THE YELLOW CAKE

ANDRZEJ ROSŁANOWSKI AND SAHARON SHELAH

Abstract. In this paper we consider the following property:

($\circledast^{\rm Da}$) For every function $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ there are functions $g^0_n, g^1_n:\mathbb{R}\longrightarrow\mathbb{R}$ (for $n<\omega$) such that

$$(\forall x,y\in\mathbb{R})(f(x,y)=\sum_{n<\omega} g^0_n(x)g^1_n(y)).$$

We show that, despite some expectation suggested by [Sh 675], ($\circledast^{\rm Da}$) does not imply $\mathbf{MA}$(σ–centered). Next, we introduce cardinal characteristics of the continuum responsible for the failure of ($\circledast^{\rm Da}$).

0. Introduction

In the present paper we will consider the following property:

($\circledast^{\rm Da}$) For every function $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ there are functions $g^0_n, g^1_n:\mathbb{R}\longrightarrow\mathbb{R}$ (for $n<\omega$) such that

$$(\forall x,y\in\mathbb{R})(f(x,y)=\sum_{n<\omega} g^0_n(x)g^1_n(y)).$$

Davies [Da74] showed that CH implies ($\circledast^{\rm Da}$) and Miller [Mixx, Problem 15.11], [Mi91] and Ciesielski [Ci97, Problem 7] asked if ($\circledast^{\rm Da}$) is equivalent to CH. It was shown in [Sh 675, §3] that the answer is negative. Namely,

Theorem 0.1. (1) *(See* [Sh 675, 3.4]*)* $\mathbf{MA}$(*σ-centered*) *implies* ($\circledast^{\rm Da}$).

(2) *(See* [Sh 675, 3.6]*) If $\mathbb{P}$ is the forcing notion for adding $\aleph_2$ Cohen reals then* $\Vdash_{\mathbb{P}} \neg(\circledast^{\rm Da})$.

The proof of [Sh 675, Conclusion 3.4]) strongly used the assumptions causing an impression that the property ($\circledast^{\rm Da}$) might be equivalent to $\mathbf{MA}$(σ-centered).

The first section introduces a strong variant of ccc which is useful in preserving unbounded families. In the second section we show that ($\circledast^{\rm Da}$) does not imply $\mathbf{MA}$(σ-centered). Finally, the in next section we show the combinatorial heart of [Sh 675, Proposition 3.6] and we introduce cardinal characteristics of the continuum closely related to the failure of ($\circledast^{\rm Da}$).

Notation Most of our notation is standard and compatible with that of classical textbooks on Set Theory (like Bartoszyński Judah [BaJu95]). However in forcing we keep the convention that *a stronger condition is the larger one.*

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Notation 0.2. (1) For two sequences η, ν we write $\nu \lhd \eta$ whenever ν is a proper initial segment of η, and $\nu \trianglelefteq \eta$ when either $\nu \lhd \eta$ or $\nu = \eta$. The length of a sequence η is denoted by $\ell g(\eta)$.

(2) The set of rationals is denoted by $\mathbb{Q}$ and the set of reals is called $\mathbb{R}$. The cardinality of $\mathbb{R}$ is called $\mathfrak{c}$ (and it is refered to as the the continuum). The dominating number (the minimal size of a dominating family in ${}^{\omega}\omega$ in the ordering of eventual dominance) is denoted by $\mathfrak{d}$ and the unbounded number (the minimal size of an unbounded family in that order) is called $\mathfrak{b}$.

(3) The quantifiers $(\forall^\infty n)$ and $(\exists^\infty n)$ are abbreviations for

$$(\exists m \in \omega)(\forall n > m) \quad \text{and} \quad (\forall m \in \omega)(\exists n > m),$$

respectively.

(4) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a dot above (e.g. $\dot{A}$, $\dot{f}$).

1. $\mathcal{F}$–sweet forcing notion

Definition 1.1. An uncountable family $\mathcal{F} \subseteq {}^{\omega}\omega$ is spread if

($\boxtimes$) for each $k^*, n^* < \omega$ and a sequence $\langle f_{\alpha,n} : \alpha < \omega_1,\ n < n^* \rangle$ of pairwise distinct elements of $\mathcal{F}$ there are an increasing sequence $\langle \alpha_i : i < \omega \rangle \subseteq \omega_1$ and an integer $k > k^*$ such that

$$(\forall i < \omega)(\forall n < n^*)(f_{\alpha_i,n}(k) < f_{\alpha_{i+1},n}(k)).$$

Remark 1.2. (1) Note that if an uncountable family $\mathcal{F} \subseteq {}^{\omega}\omega$ has the property that its every uncountable subfamily is unbounded on every $K \in [\omega]^{\omega}$ then $\mathcal{F}$ is spread.

(2) If κ is uncountable and one adds κ many Cohen reals $\langle c_\alpha : \alpha < \kappa \rangle \subseteq {}^{\omega}\omega$ then $\{c_\alpha : \alpha < \kappa\}$ is a spread family.

(3) If there is a spread family then $\mathfrak{b} = \aleph_1$ (so in particular $\mathbf{MA}_{\aleph_2}(\sigma$–centered) fails).

Definition 1.3. Let $\mathcal{F} \subseteq {}^{\omega}\omega$ be a spread family. A forcing notion $\mathbb{P}$ is $\mathcal{F}$*–sweet* if the following condition is satisfied:

$(\boxplus)^{\mathcal{F}}_{\rm sweet}$ for each sequence $\langle p_\alpha : \alpha < \omega_1 \rangle \subseteq \mathbb{P}$ there are $A \in [\omega]^{\aleph_1}$, $k^* < \omega$ and a sequence $\langle f_{\alpha,n} : n < n^*,\ \alpha \in A \rangle \subseteq \mathcal{F}$ such that $(\alpha, n) \neq (\alpha', n') \Rightarrow f_{\alpha,n} \neq f_{\alpha',n'}$ and

($\oplus$) if $\langle \alpha_i : i < \omega \rangle$ is an increasing sequence of elements of A such that for some $k \in (k^*, \omega)$

$$(\forall i < \omega)(\forall n < n^*)(f_{\alpha_i,n}(k) < f_{\alpha_{i+1},n}(k))$$

then there is $p \in \mathbb{P}$ such that $p \Vdash (\exists^\infty i \in \omega)(p_{\alpha_i} \in \Gamma_{\mathbb{P}})$.

Proposition 1.4. *Assume that $\mathcal{F} \subseteq {}^{\omega}\omega$ is a spread family and $\mathbb{P}$ is an $\mathcal{F}$–sweet forcing notion. Then*

$$\Vdash_{\mathbb{P}} \text{“ } \mathcal{F} \text{ is a spread family ”}.$$

Proof. First note that easily $\mathcal{F}$–sweetness implies the ccc.

Suppose that $k^+ < \omega$, $\langle \dot{f}_{\alpha,n} : \alpha < \omega_1,\ n < n^+\rangle$ are $\mathbb{P}$–names for elements of $\mathcal{F}$, $p \in \mathbb{P}$ and

$$p \Vdash_{\mathbb{P}} (\forall \alpha, \alpha' < \omega_1)(\forall n, n' < n^+)((\alpha, n) \neq (\alpha', n') \ \Rightarrow\ \dot{f}_{\alpha,n} \neq \dot{f}_{\alpha',n'}).$$

For $\alpha < \omega_1$ choose conditions $p_\alpha \geq p$ and functions $f_{\alpha,n} \in \mathcal{F}$ (for $n < n^+$) such that $p_\alpha \Vdash (\forall n < n^+)(\dot{f}_{\alpha,n} = f_{\alpha,n})$. Passing to a subsequence, we may assume that

$$(\alpha, n) \neq (\alpha', n') \ \Rightarrow\ f_{\alpha,n} \neq f_{\alpha',n'}.$$

Choose $k^* > k^+$, a set $A \in [\omega]^{\aleph_1}$ and a sequence $\langle f_{\alpha,n} : \alpha \in A,\ n^+ \leq n < n^*\rangle$ as guaranteed by $(\boxplus)^{\mathcal{F}}_{\rm sweet}$ of 1.3 for $\langle p_\alpha : \alpha < \omega_1\rangle$ (note that here, for notational convenience, we use the interval $[n^+, n^*)$ instead of n^* there). Shrinking the set A and possibly decreasing n^* (and reenumerating $f_{\alpha,n}$'s) we may assume that all functions in appearing in $\langle f_{\alpha,n} : \alpha \in A,\ n < n^*\rangle$ are distinct. By $(\boxtimes)$ of 1.1 we find $k > k^*$ and an increasing sequence $\langle \alpha_i : i < \omega\rangle \subseteq A$ such that

$$(\forall i < \omega)(\forall n < n^*)(f_{\alpha_i,n}(k) < f_{\alpha_{i+1},n}(k)).$$

But it follows from $(\oplus)$ of 1.3 that now we can find a condition $q \in \mathbb{P}$ such that $q \Vdash (\exists^\infty i \in \omega)(p_{\alpha_i} \in \Gamma_{\mathbb{P}})$. As all conditions p_α are stronger than p we may demand that $q \geq p$. Now use the choice of the p_{α_i}'s and $f_{\alpha_i,n}$ (for $n < n^+$) to finish the proof. $\square$

Theorem 1.5. *Assume $\mathcal{F}$ is a spread family. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \gamma\rangle$ be a finite support iteration of forcing notions such that for each $\alpha < \gamma$ we have*

(1) $\Vdash_{\mathbb{P}_\alpha}$ *" $\mathcal{F}$ is spread ", and*
(2) $\Vdash_{\mathbb{P}_\alpha}$ *" $\dot{\mathbb{Q}}_\alpha$ is $\mathcal{F}$–sweet ".*

Then $\mathbb{P}_\gamma$ is $\mathcal{F}$–sweet (and consequently, $\Vdash_{\mathbb{P}_\gamma}$ " $\mathcal{F}$ is a spread family ").

Proof. We show this by induction on γ.

CASE 1: $\gamma = \beta + 1$
Let $\langle p_\alpha : \alpha < \omega_1\rangle \subseteq \mathbb{P}_{\beta+1}$. Take a condition $p^* \in \mathbb{P}_\beta$ such that

$$p^* \Vdash_{\mathbb{P}_\beta} \text{" } \{\alpha < \omega_1 : p_\alpha{\restriction}\beta \in \Gamma_{\mathbb{P}_\beta}\} \text{ is uncountable "}$$

(there is one by the ccc). Next, use the assumption that $\dot{\mathbb{Q}}_\beta$ is $\mathcal{F}$–sweet and get $\mathbb{P}_\beta$–names $\dot{A} \in [\omega_1]^{\aleph_1}$ and $\dot{k}^*$, $\dot{n}^*$ and $\langle \dot{f}_{\alpha,n} : \alpha \in \dot{A},\ n < \dot{n}^*\rangle \subseteq \mathcal{F}$ such that the condition p^* forces that they are as guaranteed by $(\boxplus)^{\mathcal{F}}_{\rm sweet}$ of 1.3 for the sequence $\langle p_\alpha(\beta) : \alpha < \omega_1,\ p_\alpha{\restriction}\beta \in \Gamma_{\mathbb{P}_\beta}\rangle$.

Let A' be the set of all $\alpha < \omega_1$ such that there is a condition stronger than both p^* and $p_\alpha{\restriction}\beta$ which forces that $p_\alpha(\beta)$ is in $\dot{A}$. Clearly $|A'| = \aleph_1$. For each $\alpha \in A'$ choose a condition $q_\alpha \in \mathbb{P}_\beta$ stronger than both p^* and $p_\alpha{\restriction}\beta$ which forces that $p_\alpha(\beta) \in \dot{A}$ and decides the values of $\dot{k}^*$, $\dot{n}^*$ and $\langle \dot{f}_{\alpha,n} : n < \dot{n}^*\rangle$. Next we may choose $A'' \in [A']^{\aleph_1}$, k^*, n^* and $\langle f_{\alpha,n} : \alpha \in A'',\ n < n^*\rangle \subseteq \mathcal{F}$ such that (for each $\alpha \in A''$ and $n < n^*$) $q_\alpha \Vdash$" $\dot{k}^* = k^*$ & $\dot{n}^* = n^*$ & $\dot{f}_{\alpha,n} = f_{\alpha,n}$ ". Moreover we may demand that the $f_{\alpha,n}$'s are pairwise distinct (for $\alpha \in A''$, $n < n^*$).

Apply the inductive hypothesis to the sequence $\langle q_\alpha : \alpha \in A''\rangle$ (and $\mathbb{P}_\beta$) to get $A \in [A'']^{\aleph_1}$, k^+, $n^+ > n^*$ and $\langle f_{\alpha,n} : \alpha \in A,\ n^* \leq n < n^+\rangle$. For simplicity we may assume that there are no repetitions in the sequence $\langle f_{\alpha,n} : \alpha \in A,\ n < n^*\rangle$ (we may shrink A and decrease n^* reenumerating $f_{\alpha,n}$'s suitably). We claim that this sequence and $\max\{k^*, k^+\}$ satisfy the demand in $(\oplus)$ if 1.3. So suppose that

$\langle\alpha_i : i<\omega\rangle$ is an increasing sequence of elements of A such that for some $k>k^*,k^+$ we have

$$(\forall i<\omega)(\forall n<n^+)(f_{\alpha_i,n}(k)<f_{\alpha_{i+1},n}(k)).$$

Clearly, by our choices, we find a condition $p^+\in\mathbb{P}_\beta$ stronger than p^* such that $p^+\Vdash(\exists^\infty i\in\omega)(q_{\alpha_i}\in\Gamma_{\mathbb{P}_\beta})$. Next, in $\mathbf{V}^{\mathbb{P}_\beta}$, we look at the sequence $\langle p_{\alpha_i}(\beta) : q_{\alpha_i}\in\Gamma_{\mathbb{P}_\beta},\ i<\omega\rangle$. We may find a $\mathbb{P}_\beta$–name $p^+(\beta)$ such that (p^+ forces that)

$$p^+(\beta)\Vdash_{\dot{\mathbb{Q}}_\beta}(\exists^\infty i\in\omega)(q_{\alpha_i}\in\Gamma_{\mathbb{P}_\beta}\ \&\ p_{\alpha_i}(\beta)\in\Gamma_{\dot{\mathbb{Q}}_\beta}).$$

Look at the condition $p^{+\frown}p^+(\beta)$.

Case 2: γ is a limit ordinal.
If $\langle p_\alpha:\alpha<\omega_1\rangle\subseteq\mathbb{P}_\gamma$ then, under the assumption of the current case, for some $A\in[\omega_1]^{\aleph_1}$ and $\delta<\gamma$, the sets $\{\mathrm{supp}(p_\alpha)\setminus\delta:\alpha\in A\}$ are pairwise disjoint. Apply the inductive hypothesis to $\mathbb{P}_\delta$ and the sequence $\langle p_\alpha{\restriction}\delta:\alpha\in A\rangle$. □

Conclusion 1.6. Suppose that $\kappa>\aleph_1$ is a regular cardinal such that $\kappa^{<\kappa}=\kappa$ and $(\forall\mu<\kappa)(\mu^{\aleph_0}<\kappa)$. Then there is a ccc forcing notion $\mathbb{P}$ of size κ such that

$$\Vdash_{\mathbb{P}}\text{“ there is a spread family } \mathcal{F}\subseteq{}^\omega\omega \text{ of size } \kappa\ \&\ \mathfrak{c}=\kappa\ \&\ \mathbf{MA}(\mathcal{F}\text{–sweet}) \text{ ”}.$$

Proof. First note that if $\mathbb{P}$ is an $\mathcal{F}$–sweet forcing notion, $\mathcal{I}_\xi\subseteq\mathbb{P}$ (for $\xi<\mu<\kappa$) are dense subsets of $\mathbb{P}$ and $p\in\mathbb{P}$ then, under our assumptions, there is a set $\mathbb{P}^*\subseteq\mathbb{P}$ of size less than κ such that $p\in\mathbb{P}^*$ and

- if $p,q\in\mathbb{P}^*$ are incompatible in $\mathbb{P}^*$ then they are incompatible in $\mathbb{P}$,
- if $\langle p_i:i<\omega\rangle\subseteq\mathbb{P}^*$ is not a maximal antichain in $\mathbb{P}$ then it is not in $\mathbb{P}^*$,
- for each $\xi<\mu$ the intersection $\mathcal{I}_\xi\cap\mathbb{P}^*$ is dense in $\mathbb{P}^*$.

(Thus $\mathbb{P}^*\lessdot\mathbb{P}$ and so it is $\mathcal{F}$–sweet.)

Now, using standard bookkeeping arguments, build a finite support iteration $\langle\mathbb{P}_\alpha,\dot{\mathbb{Q}}_\alpha:\alpha<\kappa\rangle$ such that

(1) $\mathbb{Q}_0$ is the forcing notion adding κ many Cohen real $\mathcal{F}=\langle f_\alpha:\alpha<\kappa\rangle\subseteq{}^\omega\omega$ (with finite conditions), [so in $\mathbf{V}^{\mathbb{Q}_0}$, the family $\mathcal{F}$ is spread]
(2) for each $\alpha<\kappa$, $\Vdash_{\mathbb{P}_{1+\alpha}}$“ $\dot{\mathbb{Q}}_{1+\alpha}$ is a $\mathcal{F}$–sweet forcing notion of size $<\kappa$ ”,
(3) if $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\kappa$–name for a $\mathcal{F}$–sweet forcing notion of size $<\kappa$ then for κ many $\alpha<\kappa$, $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$–name and $\Vdash_{\mathbb{P}_\alpha}\dot{\mathbb{Q}}=\dot{\mathbb{Q}}_\alpha$.

It follows from 1.5 that in $\mathbf{V}^{\mathbb{P}_\alpha}$ (for $0<\alpha\le\kappa$) the family $\mathcal{F}$ is spread, so there are no problems with carrying out the construction. Easily $\mathbb{P}_\kappa$ is as required. □

Remark 1.7. Note the similarity of $\mathbf{MA}(\mathcal{F}$–sweet) to the methods used in [Sh:98, §4].

2. More on Davies' Problem

The aim of this section is to show that ($\circledast^{\mathrm{Da}}$) does not imply $\mathbf{MA}(\sigma$–centered).

Let $\langle\nu_n:n<\omega\rangle$ be an enumeration of ${}^{\omega>}\omega$ such that $\ell g(\nu_n)\le n$. For distinct $\rho_0,\rho_1\in{}^\omega\omega$ let $\delta(\rho_0,\rho_1)=1+\max\{m:\nu_m\vartriangleleft\rho_0\ \&\ \nu_m\vartriangleleft\rho_1\}$. (Note that $\rho_0{\restriction}\delta(\rho_0,\rho_1)\neq\rho_1{\restriction}\delta(\rho_0,\rho_1)$.)

Assume that there exists a spread family of size $\mathfrak{c}$ and let $\mathcal{F}=\langle\rho_\alpha:\alpha<\mathfrak{c}\rangle\subseteq{}^\omega\omega$ be such a family (later we will choose the one coming from adding κ many Cohen reals).

Definition 2.1. Let $\zeta<\mathfrak{c}$ be an ordinal and let $f:\zeta\times\zeta\longrightarrow\mathbb{R}$.

(1) *A ζ–approximation* is a sequence $\bar{g} = \langle g^\ell_\eta : \ell < 2,\ \eta \in {}^{\omega>}\omega\rangle$ such that:
   (a) $g^\ell_\eta : \zeta \longrightarrow \mathbb{Q}$ (for $\ell < 2$, $\eta \in {}^{\omega>}\omega$),
   (b) if $\alpha < \zeta$ then $(\forall \beta < \mathfrak{c})(\exists^\infty k \in \omega)(g^\ell_{\rho_\alpha\restriction k}(\alpha) \neq 0\ \&\ \nu_k \lhd \rho_\beta)$,
   (c) if $\alpha < \zeta$, $\eta \in {}^{\omega>}\omega$ and neither η nor $\nu_{\ell g(\eta)}$ is an initial segment of ρ_α, then $g^0_\eta(\alpha) = g^1_\eta(\alpha) = 0$.

(2) If $\zeta_0 < \zeta_1$ and $\bar{g}^k = \langle g^{\ell,k}_\eta : \ell < 2,\ \eta \in {}^{\omega>}\omega\rangle$ (for $k = 0, 1$) are ζ_k–approximations such that $g^{\ell,0}_\eta \subseteq g^{\ell,1}_\eta$ (for all $\ell < 2$ and $\eta \in {}^{\omega>}\omega$) then we say that $\bar{g}^1$ *extends* $\bar{g}^0$ (in short: $\bar{g}^0 \preceq \bar{g}^1$).

(3) We say that a ζ–approximation $\bar{g}$ *agrees with the function* f if

$$(\forall \alpha, \beta < \zeta)\big(f(\alpha,\beta) = \sum_{\eta \in {}^{\omega>}\omega} g^0_\eta(\alpha)\cdot g^1_\eta(\beta) \text{ and the series converges absolutely}\big).$$

Proposition 2.2. *If $\bar{g}^\xi$ are ζ_ξ–approximations (for $\xi < \xi^*$) such that the sequence $\langle \bar{g}^\xi : \xi < \xi^*\rangle$ is $\preceq$–increasing and $\zeta_{\xi^*} = \bigcup\limits_{\xi<\xi^*} \zeta_\xi$ then there is a ζ_{ξ^*}–approximation $\bar{g}^{\xi^*}$ such that $(\forall \xi < \xi^*)(\bar{g}^\xi \preceq \bar{g}^{\xi^*})$. Moreover, if $f : \zeta_{\xi^*} \times \zeta_{\xi^*} \longrightarrow \mathbb{R}$ and each $\bar{g}^\xi$ agrees with $f\restriction(\zeta_\xi \times \zeta_\xi)$ then $\bar{g}^{\xi^*}$ agrees with f.*

Thus if we want to show that $(\circledast^{\rm Da})$ holds we may take a function $f : \mathfrak{c}\times\mathfrak{c} \longrightarrow \mathbb{R}$ (it should be clear that we may look at functions of that type only) and try to build a $\preceq$–increasing sequence $\langle \bar{g}^\xi : \xi < \mathfrak{c}\rangle$ of approximations. If we make sure that $\bar{g}^\xi$ is a ξ–approximation that agrees with $f\restriction(\xi\times\xi)$ then the limit $\bar{g}^{\mathfrak{c}}$ of $\bar{g}^\xi$'s will give us witnesses for f. (Note that by the absolute convergence demand in 2.1(3) we do not have to worry about the order in the series.) At limit stages of the construction we use 2.2, but problems may occur at some successor stage. Here we need to use forcing.

Definition 2.3. Assume that $\zeta < \mathfrak{c}$ is an ordinal, and $f : (\zeta+1)\times(\zeta+1) \longrightarrow \mathbb{R}$. Let $\bar{g} = \langle g^\ell_\eta : \ell < 2,\ \eta \in {}^{\omega>}\omega\rangle$ be a ζ–approximation which agrees with $f\restriction\zeta\times\zeta$. We define a forcing notion $\mathbb{P}^{\bar{g},\zeta}_f$ as follows:

a condition is a tuple $p = \langle Z^p, j^p, \langle r^p_{\ell,\eta} : \ell < 2,\ \eta \in {}^{j^p>}\omega\rangle\rangle$ such that

(α) $j^p < \omega$ and Z^p is a finite subset of ζ, $r^p_{\ell,\eta} \in \mathbb{Q}$ (for $\ell < 2$, $\eta \in {}^{j^p>}\omega$),
(β) the set $\{\eta \in {}^{j^p>}\omega : r^p_{0,\eta} \neq 0 \text{ or } r^p_{1,\eta} \neq 0\}$ is finite, and if $\eta \in {}^{j^p>}\omega$ and neither η nor $\nu_{\ell g(\eta)}$ is an initial segment of ρ_ζ then $r^p_{\ell,\eta} = 0$,
(γ) if $\alpha \in Z^p$ then

$$|f(\alpha,\zeta) - \textstyle\sum\{g^0_\eta(\alpha)\cdot r^p_{1,\eta} : \eta \in {}^{j^p>}\omega\}| < 2^{-j^p},$$
$$|f(\zeta,\alpha) - \textstyle\sum\{r^p_{0,\eta}\cdot g^1_\eta(\alpha) : \eta \in {}^{j^p>}\omega\}| < 2^{-j^p}, \quad \text{and}$$
$$|f(\zeta,\zeta) - \textstyle\sum\{r^p_{0,\eta}\cdot r^p_{1,\eta} : \eta \in {}^{j^p>}\omega\}| < 2^{-j^p}$$

(note that by demand (β) all the sums above are finite),
(δ) if $\alpha,\beta \in Z^p\cup\{\zeta\}$ are distinct then $\delta(\rho_\alpha,\rho_\beta) < j^p$;

the order is defined by $p \leq q$ if and only if

(a) $j^p \leq j^q$, $Z^p \subseteq Z^q$ and $r^p_{\ell,\eta} = r^q_{\ell,\eta}$ for $\eta \in {}^{j^p>}\omega$, $\ell < 2$,

(b) if $\alpha \in Z^p$ then

$$\sum\{|r^p_{0,\eta}\cdot g^1_\eta(\alpha)| : \eta \in {}^{j^q>}\omega \setminus {}^{j^p>}\omega\} < 4\frac{1-2^{j^p-j^q}}{2^{j^p-1}},$$
$$\sum\{|g^0_\eta(\alpha)\cdot r^p_{1,\eta}| : \eta \in {}^{j^q>}\omega \setminus {}^{j^p>}\omega\} < 4\frac{1-2^{j^p-j^q}}{2^{j^p-1}}, \qquad \text{and}$$
$$\sum\{|r^p_{0,\eta}\cdot r^p_{1,\eta}| : \eta \in {}^{j^q>}\omega \setminus {}^{j^p>}\omega\} < 4\frac{1-2^{j^p-j^q}}{2^{j^p-1}}.$$

Proposition 2.4. *Suppose that $\zeta < \mathfrak{c}$, $f : (\zeta+1)\times(\zeta+1) \longrightarrow \mathbb{R}$ and $\bar{g}$ is a ζ–approximation that agrees with $f{\restriction}\zeta\times\zeta$. Then:*

1. *$\mathbb{P}^{\bar{g},\zeta}_f$ is a (non-trivial) $\mathcal{F}$–sweet forcing notion of size $|\zeta|+\aleph_0$.*
2. *In $\mathbf{V}^{\mathbb{P}^{\bar{g},\zeta}_f}$, there is a $(\zeta+1)$–approximation $\bar{g}^*$ such that $\bar{g} \prec \bar{g}^*$ and $\bar{g}^*$ agrees with f.*

Proof. (1) First note that $(\mathbb{P}^{\bar{g},\zeta}_f, \le)$ is a partial order and easily $\mathbb{P}^{\bar{g},\zeta}_f \neq \emptyset$ (remember that Z^p may be empty). Before we continue let us show the following claim that will be used later too.

Claim 2.4.1. *For each $j<\omega$, $\xi<\zeta$ and $\rho \in {}^\omega\omega$ the sets*

$$\mathcal{I}^j \stackrel{\rm def}{=} \{p \in \mathbb{P}^{\bar{g},\zeta}_f : j^p \ge j\},$$
$$\mathcal{I}_\xi \stackrel{\rm def}{=} \{p \in \mathbb{P}^{\bar{g},\zeta}_f : \xi \in Z^p\}, \qquad \textit{and}$$
$$\mathcal{I}^j_\rho \stackrel{\rm def}{=} \{p \in \mathbb{P}^{\bar{g},\zeta}_f : j<j^p \ \&\ (\forall \ell<2)(\exists k \in (j,j^p))(r^p_{\ell,\rho_\zeta{\restriction}j} \neq 0 \ \&\ \nu_j \vartriangleleft \rho)\}$$

are dense subsets of $\mathbb{P}^{\bar{g},\zeta}_f$.

Proof of the claim. Let $j<\omega$, $\xi<\zeta$, $\rho\in{}^\omega\omega$ and $p \in \mathbb{P}^{\bar{g},\zeta}_f$.

If $j \le j^p$ then $p \in \mathcal{I}^j$, so suppose that $j^p<j$. Let $\langle \xi_m : m<m^*\rangle$ enumerate Z^p. Choose pairwise distinct $\langle j_{\ell,m} : \ell<2, m<m^*\rangle \subseteq (j,\omega)$ such that $\nu_{j_{\ell,m}} \vartriangleleft \rho_\zeta$ and $g^\ell_{\rho_{\xi_m}{\restriction}j_{\ell,m}}(\xi_m) \neq 0$ (remember 2.1(1b)). Fix $j^*>j$ such that ν_{j^*} is not an initial segment of any ρ_{ξ_m} (for $m<m^*$). Let $j^q = j+\max\{j_{\ell,m} : \ell<2,\ m<m^*\}+j^*$, $Z^q=Z^p$ and define $r^q_{0,\eta}, r^q_{1,\eta}$ as follows.

1. If $\eta \in {}^{j^p>}\omega$ then $r^q_{\ell,\eta} = r^p_{\ell,\eta}$.
2. If $\eta \in {}^{j^q>}\omega \setminus {}^{j^p>}\omega \setminus \{\rho_{\xi_m}{\restriction}j_{\ell,m} : m<m^*\} \setminus \{\rho_\zeta{\restriction}j^*\}$, $\ell<2$ then $r^q_{1-\ell,\eta}=0$.
3. If $\eta = \rho_\zeta{\restriction}j^*$ then $r^q_{0,\eta}, r^q_{1,\eta} \in \mathbb{Q}\setminus\{0\}$ are such that $|r^q_{0,\eta}\cdot r^q_{1,\eta}| < 2^{-j^p}$ and
$$|f(\zeta,\zeta) - \sum\{r^p_{0,\nu}\cdot r^p_{1,\nu} : \nu \in {}^{j^p>}\omega\} - r^q_{0,\eta}\cdot r^q_{1,\eta}| < 2^{-2j^q}.$$
4. If $\eta = \rho_{\xi_m}{\restriction}j_{0,m}$, $m<m^*$ then $r^q_{1,\eta} \in \mathbb{Q}$ is such that $|g^0_\eta(\xi_m)\cdot r^q_{1,\eta}| < 2^{-j^p}$ and
$$|f(\xi_m,\zeta) - \sum\{g^0_\nu(\xi_m)\cdot r^p_{1,\nu} : \nu \in {}^{j^p>}\omega\} - g^0_\eta(\xi_m)\cdot r^q_{1,\eta}| < 2^{-2j^q};$$
if $\eta = \rho_{\xi_m}{\restriction}j_{1,m}$, $m<m^*$ then $r^q_{0,\eta} \in \mathbb{Q}$ is such that $|r^q_{0,\eta}\cdot g^1_\eta(\xi_m)| < 2^{-j^p}$ and
$$|f(\zeta,\xi_m) - \sum\{r^p_{0,\nu}\cdot g^1_\nu(\xi_m) : \nu \in {}^{j^p>}\omega\} - r^q_{0,\eta}\cdot g^1_\eta(\xi_m)| < 2^{-2j^q}.$$

One easily checks that $q = \langle Z^q, j^q, \langle r^q_{\ell,\eta} : \ell<2,\ \eta \in {}^{j^q>}\omega\rangle\rangle$ is a condition in $\mathbb{P}^{\bar{g},\zeta}_f$ stronger than p (and $q \in \mathcal{I}^j$).

Now suppose that $\xi\notin Z^p$. Take $j_0>j^p$ such that $(\forall\alpha\in Z^p\cup\{\zeta\})(\delta(\xi,\alpha)<j_0)$. Let $\langle\xi_m: m<m^*\rangle$ enumerate $Z^p\cup\{\xi\}$ and let $\langle j_{\ell,m}:\ell<2,\ m<m^*\rangle\subseteq(j_0,\omega)$ be pairwise distinct and such that $\nu_{j_{\ell,m}}\lhd\rho_\zeta$ & $g^\ell_{\xi_m\restriction j_{\ell,m}}(\xi_m)\neq 0$. Let $j^*>j^p$ be such that ν_{j^*} is not an initial segment of any ρ_{ξ_m}. Put $Z^q=Z^p\cup\{\xi\}$, $j^q=j^p+\max\{j_{\ell,m}:\ell<2,\ m<m^*\}+j^*$, and define $r^q_{\ell,\eta}$ like before, with one modification. If $\xi_m=\xi$ and $\eta=\rho_\xi\restriction j_{0,m}$ then $r^q_{1,\eta}\in\mathbb{Q}$ is such that $|f(\xi,\zeta)-g^0_\eta(\xi)\cdot r^q_{1,\eta}|<2^{-2j^q}$; if $\xi_m=\xi$ and $\eta=\rho_\xi\restriction j_{1,m}$ then $r^q_{0,\eta}\in\mathbb{Q}$ is such that $|f(\zeta,\xi)-r^q_{0,\eta}\cdot g^1_\eta(\xi)|<2^{-2j^q}$.

Similarly one builds a condition $q\in\mathcal{I}^j_\rho$ stronger than p (just choose j^* suitably). $\square$

Now we are going to show that $\mathbb{P}^{\bar g,\zeta}_f$ is $\mathcal{F}$–sweet. So suppose that $\langle p_\alpha:\alpha<\omega\rangle\subseteq\mathbb{P}^{\bar g,\zeta}_f$. Choose $A\in[\omega_1]^{\aleph_1}$ such that

- $\langle Z^{p_\alpha}:\alpha\in A\rangle$ forms a Δ–system with kernel Z,
- for each $\alpha,\beta\in A$, $|Z^{p_\alpha}|=|Z^{p_\beta}|$, $j^{p_\alpha}=j^{p_\beta}$ and
$$\langle r^{p_\alpha}_{\ell,\eta}:\ell<2,\ \eta\in{}^{j^{p_\alpha}>}\omega\rangle=\langle r^{p_\beta}_{\ell,\eta}:\ell<2,\ \eta\in{}^{j^{p_\beta}>}\omega\rangle$$
(remember 2.3(β)),
- if $\alpha,\beta\in A$ and $\pi:Z^{p_\alpha}\longrightarrow Z^{p_\beta}$ is the order preserving bijection then $\pi\restriction Z$ is the identity on Z and $(\forall\xi\in Z^{p_\alpha})(\rho_\xi\restriction j^{p_\alpha}=\rho_{\pi(\xi)}\restriction j^{p_\beta})$.

Let $k^*=j^{p_\alpha}$, $n^*=|Z^{p_\alpha}\setminus Z|$ for some (equivalently: all) $\alpha\in A$. For $\alpha\in A$ let $\langle f_{\alpha,n}:n<n^*\rangle$ enumerate $\{\rho_\xi:\xi\in Z^{p_\alpha}\setminus Z\}$. Clearly there are no repetitions in $\langle f_{\alpha,n}:n<n^*,\ \alpha\in A\rangle$. We claim that this sequence is as required in ($\oplus$) of 1.3. So suppose that $\langle\alpha_i:i<\omega\rangle\subseteq A$ is an increasing sequence such that for some $k>k^*$ we have
$$(\forall i<\omega)(\forall n<n^*)(f_{\alpha_i,n}(k)<f_{\alpha_{i+1},n}(k)).$$
Passing to a subsequence we may additionally demand that for each $m<k$, for every $n<n^*$, the sequence $\langle f_{\alpha_i,n}(m):i<\omega\rangle$ is either constant or strictly increasing. For $n<n^*$ let $k_n\geq j^p$ be such that the sequence $\langle f_{\alpha_i,n}\restriction k_n:i<\omega\rangle$ is constant but the sequence $\langle f_{\alpha_i,n}(k_n):i<\omega\rangle$ is strictly increasing. Take $j>k$ such that if $\nu_m\trianglelefteq f_{\alpha_i,n}\restriction k_n$, $n<n^*$ then $m<j$. Fix an enumeration $\langle\xi_m:m<m^*\rangle$ of $Z^{p_{\alpha_0}}$ (so $m^*=|Z|+n^*$) and choose $j^*,j_{\ell,m}>j+2$ with the properties as in the first part of the proof of 2.4.1 (with p_{α_0} in the place of p there). Put $Z^q=Z^{p_{\alpha_0}}$ and define $j^q,r^q_{\ell,\eta}$ exactly as there (so, in particular, for each $\eta\in{}^{j>}\omega\setminus{}^{j_{p_{\alpha_0}}>}\omega$ we have $r^q_{\ell,\eta}=0$). We claim that $q\Vdash(\exists^\infty i\in\omega)(p_{\alpha_i}\in\Gamma_{\mathbb{P}^{\bar g,\zeta}_f})$. So suppose that $q'\geq q$, $i_0<\omega$. Choose $i>i_0$ such that for each $n<n^*$ and $k'>k_n$, if $\nu_m=f_{\alpha_i,n}\restriction k'$ then $m>j^{q'}$. Moreover, we demand that if $k_n<k'<j^{q'}$, $n<n^*$ then $r^{q'}_{0,f_{\alpha_i,n}\restriction k'}=r^{q'}_{1,f_{\alpha_i,n}\restriction k'}=0$ (remember 2.3(β)). Then we have the effect that
$$(\forall\eta\in{}^{j^{q'}>}\omega\setminus{}^{j^{p_{\alpha_i}}>}\omega)(\forall\ell<2)(\forall\xi\in Z^{p_{\alpha_i}}\setminus Z)(r^{q'}_{\ell,\eta}\cdot g^1_\eta(\xi)=g^0_\eta(\xi)\cdot r^{q'}_{1,\eta}=0)).$$
So we may proceed as in the proof of 2.4.1 and build a condition q^+ stronger than both q' and p_{α_i}.

(2) Let $G\subseteq\mathbb{P}^{\bar g,\zeta}_f$ be generic over $\mathbf{V}$. For $\eta\in{}^{\omega>}\omega$ define
$$\begin{array}{ll} g^{\ell,*}_\eta(\zeta)=r^p_{\ell,\eta} & \text{where } p\in G\cap\mathcal{I}^{\ell g(\eta)+1},\\ g^{\ell,*}_\eta(\xi)=g^\ell_\eta(\xi) & \text{for } \xi<\zeta.\end{array}$$

It follows immediately from 2.4.1 (and the definition of the order on $\mathbb{P}^{\bar{g},\zeta}_f$) that the above conditions define a $\zeta+1$–approximation $\bar{g}^* = \langle g^{\ell,*}_\eta : \ell<2,\ \eta\in {}^{\omega>}\omega\rangle$ which agrees with f and extends $\bar{g}$. $\square$

Theorem 2.5. *Assume that κ is an uncountable cardinal such that $\kappa^{<\kappa}=\kappa$. Then there is a ccc forcing notion $\mathbb{P}$ of size κ such that*

$$\Vdash_{\mathbb{P}} \text{ “ } (\circledast^{\rm Da}) \ +\ \mathfrak{c}=\kappa\ +\ \textit{there is a spread family of size } \mathfrak{c} \text{ ”.}$$

Proof. Using standard bookkeeping argument build inductively a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha<\kappa\rangle$ and sequences $\langle \zeta_\alpha : \alpha<\kappa\rangle$, $\langle \dot{\bar{g}}_\alpha : \alpha<\kappa\rangle$ and $\langle \dot{f}_\alpha : \alpha<\kappa\rangle$ such that:

1. $\mathbb{Q}_0$ is the forcing notion adding κ many Cohen reals $\langle \rho_\xi : \xi<\kappa\rangle \subseteq {}^{\omega>}\omega$ (by finite approximations; so, in $\mathbf{V}^{\mathbb{Q}_0}$, $\mathfrak{c}=\kappa$ and the family $\mathcal{F}=\{\rho_\xi : \xi<\kappa\}$ is spread; we use it in the clauses below),
2. $\zeta_\alpha<\kappa$, $\dot{f}_\alpha$ is a $\mathbb{P}_\alpha$–name for a function from $(\zeta_\alpha+1)\times(\zeta_\alpha+1)$ to $\mathbb{R}$, $\dot{\bar{g}}_\alpha$ is a $\mathbb{P}_\alpha$–name for a ζ_α–approximation (for the family $\mathcal{F}$ added by $\mathbb{Q}_0$) which agrees with $\dot{f}_\alpha{\restriction}(\zeta_\alpha\times\zeta_\alpha)$,
3. $\Vdash_{\mathbb{P}_{1+\alpha}} \dot{\mathbb{Q}}_{1+\alpha} = \mathbb{P}^{\dot{\bar{g}}_\alpha,\zeta_\alpha}_{\dot{f}_\alpha}$ (for $\mathcal{F}$),
4. if $\dot{f}$ is a $\mathbb{P}_\kappa$ name for a function from $(\zeta+1)\times(\zeta+1)$ to $\mathbb{R}$, $\zeta<\kappa$ and $\dot{\bar{g}}$ is a $\mathbb{P}_\kappa$–name for a ζ–approximation which agrees with $\dot{f}{\restriction}(\zeta\times\zeta)$ then for some $\alpha<\kappa$, $\alpha>\omega$ we have
$$\dot{\bar{g}}=\dot{\bar{g}}_\alpha,\quad \dot{f}=\dot{f}_\alpha,\quad \zeta=\zeta_\alpha.$$

Clearly $\mathbb{P}_\kappa$ is a ccc forcing notion (with a dense subset) of size κ. It follows from 2.4(2), 2.2 that $\Vdash_{\mathbb{P}_\kappa} (\circledast^{\rm Da})$ (and clearly $\Vdash_{\mathbb{P}_\kappa} \mathfrak{c}=\kappa$). Moreover, by 2.4(1), 1.5 we know that, in $\mathbf{V}^{\mathbb{Q}_0}$, for each $\alpha\in[1,\kappa]$ the forcing notion $\mathbb{P}_\alpha{\restriction}[1,\kappa)$ is $\mathcal{F}$–sweet, so

$$\Vdash_{\mathbb{P}_\alpha} \text{ “ } \mathcal{F} \text{ is a spread family of size } \kappa \text{ ”}$$

(by 1.4). $\square$

3. When $(\circledast^{\rm Da})$ fails.

In this section we will strengthen the result of [Sh 675, 3.6] mentioned in 0.1(2) giving its combinatorial heart.

Definition 3.1. (1) For a function h such that $\mathrm{dom}(h)\subseteq\mathcal{X}\times\mathcal{Y}$ and $\mathrm{rng}(h)\subseteq\mathcal{Z}$ and a positive integer n we define

$$\begin{aligned}\kappa(h,n)=\min\{|\mathcal{A}_0|+|\mathcal{A}_1| :\ & \mathcal{A}_0\subseteq\mathcal{P}(\mathcal{X})\ \&\ \mathcal{A}_1\subseteq\mathcal{P}(\mathcal{Y})\ \&\\ & (\forall w\in[\mathcal{X}]^n)(\exists A\in\mathcal{A}_0)(w\subseteq A)\ \&\\ & (\forall w\in[\mathcal{Y}]^n)(\exists A\in\mathcal{A}_1)(w\subseteq A)\ \&\\ & (\forall A_0\in\mathcal{A}_0)(\forall A_1\in\mathcal{A}_1)(h[A_0\times A_1]\neq\mathcal{Z})\}.\end{aligned}$$

If $\mathcal{X}=\mathcal{Y}$ and h is as above, and n is a positive integer then we define

$$\begin{aligned}\kappa^-(h,n)=\min\{|\mathcal{A}| :\ & \mathcal{A}\subseteq\mathcal{P}(\mathcal{X})\ \&\ (\forall w\in[\mathcal{X}]^n)(\exists A\in\mathcal{A})(w\subseteq A)\ \&\\ & (\forall A\in\mathcal{A})(h[A\times A]\neq\mathcal{Z})\ \}.\end{aligned}$$

(2) For $\bar{c}=\langle c_n : n<\omega\rangle\in{}^{\omega}\mathbb{R}$ and $\bar{d}=\langle d_n : n<\omega\rangle\in{}^{\omega}\mathbb{R}$ let $h^{\oplus}(\bar{c},\bar{d}) = \sum\limits_{n<\omega} c_n\cdot d_n$ (defined if the series converges).

We will deal with the following variant of the property $(\circledast^{\rm Da})$.

Definition 3.2. For a function $h : {}^{\omega}\mathbb{R} \times {}^{\omega}\mathbb{R} \longrightarrow \mathbb{R}$ let $(\circledast_h^{\mathrm{Da}})$ mean:

$(\circledast_h^{\mathrm{Da}})$ For each $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ there are functions $g_n^0, g_n^1 : \mathbb{R} \longrightarrow \mathbb{R}$ (for $n < \omega$) such that

$$(\forall x, y \in \mathbb{R})\big(f(x,y) = h(\langle g_n^0(x) : n < \omega\rangle,\ \langle g_n^1(y) : n < \omega\rangle)\big).$$

(So $(\circledast^{\mathrm{Da}})$ is $(\circledast_{h^{\otimes}}^{\mathrm{Da}})$, where $h^{\oplus}$ is as defined in 3.1(2).)

Proposition 3.3. *Assume that a function* $h : {}^{\omega}\mathbb{R} \times {}^{\omega}\mathbb{R} \longrightarrow \mathbb{R}$ *is such that on of the following condition holds:*

(A) $\kappa(h,1) < 2^{\kappa(h,1)} = \mathfrak{c}$, *or*
(B) $\kappa(h,1) \leq \mu < \mathfrak{c}$ *for some regular cardinal* μ, *or*
(C) $\kappa^-(h,2) \leq \mu < \mathfrak{c}$ *for some regular cardinal* μ.

Then $(\circledast_h^{\mathrm{Da}})$ *fails.*

Proof. First let us consider the case of the assumption (A). Let $\mathcal{A}_0, \mathcal{A}_1 \subseteq \mathcal{P}({}^{\omega}\mathbb{R})$ exemplify the minimum in the definition of $\kappa(h,1)$, $\mathcal{A}_\ell = \{A_\xi^\ell : \xi < \kappa(h,1)\}$ (we allow repetitions). Choose a sequence $\langle r_\xi : \xi < \kappa(h,1)\rangle$ of pairwise distinct reals and fix enumerations $\langle s_\varepsilon : \varepsilon < \mathfrak{c}\rangle$ of $\mathbb{R}$ and $\langle \varphi_\varepsilon : \varepsilon < \mathfrak{c}\rangle$ of ${}^{\kappa(h,1)}\kappa(h,1)$. Let $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that

$$(\forall \xi < \mathfrak{c})(\forall \xi < \kappa(h,1))\big(f(s_\varepsilon, r_\xi) \notin h[A_\xi^0 \times A_{\varphi_\varepsilon(\xi)}^1]\big).$$

We claim that the function f witnesses the failure of $(\circledast_h^{\mathrm{Da}})$. So suppose that $g_n^0, g_n^1 : \mathbb{R} \longrightarrow \mathbb{R}$. For $\xi < \kappa(h,1)$ let $\bar{b}_\xi = \langle g_n^1 : n < \omega\rangle \in {}^{\omega}\mathbb{R}$ and let $\varphi(\xi) < \kappa(h,1)$ be such $\bar{b}_\xi \in A_{\varphi(\xi)}^1$. Take $\varepsilon < \mathfrak{c}$ such that $\varphi = \varphi_\varepsilon$ and let $\bar{a}_\varepsilon = \langle g_n^0(\varepsilon) : n < \omega\rangle$. Fix $\xi^* < \kappa(h,1)$ such that $\bar{a}_\varepsilon \in A_{\xi^*}^0$ and note that $h(\bar{a}_\varepsilon, \bar{b}_{\xi^*}) \in h[A_{\xi^*}^0 \times A_{\varphi_\varepsilon(\xi_0)}^1]$, so

$$f(s_\varepsilon, r_{\xi^*}) \neq h(\bar{a}_\varepsilon, \bar{b}_{\xi^*}) = h(\langle g_n^0(s_\varepsilon) : n < \omega\rangle,\ \langle g_n^1(r_{\xi^*}) : n < \omega\rangle).$$

Suppose now that we are in the situation (B). Let $c_0, c_1 : \mu^+ \times \mu^+ \longrightarrow \kappa(h,1)$ be such that for any sets $X_0, X_1 \in [\mu^+]^{\mu^+}$ we have

$$(\forall \zeta_0, \zeta_1 < \kappa(h,1))(\exists \langle \varepsilon_0, \varepsilon_1\rangle \in X_0 \times X_1)(c_0(\varepsilon_0, \varepsilon_1) = \zeta_0 \ \&\ c_1(\varepsilon_0, \varepsilon_1) = \zeta_1)$$

(see e.g. [Sh:g, ch III]). Let $\mathcal{A}_0, \mathcal{A}_1 \subseteq \mathcal{P}({}^{\omega}\mathbb{R})$ exemplify $\kappa(h,1)$, $\mathcal{A}_\ell = \{A_\zeta^\ell : \zeta < \kappa(h,1)\}$ (with possible repetitions). Choose a sequence $\langle r_\varepsilon : \varepsilon < \mu^+\rangle$ of pairwise distinct reals and a function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$(\forall \varepsilon_0, \varepsilon_1 < \mu^+)\big(f(r_{\varepsilon_0}, r_{\varepsilon_1}) \notin h[A_{c_0(\varepsilon_0,\varepsilon_1)}^0 \times A_{c_1(\varepsilon_0,\varepsilon_1)}^1]\big).$$

Now suppose that $g_n^0, g_n^1 : \mathbb{R} \longrightarrow \mathbb{R}$ and let $\bar{a}_\varepsilon^\ell = \langle g_n^\ell(r_\varepsilon) : n < \omega\rangle$. Choose $X_0, X_1 \in [\mu^+]^{\mu^+}$ and $\zeta_0, \zeta_1 < \kappa(h,1)$ such that $\bar{a}_\varepsilon^\ell \in A_{\zeta_\ell}^\ell$ whenever $\varepsilon \in X_\ell$. Take $\varepsilon_\ell \in X_\ell$ (for $\ell < 2$) such that $c_0(\varepsilon_0, \varepsilon_1) = \zeta_0$, $c_1(\varepsilon_0, \varepsilon_1) = \zeta_1$. Then $h(\bar{a}_{\varepsilon_0}^0, \bar{a}_{\varepsilon_1}^1) \in h[A_{c_0(\varepsilon_0,\varepsilon_1)}^0 \times A_{c_2(\varepsilon_0,\varepsilon_1)}^1]$, so $f(r_{\varepsilon_0}, r_{\varepsilon_1}) \neq h(\langle g_n^0(r_{\varepsilon_0}) : n < \omega\rangle,\ \langle g_n^1(r_{\varepsilon_1}) : n < \omega\rangle)$.

Now, suppose that the assumption (C) holds. Let $\{A_\xi : \xi < \kappa^-(h,2)\}$ be a family witnessing the minimum in the definition of $\kappa^-(h,2)$. Take a function $c : \mu^+ \times \mu^+ \longrightarrow \kappa^-(h,2)$ such that for every $X \in [\mu^+]^{\mu^+}$ and $\zeta < \kappa^-(h,2)$ there are $\varepsilon_0 < \varepsilon_1$, both in X, such that $c(\varepsilon_0, \varepsilon_1) = \zeta$ (see e.g. [Sh:g, ch III]). Take a sequence $\langle r_\varepsilon : \varepsilon < \mu^+\rangle$ of distinct reals and define a function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ so that

$$(\forall \varepsilon_0, \varepsilon_1 < \mu^+)(f(r_{\varepsilon_0}, r_{\varepsilon_1}) \notin h[A_{c(\varepsilon_0,\varepsilon_1)} \times A_{c(\varepsilon_0,\varepsilon_1)}]).$$

Like before, suppose that $g^0_n, g^1_n : \mathbb{R} \longrightarrow \mathbb{R}$ and let $\bar{a}^\ell_\varepsilon = \langle g^\ell_n(r_\varepsilon) : n < \omega \rangle$. For each $\varepsilon < \mu^+$ there is $\zeta_\varepsilon \in \kappa^-(h,2)$ such that $\bar{a}^0_\varepsilon, \bar{a}^1_\varepsilon \in A_{\zeta_\varepsilon}$. Take a set $X \in [\mu^+]^{\mu^+}$ and $\zeta^* < \kappa^-(h,2)$ such that $(\forall \varepsilon \in X)(\zeta_\varepsilon = \zeta^*)$. Then choose $\varepsilon_0 < \varepsilon_1$ both in X so that $c(\varepsilon_0, \varepsilon_1) = \zeta^*$. By our choices, $\bar{a}^0_{\varepsilon_0}, \bar{a}^1_{\varepsilon_1} \in A_{c(\varepsilon_0,\varepsilon_1)}$ and $h(\bar{a}^0_{\varepsilon_0}, \bar{a}^1_{\varepsilon_1}) \in A_{c(\varepsilon_0,\varepsilon_1)}$. But this implies that $h(\langle g^0_n(r_{\varepsilon_0}) : n < \omega \rangle,\ \langle g^1_n(r_{\varepsilon_1}) : n < \omega \rangle) \neq f(r_{\varepsilon_0}, r_{\varepsilon_1})$. ☐

Now the phenomenon of [Sh 675, 3.6] is described in a combinatorial way by 3.3, if one notices the following observation.

Proposition 3.4. *Let $h : {}^\omega\mathbb{R} \times {}^\omega\mathbb{R} \longrightarrow {}^\omega\mathbb{R}$ be a function with an absolute definition (with parameters from the ground model). Suppose that $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_1 \rangle$ is a finite support iteration of non-trivial forcing notions. Then for each $0 < n < \omega$*

$$\Vdash_{\mathbb{P}_{\omega_1}} \kappa(h,n) = \kappa^-(h,n) = \aleph_1.$$

Proof. Work in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. For $\alpha < \omega_1$ let $A_\alpha = \mathbf{V}^{\mathbb{P}_\alpha} \cap {}^\omega\mathbb{R}$. Clearly ${}^\omega\mathbb{R} = \bigcup\limits_{\alpha<\omega_1} A_\alpha$ and for each $\alpha, \beta < \omega_1$ we have $h[A_\alpha \times A_\beta] \neq {}^\omega\mathbb{R}$ (remember that the function h has definition with parameters in the ground model; at each limit stage of the iteration Cohen reals are added). ☐

4. Concluding remarks

One can notice some similarities between the property $(\circledast)^{\rm Da}$ and the rectangle problem.

Definition 4.1. (1) Let $\mathcal{R}_2$ be the family of all rectangles in $\mathbb{R} \times \mathbb{R}$, i.e. sets of the form $A \times B$ for some $A, B \subseteq \mathbb{R}$. Let $\mathcal{B}(\mathcal{R}_2)$ be the σ–algebra of subsets of $\mathbb{R} \times \mathbb{R}$ generated by the family $\mathcal{R}_2$ and let $\mathcal{B}_\alpha(\mathcal{R}_2)$ be defined inductively by: $\mathcal{B}_0(\mathcal{R}_2)$ consists of all elements of $\mathcal{R}_2$ and their complements, $\mathcal{B}_\alpha(\mathcal{R}_2) = \bigcup\limits_{\beta<\alpha} \mathcal{B}_\beta(\mathcal{R}_2)$ for limit α, and $\mathcal{B}_{\alpha+1}(\mathcal{R}_2)$ is the collection of all countable unions $\bigcup\limits_{n<\omega} A_n$ such that each A_n is in $\mathcal{B}_\alpha(\mathcal{R}_2)$ and of the complements of such unions. (So $\mathcal{B}(\mathcal{R}_2) = \mathcal{B}_{\omega_1}(\mathcal{R}_2)$.)

(2) Let us introduce the following properties of the family of subsets of $\mathbb{R} \times \mathbb{R}$:

$(\boxdot^{\rm Ku})$ $\quad \mathcal{P}(\mathbb{R} \times \mathbb{R}) = \mathcal{B}(\mathcal{R}_2)$,

$(\boxdot^{\rm Ku}_\alpha)$ $\quad \mathcal{P}(\mathbb{R} \times \mathbb{R}) = \mathcal{B}_\alpha(\mathcal{R}_2)$

Kunen [Ku68, §12] showed the following.

Theorem 4.2. (1) *(See* [Ku68, Thm 12.5]*)* **MA** *implies* $(\boxdot^{\rm Ku}_2)$.

(2) *(See* [Ku68, Thm 12.7]*) If $\mathbb{P}$ is the forcing notion for adding $\aleph_2$ Cohen reals then $\Vdash_{\mathbb{P}} \neg(\boxdot^{\rm Ku})$.*

The relation between $(\circledast^{\rm Da})$ and $(\boxdot^{\rm Ku})$ is still unclear, though the first implies the second.

Proposition 4.3. $\quad (\circledast^{\rm Da}) \quad \Rightarrow \quad (\boxdot^{\rm Ku}_\omega)$

Proof. Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$ and let $f : \mathbb{R} \times \mathbb{R} \longrightarrow 2$ be it characteristic function. Let g^0_n, g^1_n be given by $(\circledast^{\rm Da})$ for the function f. For a rational number q, $n < \omega$ and $\ell < 2$ put

$$A^\ell_{q,n} \stackrel{\rm def}{=} \{x \in \mathbb{R} : g^\ell_n(x) < q\}.$$

It should be clear that the set A can be represented as a Boolean combination of finite depth of rectangles $A^0_{q,n} \times A^1_{q',n}$ (we do not try to safe on counting the quantifiers). $\square$

The following questions arise naturally in this context.

Problem 4.4. (1) Does $(\boxdot^{\mathrm{Ku}}_\omega)$ (or $(\boxdot^{\mathrm{Ku}})$) imply $(\circledast^{\mathrm{Da}})$?

(2) Is it consistent that for some countable limit ordinal α we have $(\boxdot^{\mathrm{Ku}}_{\alpha+1})$ but $(\boxdot^{\mathrm{Ku}}_\alpha)$ fails?

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Department of Mathematics and Computer Science, Boise State University, Boise ID 83725, USA, and Mathematical Institute of Wroclaw University, 50384 Wroclaw, Poland

E-mail address: `roslanow@math.idbsu.edu`

URL: `http://math.idbsu.edu/~roslanow`

Institute of Mathematics, The Hebrew University of Jerusalem, 91904 Jerusalem, Israel, and Department of Mathematics, Rutgers University, New Brunswick, NJ 08854, USA

E-mail address: `shelah@math.huji.ac.il`

URL: `http://www.math.rutgers.edu/~shelah`