bar a_1)+1}(b) \in K$ and it is just b_1 which belongs to w_1, contradiction. Lastly it is also trivial to note that every member of N has this form. It is easy to check that N is really a τ^*-fim.

We next let $V^* = \mathrm{Space}_{\bar\Lambda^*}(N)$, let $C^* = \{g : g$ is a function from $\mathrm{Space}_{\bar\Lambda}(K)$ to $C\}$ and define a C^*-colouring d^* of V^*. For $\eta \in V^*$ let $d^*(\eta)$ be the following function from $\mathrm{Space}_{\bar\Lambda}(K)$ to C: for $\nu \in K$ we let $(d^*(\eta))(\nu) = d(\eta \cup \nu)$.

Clearly the function $d^*(\eta)$ is a C^*-colouring of K. How many such functions, that is members of C^* are there? The domain has clearly $\mathrm{card}(\mathrm{Space}_{\bar\Lambda}(K))$ members, (we can get slightly less if $\ell > 0$, but with no real influence). The range has at most c members, so the number of such functions is at most $c^{\mathrm{card}(\mathrm{Space}_{\bar\Lambda}(K))}$, a number which we have called c^* in the claim's statement.

Hence d^* is a c^*-colouring of V^*.

So as we have chosen $k_1 = f^7_{\bar\Lambda^*}(\mathbb{P}_{\bar\Lambda^*}, 1, c^*)$ we can apply Definition 4.5 to $V^* = \mathrm{Space}_{\bar\Lambda^*}(N)$ and d^*; so we can find a $(\mathbb{P}_{\bar\Lambda^*}, d^*)$-monochromatic V^*-line L^* (see $(*)_k$ of Definition 4.5(1)). Let h be the function from $U =: \mathrm{Space}_{\bar\Lambda}(K^+)$ to V defined as follows:

$(*)$ $h(\rho) = \nu$ iff:

(a) $\nu \in V, \rho \in U$,

(b) $\nu \restriction K = \rho \restriction K$

(c) if $b \in N \setminus \mathrm{supp}_N(L^*)$ then $\nu(b) = \eta(b)$ for every $\eta \in L^*$

(d) if $a \in \mathrm{supp}_N(L^*)$, (so $a \in N, F_{N,a} = F_{K,\bar{a}_1,\bar{a}_2}$, $\mathrm{base}_N(a) \subseteq \mathrm{supp}^P_N(L^*)$), and $b \in K^+, F_{K^+,b} = F, b = F(\bar{a}_1, b^*, \dots, b^*, \bar{a}_2)$ (with the number of cases of b^* being $\mathrm{arity}(F_{K,\bar{a}_1,\bar{a}_2})$) <u>then</u> $\rho(b) = \nu(a)$.

Let the range of h be called S. Now clearly

$\otimes_1(\alpha)$ h is a one to one function from U onto $S \subseteq V$

(β) S has $|\mathrm{Space}_{\bar{\Lambda}}(K^+)|$ members

(γ) S is a subspace of V of dimension $k_0, h(\rho) = \mathrm{pt}_S(\rho)$, see Definition 1.9(7).

Now clearly

$\otimes_2$ there is a C-colouring d° of U such that:
$d^\circ(\nu) = d(h(\nu))$ for $\nu \in U$

$\otimes_3$ d° is $(\ell+1, 1)$-base-invariant.

[Why? Let $b \in P^{K^+}$ be among the $(\ell+1)$ last members, (by $<^{K^+}$) and we should prove that d° is $(\{b\}, 1)$-base invariant, see Definition 4.1(1),(2).
If $b \in w_2$ we just use "d is $(\ell, 1)$-base invariant of the space V". If $b \notin w_2$ then necessarily $b = b^*$, translating the desired property "d^0 is a $(\{b^*\}, 1)$-base invariant", to a property of the colouring d, we get a demand on $d \restriction L^*$, which holds as L is $(\mathbb{P}_{\bar{\Lambda}^*}, d^*)$-base invariant (recalling by Definition 4.5(2), $\mathbb{P}_{\bar{\Lambda}^*} = \{(p,q) : p, q \in \mathbb{P}_{\bar{\Lambda}^*}$ such that $[F \in \tau^* \wedge \ \mathrm{arity}^{\tau^*}(F) > 1 \Rightarrow p(F) = q(F)]\})$.
So $\otimes_3$ really holds.]

Applying the definition of $k_0 = f^6_{\bar{\Lambda}}(\ell+1, c)$, that is Definition 4.2 to $\bar{\Lambda}, U, d^\circ$ we can conclude that there is a d°-monochromatic U-line L°. Let $L =: \{h(\rho) : \rho \in L^\circ\}$. It is easy to check that L is as required. $\square_{4.6}$

4.7 Claim. *1) Assume that $\bar{\Lambda}$ is a τ-alphabet sequence, and $p^* \in \mathfrak{p}_{\bar{\Lambda}}$ and $\mathbb{P}^+ = \mathbb{P} \cup \{(p^*, q) : q \in \mathfrak{p}_{\bar{\Lambda}}$ and $[F \in \tau \wedge$ arity $(F) > 1 \Rightarrow q(F) = p^*(F)]\} \subseteq \mathbb{P}_{\bar{\Lambda}}$ (see Definition 4.5) and $n = \Pi_{F \in \tau, \ \mathrm{arity}(F)=1} |\Lambda_F|$. <u>Then</u> $f^7_{\bar{\Lambda}}(\mathbb{P}^+, m, c) \le HJ(n, f^7_{\bar{\Lambda}}(\mathbb{P}, m, c), c)$ (on HJ see 0.3(2)).*
2) $f^7(\bar{\Lambda}, \mathbb{P}_{\bar{\Lambda}}, m, c)$ is in $\mathbb{E}_6$.

Proof. 1) Straight. Let M be a τ-fim of dimension $k =: \mathrm{HJ}(n, f^7_{\bar\Lambda}(\mathbb{P}, m, c), c)$ and let $V = \mathrm{Space}_{\bar\Lambda}(M)$ and let d be a c-colouring of V. Let $\tau^* = \{F \in \tau\colon \mathrm{arity}^\tau(F) = 1\}$ and let M^* be a τ^*-fim of dimension k; without loss of generality M^* is M restricted to τ^* and the universe of M^*. Let $\bar\Lambda^* = \bar\Lambda \restriction \tau^*$ and let $V^* = \mathrm{Space}_{\bar\Lambda^*}(M^*)$ and let h be the function from V^* to V defined as follows: let $\eta \in V^*$, for $b \in M$ let $(h(\eta))(b)$ be $\eta(b)$ if $b \in M^*$ and be $p^*(F_{M,b})$ if $b \in M \setminus M^*$. So h is a function as required and we define a c-colouring d^* of V^* by $d^*(\nu) = d(h(\nu))$ for $\nu \in V^*$.

Now essentially we apply the definition of $k = \mathrm{HJ}(n, f^7_{\bar\Lambda}(\mathbb{P}, m, c), c)$ to the space V^* and the colouring d^* and we get a subspace S^* of V^* on which d^* is constant and has dimension $f^7_{\bar\Lambda}(\mathbb{P}, m, c)$; we say "essentially" because as in Claim 1.13(1) we can reduce τ to a singleton id with $\Lambda = \pi\{\Lambda^*_F : F \in \tau^*\}$ that is

$(*)_0$ $f^7_{\bar\Lambda}(\mathbb{P}_{i+1}, m, c) \le \mathrm{HJ}(|\Pi_{F\in\tau,\ \mathrm{arity}(F)=1}|\Lambda_F||, f^7_{\bar\Lambda}(\mathbb{P}_i, m, c), c)$.

There is a unique subspace S' of V of dimension $f^7_{\bar\Lambda}(\mathbb{P}, m, c)$ such that $\eta \in S' \Rightarrow \eta \restriction M^* \in S^*$. Clearly:

$(*)_1$ if L is a V-line which is $\subseteq S$ and $(p, q) \in \mathbb{P}^+ \backslash \mathbb{P}$ then $d(pt_L(p)) = d(pt_L(q))$.

Now, letting $k' = f^7_{\bar\Lambda}(\mathbb{P}, m, c)$ and $d' = d \restriction S'$, we can apply the definition of $f^7_{\bar\Lambda}(\mathbb{P}, m, c)$ and get a subspace S of S' of dimension m such that

$(*)_2$ if L is a V-line which is included in S and $(p, q) \in \mathbb{P}$ then $d'(pt_L(p)) = d'(pt_L(q))$ which means that $d(pt_L(p)) = d(pt_L(q))$.

By $(*)_1 + (*)_2$, clearly S is as required.
2) Let $\{p^*_i : i < i(*)\}$ be maximal subset of $\mathbb{P}_{\bar\Lambda}$ such that $i < j < i(*) \Rightarrow p^*_i \restriction \{F \in \tau : \mathrm{arity}(F) > 1\} \neq p^*_j \restriction \{F \in \tau : \mathrm{arity}(F) > 1\}$ and let $\mathbb{P}_j = \{(p^*_i, q) : i < j$ and $q \in \mathfrak{p}_{\bar\Lambda}$ and $q \restriction \{F \in \tau\colon \mathrm{arity}(F) > 1\} = p^*_i \restriction \{F \in \tau : \mathrm{arity}(F) > 1\}\}$. By part (1) we have a recursion formula (we use 1.13 freely, more exactly, $(*)_0$ from the proof of part (1)).
As HJ belongs to $\mathbb{E}_5$ (by [Sh 329, 1.8(2),p.691]), we are done. $\square_{4.7}$

4.8 Definition. Let $f^{6,*}(\bar\Lambda, \ell, t, c)$ be defined by induction on ℓ as follows:
$f^{6,*}(\bar\Lambda, 0, t, c) = t$
$f^{6,*}(\bar\Lambda, \ell + 1, t, c)$ is equal to $k_0 + f^7_{\bar\Lambda[k_0]}(\mathbb{P}_{\bar\Lambda[k_0]}, 1, c^{\mathrm{card}(\mathrm{Space}_{\bar\Lambda}(M^\tau_{k_0}))}) - 1$ where $k_0 = \mathrm{Max}\{\ell + 1, f^{6,*}(\bar\Lambda, \ell, t, c)\}$ and $\bar\Lambda[k_0]$ is defined from $\bar\Lambda$ as in the main claim 4.6.

4.9 Claim. *$f^{6,*}$ belongs to $\mathbb{E}_7$.*

Proof. Straight.

4.10 Theorem. *1) The function $f^1(\bar{\Lambda}, c)$ is well defined, i.e. always get a value which is a natural number and is primitive recursive, in fact belongs to $\mathbb{E}_8$.*
2) Similarly the function $f^6(\bar{\Lambda}, \ell, c)$, see Definition 4.2.
3) f^4 is primitive recursive, in fact belongs to $\mathbb{E}_9$, see Definition 3.4.

Proof. 1),2) We write a proof of the existence of $f^1(\bar{\Lambda}, c), f^6(\bar{\Lambda}, \ell, c)$ which makes it clear that they are in $\mathbb{E}_8$.
Let $\tau = \tau[\bar{\Lambda}]$. The proof follows by induction, the main induction is on $t = \text{arity}^\tau(\tau_{\bar{\Lambda}})$ (or, if you prefer $\Pi_{F \in \tau[\bar{\Lambda}]}(|\Lambda_F| + 1)$).

CASE 0: $\text{arity}(\tau) = 1$.
This is Hales-Jewett theorem (on a bound see [Sh 329] or [GRS80]).

CASE 1: $\text{arity}(\tau) > 1$.
Let $\tau^{**}, \bar{\Lambda}^{**}$ be as $\tau^*, \bar{\Lambda}^*$ in Claim 4.4, so $\text{arity}(\tau^*) \leq \text{arity}(\tau)/2$ and $|\tau^*| \leq |\tau| \times 2^{\text{arity}(\tau)}$. Let $\ell^* = f^1(\bar{\Lambda}^{**}, c)$ so (by 4.4) clearly $f^6_{\bar{\Lambda}}(\ell^*, c) \leq \ell^*$ hence (by Definition 4.8) clearly $f^{6,*}(\bar{\Lambda}, 0, \ell^*, c) = \ell^* = f^1(\bar{\Lambda}^{**}, c)$; together we get $f^6_{\bar{\Lambda}}(\ell^*, c) \leq f^{6,*}(\bar{\Lambda}, 0, \ell^*, c)$. Now (by 4.6 + Definition 4.8), we shall prove by induction on $\ell \leq \ell^*$ that $f^6_{\bar{\Lambda}}(\ell^* - \ell, c) \leq f^{6,*}(\bar{\Lambda}, \ell, \ell^*, c)$; for $\ell = 0$ this holds by the previous sentence; for the induction step, i.e. the proof for $\ell + 1$ we apply Theorem 4.6 with $\ell^* - \ell, \ell^* - (\ell + 1)$ here standing for $\ell + 1, \ell$ there and letting $k_0 = \text{Max}\{\ell^* \ell, f^6_{\bar{\Lambda}}(\ell^* - \ell, c)\}$ and $\tau^*, \bar{\Lambda}^*, c^*$ defined as there, and we get that $f^6_{\bar{\Lambda}}(\ell^* - (\ell+1), c) \leq k_0 + f^7_{\bar{\Lambda}^*}(\mathbb{P}_{\bar{\Lambda}^*}, 1, c^*) - 1 \leq \text{Max}\{\ell^* \ell, f^6_{\bar{\Lambda}}(\ell^* - \ell, c)\} + f^7_{\bar{\Lambda}^*}(\mathbb{P}_{\bar{\Lambda}^*}, 1, c^*) - 1$ but the last expression is exactly $f^{6,*}(\bar{\Lambda}, \ell + 1, \ell^*, c)$.
So (using $\ell = \ell^*$) clearly $f^6_{\bar{\Lambda}}(0, c) \leq f^{6,*}(\bar{\Lambda}, \ell^*, \ell^*, c)$.

Now $f^1(\bar{\Lambda}, c) = f^6_{\bar{\Lambda}}(0, c) \leq f^{6,*}(\bar{\Lambda}, \ell^*, \ell^*, c) \leq f^{6,*}(\bar{\Lambda}, f^1(\bar{\Lambda}^{**}, c), f^1(\bar{\Lambda}^{**}, c), c)$.
As $f^{6,*}$ is from $\mathbb{E}_7$ by 4.7, this clearly give the desired conclusion.
3) Should be clear from the proof of 3.5 and the previous parts. $\square_{4.10}$

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