ALMOST DISJOINT PURE SUBGROUPS OF THE BAER-SPECKER GROUP

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Abstract. We prove in ZFC that the Baer-Specker group $\mathbf{Z}^\omega$ has $2^{\aleph_1}$ non-free pure subgroups of cardinality $\aleph_1$ which are almost disjoint: there is no non-free subgroup embeddable in any pair.

In this short paper we prove the following result.

Theorem 1. *There exists a family* $\mathbf{G} = \{G_\alpha : \alpha < 2^{\aleph_1}\}$ *of non-isomorphic non-free pure subgroups of the Baer-Specker group* $\mathbf{Z}^\omega$ *such that:*
(1.1) each G_α *has cardinality* $\aleph_1$*;*
(1.2) if $\alpha < \beta$*, then* G_α *and* G_β *are almost disjoint: if* H *is isomorphic to subgroups of* G_α *and* G_β*, then* H *is free. In particular,* $G_\alpha \cap G_\beta$ *is free.*

Recall that the Baer-Specker group $\mathbf{Z}^\omega$ is the abelian group of functions from the natural numbers into the integers (see [1] and [18]). It contains the canonical pure free subgroup $\mathbf{Z}_\omega = \oplus_{n<\omega}\mathbf{Z}$. The group $\mathbf{Z}^\omega$ is not κ-free for any cardinal $\kappa > \aleph_1$, but it is $\aleph_1$-free, so the groups G_α in Theorem 1 are almost free.

Theorem 1 answers a question of the first author, and has its place in the line of recent research dealing with the lattice structure of the pure subgroups of $\mathbf{Z}^\omega$ (see [2], [3], and [5]–[8]). For example, Irwin asked whether there is a subgroup of $\mathbf{Z}^\omega$ with uncountable dual but no free summands of infinite rank. This problem was resolved recently by Corner and Goebel [5] who proved the following stronger fact.

Theorem 2. [5] *The Baer-Specker group* $\mathbf{Z}^\omega$ *contains a pure subgroup* G *whose endomorphism ring splits as* $End(G) = \mathbf{Z} \oplus Fin(G)$*, with* $|G^*| = 2^{\aleph_0}$*, where* $\mathbf{Z}$ *is the scalar multiplication by integers and* $Fin(G)$ *is the ideal of all endomorphisms of* G *of finite rank.*

Quotient-equivalent and almost disjoint abelian groups have been studied by Eklof, Mekler and Shelah in [9]–[11], who showed that under various set-theoretic hypotheses, there exist families of maximal possible size of almost free abelian groups which are pairwise almost disjoint. Following [11], we say that two groups A and B are almost disjoint if whenever H is embeddable as a subgroup in both A and B, then H is free. Clearly if A and B are non-free and almost disjoint, then they are non-isomorphic in a very strong way. On the other hand, the intersection of two almost disjoint groups of size $\aleph_1$ need not necessarily be countable, so group-theoretic almost disjointness differs from its set-theoretic homonym. Theorem 1 establishes in ZFC that the Baer-Specker group contains large families of almost disjoint almost free non-free pure uncountable subgroups.

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Our group and set-theoretic notation is standard and can be found in [10] and [14]. For example, ${}^{\omega_1>}2$ is the set of partial functions from ω_1 into $\{0,1\}$ whose domains are at most countable; ${}^{\omega_1}2$ is the set of all functions from ω_1 into $\{0,1\}$; for a regular cardinal χ, $H(\chi)$ is the family of all sets of hereditary cardinality less than χ.

For a set $A \subseteq H(\chi)$ for χ large enough, we write $\mathrm{dcl}_{\left(H(\chi),\in,<\right)}[A]$ for the Skolem closure (Skolem hull) of A in the structure $\left(H(\chi),\in,<\right)$, where $<$ is a well-ordering of $H(\chi)$ (for details, see [16], 400-402, or [15], 165-170).

In proving Theorem 1, we shall appeal to the well-known Engelking-Karłowicz theorem from set-theoretic topology:

Theorem 3. [13] *If $|Y| = \mu = \mu^{<\sigma} < \lambda = |X| \leq 2^\mu$, then there are functions $h_\alpha : X \to Y$ for $\alpha < \mu$ such that for every partial function f from X to Y of cardinality less than σ, for some $\alpha < \mu$, $f \subseteq h_\alpha$.*

A self-contained short proof can be found in [17], 422-423. We shall need just the case when $\mu = \sigma = \aleph_0$, and $\lambda = 2^\mu$. Since it may be less familiar to algebraists, for convenience we deduce the fact to which we appeal later on (although it also appears as Corollary 3.17 in [4]).

Lemma 4. *There exists a family $\{f_\eta : \eta \in^{\omega_1>} 2\}$ such that $f_\eta : \omega \to \mathbf{Z}$, and whenever $\eta_1, \ldots, \eta_k$ are distinct and $a_1, \ldots, a_k \in \mathbf{Z}$, then $\{i < \omega : (\forall\ l \leq k)(f_{\eta_l}(i) = a_l)\}$ is infinite.*

Proof. Take $\mu = \sigma = \aleph_0$, $\lambda = 2^\mu$, $X = {}^{\omega_1>}2$ and $Y = \mathbf{Z}$ in the Engelking-Karłowicz theorem. Since $|{}^{\omega_1>}2| = 2^{\aleph_0}$ and $|\mathbf{Z}| = \aleph_0$, we know that there exist functions $h_n : {}^{\omega_1>}2 \to \mathbf{Z}$ for $n < \omega$ such that for every partial function f from ${}^{\omega_1>}2$ to $\mathbf{Z}$ whose domain is finite, there is some $m < \omega$ such that $f \subseteq h_m$. Let $\{g_i : i < \omega\}$ be an enumeration with infinitely many repetitions of each h_n for $n < \omega$.

For each $\eta \in^{\omega_1>} 2$, define $f_\eta : \omega \to \mathbf{Z}$ by $f_\eta(i) = g_i(\eta)$. The family $\{f_\eta : \eta \in^{\omega_1>} 2\}$ is as required: for if $\eta_1, \ldots, \eta_k$ are distinct and $a_1, \ldots, a_k \in \mathbf{Z}$ are given, then the set $f = \langle(\eta_1, a_1), \ldots, (\eta_k, a_k)\rangle$ is a finite function, so there is some m such that $f \subseteq h_m$ and it is now easy to see that $\{i < \omega : (\forall\ l \leq k)(f_{\eta_l}(i) = a_l)\}$ is infinite. □

A well-known algebraic fact will also be useful:

Lemma 5. *Let C be a closed unbounded subset of the regular uncountable cardinal κ. Suppose that H is an abelian group of cardinality κ, and $\langle H_\alpha : \alpha < \kappa\rangle$ is a κ-filtration of H (a continuous increasing chain of subgroups H_α, $|H_\alpha| < \kappa$, whose union is H). Let $S = \{\alpha \in C : H/H_\alpha$ is not κ-free$\}$. Then H is free if and only if S is non-stationary in κ.*

Proof. Well-known: see Proposition IV.1.7 in [10]. □

We refer the reader to [14] for the definitions of the characteristic $\chi(g)$ and the type $\tau(g)$ of an element g in a group.

Now we prove Theorem 1.

Proof. Let $\mathbf{P}$ be the set of prime numbers, and let $\{P_\eta : \eta \in^{\omega_1>} 2\}$ be a family of almost disjoint (infinite) subsets of $\mathbf{P}$: $\eta \neq \nu \in^{\omega_1>} 2 \ \Rightarrow\ |P_\eta \cap P_\nu| < \aleph_0$. By Lemma 4, there exists $\{f_\eta : \eta \in^{\omega_1>} 2\}$ such that $f_\eta : \omega \to \mathbf{Z}$, and if $\eta_1, \ldots, \eta_k$ are distinct and $a_1, \ldots, a_k \in \mathbf{Z}$, then $\{i < \omega : (\forall\ l \leq k)(f_{\eta_l}(i) = a_l)\}$ is infinite.

Define functions x_η and $x_{\eta,j}$ in $\mathbf{Z}^\omega$ as follows. Let $x_\eta = \langle \pi_{\eta,i} \cdot f_\eta(i) : i < \omega\rangle$ where $\pi_{\eta,i} = \Pi\{p \in P_\eta : p < i\}$, and let $x_{\eta,j} = \langle \pi^j_{\eta,i} \cdot f_\eta(i) : i < \omega\rangle$ where $\pi^j_{\eta,i} = \Pi\{p \in P_\eta : j \le p < i\}$ (=0 if $i \le j$). Note that $x_\eta = x_{\eta,0}$.

For $\eta \in^{\omega_1} 2$, let G_η be the subgroup of $\mathbf{Z}^\omega$ generated by $\mathbf{Z}_\omega \cup \{x_{\eta|\alpha,j} : \alpha < \omega_1, 0 \le j < \omega\}$.

We show that the family $\mathbf{G} = \{G_\eta : \eta \in^{\omega_1} 2\}$ satisfies the conclusions of Theorem 1.

Claim 1: G_η is pure in $\mathbf{Z}^\omega$.

Proof of Claim 1: Suppose that $rx = g$ for some $x \in \mathbf{Z}^\omega$, $r \in \mathbf{N}$, and $g \in G_\eta$. Say $g = y + n_1 x_{\eta|\alpha_1,j_1} + \cdots + n_m x_{\eta|\alpha_m,j_m}$, $n_l \ne 0$, with $y \in \mathbf{Z}_\omega$. Without loss of generality (adding more elements from $\mathbf{Z}_\omega$ to the RHS if necessary), $(\forall\ l \le m)(j_l = j)$ for some $j < \omega$, $j > r$, $y(i) = 0$ $(\forall\ i > j)$, and $x(i) = 0$ $(\forall i \le j)$. Relabelling (if necessary), we may assume that $\alpha_1 < \cdots < \alpha_m < \omega_1$, and because $x_{\eta|\alpha_l,j}(i) = 0$ if $i \le j$, we may write

$$rx = ry^* + n_1 x_{\eta|\alpha_1,j} + \cdots + n_m x_{\eta|\alpha_m,j}, \quad \text{for some } y^* \in \mathbf{Z}_\omega.$$

Fix $k \in \{1, \ldots, m\}$. Since $\eta|\alpha_1, \ldots, \eta|\alpha_m$ are distinct ($\alpha_1 < \cdots < \alpha_m$), letting $a_l = \delta_{kl}$ (Kronecker delta), we know that the set $N_k = \{i < \omega : (\forall\ l \ne k)(f_{\eta_l}(i) = 0,\ f_{\eta_k}(i) = 1)\}$ is infinite. For large enough i in this set $\big($e.g. $i > \max_{1 \le l \le m}\ [\min(P_{\eta|\alpha_l} \setminus \{0, \ldots, j\})]\big)$, $x_{\eta|\alpha_l,j}(i)$ is zero if and only if $l \ne k$. So for infinitely many $i < \omega$, for $l \ne k$, $x_{\eta|\alpha_l,j}(i) = 0$, and $x_{\eta|\alpha_k,j}(i) \ne 0$.

Unfix k. For each $k \le m$, for infinitely many $i \in (j,\omega) \cap N_k$, $rx(i) = n_k x_{\eta|\alpha_k,j}(i) = n_k \Pi\{p \in P_{\eta|\alpha_k} : j \le p < i\}$. Since $r < j$, we must have $rs_k = n_k$ for some s_k in $\mathbf{Z}$, and therefore $x = y^* + s_1 x_{\eta|\alpha_1,j} + \cdots + s_m x_{\eta|\alpha_m,j} \in G_\eta$ (G_η is torsion-free). Hence G_η is pure in $\mathbf{Z}^\omega$, which establishes Claim 1.

Claim 2: G_η has cardinality $\aleph_1$, so (1.1) holds.

Proof of Claim 2: If $\xi \ne \zeta \in^{\omega_1 >} 2$, then for some $j < \omega$, $P_\xi \cap P_\zeta \subseteq j$. Pick $p, q > j$ with $p \in P_\xi$ and $q \in P_\zeta$; so the set $B = \{i < \omega : f_\xi(i) = p \text{ and } f_\zeta(i) = q\}$ is infinite, and if $i \in B$ is bigger than $\max\{j, p, q\}$, then $x_{\xi,j}(i) \ne x_{\zeta,j}(i)$, since $x_{\xi,j}(i)$ is non-zero and divisible by p^2 but by no prime in P_ζ, and $x_{\zeta,j}(i)$ is non-zero and divisible by q^2 but by no prime in P_ξ. It follows that G_η has cardinality $\aleph_1$. After this observation, a second's reflection on the element types of G_η and G_ν (for $\eta \ne \nu$) should convince the reader that the groups are neither isomorphic nor free.

Claim 3: (1.2) holds: if $\eta_1 \ne \eta_2 \in^{\omega_1} 2$, then G_{η_1} and G_{η_2} are almost disjoint.

Proof of Claim 3: Suppose (towards a contradiction) that for some $\eta_1 \ne \eta_2 \in^{\omega_1} 2$, for some non-free abelian group H, there exist isomorphisms $\varphi_l : H \to \text{range}(\varphi_l) \le G_{\eta_l}$, $l = 1, 2$. Since G_{η_l} is $\aleph_1$-free, H must have cardinality $\aleph_1$. Let $\langle H_i : i < \omega_1\rangle$ be an ω_1-filtration of H. Without loss of generality, we may assume that each H_i is pure in H, so that H/H_i is torsion-free.

Let $G_{\eta,i} = \langle \mathbf{Z}_\omega \cup \{x_{\eta|\beta,j} : j < \omega,\ \beta < i\}\rangle$ for $i < \omega_1$ and $\eta \in \{\eta_1, \eta_2\}$.

Note that $\langle G_{\eta,i} : i < \omega_1\rangle$ is a ω_1-filtration of G_η, since it is increasing and continuous with union G_η, and each $G_{\eta,i}$ is countable. For large enough χ, the set C defined by $\{\delta < \omega_1 : \text{dcl}_{\left(H(\chi),\in,<\right)}[\delta \cup \{G_{\eta_1}, G_{\eta_2}, \{x_\nu, f_\nu : \nu \in^{\omega_1 >} 2\}, \eta_1, \eta_2, \varphi_1, \varphi_2, \{H_i :$

$i < \omega_1\}\}] \cap \omega_1 = \delta\}$ is a club of ω_1 (well-known, or see [16], 401). Note that if $\delta \in C$, then φ_l maps H_δ into $G_{\eta_l,\delta}$. Since H is not free, it follows by Lemma 5 that $S = \{\delta \in C : H/H_\delta$ is not $\aleph_1$-free$\}$ is stationary. By Pontryagin's Criterion, for each $\delta \in S$, H/H_δ has a non-free (torsion-free) subgroup K_δ/H_δ of finite rank $n_\delta + 1$ such that every subgroup of K_δ/H_δ of rank less than $n_\delta + 1$ is free. Let ${H_\delta}^+/H_\delta$ be a pure subgroup of K_δ/H_δ of rank n_δ. Then ${H_\delta}^+/H_\delta$ is free with basis $y_0 + H_\delta, \dots, y_{n_\delta - 1} + H_\delta$ say. So $K_\delta/{H_\delta}^+ \simeq (K_\delta/H_\delta)/({H_\delta}^+/H_\delta)$ is a torsion-free rank-1 group which is not free, and hence there is a non-zero element $y_{n_\delta} + {H_\delta}^+$ which is divisible in $K_\delta/{H_\delta}^+$ by infinitely many natural numbers. Call this set of natural numbers A.

For $l = 1, 2$, for large enough $j_l(*) < \omega$, and ${\beta^l}_0 < \dots < {\beta^l}_{k_l} < \omega_1$, $\varphi_l(y_m)$ is an element of the subgroup of G_{η_l} generated by $G_{\eta_l,\delta} \cup \{x_{\eta_l|{\beta^l}_0, j_l(*)}, \dots, x_{\eta_l|{\beta^l}_{k_l}, j_l(*)}\}$ for all $m \le n_\delta$.

Taking large enough $\delta \in S$, we may assume that $\min\{\alpha : \eta_1|\alpha \ne \eta_2|\alpha\} < {\beta^l}_0$, $l = 1, 2$. Since $\delta \in C$, we can show the following claims:
$(*)_1$: The set A does not contain infinitely many powers of one prime.
$(*)_2$: The set $Q = (\mathbf{P} \cap A) \subseteq P_{\eta_l|{\beta^l}_0} \cup \dots \cup P_{\eta_l|{\beta^l}_{k_l}}$.

Now $(*)_1$ is true because non-zero sums of elements in $G_{\eta_l,\delta} \cup \{x_{\eta_l|{\beta^l}_0, j_l(*)}, \dots, x_{\eta_l|{\beta^l}_{k_l}, j_l(*)}\}$ are divisible by at most finitely many powers of any given prime (by the definition of the elements $x_{\eta_l|\beta,j}$). Note that $\chi(y_{n_\delta} + {H_\delta}^+) = \cup_{\{y \in y_{n_\delta} + {H_\delta}^+\}} \chi(y) \le cup_{\{y \in y_{n_\delta} + {H_\delta}^+\}} \chi(\varphi_l(y))$, where the characteristics are taken relative to $K_\delta/{H_\delta}^+$, K_δ and
$G_{\eta_l,\delta} \cup \{x_{\eta_l|{\beta^l}_0, j_l(*)}, \dots, x_{\eta_l|{\beta^l}_{k_l}, j_l(*)}\}$ respectively. Hence $(*)_1$ holds. By $(*)_1$, since A is infinite, the set $Q = \mathbf{P} \cap A$ is infinite.

Also, the same characteristic inequality implies that $Q \subseteq P_{\eta_l|{\beta^l}_0} \cup \dots \cup P_{\eta_l|{\beta^l}_{k_l}}$. So $(*)_2$ is true. Hence, $Q \subseteq \cap_{l=1,2}(\cup_{k \le k_l} P_{\eta_l|{\beta^l}_k})$ which is finite (since the family $\{P_\eta : \eta \in {}^{\omega_1 >} 2\}$ is almost disjoint). This is a contradiction, and so Claim 3 follows, completing the proof of Theorem 1. □

Corollary 6. *Every non-slender $\aleph_1$-free abelian group G has a family $\{G_\alpha : \alpha < 2^{\aleph_1}\}$ of non-free subgroups such that:*
1. each G_α is almost free of cardinality $\aleph_1$;
2. if $\alpha < \beta$, then G_α and G_β are almost disjoint.

Proof. By Nunke's characterisation of slender groups (see Corollary IX.2.5 in [10] for example), G must contain a copy of the Baer-Specker group. □

Remark: For the same reason, the corollary is true for any non-slender cotorsion-free abelian group.

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