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UNIFORMLY ANTISYMMETRIC FUNCTIONS WITH BOUNDED RANGE

Abstract

The goal of this note is to construct a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ with a bounded countable range. This answers Problem 1(b) of Ciesielski and Larson [6]. (See also the list of problems in Thomson [9] and Problem 2(b) from Ciesielski's survey [5].) A problem of existence of uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{R}$ with finite range remains open.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *uniformly antisymmetric* [6] (or *nowhere weakly symmetrically continuous* [9]) provided for every $x \in \mathbb{R}$ the limit $\lim_{n\to\infty}(f(x+s_n) - f(x-s_n))$ equals 0 for no sequence $\{s_n\}_{n<\omega}$ converging to 0. Uniformly antisymmetric functions have been studied by Kostyrko [7], Ciesielski and Larson [6], Komjáth and Shelah [8], and Ciesielski [1, 2]. (A connection of some of these results to the paradoxical decompositions of the Euclidean space $\mathbb{R}^n$ is described in Ciesielski [3].) In particular in [6] the authors constructed a uniformly antisymmetric function $f\colon \mathbb{R} \to \mathbb{N}$ and noticed that the existence of a uniformly antisymmetric function cannot be proved without an essential use of the axiom of choice.

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The terminology and notation used in this note is standard and follows [4]. In particular for a set X we will write $|X|$ for its cardinality and $\mathcal{P}(X)$ for its power set. Also 2^ω will stand for the set of all functions from $\omega = \{0, 1, 2, \ldots\}$ into $2 = \{0, 1\}$. We consider 2^ω as ordered lexicographically.

Theorem 1. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ with countable bounded range such that for every $x \in \mathbb{R}$ there exists an $\varepsilon_x > 0$ with the property that the set*

$$S_x = \{s \in \mathbb{R} \colon |f(x - s) - f(x + s)| < \varepsilon_x\}$$

is finite. In particular f is uniformly antisymmetric.

PROOF. First notice that it is enough to find a compact zerodimensional metric space $\langle T, d\rangle$ and a function g from $\mathbb{R}$ into a countable subset T_0 of T such that for every $x \in \mathbb{R}$ there is a $\delta_x > 0$ for which the set

$$\hat{S}_x = \{s \in \mathbb{R} \colon d(g(x - s), g(x + s)) < \delta_x\}$$

is finite.

To see this assume that such a function $g : \mathbb{R} \to T$ exists and take a homeomorphic embedding h of T into $\mathbb{R}$. We claim that $f = h \circ g : \mathbb{R} \to \mathbb{R}$ is as desired. Indeed, $f[\mathbb{R}] = h[g[\mathbb{R}]]$ is countable, as it is a subset of a countable set $h[T_0]$, and it is bounded, since it is a subset of a compact set $h[T]$. So take $x \in \mathbb{R}$ and $\delta_x > 0$ for which $\hat{S}_x$ is finite. Since $h^{-1} : h[T] \to T$ is uniformly continuous, we can find an $\varepsilon_x > 0$ such that

$$|y_1 - y_2| < \varepsilon_x \text{ implies } d(h^{-1}(y_1), h^{-1}(y_1)) < \delta_x$$

for every $y_1, y_2 \in h[T]$. But for such a choice of ε_x we have

$$S_x = \{s \in \mathbb{R} \colon |h(g(x - s)) - h(g(x + s))| < \varepsilon_x\} \subset \hat{S}_x$$

proving that S_x is finite.

Thus, we proceed to construct a function g described above. The value of $g(x)$ will be defined with help of a representation of x in a Hamel basis; i.e., a linear basis of $\mathbb{R}$ over $\mathbb{Q}$. For this we will use the following notation. Let $\{y_\eta : \eta \in 2^\omega\}$ be a one-to-one enumeration of a Hamel basis $\mathcal{H}$. For every $x \in \mathbb{R}$ let $\sum_{\eta \in 2^\omega} q_{x,\eta} y_\eta$, with $q_{x,\eta} \in \mathbb{Q}$ for $\eta \in 2^\omega$, be the unique representation of x in basis $\mathcal{H}$ and let $w_x = \{\eta \in 2^\omega : q_{x,\eta} \neq 0\}$. Thus w_x is finite and

$$x = \sum_{\eta \in w_x} q_{x,\eta} y_\eta.$$

The definition of the space T is considerably more technical since it reflects several different cases of the proof that the sets $\hat{S}_x$ are indeed finite. To this end let $\{q_j\colon j<\omega\}$ be a one-to-one enumeration of $\mathbb{Q}$ with $q_0=0$. For $i<\omega$ let $\mathcal{P}_i=\mathcal{P}(\{q_j\colon j<i\})$ and put $P_i=\mathcal{P}(2^i\times\{0,1\}\times\mathcal{P}_i\times\mathcal{P}_i)$. Note that each P_i is finite, so $T=\prod_{i<\omega}P_i$, considered as the standard product of discrete spaces, is compact zerodimensional. We equip T with a distance function d defined between different $s,t\in T$ by $d(s,t)=2^{-\min\{i<\omega\colon s(i)\neq t(i)\}}$ and let

$$T_0=\{t\in T\colon (\exists n<\omega)(\forall i\geq n)\ t(i)=\emptyset\}.$$

Clearly T_0 is countable.

Now we are ready to define $g\colon\mathbb{R}\to T_0\subset T$. For this, however, we will need few more definitions. For $x\in\mathbb{R}$, $q\in\mathbb{Q}$, $i<\omega$, and $\zeta\in 2^i$ such that $\zeta\in\{(\eta\restriction i)\colon \eta\in w_x\}$ we define:

- $p(i)\in\{0,1\}$ as the parity of i, i.e., $p(i)=i \bmod 2$;
- $k_i(q)=\{q_j\in\mathbb{Q}\colon q_j<q\ \&\ j<i\}\in\mathcal{P}_i$;
- $\eta(x,\zeta)$ to be the minimum of $\{\eta\in w_x\colon \zeta\subset\eta\}$ (in the lexicographical order);
- $\xi(x,\zeta)$ to be the minimum of $\{\eta\in w_x\colon \zeta\subset\eta\}\setminus\{\eta(x,\zeta)\}$ provided $|\{\eta\in w_x\colon\zeta\subset\eta\}|\neq 1$; otherwise we put $\xi(x,\zeta)=\eta(x,\zeta)$;
- $n_x<\omega$ to be the smallest number $n>0$ such that

  (i) $\eta\restriction n\neq\xi\restriction n$ for any different $\eta,\xi\in w_x$, and

  (ii) $q_{x,\eta}\in\{q_j\colon j<n\}$ for every $\eta\in w_x$.

Consider the function $g\colon\mathbb{R}\to T_0$ defined as follows. For every $x\in\mathbb{R}$ and $i<\omega$ we define $g(x)(i)\in P_i$ as

$$\left\{\langle\zeta,p(|\{\eta\in w_x\colon\zeta\subset\eta\}|),k_i(q_{x,\eta(x,\zeta)}),k_i(q_{x,\xi(x,\zeta)})\rangle\colon \zeta\in\{(\eta\restriction i)\colon\eta\in w_x\}\right\}$$

provided $i\leq n_x$ and we put $g(x)(i)=\emptyset$ for $n_x<i<\omega$. In the argument below the key role will be played by the function k_i in general, and the coordinate $k_i(q_{x,\eta(x,\zeta)})$ in particular.

The key step in the proof that g has the desired property is that for every $x\in\mathbb{R}$ and $s\neq 0$

$$\text{if}\quad n_x\leq\max\{n_{x-s},n_{x+s}\}\quad\text{then}\quad g(x-s)(n_x)\neq g(x+s)(n_x). \tag{1}$$

To see (1) assume that $n_x\leq n_{x+s}$. If $n_{x-s}<n_x$ then $g(x-s)(n_x)=\emptyset\neq g(x+s)(n_x)$, where $g(x+s)(n_x)\neq\emptyset$ since $w_{x+s}\neq\emptyset$ as $n_{x-s}<n_x\leq n_{x+s}$ implies $x+s\neq 0$. Thus, we can assume that $n_x\leq\min\{n_{x-s},n_{x+s}\}$. Take an

$\hat{\eta} \in w_{x-s} \cup w_{x+s}$ such that $q_{x-s,\hat{\eta}} \neq q_{x+s,\hat{\eta}}$ and let $\zeta = \hat{\eta} \restriction n_x$. Note that, by the definition of n_x, the set $S = \{\eta \in w_x : \zeta \subset \eta\}$ has at most one element.

If $S = \emptyset$ then $\{\eta \in w_{x-s} : \zeta \subset \eta\} = \{\eta \in w_{x+s} : \zeta \subset \eta\} \neq \emptyset$ and so $\eta(x-s,\zeta) = \eta(x+s,\zeta) \notin w_x$ while $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$. Thus $q_0 = 0$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$ implying that $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$. Therefore $g(x-s)(n_x) \neq g(x+s)(n_x)$.

So, assume that $S \neq \emptyset$ and let η' be the only element of S. Then $\eta' \in w_{x-s} \cup w_{x+s}$. If η' belongs to precisely one of the sets w_{x+s} and w_{x-s}, say w_{x+s}, then $\{\eta \in w_{x+s} : \zeta \subset \eta\} = \{\eta \in w_{x-s} : \zeta \subset \eta\} \cup \{\eta'\}$. In particular, $p(|\{\eta \in w_{x+s} : \zeta \subset \eta\}|) \neq p(|\{\eta \in w_{x-s} : \zeta \subset \eta\}|)$ implying that $g(x-s)(n_x) \neq g(x+s)(n_x)$.

So, we can assume that $\eta' \in w_{x-s} \cap w_{x+s}$. Then $\{\eta \in w_{x-s} : \zeta \subset \eta\} = \{\eta \in w_{x+s} : \zeta \subset \eta\}$ and $\eta(x-s,\zeta) = \eta(x+s,\zeta)$. We will consider three cases.

CASE 1: $\eta' \neq \eta(x-s,\zeta) = \eta(x+s,\zeta)$. Then $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 0$, so $q_0 = 0$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$. Thus $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

CASE 2: $\eta' = \eta(x-s,\zeta) = \eta(x+s,\zeta)$ and $q_{x-s,\eta(x-s,\zeta)} \neq q_{x+s,\eta(x+s,\zeta)}$. Then $q_{x-s,\eta(x-s,\zeta)} + q_{x+s,\eta(x+s,\zeta)} = 2q_{x,\eta'}$ and, by the definition of n_x, $q_{x,\eta'} \in \{q_j : j < n_x\}$. Since $q_{x,\eta'}$ separates $q_{x-s,\eta(x-s,\zeta)}$ and $q_{x+s,\eta(x+s,\zeta)}$ we conclude that $k_{n_x}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

CASE 3: $\eta' = \eta(x-s,\zeta) = \eta(x+s,\zeta)$ and $q_{x-s,\eta(x-s,\zeta)} = q_{x+s,\eta(x+s,\zeta)}$. Then $Z = \{\eta \in w_{x-s} : \zeta \subset \eta\} \setminus \{\eta(x-s,\zeta)\} = \{\eta \in w_{x+s} : \zeta \subset \eta\} \setminus \{\eta(x+s,\zeta)\}$ is non-empty, since it contains $\hat{\eta}$, and so $\xi(x-s,\zeta) = \xi(x+s,\zeta) \notin w_x$. Therefore, as in Case 1, $q_{x-s,\xi(x-s,\zeta)} + q_{x+s,\xi(x+s,\zeta)} = 0$, so $q_0 = 0$ separates $q_{x-s,\xi(x-s,\zeta)}$ and $q_{x+s,\xi(x+s,\zeta)}$. Thus $k_{n_x}(q_{x-s,\xi(x-s,\zeta)}) \neq k_{n_x}(q_{x+s,\xi(x+s,\zeta)})$ and $g(x-s)(n_x) \neq g(x+s)(n_x)$.

This finishes the proof of (1).

Next, for every $x \in \mathbb{R}$ put $\delta_x = 2^{-n_x}$. To finish the proof of the theorem it is enough to show that every $\hat{S}_x$ defined for such a choice of δ_x is a subset of a finite set

$$Z_x = \{s \in \mathbb{R} : w_{x+s} \subset w_x \ \&\ n_{x+s} < n_x\} = \left\{ \sum_{\eta \in w_x} p_\eta y_\eta : p_\eta \in \{q_j : j < n_x\} \right\}.$$

Indeed, take an $s \in \hat{S}_x$. Then, by (1) and the definition of the distance function d, we have $\max\{n_{x-s}, n_{x+s}\} < n_x$. Notice also that if $n_{x-s} \neq n_{x+s}$, say $n_{x-s} < n_{x+s}$, then $g(x-s)(n_{x+s}) = \emptyset \neq g(x+s)(n_{x+s})$ implying that $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$, which contradicts $s \in \hat{S}_x$. So, we have $n_{x-s} = n_{x+s}$. To prove that $s \in Z_x$ it is enough to show that

$w_{x+s} \subset w_x$. But if it is not the case then there exists an $\eta \in w_{x+s} \setminus w_x$. Moreover, $q_{x+s,\eta} = -q_{x-s,\eta} \neq 0$ and $\eta = \eta(x+s,\zeta) = \eta(x-s,\zeta)$, where $\zeta = \eta \restriction n_{x+s}$. In particular, $q_0 = 0$ separates $q_{x+s,\eta(x+s,\zeta)}$ and $q_{x-s,\eta(x-s,\zeta)}$. Therefore $k_{n_{x+s}}(q_{x-s,\eta(x-s,\zeta)}) \neq k_{n_{x+s}}(q_{x+s,\eta(x+s,\zeta)})$ and $g(x-s)(n_{x+s}) \neq g(x+s)(n_{x+s})$. So $d(g(x+s), g(x-s)) \geq 2^{-n_{x+s}} > 2^{-n_x} = \delta_x$ again contradicting $s \in \hat{S}_x$. Thus, $w_{x+s} \subset w_x$ and $s \in Z_x$.

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