Generalized E-Rings

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Abstract

A ring R is called an E-ring if the canonical homomorphism from R to the endomorphism ring $\mathrm{End}\,(R_{\mathbb{Z}})$ of the additive group $R_{\mathbb{Z}}$, taking any $r \in R$ to the endomorphism left multiplication by r turns out to be an isomorphism of rings. In this case $R_{\mathbb{Z}}$ is called an E-group. Obvious examples of E-rings are subrings of $\mathbb{Q}$. However there is a proper class of examples constructed recently, see [8]. E-rings come up naturally in various topics of algebra, see the introduction. So its not surprising that they were investigated thoroughly in the last decade, see [7, 21, 4, 10, 18]. This also led to a generalization: an abelian group G is an $\mathbb{E}$-group if there is an epimorphism from G onto the additive group of $\mathrm{End}\,(G)$. If G is torsion-free of finite rank, then G is an E-group if and only if it is an $\mathbb{E}$-group, see [14]. The obvious question was raised a few years ago which we will answer by showing that the two notions do not coincide. We will apply combinatorial machinery to non-commutative rings to produce an abelian group G with (non-commutative) $\mathrm{End}\,(G)$ and the desired epimorphism with prescribed kernel H. Hence, if we let $H = 0$, we obtain a non-commutative ring R such that $\mathrm{End}\,(R_{\mathbb{Z}}) \cong R$ but R is not an E-ring.

1 Introduction

Easy examples for E-rings are subrings of $\mathbb{Q}$ and further examples up to size $2^{\aleph_0}$ coming from p-adic integers are known for a long time. Details on those rings can be found in

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the two monographs [12, 13]. Arbitrarily large E-rings are constructed more recently first in [8] and then in [7]. The examples in [7] share additional properties related with their automorphism groups. E-rings are important in connection with (strongly) indecomposable modules, see [1], Pierce [21] and Paras [20]. They are also crucial for investigating problems in homotopy theory, see the work of Farjoun [10], [18] and Casacuberta, Rodriguez, Tai [4] or [22]. So it is not surprizing that they were studied in detail over the last decade. In this connection, some years ago Feigelstock, Hausen and Raphael extended the notion of E-rings and called an abelian group G an (EE-group, we use the shortcut) $\mathbb{E}$-group if we find an epimorphism from G onto the additive group of End (G). The results are published only recently in [14]. In particular the authors show that, if G is torsion-free of finite rank, then G is an E-group if and only if G is an $\mathbb{E}$-group - the two notions coincide for groups of finite rank. The obvious question was clear: It was asked whether this is true in general. Here we want to answer this open question. The nice feature about this problem is the fact that combinatorial arguments are needed and must be applied to non-commutative ring theory to produce the required examples.

We recall the two definitions.

Definition 1.1 *If R is a ring, then $\delta : R \longrightarrow End(R_{\mathbb{Z}})$ denotes the homomorphism which takes any $r \in R$ to the $\mathbb{Z}$-endomorphism $\delta(r)$ which is multiplication by r on the left. If this homomorphism is an isomorphism, then R is called an E-ring and $R_{\mathbb{Z}}$ is called an E-group.*

Recall from Schultz [23] that E-rings are necessarily commutative. The rings R we will construct are definitely not commutative and so not E-rings. However they are close to E-rings in the sense that the following definition holds.

Definition 1.2 *If R is the endomorphism ring of some abelian group G and there is an epimorphism $G \longrightarrow R \longrightarrow 0$ with kernel H, then G is called an $\mathbb{E}(H)$-group. Moreover, G is called an $\mathbb{E}$-group if G is an $\mathbb{E}(H)$-group for some abelian group H and G is a strong $\mathbb{E}$-group if G is an $\mathbb{E}(0)$-group, i.e. the epimorphism is an ismorphism.*

Note that trivially a strong $\mathbb{E}$-group G is in fact a ring R with $R_{\mathbb{Z}} = G$ and such that End $(R_{\mathbb{Z}}) \cong R$ as rings.

One of the main results in a paper by Feigelstock, Hausen and Raphael [14] is the following

Theorem 1.3 *Let G be any torsion-free abelian group of finite rank. Then G is an E-group if and only if G is an $\mathbb{E}$-group.*

We want to prove the following result which complements the theorem by Feigelstock, Hausen and Raphael and answers their problem.

Theorem 1.4 *For any infinite cardinal $\lambda = \mu^+$ with $\mu^{\aleph_0} = \mu$ we will find an $\aleph_1$-free abelian group G of cardinality $|G| = \lambda$ which is a (strong) $\mathbb{E}$-group.*

Remark 1.5 *A modification of our construction would also ensure that the constructed $\mathbb{E}(H)$-groups are proper in the sense that they are not strong $\mathbb{E}$-groups. All one has to do is to satisfy that $G_\alpha/H \not\cong G_\alpha$ for all $\alpha < \lambda$ where $G = \bigcup_{\alpha<\lambda} G_\alpha$ is the $\mathbb{E}(H)$-group. Then the group G can not be a strong $\mathbb{E}$-group.*

An R-module is $\aleph_1$-free if all its countable subgroups are free. Recall that endomorphism rings of $\aleph_1$-free abelian groups have $\aleph_1$-free additive group. The key tool of this paper can be found in Section 2, where we invesitgate non-commutative polynomial rings over rings. The construction of G is based on a strong version of (Shelah's) Black Box as stated in [24] or slightly modified in [17], see also [5]. The paper is based on our notes from 1998 which just haven't been in final form for publication.

2 Almost free rings over non-commuting variables

Polynomial rings $R[X]$ over a ring R with commuting variable X are obviously free R-modules. We will extend this to the non-commutative case, showing that the non-commutative polynomial ring $R\langle X\rangle$ as abelian group is $\aleph_1$-free if $R_{\mathbb{Z}}$ is so. This will be needed in Section 3. The ring construction is easy and well-known, see Bourbaki [3, pp. 216, 446 ff.]. Let R be a ring of characteristic 0, then M denotes all monomials, the elements

$$r_1X^{i_1}\cdots r_nX^{i_n},\ i_j \neq 0\ (j<n),\ r_j \in R\setminus\{1\}\ (1<j\leq n) \text{ and } r_j \neq 0, (j \leq n). \tag{2.1}$$

The ring $R\langle X\rangle$ is formally generated by all sums of the monomials in M. First we assume that $R_{\mathbb{Z}} = \bigoplus_{t\in T} \mathbb{Z}t$ is freely generated as abelian group by $T = \{t_i \quad : i \in I\}$ and $t_0 = 1$ without loss of generality. Then the multiplication on $R_{\mathbb{Z}}$ is coded into the so-called "constants of structure", see Bourbaki [3, p. 437], which are

$$\gamma^i_{jk} \in \mathbb{Z}\ (i,j,k \in \mathbb{Z}) \text{ with } t_j \cdot t_k = \sum_{i\in I} \gamma^i_{jk} t_i \tag{2.2}$$

The constants are well-defined by independence. We want to use T to find a representation for elements in $R\langle X\rangle$. Any element in R can be expressed as a sum over

T with coefficients in $\mathbb{Z}$. If $r \in R\langle X\rangle$ is a sum of monomials in M, then we may substitute any $r_i \in R$ from (2.1) as indicated, and the polynominal in M turns into a sum of what we will call T-monomials.

$$m = t_{i_1}X^{j_1}\cdots t_{i_n}X^{j_n} \in TM, \text{ with } j_k \neq 0 \ (k < n), \ t_{i_k} \neq 1 \ (1 < k \leq n). \tag{2.3}$$

Hence $R\langle X\rangle$ is generated as abelian group by all T-monomials and multiplication is ruled by the structure constants (2.2) restricted to T-monomials. Hence it seems plausible that the next lemma holds. We use the structure constants on R to define the ring multiplication on $R\langle X\rangle$. If $m, m' \in TM$, let $m = \overline{m}t, m' = t'\overline{m'}$ with $\overline{m}, \overline{m'} \in TM$ and $\overline{m}$ ending with $X^{\tilde{n}}, \overline{m'}$ beginning with $X^{\tilde{m'}}$. Then the product is defined by cases.

$$\text{If } t \neq 1 = t', \text{ then } mm' = \overline{m}tm', \text{ and if } t' \neq 1 = t, \text{ then } mm' = \overline{m}t'm', \tag{2.4}$$

$$\text{if } t' = 1 = t, \text{ then } mm' = \overline{m}m', \text{ with `middle term' } X^{\tilde{m}+\tilde{m'}} \tag{2.5}$$

$$\text{if } t' \neq 1 \neq t, \ (2.2) \text{ for } tt' = \sum_{i\in I}\gamma_{jk}^i t_i, t = t_j, t' = t_k, \text{ is } mm' = \sum_{i\in I}\gamma_{jk}^i m t_i \overline{m'}. \tag{2.6}$$

If $t_0 = 1$ is involved in the last equation, then the summand $\gamma_{jk}^0 m\overline{m'}$ has a 'middle term' $X^{\tilde{m}+\tilde{m}}$ as in case $t = t' = 1$. So clearly mm' is a sum of T-monomials.

Lemma 2.1 *If R is a ring with $R_{\mathbb{Z}} = \bigoplus_{t\in T}\mathbb{Z}t, \ T = \{t_i : i \in I\}$ and structure constants (2.2) as above, then let $R'_{\mathbb{Z}} = \bigoplus_{m\in TM}\mathbb{Z}m$ be the direct sum taken over all T-monomials TM as above. The multiplication on R' is now defined by (2.4) - (2.6) and it follows that $R' = R\langle X\rangle$ as rings with center $\mathfrak{z}R' = 1\mathbb{Z}$.*

Proof: Reducing elements in $R\langle X\rangle$ to sums of T-monomials we have seen that $R' = R\langle X\rangle$ as sets. Moreover, considering sums of T-monomials it is obvious that $R'_{\mathbb{Z}} = R\langle X\rangle_{\mathbb{Z}}$ as abelian groups. We finally must show equality as rings. There are two natural ways to do this. Either we extend the structure constants on R to $R', R\langle X\rangle$ and check their ring properties (see Bourbaki [3, p.438]) or we check that R'-multiplication is $R\langle X\rangle$-multiplication. We prefer the second way. Hence we must show that (2.2), (2.4) - (2.6) and its linear extension define uniquely the ring structure on R' which is the same as $R\langle X\rangle$. Any case of the ring laws reduces to consider distributivity of a product

$$(zm + z''m'')m' \tag{2.7}$$

with $z, z'' \in \mathbb{Z},\ m, m'$ as in (2.4) - (2.6) and $m'' = \overline{m''}t''$ similar to m above. If $t = t'' = 1$ or $t' = 1$, then (2.7) becomes immediately the unique T-monomial $zmm' + z''m''m'$ by (2.4) - (2.6) and linear extension. If $\overline{m} = \overline{m''}$, then (2.7) can be treated with (2.4) and distributivity in T given by (2.2):

$$\overline{m}(zt + z''t'')t'\overline{m'} = \overline{m}(ztt' + z''t''t')\overline{m'} = z\overline{m}tt'\overline{m'} + z''\overline{m''}t''t'\overline{m'} = zmm' + z''m''m'$$

showing the unique $R\langle X\rangle$-multiplication (by m') in this case. If $\overline{m} \neq \overline{m'}$, then zmm' and $z''m''m'$ are sums of **distinct** T-monomials, hence (2.7) defines the unique $R\langle X\rangle$-multiplication by linear extension. $\square$

Our main interest in Lemma 2.1 is extracted as the following

Corollary 2.2 *(a) If R is the ring above with $R_{\mathbb{Z}}$ free, then $R\langle X\rangle_{\mathbb{Z}}/R_{\mathbb{Z}}$ is free.*
(b) If $R_{\mathbb{Z}}$ is $\aleph_1$-free, then $R\langle X\rangle_{\mathbb{Z}}/R_{\mathbb{Z}}$ is $\aleph_1$-free.

Proof: By Lemma 2.1 we have $T \leq TM$ and TM is a $\mathbb{Z}$-basis of $R\langle X\rangle_{\mathbb{Z}}$, hence (a) follows. For (b) choose any countable set C of R and let $R_C = \langle\langle C\rangle\rangle$ be the subring generated by C and $R_C\langle X\rangle$ the subring of $R\langle X\rangle$ generated by C and X respectively. The ring R_C is free by hypothesis and $R_C\langle X\rangle/R_C$ is free by (a), hence (b) follows. $\square$

If J denotes the sum of all those monomials with at least one factor X, then J is a two sided ideal of $R\langle X\rangle$ (of 'non-constant' polynomials). We have $R\langle X\rangle = R \oplus J$ and if R is a field, then J is a maximal ideal of $R\langle X\rangle$. Separating summands of higher order becomes more complicated and fortunately is not needed.

Corollary 2.3 *Let R be as above. The set J of sums of monomials which are not constant are a two-sided ideal of $R\langle X\rangle$ and $R\langle X\rangle = R \oplus J$ is a ring split-extension.*

3 A class of quadruples for constructing $\mathbb{E}$-groups

We want to find an abelian group G with $R = \operatorname{End} G$ and $\sigma : G \longrightarrow R$ an epimorphism with prescribed kernel $\ker(\sigma) = H$ and pure image $\operatorname{Im}(\sigma) \subseteq_* R$. Hence G is a left R-module and the epimorphism σ induces a $\mathbb{Z}$-homomorphism

$$\sigma_* : G \longrightarrow R$$

such that $\sigma_*(x) \in R = \operatorname{End} G$ is defined by

$$\sigma_*(x)(y) = \sigma(y)x \text{ for all } y \in G. \tag{3.1}$$

The construction of (G, R, σ, σ_*) is inductively extending approximations of such quadruples such that the final one is as required. Let H be a fixed but arbitrary abelian group which is $\aleph_1$-free. We say that

Definition 3.1 *The quadruple* $q = (R_q, G_q, \sigma_q, \sigma_{q*}) = (R, G, \sigma, \sigma_*)$ *belongs to the class* $\mathfrak{K}_H$ *if the following holds.*
(a) R *is a unital ring of characteristic* 0.
(b) G *is a left* R*-module,*
(c) $G_{\mathbb{Z}}, R_{\mathbb{Z}}$ *are* $\aleph_1$*-free,*
(d) $\sigma : G \longrightarrow R$ *is a* $\mathbb{Z}$*-homomorphism with* $\ker \sigma = H$, *and* $1_R \in \mathrm{Im}\,(\sigma) \subseteq_* R$
(e) $\sigma_* : G \longrightarrow \mathrm{End}\, G$ *is a* $\mathbb{Z}$*-homomorphism defined by (3.1) and* $\sigma_*(g) \in R$ *for each* $g \in G$, *hence* $\sigma_* : G \longrightarrow R$,
(f) $\mathrm{Ann}\,_R G = 0$ *and* $R \subseteq_* \mathrm{End}\,(G)$.

We often use $q_\alpha = (R_\alpha, G_\alpha, \sigma_\alpha, \sigma_{\alpha*}) \in \mathfrak{K}_H$ for those quadruples. The next lemma is used to prove Proposition 3.3.

Lemma 3.2 *(a) If* G *is a free abelian group of finite rank, then* $\mathrm{End}\, G$ *is a free abelian group.*
(b) If G *is* $\aleph_1$*-free, then* $\mathrm{End}\, G$ *is an* $\aleph_1$*-free abelian group as well.*

Proof: Any endomorphism of G can be represented as an element of the cartesian product G^G and $\mathrm{End}\, G \subseteq G^G$ as abelian group. However $\aleph_1$-freeness is closed under cartesian products and subgroups, hence $\mathrm{End}\, G$ is $\aleph_1$-free. If G is freely generated by a finite set E, we may replace G^G by G^E which is free and (a) follows. □

The class $\mathfrak{K}_H$ of quadruples is partially ordered by inclusion, i.e. if

$$q = (R, G, \sigma, \sigma_*), q' = (R', G', \sigma', \sigma'_*) \in \mathfrak{K}_H,$$

then

$$q \le q' \text{ if and only if } R \subseteq R', G \subseteq G', \sigma \subseteq \sigma' \text{ and (hence) } \sigma_* \subseteq \sigma'_*.$$

The following proposition will ensure that our construction of an $\mathbb{E}(H)$-group will take place within $\mathfrak{K}_H$.

Proposition 3.3 *(a) If* $q_i = (R_i, G_i, \sigma_i, \sigma_{i*})$, $(i \in I)$ *is a continuous, ascending chain of quadruples in* $\mathfrak{K}_H$, *and* $G_i \subseteq_* G_{i+1}$, $1_{R_i} = 1_{R_{i+1}}$ *for all* $i \in I$, *then*

$$\bigcup_{i \in I} q_i = (\bigcup_{i \in I} R_i, \bigcup_{i \in I} G_i, \bigcup_{i \in I} \sigma_i, \bigcup_{i \in I} \sigma_{i*}) \in \mathfrak{K}_H.$$

(b) $(\mathbb{Z}, \mathbb{Z} \oplus H, \sigma, \sigma_*) \in \mathfrak{K}_H$ *where* $\sigma : \mathbb{Z} \oplus H \to \mathbb{Z}$ *is the canonical projection.*
(c) If $q \in \mathfrak{K}_H$, *then there exists* $q \le q' \in \mathfrak{K}_H$ *such that* $R_q \subseteq \sigma_{q'}(G_{q'})$ *and* $G_q \subseteq_* G_{q'}$.

Proof: (a) By continuity and (3.1) we have $\sigma_* = (\bigcup_{i\in I} \sigma_i)_* = \bigcup_{i\in I} \sigma_{i*}$, hence $\sigma_* : G \to R$ with pure image, where $G = \bigcup_{i\in I} G_i$ and $R = \bigcup_{i\in I} R_i$. Moreover, R is a unital ring with $1_R = 1_{R_i}$ for all $i \in I$ and G is a left R-module. $G_{\mathbb{Z}}$ is $\aleph_1$-free since G_i is pure in G_{i+1} for all $i \in I$ and therefore $R \subseteq \mathrm{End}_{\mathbb{Z}} G$ is $\aleph_1$-free as well by Lemma 3.2 (b). Note that $\mathrm{Ann}_R G = 0$. Finally, $\ker \sigma = H$ and $1_R \in \mathrm{Im}(\sigma)$ is clear. Hence it remains to show that R is pure in $\mathrm{End}_{\mathbb{Z}} G$. Assume that $\varphi \in \mathrm{End}_{\mathbb{Z}} G$ and $n\varphi \in R$ for some integer n. Thus there is some $i \in I$ such that $n\varphi = r' \in R_i \subseteq_* \mathrm{End}_{\mathbb{Z}} G_i$. We claim that $\varphi \restriction_{G_j} \in \mathrm{End}_{\mathbb{Z}} G_j$ for all $j \geq i$. Let $g \in G_j$, then $n\varphi(g) = r'g \in G_j$ and hence $\varphi(g) \in G_j$ by purity. Thus $\varphi \restriction_{G_j} = r_j \in R_j$ for all $i \leq j \in I$ because $R_j \subseteq_* \mathrm{End}_{\mathbb{Z}} G_j$. Since $n\varphi = r'$, we obtain $nr_j = r'$ for all j and therefore $r := r_j = r_k$ for all $j, k \geq i$ by torsion-freeness. Thus $\varphi = r \in R$.
(b) is obvious, and
(c) needs some work: Let $R'' = Rx_0 \oplus Rx$ be a ring direct sum, x, x_0 two central orthogonal idempotents. We put $Rx_0 = R$ if there is no ambiguity. Let $G' = G \oplus Re$ an R''-module with R'' acting component-wise, hence $\mathrm{Ann}_{R''} e = Rx_0 = \mathrm{Ann}\, x$ and $\mathrm{Ann}_{R''} G = Rx$. Next we extend σ and let $\sigma \subseteq \sigma'$ such that $\sigma'(re) = rx$ for all $r \in R$ (and $\sigma'(g) = \sigma(g)x_0, g \in G$). Hence $\sigma' : G' \longrightarrow R''$ satisfies $\ker \sigma' = \ker \sigma = H$ and $R \subseteq \mathrm{Im}\, \sigma'$ as required. Clearly, $\mathrm{Im}\, \sigma' = \mathrm{Im}\, \sigma x_0 \oplus Rx \subseteq_* R''$ since $\mathrm{Im}\, \sigma$ was pure in R. Note that σ'_* is actually defined by (3.1), such that $\sigma'_* : G' \longrightarrow \mathrm{End}_{\mathbb{Z}} G'$, hence R'' must be enlarged for $\sigma'_* : G' \longrightarrow R'$ in Definition 3.1(e). The ring R'' acts by scalar multiplication (on the left) on G'. The action is faithful by hypothesis (Definition 3.1 (f)), hence R'' can be viewed as a subring of $\mathrm{End}_{\mathbb{Z}} G'$. Let $R' = R''[\mathrm{Im}\, \sigma'_*]_*$ be the pure unital subring of $\mathrm{End}_{\mathbb{Z}} G'$ generated by R'' and $\mathrm{Im}\, \sigma'_*$.

Obviously $q \leq q'$ for $q' = (R', G', \sigma', \sigma'_*)$ and $\sigma' = \sigma'_q$ satisfies (c) of Proposition 3.3. It remains to show that $q' \in \mathfrak{K}_H$. We must check Definition 3.1 (c), (d) and (f). We have $\ker \sigma' = \ker \sigma = H$. Moreover, if $g \in G'$, then $\sigma'_*(g)(G') = 0$ implies $\sigma'_*(g) = 0$, hence $\mathrm{Ann}_{R'} G' = 0$ and (f) follows. It remains to show that $\mathrm{Im}\, \sigma'$ is pure in R'. As shown above, $\mathrm{Im}\, \sigma'$ is pure in R'', hence it suffices to show that R'' is pure in R'. We will even show that R'' is pure in $\mathrm{End}_{\mathbb{Z}} G'$. Let $\varphi \in \mathrm{End}_{\mathbb{Z}} G'$ and assume that $n\varphi = rx_0 \oplus r'x \in R''$ for some integer n. By the torsion-freeness of G' and $R_{\mathbb{Z}}$ it follows that φ is of the form $\varphi' \oplus r^*$ for some $\varphi' \in \mathrm{End}_{\mathbb{Z}} G$ and $r^* \in R$. Hence $r' = nr^*$ and $n\varphi' = r$ and therefore $\varphi' = r'' \in R$ since R is pure in $\mathrm{End}_{\mathbb{Z}} G$. Thus $\varphi = r''x_0 \oplus r^*x \in R''$ and the purity of R'' is established. Finally, the abelian groups $G, R_{\mathbb{Z}}$ are $\aleph_1$-free by hypothesis on q, hence G' is $\aleph_1$-free and $\mathrm{End}_{\mathbb{Z}} G'$ is $\aleph_1$- free by Lemma 3.2, and therefore $q' \in \mathfrak{K}_H$. $\square$

From the last proof we extract a useful

Definition 3.4 *If $\sigma_* : G \longrightarrow \mathrm{End}_{\mathbb{Z}} G$ is a $\mathbb{Z}$-homomorphism and R acts faithful on the left R-module G by scalar multiplication, then we denote by*

$$R_{\sigma_*} \subseteq_* \mathrm{End}_{\mathbb{Z}} G$$

the pure unital subring of $\mathrm{End}_{\mathbb{Z}} G$ *generated by* $R \subseteq \mathrm{End}_{\mathbb{Z}} G$ *and* $\mathrm{Im}\, \sigma_*$.

The next proposition provides the link to our construction in Section 4.

Proposition 3.5 *Let $q = (R, G, \sigma, \sigma_*) \in \mathfrak{K}_H$. Then there exists a 'transcendental extension' $q \leq q' \in \mathfrak{K}_H$ such that the following holds for $q' = (R', G', \sigma', \sigma'_*)$.*
(a) $G' = G \oplus R\langle X\rangle e$ *as $R\langle X\rangle$-module where X acts as identity on G.*
(b) $\sigma'(g) = \sigma(g)Xe$, $\sigma'(re) = re$ *if $r \in R$ and* $\sigma'(re) = rXe$ *else.*
(c) $R' = (R\langle X\rangle)_{\sigma'_*}$ *with $R\langle X\rangle$ from Section 2.*
(d) $R \subseteq \mathrm{Im}\,(\sigma)$.

Proof: We must show that $q' \in \mathfrak{K}_H$. The ring R is $\aleph_1$-free by hypothesis, hence $R\langle X\rangle$ is $\aleph_1$-free by Corollary 2.2(b) and R' is $\aleph_1$-free by Lemma 3.2. Clearly, $\ker \sigma' = \ker \sigma = H$ and $\mathrm{Im}\, \sigma' \subseteq_* R'$ with $R \subseteq \mathrm{Im}\,(\sigma)$ follow as in the proof of Proposition 3.3. Moreover, G' is $\aleph_1$-free and $\mathrm{Ann}_{R'} G' = 0$, so the proposition follows. □

We have an immediate corollary-definition.

Corollary 3.6 *If $K^o_H = \{q \in \mathfrak{K}_H, \sigma_q$ maps onto $R_q\}$, then $\mathfrak{K}^o_H$ is dense in $\mathfrak{K}_H$.*

Proof: Apply Proposition 3.3(c) ω times and note that Proposition 3.3(a) can be used because the union of the countable sequences of rings and modules respectively are $\aleph_1$-free by construction. Hence any $q \in \mathfrak{K}_H$ is below some $q' \in \mathfrak{K}^o_H$. □

4 Construction of $\mathbb{E}$-groups by black box arguments

The combinatorial ideas of Sections 4 and 5 can be found in Shelah [24], see also the appendix *Shelah's Black Box* in Corner, Göbel [5] and this Black Box could be used. However, we will use a stronger version of the Black Box as developed in [17]. The Black Box needs, as usually, some harmless alterations, which are obvious and the proof is left to the reader.

Let λ be some infinite cardinal, and $\{X_\alpha : \alpha < \lambda\}$ a sequence of transcendental elements which will be used to define ring extensions:

First we define ring extensions 'locally' and let $R_{\alpha+1} \supseteq R_\alpha(X_\alpha) = R_\alpha\langle X_\alpha\rangle = R_\alpha c_\alpha \oplus J_\alpha c'_\alpha$ a ring direct sum with central idempotents c_α and c'_α where J_α is defined as in Corollary 2.3. Since there is no danger of confusion we usually will omit the 'place holders' c_α and c'_α. The sequence of rings R_α with $|R_\alpha| < \lambda$ ($\alpha < \lambda$) will be completed during the construction of the abelian group G (with $\mathrm{End}_{\mathbb{Z}} G = \bigcup_{\alpha<\lambda} R_\alpha$), taking unions at limit steps, i.e. $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$ if α is a limit ordinal. Note that $\bigcup_{\alpha<\lambda} R_\alpha = \bigcup_{\alpha<\lambda} R_\alpha(X_\alpha)$.
Similarly we define

$$B = \bigoplus_{\alpha<\lambda} R_\alpha(X_\alpha)\, e_\alpha \subseteq G \subseteq \widehat{B}$$

where $\widehat{B}$ denotes the p-adic completion of B (as an abelian group) for some fixed prime p. Since we want to apply the Black Box later on we need a free basis-module inside B. Therefore, assume that our rings R_α (and hence $R_\alpha(X_\alpha)$) ($\alpha < \lambda$) are $\aleph_1$-free, hence homogeneous of type $\mathbb{Z}$. By a well-known result [16, Theorem 128] there exists for each $\alpha < \lambda$ a completely decomposable group $F_\alpha \subseteq_* R_\alpha(X_\alpha)$ such that $|R_\alpha(X_\alpha)| \leq |F_\alpha|^{\aleph_0}$ and

$$F_\alpha \subseteq_* R_\alpha(X_\alpha) \subseteq_* \widehat{F_\alpha}$$

where $\widehat{F_\alpha}$ is the p-adic completion of F_α ($\alpha < \lambda$). Note that F_α is a free abelian group since $R_\alpha(X_\alpha)$ is $\aleph_1$-free. We collect all those F_α ($\alpha < \lambda$) and define $F := \bigoplus_{\alpha<\lambda} F_\alpha e_\alpha$. Thus F is a free abelian group of cardinality at most λ such that

$$F \subseteq_* B \subseteq_* \widehat{F}.$$

Let $F_\alpha = \bigoplus_{\varepsilon<\rho} \mathbb{Z} a_\varepsilon$ where $\rho = |F|$. Writing $e_{(\varepsilon,\alpha)}$ for $a_\varepsilon e_\alpha$ it follows that $B' = \bigoplus_{(\varepsilon,\alpha)\in\rho\times\lambda} \mathbb{Z} e_{(\varepsilon,\alpha)}$ satisfies $\widehat{B} = \widehat{B'}$. For later use we put the lexicographic ordering on $\rho \times \lambda$; since ρ, λ are ordinals $\rho \times \lambda$ is well ordered.
We are ready to define supports of elements in $\hat{B}$. If $0 \neq g \in \widehat{B}$ then we can write $g = \sum_{\alpha\in I} g_\alpha e_\alpha$ and $I \subseteq \lambda, |I| \leq \aleph_0, g_\alpha \in R_\alpha(X_\alpha)$. Moreover, each $g_\alpha = g'_\alpha + g''_\alpha$ with $g'_\alpha \in R_\alpha$ and $g''_\alpha \in J_\alpha$. We define:

$$\text{The } \lambda\text{-support of } g \text{ is the set } [g]_\lambda = \{\alpha < \lambda : g_\alpha \neq 0\}. \tag{4.1}$$

$$\tag{4.2}$$

The notion of λ-support naturally extends to subsets of $\widehat{B}$, see again [17].
On the other hand, any element $0 \neq g \in \widehat{B} = \widehat{B'}$ can be written as

$$g = (g_{(\varepsilon,\alpha)} e_{(\varepsilon,\alpha)})_{(\varepsilon,\alpha)\in\rho\times\lambda} \in \widehat{B'} \subseteq \prod_{(\varepsilon,\alpha)\in\rho\times\lambda} \widehat{Z} e_{(\varepsilon,\alpha)}$$

and we define the 'usual' support of g by $[g] = \{(\varepsilon,\alpha) \in \rho\times\lambda | g_{(\varepsilon,\alpha)} \neq 0\}$. Note that $||[g]|| \leq \aleph_0$ and that the λ-support of g is $[g]_\lambda = \{\alpha < \lambda | \exists\varepsilon < \rho : (\varepsilon,\alpha) \in [g]\}$. As usual we may define a norm on $\widehat{B'}$ by $||\alpha|| = \alpha+1$ ($\alpha < \lambda$), $||M|| = \sup_{\alpha\in M}||\alpha||$ ($M \subseteq \lambda$) and $||g|| = ||[g]_\lambda||$ ($g \in \widehat{B'}$), i.e. $||g|| = \min\{\beta < \lambda | [g]_\lambda \subseteq \beta\}$. Note, $[g]_\lambda \subseteq \beta$ holds if and only if $g \in \widehat{B'_\beta}$ where $B'_\beta = \bigoplus_{\alpha<\beta} R_\alpha\,(X_\alpha)\,e_\alpha$. Finally, we also have a ring support and ring norm defined by $[g]_{ring} = \{\varepsilon < \rho | \exists\alpha < \lambda : (\varepsilon,\alpha) \in [g]\}$ and $||g||_{ring} = ||[g]_{ring}||$.
We will now state a suitable version of the Strong Black Box as developed in [17]. The proof is almost identical with the one in [17, Section 1] and will therefore be left to the reader but we will give all the necessary definitions and results.
Fix cardinals $\kappa \geq \aleph_0$, $\mu = \mu^\kappa$ such that $\lambda = \mu^+$. We need to say what we mean by a canonical homomorphism. For this we fix bijections $g_\gamma : \mu \to \gamma$ for all γ with $\mu \leq \gamma < \lambda$ where we put $g_\mu = id_\mu$. For technical reasons we also put $g_\gamma = g_\mu$ for $\gamma < \mu$. Moreover, let $g_{(\varepsilon,\alpha)} = g_\varepsilon \times g_\alpha$ for all $(\varepsilon,\alpha) \in \rho\times\lambda$.

Definition 4.1 *We define P to be a* canonical summand *of B' if $P = \bigoplus_{(\varepsilon,\alpha)\,\in\,I} \mathbb{Z}e_{(\varepsilon,\alpha)}$ for some $I \subseteq \rho\times\lambda$ with $|I| \leq \kappa$ such that:*

- *if $(\varepsilon,\alpha) \in I$ then $(\varepsilon,\varepsilon) \in I$;*
- *if $(\varepsilon,\alpha) \in I, \alpha\in\rho$ then $(\alpha,\varepsilon) \in I$;*
- *if $(\varepsilon,\alpha) \in I$ then $(I \cap (\mu\times\mu))\,g_{\varepsilon,\alpha} = I \cap \operatorname{Im} g_{\varepsilon,\alpha}$; and*
- *$||P|| < \lambda^o$, where $\lambda^0 = \{\alpha < \lambda |\ cf(\alpha) = \aleph_0\}$.*

Accordingly, $\varphi : P \to \widehat{B'}$ is said to be a canonical homomorphism *if P is a canonical summand of B' and $\operatorname{Im}\varphi \subseteq \widehat{P}$; we put $[\varphi] = [P]$, $[\varphi]_\lambda = [P]_\lambda$ and $||\varphi|| = ||P||$.*

If we denote by $\mathcal{C}$ the set of all canonical homomorphisms, then $|\mathcal{C}| = \lambda$ holds (see [17]). Our version of the Strong Black Box reads as follows (compare [17, Theorem 1.1.2.]):

Black Box Theorem 4.2 *Let $E \subseteq \lambda^0$ be a stationary subset of λ with $\lambda = \mu^+$, $\mu^\kappa = \mu$.*
Then there exists a family $\mathcal{C}^*$ of canonical homomorphisms with the following properties:

(i) If $\varphi \in \mathcal{C}^*$, then $||\varphi|| \in E$.

(ii) If φ, φ' are two different elements of $\mathcal{C}^*$ of the same norm α then $||[\varphi]_\lambda \cap [\varphi']_\lambda|| < \alpha$.

(iii) *PREDICTION: For any homomorphism* $\psi : B' \to \widehat{B'}$ *and for any subset* I *of* λ *with* $|I| \leq \kappa$ *the set*

$$\{\alpha \in E | \exists \varphi \in \mathcal{C}^* : ||\varphi|| = \alpha, \varphi \subseteq \psi, I \subseteq [\varphi]\}$$

is stationary.

For the proof of the above Theorem we have to define an equivalence relation on $\mathcal{C}$:

Definition 4.3 *Canonical homomorphism* φ, φ' *are said to be* equivalent *or* of the same type (*notation:* $\varphi \equiv \varphi'$), *if* $[\varphi] \cap (\mu \times \mu) = [\varphi'] \cap (\mu \times \mu)$ *and there exists an order-isomorphism* $f : [\varphi] \to [\varphi']$ *such that* $(x\bar{f})\varphi' = (x\varphi)\bar{f}$ *for all* $x \in \operatorname{dom}\varphi$ *where* $\bar{f} : \widehat{\operatorname{dom}\varphi} \to \widehat{\operatorname{dom}\varphi'}$ *is the unique extension of the R-homomorphism defined by* $e_{(\varepsilon,\alpha)}\bar{f} = e_{(\varepsilon,\alpha)f}$ $((\varepsilon, \alpha) \in [\varphi])$.

As in [17] it is easy to see that there are at most μ different types. Next we have to recall the definition of an admissible sequence.

Definition 4.4 *Let* $\varphi_0 \subset \varphi_1 \subset\subset \cdots \subset \varphi_n \subset \cdots (n < \omega)$ *be an increasing sequence of canonical homomorphisms.*
Then $(\varphi_n)_{n<\omega}$ *is said to be admissible if* $[\varphi_0] \cap (\mu \times \mu) = [\varphi_n] \cap (\mu \times \mu)$ *for all* $n < \omega$. *Also, we say that* $(\varphi_n)_{n<\omega}$ *is admissible for a sequence* $(\beta_n)_{n<\omega}$ *of ordinals in* λ *if* $(\varphi_n)_{n<\omega}$ *is admissible satisfying* $||\varphi_n|| \leq \beta_n < ||\varphi_{n+1}||$ *and* $[\varphi_n] = [\varphi_{n+1}] \cap (\beta_n \times \beta_n)$ *for all* $n < \omega$. *Moreover, two admissible sequences* $(\varphi_n)_{n<\omega}$ *and* $(\varphi'_n)_{n<\omega}$ *are said to be equivalent or of the same type if* $\varphi_n \equiv \varphi'_n$ *for all* $n < \omega$.

Note that the union $\bigcup_{n<\omega} \varphi_n$ of an admissible sequence $(\varphi_n)_{n<\omega}$ is also an element of $\mathcal{C}$. Moreover, if we let $\mathcal{T}$ be the set of all possible types of admissible sequences of canonical homomorphisms, then clearly $|\mathcal{T}| \leq \mu^\kappa = \mu$. If $(\varphi_n)_{n<\omega}$ is admissible of type τ , then we also use the notion of τ-admissible. As in [17], the following proposition is the main ingredient of the proof of the Black Box Theorem 4.2.

Proposition 4.5 *Let* $\psi : B' \to \widehat{B'}$ *be a homomorphism,* $I \subseteq \rho \times \lambda$ *a set of cardinality at most* κ *and* $\mathcal{H} = \mathcal{H}_{\psi,I} = \{\varphi \in \mathcal{C} | \varphi \subseteq \psi, I \subseteq [\varphi]\}$.
Then there exists a type $\tau \in \mathcal{T}$ *such that*

$$\exists \varphi_0 \in \mathcal{H} \forall \beta_0 \geq ||\varphi_0|| \cdots \exists \varphi_n \in \mathcal{H} \forall \beta_n \geq ||\varphi_n|| \cdots$$

with $(\varphi_n)_{n<\omega}$ *is* τ*-admissible.*

The proof is contained in [17] and also for the proof of the Black Box Theorem 4.2 we refer to [17, Section 1]. Finally we have a corollary suitable for application.

Corollary 4.6 *Let the assumption be the same as in the Black Box Theorem 4.2. Then there exists an ordinal $\lambda^* \geq \lambda$ with $|\lambda^*| = \lambda$ and a family $(\varphi_\beta)_{\beta<\lambda^*}$ of canonical homomorphisms such that*

(i) $\varphi_\beta \in \mathcal{C}^*$ *and* $||\varphi_\beta|| \in E$ *for all* $\beta < \lambda^*$.

(ii) $||\varphi_\gamma|| \leq ||\varphi_\beta||$ *for all* $\gamma \leq \beta < \lambda^*$.

(iii) $||[\varphi_\beta]_\lambda \cap [\varphi_\gamma]_\lambda|| < ||\varphi_\beta||$ *for all* $\gamma < \beta < \lambda^*$.

(iv) PREDICTION: For any homomorphism $\psi : B' \to \widehat{B'}$ *and for any subset* I *of* λ *with* $|I| \leq \kappa$ *the set*

$$\{\alpha \in E | \exists \beta < \lambda^* : ||\varphi_\beta|| = \alpha, \varphi_\beta \subseteq \psi, I \subseteq [\varphi_\beta]\}$$

is stationary.

5 The inductive steps in the construction of $q = (R, G, \sigma, \sigma_*)$.

We now use induction along $\alpha < \lambda^*$ given by the Black Box to find quadruples $q_\alpha = (R_\alpha, G_\alpha, \sigma_\alpha, \sigma_{\alpha*}) \in \mathfrak{K}_H$ for a fixed $\aleph_1$-free group H, see Definition 3.1. Let H be given and choose $q_0 \in \mathfrak{K}_H$ arbitrary, e.g. $q_0 = (\mathbb{Z}, \mathbb{Z} \oplus H, \sigma, \sigma_*)$ which is in $\mathfrak{K}_H$ by Proposition 3.3 (b). Necessarily we have to assume that $|H| < \lambda$. Let $(\varphi_\beta)_{\beta<\lambda^*}$ be a family of canonical homomorphisms as given by Corollary 4.6. For any $\beta < \lambda^*$ let $P_\beta = \operatorname{dom} \varphi_\beta$. Suppose that the quadruples $q_\beta = (R_\beta, G_\beta, \sigma_\beta, \sigma_{\beta*})$ are constructed for all $\beta < \alpha$ subject to the following conditions:

(i) R_β is a unital ring with $1_{R_\beta} = 1_{R_\gamma}$ for all $\gamma \leq \beta$

(ii) R_β, G_β are $\aleph_1$-free

(iii) $\sigma_\beta(G_\beta) \subseteq R_\beta$

(iv) $G_\beta \subseteq_* G_{\beta+1}$

(v) $R_\beta \subseteq \operatorname{Im} \sigma_{\beta+1}$ if $\beta \notin E$

(vi) $G_\beta = \bigcup_{\gamma<\beta} G_\gamma$ if β is a limit ordinal

(vii) $B_\beta = \bigoplus_{\gamma<\beta} R_\gamma(X_\gamma)\, e_\gamma \subseteq_* G_\beta \subseteq_* \widehat{B_\beta}$ where $\widehat{B_\beta}$ is the p-adic completion of B_β.

We first have to prove a Step Lemma.

Step Lemma 5.1 *Let* $P = \bigoplus_{(\varepsilon,\alpha)\in I^*} \mathbb{Z}e_{(\varepsilon,\alpha)}$ *for some* $I^* \subseteq \rho\times\lambda^*$ *and let* M *be a subgroup of* $\widehat{B'}$ *with* $P \subseteq_* M \subseteq_* \widehat{B'}$ *which is* $\aleph_1$*-free and an* R*-module, where* $R = \bigcup_{\alpha\in I'} R_\alpha(X_\alpha)$ *with* $I' = \{\alpha < \lambda : \exists\varepsilon < \rho, (\varepsilon,\alpha) \in I^*\}$*. Assume that* $q = (R, M, \sigma, \sigma_*) \in \mathfrak{K}_H$*. Also suppose that there is a set* $I = \{(\varepsilon_n, \alpha_n) : n < \omega\} \subseteq [P] = I^*$ *such that* $\alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots (n < \omega)$ *and*

(i) $M = \bigcup_{n<\omega} G_{\alpha_n}$, $R = \bigcup_{n<\omega} R_{\alpha_n}$;

(ii) $(R_{\alpha_n}, G_{\alpha_n}, \sigma_{\alpha_n}, \sigma_{\alpha_n *}) \in \mathfrak{K}_H$ *for all* $n < \omega$

(iii) $I_\lambda \cap [g]_\lambda$ *is finite for all* $g \in M$ $(I_\lambda = [I]_\lambda)$*.*

Moreover, let $\varphi : P \to M$ *be a homomorphism which is not multiplication by an element from* R*.*
Then there exists an element $y \in \widehat{P}$ *and an element* $q \subseteq q' = (R', M', \sigma', \sigma'_*) \in \mathfrak{K}_H$ *such that* $M \subseteq_* M' \subseteq_* \widehat{B'}$*,* $y \in M'$ *and* $y\varphi \notin M'$*.*

Proof: By assumption M is an R-module and hence the completion $\widehat{M}$ is an $\widehat{R}$-module. Thus for any $y \in \widehat{P}$ and $r \in \widehat{R}$ it follows that ry is defined inside $\widehat{M}$ and

$$[ry] \subseteq [y].$$

We construct a new group $M \subseteq_* M_y$ for $y \in \widehat{P}$ as follows:
Put $M_y^1 = \langle M, Ry\rangle_* \subseteq \widehat{B'}$. Since $\operatorname{Im}\sigma$ is a pure subgroup of R there is a unique extension

$$\widehat{\sigma} : \widehat{M} \to \widehat{R}$$

of σ such that $\ker\widehat{\sigma} = \ker\sigma = H$. Moreover, $\operatorname{Im}\widehat{\sigma}$ is pure in $\widehat{R}$. We choose $R_y^1 = R[\operatorname{Im}\widehat{\sigma}\restriction_{M_y^1}]_* \subseteq \widehat{R}$ and let

$$M_y^2 = \left\langle R_y^1 M, R_y^1 y\right\rangle_* \subseteq \widehat{B'}.$$

Again we may choose $R_y^2 = R_y^1[\operatorname{Im}\widehat{\sigma}\restriction_{M_y^2} \subseteq \widehat{P}]_* \subseteq \widehat{R}$. Continuing this way we obtain a sequence of groups M_y^n and rings R_y^n $(n \in \omega)$ such that

$$M_y^{n+1} \text{ is an } R_y^n\text{-module and } \widehat{\sigma}(M_y^n) \subseteq R_y^n.$$

Taking $M_y = \bigcup_{n\in\omega} M_y^n$ and $R'_y = \bigcup_{n\in\omega} R_y^n$ we get that M_y is an R'_y-module and $\sigma_y = \widehat{\sigma}\restriction_{M_y}\colon M_y \to R'_y$. It is a standard support argument to see that M_y and (hence) R'_y are still $\aleph_1$-free. Finally take $R_y = \left(R'_y\right)_{\sigma_y^*} \subseteq \mathrm{End}\,_{\mathbb{Z}}(M_y)$. By construction $\mathrm{Im}\,\sigma_y$ is pure in R_y and hence $q_y = (R_y, M_y, \sigma_y, \sigma_y^*) \in \mathfrak{K}_H$.
At this stage we determine y more specific in order to obtain that $\varphi \notin \mathrm{End}\,_{\mathbb{Z}}M_y$. Let $y = \sum_{n\in\omega} p^n e_{(\varepsilon_n,\alpha_n)}$ and $x = \varphi(y) \in \widehat{M}$. If $x \notin M_y$, then choose $M' = M_y$ and $R' = R_y$ and hence $q_y \in \mathfrak{K}_H$ with $\varphi \notin \mathrm{End}\,_{\mathbb{Z}}M_y$.
If $x \in M_y$, then there are integers k and n such that

$$p^k\varphi(y) = r_n g + r'_n y$$

for some $r_n, r'_n \in R_y^n$ and $g \in M$. It follows that

$$\left(p^k\varphi - r'_n\right) y = r_n g.$$

Since $\left(p^k\varphi - r'_n\right) \neq 0$ there is $b' \in P$ such that

$$\left(p^k\varphi - r'_n\right) b' \neq 0.$$

Note that b' has finite support. Moreover, by the cotorsion-freeness of R there exists $\pi \in \widehat{R}$ such that $\pi b \notin M$ with $b = \left(p^k\varphi - r'_n\right) b'$. Let $y' = y + \pi b'$. We claim that $\varphi \notin \mathrm{End}\,_{\mathbb{Z}}(M_{y'})$. By way of contradiction assume that

$$p^l\varphi(y + \pi b') = r_m^* g^* + r_m^{'*}(\pi b' + y)$$

for some integer $l \geq k$ and elements $r_m^*, r_m^{'*} \in R_{y'}^n$ and $g^* \in M$. Without loss of generality, we may assume $n = m$, hence

$$p^l\varphi(y + \pi b') = r_n^* g^* + r_n^{'*}(\pi b' + y).$$

Let $s = p^l/p^k$. Hence

$$p^l\varphi(\pi b') = p^l\varphi(y + \pi b') - sp^k\varphi(y) = r_n^* g^* + r_n^{'*}(\pi b' + y) - s(r_n g + r'_n y) =$$
$$= (r_n^* g^* - s r_n g) + r_n^{'*}\pi b' + (r_n^{'*} - s r'_n) y.$$

Since $[\pi b'] = [b']$, $[\varphi(\pi b')] = [\varphi(b')]$ and $g^*, g \in M$ an easy support argument shows that $r_n^{*'} = s r'_n$ and hence

$$sp^k\varphi(\pi b') = (r_n^* g^* - s r_n g) + s r'_n \pi b'$$

and thus

$$s\pi(p^k\varphi(b') - r'_n b') = (r_n^* g^* - s r_n g) \in M.$$

By purity we get $\pi(p^k\varphi(b') - r_n' b') = \pi b \in M$ - a contradiction. Finally we put $M' = M_{y'}$, $R' = R_{y'}$ and $q' = q_y \in \mathfrak{K}_H$. Thus

$$M' = R'M + \sum_{k<\omega} R' y^{(k)},$$

where $y^{(k)} = \sum\limits_{n \geq k} \frac{p^n}{p^k} e_{(\varepsilon_n, \alpha_n)}$ or $y^{(k)} = \sum\limits_{n\geq k} \frac{p^n}{p^k} e_{(\varepsilon_n,\alpha_n)} + \pi^{(k)} b$. □

We will now carry on the construction to α and distinguish three cases.

Case 1: Suppose α is a limit ordinal. Then $G_\alpha = \bigcup\limits_{\beta<\alpha} G_\beta$ is $\aleph_1$-free by (iii) and hence Proposition 3.3 (a) shows that we can take unions, i.e. $q_\alpha = \bigcup\limits_{\beta<\alpha} q_\beta$.

Case 2: Suppose $\alpha = \beta + 1$, then $||\varphi_\beta|| \in \lambda^0$. Assume that $\operatorname{Im}\varphi_\beta \nsubseteq G_\beta$ or $\varphi_\beta \in R_\beta$. In this case we let $G_{\beta+1} = G_\beta \oplus R_\beta \langle X_\beta \rangle e_\beta$ as in Proposition 3.5 with $R'_\alpha = R_\beta \langle X_\beta \rangle = R_\beta (X_\beta)$ with X acting as identity on G_β. We let $R_\alpha = (R'_\alpha)_{\sigma^*_\alpha}$ and by Proposition 3.5 it follows that $(R_\alpha, G_\alpha, \sigma_\alpha, \sigma^*_\alpha) \in \mathfrak{K}_H$ where σ_α and $\sigma^*_\alpha = \sigma_{\alpha*}$ are taken from Proposition 3.5. Put $y_\beta = 0$.

Case 3: Suppose that $\alpha = \beta + 1$ and $\operatorname{Im}\varphi_\beta \subseteq G_\beta$, $\varphi_\beta \notin R_\beta$. In this case we try to 'kill' our undesired homomorphism φ_β which comes from the Black Box prediction. Recall that $||\varphi_\beta|| \in \lambda^0$, hence there are $(\varepsilon_n, \beta_n) \in [\varphi_\beta]$ $(n \in \omega)$ such that $\beta_0 < \beta_1 < \cdots < \beta_n < \cdots$ and $\sup_{n\in\omega} \beta_n = ||\varphi_\beta||$. Without loss of generality we may assume that $\beta_n \notin E$ for all $n \in \omega$ and hence $G_{\beta_n+1} = G_{\beta_n} \oplus R_{\beta_n} \langle X_{\beta_n} \rangle e_{\beta_n}$. We put $I = \{(\varepsilon_n, \beta_n) | n < \omega\}$. Then $I_\lambda \cap [g]_\lambda$ is finite for all $g \in G_\beta$. We apply the Step Lemma 5.1 to I as above, $P = \operatorname{dom}\varphi_\beta$ and $M = G_\beta$. Therefore there exists an extension $q_\alpha = q_{\beta+1}$ of q_β and an element $y_\beta \in G_\alpha$ such that $y_\beta\varphi_\beta \notin G_\alpha$ and $||y_\beta|| = ||\varphi_\beta|| = ||P_\beta||$.

Finally put $q_H = (G_H, R_H, \sigma_H, \sigma_{H*}) = \bigcup\limits_{\alpha<\lambda} q_\alpha \in \mathfrak{K}_H$. Obviously, G_H has cardinality λ and $R \subseteq \operatorname{End}_{\mathbb{Z}} G$. Moreover,

$$G_H = B + \sum_{\beta<\lambda^*} \sum_{k<\omega} R_\beta y_\beta^{(k)}.$$

Next we describe the elements of G_H.

Lemma 5.2 *Let G_H be as above and let $g \in G_H \setminus B$. Then there are $k < \omega$ and a finite subset N of λ^* such that $g \in B + \sum\limits_{\beta\in N} R_\beta y_\beta^{(k)}$ and $[g]_\lambda \cap [y_\beta]_\lambda$ is infinite if and only if $\beta \in N$. In particular, if $||g||$ is a limit ordinal then $||g|| = ||y_{\beta_*}|| = ||\varphi_{\beta_*}||$ for $\beta_* = \max N$.*

Proof: Let $g \in G_H = B + \sum_{\beta<\lambda^*} \sum_{n<\omega} R_\beta y_\beta^{(n)}$. Then there are a finite subset N' of λ^*, $b \in B$, $k \in \omega$, $a_{\beta,n} \in R_\beta$ ($\beta \in N'$, $n \le k$) such that

$$g = b + \sum_{\beta\in N'} \sum_{n\le k} a_{\beta,n} y_\beta^{(n)}.$$

Since $y_\beta^{(n)} - \frac{p^k}{p^n} y_\beta^{(k)} \in B' \subseteq B$ this expression reduces to

$$g = b' + \sum_{\beta\in N'} a_\beta y_\beta^{(k)}$$

for some $a_\beta \in R_\beta$ ($\beta \in N'$), $b' \in B'$. Putting $N = \{\beta \in N' \mid a_\beta \neq 0\}$ ($N \neq \emptyset$ for $g \notin B$) the conclusion of the lemma follows since $[y_\beta]_\lambda \cap [y_{\beta'}]_\lambda$ is finite for $\beta \neq \beta'$ by Corollary 4.6 (iii). □

Using the above lemma we prove further properties of G_H.

Lemma 5.3 *Let G_H be as above and define G^α ($\alpha < \lambda$) by $G^\alpha := \{g \in G_H \mid ||g|| < \alpha,\ ||g||_{ring} < \alpha\}$. Then:*

(a) *$G_H \cap \widehat{P_\beta} \subseteq G_{\beta+1}$ for all $\beta < \lambda^*$;*

(b) *$\{G^\alpha \mid \alpha < \lambda\}$ is a λ-filtration of G_H; and*

(c) *if $\beta < \lambda^*$, $\alpha < \lambda$ are ordinals such that $||\varphi_\beta|| = \alpha$ then $G^\alpha \subseteq G_\beta$.*

Note, we used the lower index ($\beta < \lambda^*$) for the construction while we use the upper index ($\alpha < \lambda$) for the filtration.

Proof: First we show (a). Let $g \in G_H \cap \widehat{P_\beta}$ for some $\beta < \lambda^*$. Since $B_{\beta+1} \subseteq G_{\beta+1}$ we assume $g \in G_H \setminus B_{\beta+1}$. Then, by Lemma 5.2, $g \in B + \sum_{\gamma\in N} R_\gamma y_\gamma^{(k)}$ for some finite $N \subseteq \lambda^*$, $k < \omega$ such that $[g]_\lambda \cap [_\lambda y_\gamma]_\lambda$ is infinite for $\gamma \in N$.

Since $g \in \widehat{P_\beta}$ we also have $[g]_\lambda \subseteq [P_\beta]_\lambda$ ($= [\widehat{P_\beta}]_\lambda$).

If $||g|| < ||P_\beta||$ then $N \subseteq \beta$ by Corollary 4.6(ii) and thus $g \in G_\beta \subseteq G_{\beta+1}$.

Otherwise, if $||g|| = ||P_\beta||$ ($\in \lambda^o$) then $||g|| = ||y_{\gamma_*}|| = ||\varphi_{\gamma_*}||$ for $\gamma_* = \max N$ and $[g]_\lambda \cap [y_{\gamma_*}]_\lambda \subseteq [\varphi_\beta]_\lambda \cap [\varphi_{\gamma_*}]_\lambda$ is infinite. Hence $\beta = \gamma_*$ by condition (iii) of Corollary 4.6 and so $g \in G_{\beta+1}$ as required.

Condition (b) is obvious.

To see (c) let $\beta < \lambda^*$, $\alpha < \lambda$ with $||\varphi_\beta|| = \alpha$ and let $g \in G^\alpha$. If $g \in B_\beta$ we are finished. Otherwise, by Lemma 5.2, we have $g \in B + \sum_{\gamma\in N} R_\gamma y_\gamma^{(k)}$ ($N \subseteq \lambda^*$ finite, $k \in \omega$) with $[g]_\lambda \cap [y_\gamma]_\lambda$ is infinite for $\gamma \in N$. This implies $||\varphi_\gamma|| = ||y_\gamma|| \le ||g|| < \alpha = ||\varphi_\beta||$ for all $\gamma \in N$ and thus $N \subseteq \beta$ by Corollary 4.6(ii), i.e. $g \in G_\beta$, which finishes the proof. □

6 Proof of the Main Theorem

In this final section we want to prove our Main Theorem which reads as follows:

Main Theorem 6.1 *Let λ be an infinite cardinal such that $\lambda = \mu^+$ with $\mu^{\aleph_0} = \mu$ and let H be an $\aleph_1$-free abelian group of size less than λ. Then there exists an $\aleph_1$-free $\mathbb{E}(H)$-group of cardinality λ with non-commutative endomorphism ring. In particular, there is a strong $\mathbb{E}$-group of cardinality λ.*

Proof: Let λ and H be given as stated in the theorem and choose a stationary subset E of λ whose members have cofinality ω. We construct $q_H = (R_H, G_H, \sigma_H, \sigma_{H*}) = \bigcup_{\alpha<\lambda} q_\alpha$ as in the previous section. Thus $R_H \subseteq_* \operatorname{End}_{\mathbb{Z}} G_H$ and $\sigma : G_H \to R_H$ with kernel $\ker \sigma_H = H$. Moreover, G_H is $\aleph_1$-free and R_H is obviously non-commutative. We first claim that σ_H is surjective. Therefore let $r \in R_H$, hence there exists $\alpha < \lambda$ such that $r \in R_\alpha$. Without loss of generality we may assume that $\alpha \notin E$. By (v) we conclude that $r \in R_\alpha \subseteq \operatorname{Im} \sigma_{\alpha+1}$. Since $\sigma_{\alpha+1} \subseteq \sigma_H$ as functions it follows that $r \in \operatorname{Im} \sigma_H$ and thus σ_H is surjective.
It remains to prove that $R_H = \operatorname{End}_{\mathbb{Z}} G_H$. Assume that $\varphi \in \operatorname{End}_{\mathbb{Z}} G_H \backslash R$. Let $\varphi' = \varphi \restriction_{B'}$, hence $\varphi' \notin R$. Let $I = \{(\varepsilon_n, \alpha_n) | n < \omega\} \subseteq \rho \times \lambda$ such that $\alpha_0 < \alpha_1 < \cdots \alpha_n < \cdots$ and $I_\lambda \cap [g]_\lambda$ is finite for all $g \in G_H$. Note that the existence of I can be easily arranged, e.g. let $E \subsetneq \lambda^0, \alpha \in \lambda^0 \backslash E, \varepsilon_n \in \rho (n < \omega)$ arbitrary and $(\alpha_n)_{n<\omega}$ any ladder on α.
By Lemma 5.1 there exists an element $y \in \widehat{B'}$ such that $y\varphi' \notin G'_H$ which is an extension of G_H. By the Black Box Theorem 4.2 the set

$$E' = \{\alpha \in E | \exists \beta < \lambda^* : ||\varphi_\beta|| = \alpha, \varphi_\beta \subseteq \varphi', [y] \subseteq [\varphi_\beta]\}$$

is stationary since $||[y]|| < \aleph_0$. Note, $[y] \subseteq [\varphi_\beta]$ implies that $y \in \widehat{\operatorname{dom} \varphi_\beta}$. Moreover, the set $C = \{\alpha < \lambda : \varphi(G^\alpha) \subseteq G^\alpha\}$ is a cub in λ, hence $E' \cap C \neq \emptyset$. Let $\alpha \in E' \cap C$. Then $G^\alpha \varphi' \subseteq G^\alpha$ and there exists an ordinal $\beta < \lambda^*$ such that $||\varphi_\beta|| = \alpha$, $\varphi_\beta \subseteq \varphi$ and $y \in \widehat{\operatorname{dom} \varphi_\beta}$. The first property implies that $G^\alpha \subseteq G_\beta$ by Lemma 5.3 and the latter properties imply that $\varphi_\beta \notin R$. Moreover, $\operatorname{dom} \varphi_\beta \subseteq B'$ with $||\operatorname{dom} \varphi_\beta||_{ring} \leq ||\operatorname{dom} \varphi_\beta|| = \alpha$ and hence $\operatorname{dom} \varphi_\beta$, and also $(\operatorname{dom} \varphi_\beta)\varphi$ are contained in $G^\alpha \subseteq G_\beta$.
Therefore $\varphi_\beta : \operatorname{dom} \varphi_\beta \to G_\beta$ with $\varphi_\beta \notin R_\beta$ and thus it follows form the construction that $y_\beta \varphi_\beta \notin G_{\beta+1}$. On the other hand it follows from Lemma 5.3 that $y_\beta \varphi_\beta = y_\beta \varphi \in G_H \cap \widehat{P_\beta} \subseteq G_{\beta+1}$ - a contradiction. Thus $\operatorname{End}_{\mathbb{Z}} G_H = R$. $\square$

Corollary 6.2 *Let λ be an infinite cardinal such that $\lambda^{\aleph_0} = \lambda$. Then there is a non-commutative ring R such that* $\operatorname{End}_{\mathbb{Z}}(R_{\mathbb{Z}}) \cong R$.

Proof: Let $H = \{0\}$ and apply Theorem 6.1 to obtain a strong $\mathbb{E}$-group G. Thus $\sigma : G \to \operatorname{End}_{\mathbb{Z}} G$ is an isomorphism, where $\operatorname{End}_{\mathbb{Z}} G$ is non-commutative. Hence, $R := G$ has a non-commutative ring structure such that $\operatorname{End}_{\mathbb{Z}}(R_{\mathbb{Z}}) \cong R$. $\square$

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