

# On $\triangleleft^*$ -maximality

Mirna Džamonja  
School of Mathematics  
University of East Anglia  
Norwich, NR4 7TJ, UK

M.Dzamonja@uea.ac.uk

<http://www.mth.uea.ac.uk/people/md.html>

and

Saharon Shelah  
Mathematics Department  
Hebrew University of Jerusalem  
91904 Givat Ram, Israel

shelah@sunset.huji.ac.il

<http://shelah.logic.at/>

May 23, 2011

## Abstract

This paper investigates a connection between the semantic notion provided by the ordering  $\triangleleft^*$  among theories in model theory and the syntactic (N)SOP<sub>n</sub> hierarchy of Shelah. It introduces two properties which are natural extensions of this hierarchy, called SOP<sub>2</sub> and SOP<sub>1</sub>. It is shown here that SOP<sub>3</sub> implies SOP<sub>2</sub> implies SOP<sub>1</sub>. In [Sh 500] it was shown that SOP<sub>3</sub> implies  $\triangleleft^*$ -maximality and we prove here that  $\triangleleft^*$ -maximality in a model of GCH implies a property called SOP<sub>2</sub>''. It has been subsequently shown by Shelah and Usvyatsov that SOP<sub>2</sub>' and SOP<sub>2</sub> are equivalent, so obtaining an implication between  $\triangleleft^*$ -maximality and SOP<sub>2</sub>. It is not known if SOP<sub>2</sub> and SOP<sub>3</sub> are equivalent.

Together with the known results about the connection between the  $(N)SOP_n$  hierarchy and the existence of universal models in the absence of GCH, the paper provides a step toward the classification of unstable theories without the strict order property.<sup>1</sup>

*Changes from the published version:*

*In the published version of this paper it is claimed that witnesses to being  $SOP_1$  can be chosen to be highly indiscernible, and this is justified by a certain notion of 1-fbti. The definition of this notion (Definition 2.10) has a typo in a crucial place, and in addition Claim 2.11 for  $t = 2$  is incorrect and for  $t = 1$  the proof is incomplete. In this version we clarify these statements and proofs by introducing a new notion of indiscernibility 3-fbti. The corrected statement is that witnesses to being  $SOP_1$  can be chosen to be 3-fbti.*

*That there are inconsistencies in the notions we used in the original paper was first observed by Lynn Scowl (September 2008), Byunghan Kim (May 2009) and Enrique Casanovas and Martin Ziegler (July 2010). Whilst a Ph.D. student at UEA in 2008, Mark Wong also observed some inconsistencies and made partial progress in rectifying them. There is a paper by Kim and Kim (to appear in APAL as of March 2011) which gives a different notion of 1-fbti and shows that witnesses can be chosen with that kind of indiscernibility.*

## 0 Introduction

This paper investigates a connection between the ordering  $\triangleleft^*$  among theories in model theory and the  $(N)SOP_n$  hierarchy of Shelah and as such provides a step toward the classification of unstable theories without the strict order property. The thesis we pursue is that the syntactic property  $SOP_2$  is closely

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<sup>1</sup>This publication is numbered 692 in the list of publications of Saharon Shelah. The authors thank the United States-Israel Binational Science Foundation for a grant supporting this research and the NSF USA for their grant numbered NSF-DMS97-04477. Mirna Džamonja thanks EPSRC for their support through grant number GR/M71121, as well as the Royal Society for their support through grant number SV/ISR/NVB. We would also like to acknowledge the support of the Erdős Research Center in Budapest, during the Workshop on Set-theoretic Topology, July 1999, and support of the Hebrew University of Jerusalem and the Academic Study Group during July 1999. Finally, warm thanks are due to Alex Usvyatsov for his comments and improvements to the manuscript.

Keywords: classification theory, unstable theories, SOP hierarchy, oak property.  
AMS 2000 Classification: 03C45, 03C55.

related to the semantic property of being maximal in the  $\triangleleft^*$ -order. We shall now give the relevant definitions and explain the motivation behind the paper as well as noting our main results. For the purpose of this introductory discussion we shall limit ourselves to countable (complete first order) theories.

The following order among theories was introduced and investigated by Keisler in [Ke].

**Definition 0.1** (1) For any cardinal  $\lambda$ , the Keisler order  $\triangleleft_\lambda$  among theories is defined as follows:  $T_0 \triangleleft_\lambda T_1$  if whenever  $M_l (l < 2)$  is a model of  $T_0, T_1$  respectively and  $\mathcal{D}$  is a regular ultrafilter over  $\lambda$ , then the  $\lambda^+$ -compactness of  $M_1^\lambda/\mathcal{D}$  implies the  $\lambda^+$ -compactness of  $M_0^\lambda/\mathcal{D}$ .

(2) We say  $T_0 \triangleleft T_1$  if for all  $\lambda$  we have  $T_0 \triangleleft_\lambda T_1$ .

The relevance of this order to the project of classifying unstable theories without strict order property lies in the two following theorems of Shelah (note that the second one implies the first).

**Theorem 0.2** (Shelah [Sh c], VI4.3) Any (countable) theory with the strict order property is  $\triangleleft$ -maximal.

As stated in [Sh c], pg xiv, Ch VI of [Sh c] gives a rather complete picture of Keisler’s order and to complete it we should know more about unstable theories without the strict order property. Paper [Sh 500] started a classification of such theories by introducing the hierarchy  $SOP_n$  for  $n \geq 3$  and in particular it is stated there that being maximal in the Keisler order is not a characterisation of theories with the strict order property,

**Theorem 0.3** (Shelah [Sh 500], see also [ShUs 844]) Any theory with  $SOP_3$  is  $\triangleleft$ -maximal.

Details of the proof are given in [ShUs 844]. Precise definitions of properties  $SOP_n$  for  $n \geq 3$  will be repeated below in §2 but for the moment we note that it was proved in [Sh 500] that for  $n \geq 3$

$$\text{strict order property} \implies SOP_{n+1} \implies SOP_n \implies \text{not simple}$$

and that all the implications are irreversible. One may now wonder if having  $SOP_3$  is a characterisation of theories that are maximal in the Keisler order,

giving us a semantic equivalent to the syntactic notion of  $\text{SOP}_3$ . This would be consistent with what is known about this order, see the Introduction to Ch VI of [Sh c]. This question remains open but instead one may attempt to give a characterisation of  $\text{SOP}_3$  or  $\text{SOP}_n$  in terms of some other similarly defined order. This is suggested by [Sh 500] which in fact gives a theorem stronger than 0.3, namely

**Theorem 0.4** (Shelah [Sh 500], see also [ShUs 844]) Any theory with  $\text{SOP}_3$  is  $\triangleleft^*$ -maximal.

The definition of this order will be recalled in §1 where we shall also prove that being  $\triangleleft^*$ -maximal implies being maximal in the Keisler order. Given this fact one may now ask if being  $\triangleleft^*$ -maximal characterises theories with  $\text{SOP}_3$ . To test this claim it is natural to investigate a prototypical example of an  $\text{NSOP}_3$  theory that is still not simple, which is  $T_{\text{feq}}^*$ . In §1 we shall recall the definition of this theory and show that in fact it is not  $\triangleleft^*$ -maximal, as it is consistently strictly below the theory of a dense linear order with no first or last element (all we need for the consistency is a partial GCH assumption).

This naturally leads to the question of the possibility of refining the distinction between simplicity and  $\text{SOP}_3$ . Definition of the  $\text{SOP}_n$  hierarchy from [Sh 500] does not immediately give way to such a refinement as  $\text{SOP}_n$  is roughly speaking, defined in terms of omitting loops of size  $n$ . However in §2 we introduce two properties  $\text{SOP}_2$  and  $\text{SOP}_1$  that in fact satisfy

$$\text{SOP}_3 \implies \text{SOP}_2 \implies \text{SOP}_1 \implies \text{not simple.}$$

We then ask if these properties in any way characterise the maximality in  $\triangleleft^*$ . To this end in §3 we prove that any theory that is  $\triangleleft^*$ -maximal in a model of a sufficient amount of GCH must satisfy a syntactic property  $\text{SOP}_2''$ . Together with a subsequent result of Shelah and Usvyatsov in [ShUs 844] that proved that  $\text{SOP}_2''$  is equivalent to  $\text{SOP}_2$  we hence obtain that  $\triangleleft^*$ -maximality in any model of a sufficiently rich fragment of GCH implies  $\text{SOP}_2$ . (See §3 for the definition of  $\text{SOP}_2''$  and the exact reference from [ShUs 844]). To summarise, our main result, appearing as Corollary 3.9(1) below is

**Theorem 0.5** Suppose that  $T$  is a theory that is  $\triangleleft^*$ -maximal in some universe of set theory in which  $2^\lambda = \lambda^+$  holds for all large enough regular  $\lambda$ . Then  $T$  has  $\text{SOP}_2$ .

Several questions remain open. The main one of course is if  $\text{SOP}_2$  is actually equivalent to  $\triangleleft^*$ -maximality. Recall from the discussion above that we know that  $\text{SOP}_3$  implies  $\triangleleft^*$ -maximality. It is not known if  $\text{SOP}_3$  and  $\text{SOP}_2$  are actually equivalent. We also note that Shelah and Usvyatsov have proved in [ShUs 844] a local version of the implication  $\text{SOP}_2 \implies \triangleleft^*$ -maximality, see §3 for a more detailed discussion.

A burning question also is that we in fact do not know almost anything about the reverse of other implications in the (consistent) diagram

$$\text{SOP}_3 \implies \triangleleft^* \text{-maximality} \implies \text{SOP}_2 \implies \text{SOP}_1 \implies \text{not simple},$$

apart that not all of them may be equivalences, as  $T_{\text{feq}}^*$  is not simple but is  $\text{NSOP}_3$ . In fact [ShUs 844] proves that  $T_{\text{feq}}^*$  is not even  $\text{SOP}_1$ .

Before laying down the organisation of the paper let us also mention the connection of the  $\text{SOP}_n$  hierarchy with another semantic property, which is the possibility of having a universal model at  $\lambda$  in some universe of set theory where a sufficient amount of GCH fails (under GCH every countable first order theory has a universal model in every uncountable cardinal). The connection between this property and unstable theories without the strict order property has been investigated in a series of papers, notably in [KjSh 409] where it is proved that if GCH fails sufficiently then there are no universal dense linear orders. It was proved in [Sh 500] that  $\text{SOP}_4$  is already sufficient for such a negative universality result. The question of universality is interesting also for classes that are not elementary classes of models of a first order theory, for example for classes without amalgamation the most interesting case is the strong limit singular  $\mu$  of cofinality  $\aleph_0$ . In [GrSh 174] it is proved that for such  $\mu$  and  $\lambda < \mu$  a strongly compact cardinal the class of models of any  $L_{\lambda,\mu}$ -theory of cardinality  $< \mu$  admits a universal model of cardinality  $\mu$ . A rather detailed description of what is known about the connection of unstable theories without the strict order property and the universality problem may be found in the introduction to [DjSh 710].

The paper is organised as follows. In the first section we investigate the theory  $T_{\text{feq}}^*$ . This is simply the model completion of the theory of infinitely many parametrised equivalence relations. We show that under a partial GCH assumption, this theory is not maximal with respect to  $\triangleleft_\lambda^*$ , as it is strictly below the theory of a dense linear order. In the second section of the paper we extend Shelah’s NSOP $_n$  hierarchy by introducing two further properties SOP $_1$  and SOP $_2$ , and we show that their names are justified by their position in the hierarchy. Namely  $\text{SOP}_3 \implies \text{SOP}_2 \implies \text{SOP}_1$ . Furthermore, SOP $_1$  theories are not simple. The last section of the paper contains the main result showing that  $\triangleleft^*$ -maximality in a model of a sufficiently rich fragment of GCH implies SOP $_2'$ , and hence SOP $_2$  by Shelah-Usvyatsov.

The following conventions will be used in the paper.

**Convention 0.6** Unless specified otherwise, a “theory” stands for a first order complete theory. An unattributed  $T$  stands for a theory. We use  $\tau(T)$  to denote the vocabulary of a theory  $T$ , and  $\mathcal{L}(T)$  to denote the set of formulae of  $T$ .

By  $\mathfrak{C} = \mathfrak{C}_T$  we denote a  $\bar{\kappa}$ -saturated model of  $T$ , for a large enough regular cardinal  $\bar{\kappa}$  and we assume that any models of  $T$  that we mention are elementary submodels of  $\mathfrak{C}$ .

$\lambda, \mu, \kappa$  stand for infinite cardinals.

## 1 On the order $\triangleleft_\lambda^*$

**Definition 1.1** (1) For (first order complete) theories  $T_0$  and  $T$  we say that

$\bar{\varphi} = \langle \varphi_R(\bar{x}_R) : R \text{ a predicate of } \tau(T_0) \text{ or a function symbol of } \tau(T_0) \text{ or } = \rangle,$

(where we have  $\bar{x}_R = (x_0, \dots, x_{n(R)-1})$ ), *interprets*  $T_0$  in  $T$ , or that  $\bar{\varphi}$  is an *interpretation* of  $T_0$  in  $T$ , or that

$$T \vdash \text{“}\bar{\varphi} \text{ is a model of } T_0\text{”},$$

if each  $\varphi_R(\bar{x}_R) \in \mathcal{L}(T)$ , and for any  $M \models T$ , the model  $M^{[\bar{\varphi}]}$  described below is a model of  $T_0$ . Here,  $N = M^{[\bar{\varphi}]}$  is a  $\tau(T_0)$  model, whose set of elements

is  $\{a : M \models \varphi_=(a, a)\}$  (so  $M^{[\bar{\varphi}]} \subseteq M$ ) and  $R^N = \{\bar{a} : M \models \varphi_R[\bar{a}]\}$  for a predicate  $R$  of  $T_0$ .

For any function symbol  $f$  of  $\tau(T_0)$  we have that  $N \models "f(\bar{a}) = b"$  iff  $M \models \varphi_f(\bar{a}, b)$ , while

$$M \models "\varphi_f(\bar{a}, b) = \varphi_f(\bar{a}, c) \implies b = c"$$

for all  $\bar{a}, b, c$ .

(2) We say that the interpretation  $\bar{\varphi}$  is *trivial* if  $\varphi_R(\bar{x}_R) = R(\bar{x}_R)$  for all  $R \in \tau(T_0)$ , so  $M^{[\bar{\varphi}]} = M \upharpoonright \tau(T_0)$ , for any model  $M$  of  $T$ .

(The last clause in Definition 1.1(1) shows that we can in fact restrict ourselves to vocabularies without function or constant symbols.)

We use the notion of interpretations to define a certain relation among theories. This relation was introduced by S. Shelah in [Sh 500], section §2 and one can see [ShUs 844] for a more detailed exposition. The reason we are interested in this ordering is Shelah's Theorem 0.3 quoted in the Introduction and we shall now start developing methods for the proof of our main result 3.9.

**Definition 1.2** For (complete first order) theories  $T_0, T_1$  we define:

- (1) A triple  $(T, \bar{\varphi}_0, \bar{\varphi}_1)$  is called a  $(T_0, T_1)$ -*superior* iff  $T$  is a theory and  $\bar{\varphi}_l$  is an interpretation of  $T_l$  in  $T$ , for  $l < 2$ .
- (2) For a cardinal  $\kappa$ , a  $(T_0, T_1)$ -superior  $(T, \bar{\varphi}_0, \bar{\varphi}_1)$  is called  $\kappa$ -*relevant* iff  $|T| < \kappa$ .
- (3) For regular cardinals  $\lambda, \mu$  we say  $T_0 \triangleleft_{\lambda, \mu}^* T_1$  if there is a  $\min(\mu, \lambda)$ -relevant  $(T_0, T_1)$ -superior triple  $(T, \bar{\varphi}_0, \bar{\varphi}_1)$  such that in every model  $M$  of  $T$  in which  $M^{[\bar{\varphi}_1]}$  is  $\mu$ -saturated, the model  $M^{[\bar{\varphi}_0]}$  is  $\lambda$ -saturated. If this happens, we call the triple a witness for  $T_0 \triangleleft_{\lambda, \mu}^* T_1$ .
- (4) We say that  $T_0 \triangleleft_{\lambda, \mu}^* T_1$  *over*  $\theta$  if  $\theta \leq \lambda, \theta \leq \mu$  and  $T_0 \triangleleft_{\lambda, \mu}^* T_1$  as witnessed by a  $(T, \bar{\varphi}_0, \bar{\varphi}_1)$  with  $|T| < \theta$ .
- (5) If  $\lambda = \mu$ , we write  $\triangleleft_{\lambda}^*$  in place of  $\triangleleft_{\lambda, \mu}^*$ .

- (6) We say that  $T_1 \triangleleft^* T_2$  iff  $T_1 \triangleleft_\lambda^* T_2$  holds for all large enough regular  $\lambda$ .
- (6)  $T^*$  is  $\triangleleft_\lambda^*$ -maximal iff  $T \triangleleft_\lambda^* T^*$  holds for all  $T$ . The notion of  $\triangleleft^*$ -maximality is defined analogously.
- (7) We say  $T_0 \triangleleft_{\lambda, \neq}^* T_1$  iff  $T_0 \triangleleft_\lambda^* T_1$  but  $\neg(T_1 \triangleleft_\lambda^* T_0)$ .

Although in this paper we do not consider this in its own right, it is natural to define the local versions of the  $\triangleleft^*$ -relation. This is used by Shelah and Usvyatsov in [ShUs 844] to obtain their local converse to the implication  $\triangleleft^*$ -maximality  $\implies$  SOP<sub>2</sub>, see §3 for more discussion on this.

**Definition 1.3** Relations  $\triangleleft_{\lambda, \mu}^{*,1}$  and  $\triangleleft_\lambda^{*,1}$  are the local versions of  $\triangleleft_{\lambda, \mu}^*$  and  $\triangleleft_\lambda^*$  respectively, where by a local version we mean that in the definition of the relations, only types of the form

$$p \subseteq \{\pm\vartheta(x, \bar{a}) : \bar{a} \in {}^{lg(\bar{y})}M\}$$

for some fixed  $\vartheta(x, \bar{y})$  are considered.

**Observation 1.4** (0) If  $T_0 \triangleleft_{\lambda, \mu}^* T_1$  and  $l < 2$ , then there is a witness  $(T, \bar{\varphi}^0, \bar{\varphi}^1)$  such that  $\bar{\varphi}^l$  is trivial, hence  $T_l \subseteq T$ .

(1)  $\triangleleft_\lambda^*$  is a partial order among theories (note that  $T \triangleleft_\lambda^* T$  for every complete  $T$  of size  $< \lambda$ , and that the strict inequality is written as  $T_1 \triangleleft_{\lambda, \neq}^* T_2$ ).

(2) If  $T_0 \triangleleft_{\lambda, \mu}^* T_1$  over  $\theta$  and  $T_1 \triangleleft_{\mu, \kappa}^* T_2$  over  $\theta$ , then  $T_0 \triangleleft_{\lambda, \kappa}^* T_2$  over  $\theta$ .

[Why? (0) Trivial.

(1) Suppose that  $T_l \triangleleft_\lambda^* T_{l+1}$  for  $l < 2$  over  $\theta$ , as exemplified by  $(T^*, \bar{\varphi}_0, \bar{\varphi}_1)$  and  $(T^{**}, \bar{\psi}_1, \bar{\psi}_2)$  respectively. Without loss of generality,  $\bar{\varphi}_1$  is trivial (apply part (0)), so as  $T^*$  is complete we have  $T_1 \subseteq T^*$ . Similarly, without loss of generality,  $\bar{\psi}_1$  is trivial and so, as  $T^{**}$  is complete, we have  $T_1 \subseteq T^{**}$ . As  $T_1$  is complete, without loss of generality,  $T^*$  and  $T^{**}$  agree on the common part of their vocabularies, and hence by Robinson Consistency Criterion,  $T \stackrel{\text{def}}{=} T^* \cup T^{**}$  is consistent. Also  $|T^*| + |T^{**}| < \theta$ , hence  $|T| < \theta$ . Clearly  $T$  interprets  $T_0, T_1, T_2$  by  $\bar{\varphi}_0, \bar{\varphi}_1 = \bar{\psi}_1$  and  $\bar{\psi}_2$  respectively and  $T$  is complete. We now show that the triple  $(T, \bar{\varphi}_0, \bar{\psi}_2)$  is a  $(T_0, T_2)$ -superior which witnesses  $T_0 \triangleleft_\lambda^* T_2$  over  $\theta$ . So suppose that  $M$  is a model of  $T$  in which



$M^{[\bar{\psi}_2]}$  is  $\lambda$ -saturated. As  $(T^{**}, \bar{\psi}_1, \bar{\psi}_2)$  witnesses  $T_1 \triangleleft_{\lambda}^* T_2$ , we can conclude that  $M^{[\bar{\varphi}_1]} = M^{[\bar{\psi}_1]}$  is  $\lambda$ -saturated. We can argue similarly that  $M^{[\bar{\varphi}_0]}$  is  $\lambda$ -saturated.

(2) is proved similarly to (1).]

In this section we consider an example of a theory which is a prototypical example of an NSOP<sub>3</sub> theory that is not simple (see [Sh 457]). It is the model completion of the theory of infinitely many (independent) parametrised equivalence relations, formally defined below. We shall prove that for  $\lambda$  such that  $\lambda = \lambda^{<\lambda}$  and  $2^\lambda = \lambda^+$ , this theory is strictly  $\triangleleft_{\lambda^+}^*$ -below the theory of a dense linear order with no first or last element.

**Definition 1.5** (1)  $T_{\text{feq}}$  is the following theory in  $\{P, Q, E, R, F\}$

- (a) Predicates  $P$  and  $Q$  are unary and disjoint, and  $(\forall x) [P(x) \vee Q(x)]$ ,
- (b)  $E$  is an equivalence relation on  $Q$ ,
- (c)  $R$  is a binary relation on  $Q \times P$  such that

$$[x R z \ \& \ y R z \ \& \ x E y] \implies x = y.$$

(so  $R$  picks for each  $z \in P$  (at most one) representative of any  $E$ -equivalence class).

- (d)  $F$  is a (partial) binary function from  $Q \times P$  to  $Q$ , which satisfies

$$F(x, z) \in Q \ \& \ F(x, z) R z \ \& \ x E F(x, z).$$

(so for  $x \in Q$  and  $z \in P$ , the function  $F$  picks the representative of the  $E$ -equivalence class of  $x$  which is in the relation  $R$  with  $z$ ).

- (2)  $T_{\text{feq}}^+$  is  $T_{\text{feq}}$  with the requirement that  $F$  is total.
- (3) For  $n < \omega$ , we let  $T_{\text{feq}}^n$  be  $T_{\text{feq}}^+$  enriched by the sentence saying that over any  $n$  elements, any (not necessarily complete) quantifier free type consisting of basic (atomic and negations of the atomic) formulae with no direct contradictions, is realised.

**Note 1.6** One may easily check that every model of  $T_{\text{feq}}$  can be extended to a model of  $T_{\text{feq}}^+$  and that  $T_{\text{feq}}^+$  has the amalgamation property and the joint embedding property. This theory also has a model completion, which can be constructed directly, and which we denote by  $T_{\text{feq}}^*$ . It follows that  $T_{\text{feq}}^*$  is a complete theory with infinite models, in which  $F$  is a full function.

**Remark 1.7** Notice that  $T_{\text{feq}}$  has been defined somewhat differently than in [Sh 457, §1], but the difference is non-essential, as the following Claim 1.8 shows that the two theories have the same model completion. This claim also shows the origin of the name “infinitely many independent equivalence relations” for  $T_{\text{feq}}^*$ .

**Claim 1.8** Let  $T$  be the theory defined (in [Sh457, 1]) by

- (a)  $T$  has unary predicates  $P$  and  $Q$  and a three place relation  $E$  written as  $y E_x z$ ,
- (b) the universe of any model of  $T$  is a disjoint union of  $P$  and  $Q$ ,
- (c)  $y E_x z \implies P(x) \ \& \ Q(y), Q(z)$ ,
- (d) for any fixed  $x \in P$  the relation  $E_x$  is an equivalence relation on  $Q$ .

Then  $T_{\text{feq}}^*$  is the model completion of  $T$ .

**Proof of the Claim.** Let  $M$  be a model of  $T_{\text{feq}}$ , we shall extend  $M$  to a model of  $T$  as follows. Each  $E$ -equivalence class  $e = a/E$  gives rise to an equivalence relation  $E_e$  on  $P$  given by:

$$z_1 E_e z_2 \text{ iff } z_1, z_2 \in P \text{ and } F(a, z_1) = F(a, z_2).$$

This definition does not depend on  $a$ , just on  $a/E$ . Let  $P^N$  and  $Q^N$  be  $Q^M$  and  $P^M$  respectively. Define  $y E_x^N z$  iff  $y E_e z$  where  $e = x/E^M$ . Clearly  $N$  is a model of  $T$ .

Now suppose that we have a model  $M$  of  $T$  and we shall extend it to a model  $N$  of  $T_{\text{feq}}$ . Let  $P^N$  and  $Q^N$  be  $Q^M$  and  $P^M$  respectively. Define  $x E^N x'$  iff for every  $y, z$  we have  $y E_x z$  iff  $y E_{x'} z$ . Choose a representative of each  $E$ -equivalence class and for any  $z \in Q^N$  and such a representative  $x$  let

$F(x, z) = x$ . Then for  $x' \in Q^N$  which has not been chosen as a representative of any equivalence class, find  $x$  which has been chosen as its representative and define  $F(x', z) = F(x, z)$  for all  $z \in P^N$ .

This shows that  $T_{\text{feq}}$  and  $T$  are cotheories ([ChKe], 3.5.6(2)). Being the model completion of  $T_{\text{feq}}$ ,  $T_{\text{feq}}^*$  is its cotheory, and hence a cotheory of  $T$ . Hence  $T_{\text{feq}}^*$  is a model companion of  $T$ . In order to prove that it is the model completion of  $T$  it suffices to show that  $T$  has the amalgamation property ([ChKe], 3.5.18) which is easily seen directly. ★<sub>1.8</sub>

**Observation 1.9**  $T_{\text{feq}}^*$  has elimination of quantifiers and for any  $n$ , any model of  $T_{\text{feq}}^*$  is a model of  $T_{\text{feq}}^n$ .

**Notation 1.10**  $T_{\text{ord}}$  stands for the theory of a dense linear order with no first or last element.

The following convention will make the notation used in this section simpler.

**Convention 1.11** Whenever considering  $(T_{\text{ord}}, T_{\text{feq}}^*)$ -superiors  $(T, \bar{\varphi}, \bar{\psi})$  we shall abuse the notation and assume  $\bar{\varphi} = (I, <_0)$  and  $\bar{\psi} = (P, Q, E, R, F)$ . In such a case we may also write  $P^M$  in place of  $P^{M[\bar{\psi}]}$  etc., and we may simply say that  $T$  is a  $(T_{\text{ord}}, T_{\text{feq}}^*)$ -superior.

We intend to prove that for  $\lambda$  satisfying  $\lambda^{<\lambda}$  and  $2^\lambda = \lambda^+$  the theory  $T_{\text{feq}}^*$  is strictly  $\triangleleft_{\lambda^+}^*$ -below  $T_{\text{ord}}$  (Theorem 1.17 below). This will be done by a diagonalisation argument where for a given  $\lambda$ -relevant  $(T_{\text{ord}}, T_{\text{feq}}^*)$ -superior  $T$  we inductively construct a model of  $T$  that is saturated for  $T_{\text{feq}}^*$  but not for  $T_{\text{ord}}$ . Main Claim 1.13 provides one step in the required induction. In Stage A of its proof we use the elimination of quantifiers in  $T_{\text{feq}}^*$  to reduce the situation to  $T_{\text{feq}}$ -types of four prescribed kinds, and then we show that we may in fact work only with three of them. Stage B contains the main point of the proof, which is the construction of a certain tree of models and embeddings. Once this is done in Stage C we use the analysis from Stage A to show that the  $T_{\text{feq}}^*$ -type defined by the union of the embeddings is consistent. In Stage D we take  $N \prec \mathfrak{C}$  of size  $\lambda$  that realises this type and show that

$N$  must omit most of the Dedekind cuts induced by the tree of embeddings, and that most of these cuts are not definable over  $N$ . After an application of an appropriate automorphism of  $\mathfrak{C}$  this finishes the proof of the Main Claim. The proof of the theorem then follows by induction. The cardinal arithmetic assumptions are used in Stage D and in the inductive proof of the theorem.

**Definition 1.12** For a  $\lambda$ -relevant  $(T_{\text{ord}}, T_{\text{feq}}^*)$ -superior  $T$ , the statement

$$*[M, \bar{a}, \bar{b}] = *[M, \bar{a}, \bar{b}, T, \lambda]$$

means:

- (i)  $M$  is a model of  $T$  of size  $\lambda$ ,
- (ii)  $\bar{a} = \langle a_i : i < \lambda \rangle$ ,  $\bar{b} = \langle b_i : i < \lambda \rangle$ , are sequences of elements of  $I^{M^{[\bar{\varphi}]}}$  such that

$$i < j < \lambda \implies a_i <_0 a_j <_0 b_j <_0 b_i,$$

- (iii) there is no  $x \in M^{[\bar{\varphi}]}$  such that for all  $i$  we have  $a_i <_0 x <_0 b_i$ ,
- (iv) the Dedekind cut  $\{x : \bigvee_{i < \lambda} x <_0 a_i\}$  is not definable by any formula of  $\mathcal{L}(M)$  with parameters in  $M$ .

**Main Claim 1.13** Assume  $\lambda^{<\lambda} = \lambda$  and  $(T, \bar{\varphi}, \bar{\psi})$  is a  $\lambda$ -relevant  $(T_{\text{ord}}, T_{\text{feq}}^*)$ -superior. Further assume that  $*[M, \bar{a}, \bar{b}]$  holds, and  $p = p(z)$  is a (consistent)  $T_{\text{feq}}^*$ -type over  $M^{[\bar{\psi}]}$ . Then there is  $N \models T$  with  $M \prec N$ , such that  $p(z)$  is realised in  $N^{[\bar{\psi}]}$  and  $*[N, \bar{a}, \bar{b}]$  holds.

**Proof of the Main Claim.**

**Stage A.** Without loss of generality,  $p$  is complete in the  $T_{\text{feq}}^*$ -language over  $M^{[\bar{\psi}]}$ . (By Convention 1.11, we can consider  $p$  to be a type over  $M$  (rather than  $M^{[\bar{\psi}]}$ ). We shall use this Convention throughout the proof). If  $p$  is realised in  $M$ , our conclusion follows by taking  $N = M$ , so let us assume that this is not the case. Using the elimination of quantifiers for  $T_{\text{feq}}^*$ , we can without loss of generality assume that  $p(z)$  consists of quantifier free formulae with parameters in  $M$ . This means that one of the following four cases must happen:

Case 1. (This will be the main case)  $p(z)$  implies that  $z \in P$  and it determines which elements of  $Q^M$  are  $R$ -connected to  $z$ . Hence for some function  $f : Q^M \rightarrow Q^M$  which respects  $E$ , i.e.

$$a E b \implies f(a) = f(b),$$

and

$$f(a) \in a/E^M;$$

we have

$$p(z) = \{P(z)\} \cup \{b R z : b \in \text{Rang}(f)\}$$

and no  $a \in P^M$  satisfies  $p$ .

Case 1A. Like Case 1, but  $f$  is a partial function and

$$\begin{aligned} p(z) = \{P(z)\} \cup \{f(b) R z : b \in \text{Dom}(f)\} \\ \cup \{\neg(b R z) : (b/E^M \cap \text{Rang}(f)) = \emptyset\}. \end{aligned}$$

(This Case will be reduced to Cases 1-3 in Subclaim 1.15).

Case 2.  $p(z)$  determines that  $z \in Q$  and that it is  $E$ -equivalent to some  $a^* \in Q^M$ , but not equal to any “old” element. Note that in this case if  $b^*$  realises  $p(z)$ , we cannot have  $b^* R c$  for any  $c \in P^M$ , as this would imply  $F(a^*, c) = b^* \in M^{[\bar{\vartheta}]}$  (and we have assumed that  $p$  is not realised in  $M^{[\bar{\vartheta}]}$ ). Hence, the complete  $M$ -information is given by

$$p(z) = \{Q(z)\} \cup \{a^* E z\} \cup \{a \neq z : a \in a^*/E^M\}.$$

Case 3.  $p(z)$  determines that  $z \in Q$ , but it has a different  $E$ -equivalence class than any of the elements of  $Q^M$ . As  $p$  is complete, it must determine for which  $c \in P^M$  we have  $z R c$ , and for which  $c, d \in P^M$  we have  $F(z, c) = F(z, d)$ . Hence, for some  $f : P^M \rightarrow \{\text{yes}, \text{no}\}$  and some  $\mathcal{E}$ , an equivalence relation on  $P^M$  such that  $c \mathcal{E} d \implies f(c) = f(d)$ , we have

$$\begin{aligned} p = \{Q(z)\} \cup \{\neg(a E z) : a \in Q^M\} \cup \{(z R b)^{f(b)} : b \in P^M\} \\ \cup \{(F(z, c) = F(z, d))^{ifc \mathcal{E} d} : c, d \in P^M\}. \end{aligned}$$

In the above description, we have used

**Notation 1.14** For a formula  $\vartheta$  we let  $\vartheta^{\text{yes}} \equiv \vartheta$  and  $\vartheta^{\text{no}} \equiv \neg\vartheta$ .

**Subclaim 1.15** It suffices to deal with Cases 1,2,3, ignoring the Case 1A.

**Proof of the Subclaim.** Suppose that we are in the Case 1A. Let

$$\{d_i/E^M : i < i^* \leq \lambda\}$$

list the  $d/E^M$  for  $d \in Q^M$  such that  $d' \in d/E^M \implies \neg(d' R z) \in p(z)$ . We choose by induction on  $i \leq i^*$  a pair  $(M_i, c_i)$  such that

- (a)  $M_0 = M$ ,  $||M_i|| = \lambda$ ,
- (b)  $\langle M_i : i \leq i^* \rangle$  is an increasing continuous elementary chain,
- (c)  $*[M_i, \bar{a}, \bar{b}]$
- (d)  $c_i \in (d_i/E^{M_{i+1}}) \setminus M_i$ , for  $i < i^*$ .

For  $i$  limit or  $i = 0$ , the choice is trivial. For the situation when  $i$  is a successor, we use Case 2.

Let  $\langle c_i/E^{M_{i^*}} : i \in [i^*, i^{**}] \rangle$  list without repetitions the  $c/E^{M_{i^*}}$  which are disjoint to  $M$ . Note that  $|i^{**}| \leq \lambda$ . Let

$$p^+(z) \stackrel{\text{def}}{=} p(z) \cup \{c_i R z : i < i^{**}\}.$$

Then  $p^+(z)$  is a complete type (for  $M_{i^*}^{[\bar{a}, \bar{b}]}$ ), and  $*[M_{i^*}, \bar{a}, \bar{b}]$  holds by (c). If  $p^+(z)$  is realised in  $M_{i^*}$ , we can let  $N = M_{i^*}$  and we are done. Otherwise,  $p^+(z)$  is not realised in  $M_{i^*}$  and is a type of the form required in Case 1, so we can proceed to deal with it using the assumptions on Case 1.  $\star_{1.15}$

**Stage B.** Let us assume that  $p$  is a type as in one of the Cases 1,2 or 3, which we can do by Subclaim 1.15. We shall define  $\langle M_\alpha : \alpha < \lambda \rangle$ , an  $\prec$ -increasing continuous sequence of elementary submodels of  $M$ , each of size  $< \lambda$ , and with union  $M$ , such that:

- (a) In Case 1, each  $M_\alpha$  is closed under  $f$ ,
- (b) In Case 2,  $a^* \in M_0$ ,
- (c) For every  $\alpha < \lambda$ ,

$$(M_\alpha, \{(a_j, b_j) : j < \lambda\} \cap M_\alpha) \prec (M, \{(a_j, b_j) : j < \lambda\}),$$

Hence, for some club  $C$  of  $\lambda$  consisting of limit ordinals  $\delta$ , we have that for all  $\delta \in C$ ,

$$a_j \in M_\delta \iff b_j \in M_\delta \iff j < \delta, \\ (\forall c \in I^{M_\delta})(\exists j < \delta) [c <_0 a_j \vee b_j <_0 c].$$

Let  $C = \{\delta_i : i < \lambda\}$  be an increasing continuous enumeration.

Now we come to the **main point** of the proof.

By induction on  $i = \text{lg}(\eta) < \lambda$  we shall choose  $\bar{h} = \langle h_\eta : \eta \in {}^{\lambda>}2 \rangle$ , a sequence such that

( $\alpha$ )  $h_\eta$  is an elementary embedding of  $M_{\delta_{\text{lg}(\eta)}}$  into  $\mathfrak{C}_T$ , whose range will be denoted by  $N_\eta$ .

( $\beta$ )  $\nu \triangleleft \eta \implies h_\nu \subseteq h_\eta$ .

( $\gamma$ ) If  $\eta_l \in {}^{\lambda>}2$  for  $l = 0, 1$  and  $\eta_0 \cap \eta_1 = \eta$ , then:

(i)  $N_{\eta_0} \cap N_{\eta_1} = N_\eta$ .

(ii) In addition, if  $a_l \in Q^{N_{\eta_l}}$  for  $l = 0, 1$  and  $a_0 E^{\mathfrak{C}_T} a_1$ , then for some  $a \in Q^{N_\eta}$  we have  $a_l E^{\mathfrak{C}_T} a$  for  $l = 0, 1$ . (Equivalently, if  $a_l \in Q^{N_{\eta_l}}$  and  $\neg(\exists a \in N_\eta)(\wedge_{l < 2} a_l E a)$ , then  $\neg(a_0 E a_1)$ ).

( $\delta$ )  $\models$  “ $h_{\eta \smallfrown \langle 0 \rangle}(b_{\delta_{\text{lg}(\eta)}}) <_0 h_{\eta \smallfrown \langle 1 \rangle}(a_{\delta_{\text{lg}(\eta)}})$ ” (see Convention 1.11 on  $<_0$ ).

Note that the requirement of  $h_\eta$  being elementary and onto  $N_\eta$  in particular implies that

( $\delta'$ ) If for some  $l < 2$  and  $\eta \in {}^{\lambda>}2$  we have  $a \in N_{\eta \smallfrown \langle l \rangle} \setminus N_\eta$  and  $b \in N_\eta$ , then  $a E^{\mathfrak{C}_T} b$  iff  $a = h_{\eta \smallfrown \langle l \rangle}(a')$  for some  $a'$  such that  $a' E^{\mathfrak{C}_T} h_\eta^{-1}(b)$ .

We now describe the inductive choice of  $h_\eta$  for  $\eta \in {}^{\lambda>}2$ , the induction being on  $i = \text{lg}(\eta)$ . Let  $h_\emptyset = \text{id}_{M_0}$ . If  $i$  is a limit ordinal, we just let  $h_\eta \stackrel{\text{def}}{=} \bigcup_{j < \text{lg}(\eta)} h_{\eta \smallfrown j}$ . Hence, the point is to handle the successor case.

Fixing  $i < \lambda$ , let  $\langle \eta_{i,\alpha} : \alpha < \alpha^* \leq \lambda \rangle$  list  ${}^{i+1}2$ , in such a manner that  $\eta_{i,2\alpha} \upharpoonright i = \eta_{i,2\alpha+1} \upharpoonright i$  and  $\eta_{i,2\alpha+l}(i) = l$  for  $l < 2$  (we are using the assumption  $\lambda^{<\lambda} = \lambda$ ). Now we choose  $h_{\eta_{i,2\alpha+l}}$  by induction on  $\alpha$ . Hence, coming to  $\alpha$ , let us denote by  $\eta_l$  the sequence  $\eta_{i,2\alpha+l}$ , and let  $\eta_0 \cap \eta_1 = \eta$  (so  $\eta_0 \upharpoonright i = \eta_1 \upharpoonright i = \eta$ ). Let  $M_{\delta_{i+1}} \setminus M_{\delta_i} = \{d_j^i : j < j_i^*\}$ , so that  $d_0^i = a_{\delta_i}$  and  $d_1^i = b_{\delta_i}$ . We consider the type  $\Gamma$ , which is the union of

(a)

$$\Gamma_0 \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \varphi(x_{j_0}^0, \dots, x_{j_{n-1}}^0; h_\eta(\bar{c})) : n < \omega \ \& \ \bar{c} \subseteq M_{\delta_i} \ \& \ j_0, \dots, j_{n-1} < j_i^* \ \& \ \mathcal{X} \\ M_{\delta_{i+1}} \models \varphi(d_{j_0}^i, \dots, d_{j_{n-1}}^i; \bar{c}) \end{array} \right\},$$

(taking care of one “side” (for  $\eta_0$  or  $\eta_1$ ) of the part ( $\alpha$ ) above)

(b)  $\Gamma_1$ , defined analogously to  $\Gamma_0$ , but with  $x_{j_0}^0, \dots, x_{j_{n-1}}^0$  replaced everywhere by  $x_{j_0}^1, \dots, x_{j_{n-1}}^1$ ,

(taking care of the remaining “side” of ( $\alpha$ ) above, interchanging  $\eta_0$  and  $\eta_1$ )

(c)  $\Gamma_2 = \{(x_0^0, x_1^0)_I \cap (x_0^1, x_1^1)_I = \emptyset\}$ ,

(this says that the above intervals in  $<_0$  are disjoint, which after the right choice of  $h_{\eta_0}(d_j^i) =$  a realisation of  $x_j^0$  or  $h_{\eta_0}(d_j^i) =$  a realisation of  $x_j^1$  ( $j < 2$ ), and similarly for  $h_{\eta_1}$ , will take care of the part ( $\delta$ ) above.)

(d)  $\Gamma_3 = \Gamma_3^0 \cup \Gamma_3^1$ , where for  $l < 2$

$$\Gamma_3^l = \{x_j^l \neq c : j < j_i^*, c \in \cup\{\text{Rang}(h_\rho) : h_\rho \text{ already defined}\}\}.$$

(e)

$$\Gamma_4 = \{x_{j_1}^0 \neq x_{j_2}^1 : j_0, j_1 < j_i^*\}$$

((d)+ (e) are taking care of ( $\gamma$ ) above, part (i)).

(f)

$$\Gamma_5 = \left\{ \begin{array}{l} \neg(x_{j_0}^0 E x_{j_1}^1) : \text{if } j_0, j_1 < j_i^* \\ \text{but there is no } a \in M_{\delta_i} \text{ with } [d_{j_0}^i E a \ \vee \ d_{j_1}^i E a] \end{array} \right\}.$$

(together with  $\Gamma_6$  below, taking care of part ( $\gamma$ )(ii), see below. Note that the type is defined using  $\vee$  rather than  $\wedge$ , but this will turn out to be sufficient.)

(g)  $\Gamma_6 = \Gamma_6^0 \cup \Gamma_6^1$ , where

$$\Gamma_6^l = \left\{ \begin{array}{l} \neg(x_j^l E b) : j < j_i^* \text{ and } b \text{ is an element of the set} \\ \cup \{\text{Rang}(h_\rho) : h_\rho \text{ already defined and } \neg(\exists c \in N_\eta)[b E c]\} \end{array} \right\}.$$



First note that requiring  $\Gamma_5 \cup \Gamma_6$  throughout the construction indeed guarantees that  $(\gamma)(ii)$  can be satisfied. Namely, suppose that the realisations of  $x_{j_0}^0$  and  $x_{j_1}^1$  are  $E$ -equivalent. Then by  $\Gamma_5$  we must have that for some  $l < 2$  and  $a \in M_{\delta_i}$  we have that  $d^i j_l E a$ . By transitivity then the realisation of  $x_{j_{1-l}}^{1-l}$  would have to be  $E$ -equivalent to  $h_\eta(a)$ , which might only be precluded by  $d_{j_{1-l}}^i$  being  $E$  equivalent to some  $c$  such that  $a$  and  $c$  are not  $E$ -equivalent. This cannot happen by  $\Gamma_6$ .

We conclude that, if  $\Gamma$  is consistent, as  $\mathfrak{C}$  is  $\bar{\kappa}$ -saturated, the functions  $h_{\eta_l}$  can be defined. Namely, for a realisation  $\{c_j^l : j < j_i^*, l < 2\}$  of  $\Gamma$ , we can define  $g_l(d_j^i) = c_j^l$ , and then we let  $h_{\eta_0} = g_0$  if  $c_1^0 <_0 c_0^1$ , and  $g_1$  otherwise. We let  $h_{\eta_1} = g_{1-l}$  if  $h_{\eta_0} = g_l$ . This guarantees that  $(\delta)$  above is satisfied.

Let us then show that  $\Gamma$  is consistent. Suppose for contradiction that this is not so, so we can find a finite  $\Gamma' \subseteq \Gamma$  which is inconsistent. Let  $\{j_0, \dots, j_{n-1}\}$  be an increasing enumeration of a set including all  $j < j_i^*$  such that  $x_j^l$  is mentioned in  $\Gamma'$  for some  $l < 2$  and let  $\bar{d} = \langle d_{j_0}^i, \dots, d_{j_{n-1}}^i \rangle$ . Without loss of generality, 0 and 1 appear in the list  $\{j_0, \dots, j_{n-1}\}$  and hence  $j_0 = 0$  while  $j_1 = 1$ . By closing under conjunctions and increasing  $\Gamma'$  (retaining that  $\Gamma' \subseteq \Gamma$  is finite) if necessary, we may assume that for some formula  $\sigma(x_0, \dots, x_{n-1}; \bar{c}) \in \text{tp}(\bar{d}/M_{\delta_i})$ , we have

$$\Gamma' \cap \Gamma_l = \{\vartheta_l(x_{j_0}^l, \dots, x_{j_{n-1}}^l; h_\eta(\bar{c}))\}$$

for  $l < 2$ , where  $\vartheta_l(x_{j_0}^l, \dots, x_{j_{n-1}}^l; h_\eta(\bar{c}))$  is the formula obtained from  $\sigma$  by replacing  $x_k$  by  $x_{j_k}^l$  and  $\bar{c}$  by  $h_\eta(\bar{c})$ .

Let  $\vartheta_2$  be the formula comprising  $\Gamma_2$  and  $\vartheta_3(\bar{x}^l; \bar{c}_3^l) = \bigwedge(\Gamma_3^l \cap \Gamma')$ , while  $\vartheta_4 = \bigwedge(\Gamma_4 \cap \Gamma')$  and  $\vartheta_5 = \bigwedge(\Gamma_5 \cap \Gamma')$ . Let  $\vartheta_3 = \vartheta_3^0 \wedge \vartheta_3^1$  and  $\vartheta = \bigwedge_{k < 6, k \neq 2, 3} \vartheta_k$ . Without loss of generality,  $\vartheta$  includes statements  $x_0^l \neq \dots \neq x_{n-1}^l$  and  $x_0^l <_0 x_1^l$  for  $l < 2$ . We may also assume that  $(x_0^0, x_1^0)_I \cap (x_0^1, x_1^1)_I = \emptyset$  is included in  $\Gamma'$ . The choice of  $n$  may be assumed to have been such that for some  $c_0^l, \dots, c_{n-1}^l$  (for  $l < 2$ ) from  $\bigcup\{\text{Rang}(h_\rho) : h_\rho \text{ already defined}\}$ , we have

$$\Gamma' \cap \Gamma_3 = \{x_{j_m}^l \neq c_k^l : l < 2, k < n, m < n\},$$

and finally that

$$\Gamma' \cap \Gamma_5 = \{\neg(x_{j_k}^0 E x_{j_m}^1) : k, m < n \ \& \ \neg(\exists a \in M_{\delta_i})[d_{j_k}^i E a \ \vee \ d_{j_m}^i E a]\}.$$

By extending  $h_\eta$  to an automorphism  $\hat{h}_\eta$  of  $\mathfrak{C}$ , and applying  $(\hat{h}_\eta)^{-1}$ , we may assume that  $h_\eta = \text{id}_{M_{\delta_i}}$ . We can also assume that no  $c_k^l$  is an element of  $M_{\delta_i}$ , as otherwise the relevant inequalities can be absorbed by  $\sigma$ .

We shall use the following general

**Fact 1.16** Suppose that  $N \prec \mathfrak{C}$  and  $\bar{e} \in {}^m\mathfrak{C}$  is disjoint from  $N$ , while  $N \subseteq A$ .  
Then

$$r(\bar{x}) \stackrel{\text{def}}{=} \text{tp}(\bar{e}, N) \cup \{x_k \neq d : d \in A \setminus N, k < m\} \\ \cup \{\neg(x_k E d) : d \in A \ \& \ (d/E) \cap N = \emptyset, k < m\},$$

is consistent (and in fact, every finite subset of it is realised in  $N$ ).

**Proof of the Fact.** Otherwise, there is a finite  $r'(\bar{x}) \subseteq r(\bar{x})$  which is inconsistent. Without loss of generality,  $r'(\bar{x})$  is the union of sets of the following form (we have a representative type of the sets for each clause)

- $\{\varrho(\bar{x}, \bar{c})\}$  for some  $\bar{c} \subseteq N$  and  $\varrho$  such that  $\models \varrho[\bar{e}, \bar{c}]$ .
- $\{x_k \neq \hat{c}_k^s : k < m\}$  for some  $\hat{c}_0^s, \dots, \hat{c}_{m-1}^s \in A \setminus N$  and  $s < s^* < \omega$ ,
- $\{\neg(x_k E \hat{d}_k^t) : k < m\}$  for some  $\hat{d}_0^t, \dots, \hat{d}_{m-1}^t \in A \setminus N$  and  $t < t^* < \omega$  such that  $(\hat{d}_k^t/E) \cap N = \emptyset$ .

By the elementarity of  $N$ , there is  $\bar{e}' \in N$  with  $N \models \varrho[\bar{e}', \bar{c}]$ . By the choice of the rest of the formulae in  $r'(\bar{x})$ , we see that  $\bar{e}'$  satisfies them as well, which is a contradiction.  $\star_{1.16}$

Let  $\bar{x}^l = (x_0^l, \dots, x_{n-1}^l)$  for  $l < 2$ . Let

$$\Phi_0 \stackrel{\text{def}}{=} \{\varphi(\bar{x}^0) : \varphi(x_{j_0}^0, \dots, x_{j_{n-1}}^0) \in \Gamma' \cap (\Gamma_0 \cup \Gamma_3^0 \cup \Gamma_6^0)\}.$$

Applying the last phrase in the above Fact to  $\Phi_0(\bar{x}^0)$ , the model  $M_{\delta_i}$  and  $\bar{d}$ , we obtain a sequence  $\bar{e}^0 = (e_0^0, \dots, e_{n-1}^0) \in M_{\delta_i}$  which realises  $\Phi_0(\bar{x}^0)$ . For  $k, m$  such that  $\neg(x_{j_k}^0 E x_{j_m}^1) \in \Gamma_5$  we have  $\neg(\exists a \in M_{\delta_i})(a E d_{j_k}^i \vee a E d_{j_m}^i)$ . So

$$\neg(x_k E e_m^0) \wedge \neg(e_m^0 E x_m) \in \text{tp}(\bar{d}/M_{\delta_i}).$$

Let now

$$\begin{aligned} \Phi_1(\bar{x}^1) = & \{ \neg(x_k^1 E e_m^0) \wedge \neg(e_k^0 E x_m^1) : \neg(x_{j_k}^0 E x_{j_m}^1) \in \Gamma_5 \} \\ & \cup \{ x_k^1 \neq e_m^0 : k, m < n \} \cup \{ \varphi(\bar{x}^1) : \varphi(x_{j_0}^1, \dots, x_{j_{n-1}}^1) \in \Gamma' \cap (\Gamma_1 \cup \Gamma_3^1 \cup \Gamma_6^1) \}. \end{aligned}$$

$\Phi_1(\bar{x}^1)$  is a finite set to which we can apply the last phrase of Fact 1.16. In this way we find  $\bar{e}^1 = (e_0^1, \dots, e_{n-1}^1) \in M_{\delta_i}$  realising  $\Phi_1(\bar{x}^1)$ . Now we show that  $\bar{e}^0 \frown \bar{e}^1$  realises  $\Gamma' \setminus \Gamma_2$ . So suppose  $\neg(x_{j_k}^0 E x_{j_m}^1) \in \Gamma' \cap \Gamma_5$ , then  $\neg(x_k^1 E e_m^0) \in \Phi_1$ , hence  $\neg(e_k^1 E e_m^0)$ . Also  $\wedge_{k,m < n} (e_k^1 \neq e_m^0)$  holds, by the choice of  $\Phi_1$ , so  $\bar{e}^0 \frown \bar{e}^1$  realises  $\Gamma' \cap \Gamma_4$ . Now we need to deal with  $\Gamma_2$ . Let

$$\mathcal{D} \stackrel{\text{def}}{=} \{ (\bar{u}^0, \bar{u}^1) : (\bar{u}^0, \bar{u}^1) \text{ satisfies } \vartheta \}.$$

So  $\mathcal{D}$  is first order definable with parameters in  $M_{\delta_i}$  and we have just shown that  $\mathcal{D} \cap M_{\delta_i} \neq \emptyset$ . Also if  $\bar{e}^0 \frown \bar{e}^1 \subseteq M_{\delta_i}$  satisfies  $\vartheta$ , it necessarily realises  $\Gamma' \setminus \Gamma_2$  (as no  $c_k^l \in M_{\delta_i}$ , see the definition). As  $\Gamma'$  is presumed to be inconsistent, no  $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \cap M_{\delta_i}$  can realise  $\Gamma'$ , i.e. satisfy  $\vartheta_2$ , and hence for no  $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \cap M_{\delta_i}$  are the intervals  $(u_0^0, u_1^0)_I$  and  $(u_0^1, u_1^1)_I$  disjoint. Now we claim that if  $(\bar{u}^0, \bar{u}^1) \in M_{\delta_i} \cap \mathcal{D}$ , then  $(u_0^0, u_1^0)_I \cap (a_{\delta_i}, b_{\delta_i})_I \neq \emptyset$ .

Indeed, suppose otherwise, say  $u_1^0 <_0 d_0^i = a_{\delta_i}$ , so  $u_1^0 <_0 x_0 \in \text{tp}(\bar{d}/M_{\delta_i})$ . Arguing as above, with  $\bar{u}^0$  in place of  $\bar{e}^0$  and  $\Phi_1(\bar{x}) \cup \{u_1^0 <_0 x_0\}$  in place of  $\Phi_1(\bar{x}^1)$ , we can find  $\bar{u} \in M_{\delta_i}$  satisfying  $(u_1^0 <_0 u_0)$  and such that  $(\bar{u}^0, \bar{u})$  satisfies  $\vartheta$ . So  $(\bar{u}^0, \bar{u}) \in \mathcal{D} \cap M_{\delta_i}$  and the intervals  $(u_0^0, u_1^0)_I$  and  $(u_0, u_1)_I$  are disjoint, a contradiction. A similar contradiction can be derived from the assumption  $b_{\delta_i} = d_1^{\delta_i} <_0 u_0^0$ . Now note that  $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \implies (\bar{u}^1, \bar{u}^0) \in \mathcal{D}$ , so if  $(\bar{u}^0, \bar{u}^1) \in \mathcal{D} \cap M_{\delta_i}$  we also have  $(u_0^1, u_1^1)_I \cap (a_{\delta_i}, b_{\delta_i})_I \neq \emptyset$ .

By the choice of  $C$ , there is no  $x \in M_{\delta_i}$  with  $d_0^{\delta_i} \leq_0 x \leq_0 d_1^{\delta_i}$ , hence

$$\text{if } (\bar{u}^0, \bar{u}^1) \in M_{\delta_i} \cap \mathcal{D} \text{ and } l < 2, \text{ we have } u_0^l <_0 d_0^{\delta_i} <_0 d_1^{\delta_i} <_0 u_1^l. \quad (*)$$

Let  $\sigma^*(\bar{y})$  be  $\exists \bar{x}((\bar{x}, \bar{y}) \in \mathcal{D})$ . Hence if

$$\varrho_0(z) = (\exists \bar{y})[\sigma^*(\bar{y}) \wedge z \leq_0 y_0]$$

and

$$\varrho_1(z) = (\exists \bar{y})[\sigma^*(\bar{y}) \wedge y_1 \leq_0 z],$$

then

$$M_{\delta_i} \models (\forall z_0, z_1)[\varrho_0(z_0) \wedge \varrho_1(z_1) \implies z_0 <_0 z_1].$$

Of course, this holds in  $\mathfrak{C}$  as well, so

- (a)  $\varrho_0(z)$  defines an initial segment of  $I$ ,
- (b)  $\varrho_1(z)$  defines an end segment of  $I$ ,
- (c) the segments defined by  $\varrho_0(z)$  and  $\varrho_1(z)$  are disjoint,
- (d)  $\varrho_0(M_{\delta_i}) \cup \varrho_1(M_{\delta_i}) = I \cap M_{\delta_i}$ .

[Why? Note that  $(\bar{e}^0, \bar{d}) \in \mathcal{D}$ . Hence  $\sigma^*(\bar{d})$  holds. As for every  $a \in I \cap M_{\delta_i}$  we have  $a <_I a_{\delta_i}$  or  $a >_I b_{\delta_i}$ , the conclusions follows.]

- (e)  $\varrho_0(a_{\delta_i})$  and  $\varrho_1(b_{\delta_i})$  hold.

[Why? Again because  $\sigma^*(\bar{d})$  holds.]

The above arguments show that  $\{x : (\exists \bar{y})[(\sigma^*(\bar{y}) \wedge x <_0 y_0)]\}$  defines the Dedekind cut  $\{x : x <_0 a_{\delta_i}\}$  over  $M_{\delta_i}$ , which contradicts the choice of  $C$  and the fact that the Dedekind cut induced by  $(\bar{a}, \bar{b})$  is not definable (which is a part of the definition of  $*[M, \bar{a}, \bar{b}]$ ).

**Stage C.** Now we have shown that the trees  $\langle N_\eta : \eta \in {}^{\lambda>}2 \rangle, \langle h_\eta : \eta \in {}^{\lambda>}2 \rangle$  of models and embeddings can be defined as required, and we consider

$$p^* \stackrel{\text{def}}{=} \bigcup_{\eta \in {}^{\lambda>}2} h_\eta(p \upharpoonright M_{\delta_{\text{lg}(\eta)}}).$$

We shall show that  $p^*$  is finitely satisfiable, hence satisfiable. Let  $\Gamma' \subseteq p^*$  be finite. Recalling the analysis of  $p$  from Stage A, we consider each of the cases by which  $p$  could have been defined (ignoring Case 1A, as justified by Subclaim 1.15.)

Case 1. In this case there is a function  $f : Q^M \rightarrow Q^M$  respecting  $E$ , with  $aEf(a)$  for all  $a \in Q^M$ , and without loss of generality there are some  $\eta_0, \dots, \eta_{m-1} \in {}^{\lambda>}2$  and  $\{b_i^j : j < m, i < n_j\} \subseteq \text{Rang}(f)$  such that

$$\Gamma' = \{P(z)\} \cup \bigcup_{j < m} \{h_{\eta_j}(b_i^j)Rz : i < n_j\},$$

and for each  $j$  we have  $\{b_i^j : i < n_j\} \subseteq M_{\delta_{\text{lg}(\eta_j)}}$ . Let  $n \stackrel{\text{def}}{=} \sum_{j < m} n_j$ , hence  $\Gamma'$  is a quantifier free (partial) type over  $n$  variables in  $\mathfrak{C}^{[\bar{\psi}]}$ . By Observation 1.9, we only need to check that in  $\Gamma'$  there are no direct contradictions with the axioms of  $T_{\text{feq}}^+$ .

The only possibility for such a contradiction is that for some  $j_0, j_1$  and  $b_i^{j_0}, b_k^{j_1}$  we have

$$h_{\eta_{j_0}}(b_i^{j_0}) \neq h_{\eta_{j_1}}(b_k^{j_1}) \wedge h_{\eta_{j_0}}(b_i^{j_0})E h_{\eta_{j_1}}(b_k^{j_1})$$

and  $h_{j_0}(b_i^{j_0})Rz, h_{j_1}(b_k^{j_1})Rz \in \Gamma'$ . In such a case, any  $c$  which would realise  $\Gamma'$  would contradict part (c) of the definition of  $T_{\text{feq}}^+$ . Suppose that  $b_i^{j_0}, b_k^{j_1}$  and  $\eta_0, \eta_1$  are as above. Let  $\eta_l \stackrel{\text{def}}{=} \eta_{j_l}$  for  $l < 2$  and let  $\eta = \eta_0 \cap \eta_1$ . By part  $(\gamma)$ (ii) in the definition of  $\bar{h}$ , there is  $\hat{b} \in N_\eta$  such that  $h_{\eta_0}(b_i^{j_0})E \hat{b}$  and  $h_{\eta_1}(b_k^{j_1})E \hat{b}$ . For some  $b \in M_{\delta_{\text{lg}(\eta)}}$  we have

$$h_{\eta_0}(b) = h_{\eta_1}(b) = h_\eta(b) = \hat{b},$$

so applying the elementarity of the maps, we obtain

$$b_i^{j_0}E bE b_k^{j_1}.$$

On the other hand, by the definition of  $p^*$  we have  $b_i^{j_0}Rz \in p(z)$  and  $b_k^{j_1}Rz \in p(z)$ . By the demands on  $p$  this implies that  $b_i^{j_0} = b_k^{j_1} \notin M_{\delta_{\text{lg}(\eta)}}$  and  $f(b) = b_i^{j_0}$ , contradicting the fact that  $M_{\delta_{\text{lg}(\eta)}}$  is closed under  $f$ .

Case 2. For a fixed  $a^* \in M_0$  we have

$$p(z) = \{Q(z)\} \cup \{a^*Ez\} \cup \{z \neq c : c \in a^*/E^M\},$$

so without loss of generality

$$\Gamma' = \{Q(z)\} \cup \{a^*Ez\} \cup \{z \neq h_{\eta_j}(c_j) : j < m\}$$

for some  $c_0, \dots, c_{m-1} \in a^*/E^M$  and  $\eta_0, \dots, \eta_{m-1} \in {}^{\lambda > 2}$ , as  $h_\emptyset = \text{id}_{M_0}$ . As  $a^*/E$  is infinite in any model of  $T_{\text{feq}}^*$ , the set  $\Gamma'$  is consistent.

Case 3. We may assume that for some equivalence relation  $\mathcal{E}$  on  $P^M$ , a function  $f$  from  $P^M$  into  $\{\text{yes}, \text{no}\}$ , sequences  $\eta_0, \dots, \eta_{m-1} \in {}^{\lambda > 2}$ , and  $\{a_i^k : i < m, k < n\} \subseteq Q^M$  and  $\{b_i^k, c_i^k, d_i^k : i < m, k < n\} \subseteq P^M$  we have  $e_1 \mathcal{E} e_2 \implies f(e_1) = f(e_2)$  and

$$\begin{aligned} \Gamma'(z) = & \{Q(z)\} \cup \bigcup_{k < n} \{- (h_{\eta_k}(a_i^k)Ez) : i < m\} \cup \bigcup_{k < n} \{(zR h_{\eta_k}(b_i^k))^{f(b_i^k)} : i < m\} \\ & \cup \bigcup_{k < n} \{[F(z, h_{\eta_k}(c_i^k)) = F(z, h_{\eta_k}(d_i^k))]^{\text{if } c_i^k \mathcal{E} d_i^k} : i < m\}. \end{aligned}$$

We could have a contradiction if for some  $k_1, k_2, i_1, i_2$  we had  $f(b_{i_1}^{k_1}) = \text{yes}$ ,  $f(b_{i_2}^{k_2}) = \text{no}$ , but  $h_{\eta_{k_1}}(b_{i_1}^{k_1}) = h_{\eta_{k_2}}(b_{i_2}^{k_2})$ , which cannot happen by  $\gamma(i)$  and the fact that each  $h_\eta$  is 1-1. Another possibility is that for some  $b_{i_1}^{k_1}, b_{i_2}^{k_2}$  we have  $f(b_{i_1}^{k_1}) = f(b_{i_2}^{k_2}) = \text{yes}$ , but  $h_{\eta_{k_1}}(b_{i_1}^{k_1}) \neq h_{\eta_{k_2}}(b_{i_2}^{k_2})$  while  $h_{\eta_{k_1}}(b_{i_1}^{k_1}) E h_{\eta_{k_2}}(b_{i_2}^{k_2})$ . To see that this cannot happen, we distinguish various possibilities for  $b_{i_1}^{k_1}, b_{i_2}^{k_2}$  and use part  $(\gamma)(ii)$  in the choice of  $\bar{h}$ .

Yet another possible source of contradiction could come from a similar consideration involving the last clause in the definition of  $\Gamma'(z)$ , which cannot happen for similar reasons.

**Stage D.** Now we can conclude, using  $\lambda = \lambda^{<\lambda}$  and  $|T| < \lambda$ , that there is a model  $N^* \prec \mathfrak{C}$  of size  $\lambda$  with  $\bigcup_{\eta \in \lambda > 2} N_\eta \subseteq N^*$ , such that  $p^*$  is realised in  $N^*$ . For  $\nu \in \lambda^2$ , let  $h_\nu \stackrel{\text{def}}{=} \bigcup_{i < \lambda} h_{\nu|_i}$ , and let  $N_\nu \stackrel{\text{def}}{=} \text{Rang}(h_\nu)$ , while  $p_\nu \stackrel{\text{def}}{=} h_\nu(p)$ .

For such  $\nu$ , let

$$q_\nu(x) \stackrel{\text{def}}{=} \{I(x)\} \cup \{h_\nu(a_i) <_0 x <_0 h_\nu(b_i) : i < \lambda\}.$$

Hence we have that for  $\nu \neq \rho$  from  $\lambda^2$ , the types  $q_\nu$  and  $q_\rho$  are contradictory, by  $(\delta)$  above. As  $\|N^*\| + |L(T)| \leq \lambda$ , there are only  $\leq \lambda$  definable Dedekind cuts of  $<_0$  over  $N^*$ , and only  $\leq \lambda$  types  $q_\nu$  are realised in  $N^*$ . Hence there is  $\nu \in \lambda^2$  (actually  $2^\lambda$  many) such that the Dedekind cut  $\{x : \forall i < \lambda x <_0 h_\nu(a_i)\}$  is not definable over  $N^*$  and  $q_\nu$  is not realised in  $N^*$ . So  $N^*$  omits  $q_\nu$  and realises  $p_\nu$ . We let  $N = h(N^*)$ , where  $h$  is an automorphism of  $\mathfrak{C}$  extending  $h_\nu^{-1}$ . ★<sub>1.13</sub>

**Theorem 1.17** Assume that  $\lambda^{<\lambda} = \lambda$  and  $2^\lambda = \lambda^+$ .

(1) For any  $\lambda$ -relevant  $(T_{\text{ord}}, T_{\text{feq}}^*)$ -superior  $(T, \bar{\varphi}, \bar{\psi})$ , the theory  $T$  has a model  $M^*$  of cardinality  $\lambda^+$  such that

- (i)  $\bar{\varphi}^{M^*}$  is not  $\lambda^+$ -saturated,
- (ii)  $\bar{\psi}^{M^*}$  is  $\lambda^+$ -saturated.

(2) We can strengthen the claims in (i) and (ii) to include any interpretations of a dense linear order and  $T_{\text{feq}}^*$ -respectively in  $M^*$ , even with parameters.

**Proof.** We prove (1), and (2) is proved similarly. Using the Main Claim 1.13, we can construct  $M^*$  of size  $\lambda^+$ , by an  $\prec$ -increasing continuous sequence  $\langle M_i^* : i \leq \lambda^+ \rangle$ , with  $\|M_i^*\| = \lambda$  satisfying  $*[M_i, \bar{a}, \bar{b}]$  for each  $i \leq \lambda^+$ , and letting  $M^* = M_{\lambda^+}$ . The Main Claim 1.13 is used in the successor steps. To assure that  $M^*$  is  $\lambda^+$ -saturated for  $T_{\text{feq}}^*$ , we use the assumption  $2^\lambda = \lambda^+$ , to do the bookkeeping of all  $T_{\text{feq}}^*$ -types involved.  $\star_{1.17}$

**Conclusion 1.18** Under the assumptions of Theorem 1.17, the theory  $T_{\text{feq}}^*$  is  $\triangleleft_{\lambda^+}^*$  strictly below the theory of a dense linear order with no first or last elements.

[Why? It is below by Shelah’s Theorem 0.4 above.]

We recall that our motivation for studying  $\triangleleft^*$  is to try to characterise  $\text{SOP}_3$  (or  $\text{SOP}_2$ ) theories by the  $\triangleleft^*$ -maximality. As we explained in the Introduction this has origins in the connection between the maximality in the Keisler order and having the strict order property, so we should show here what is the connection between the maximality in Keisler’s order and the maximality in the order  $\triangleleft^*$ . The following Claim 1.19 does that for countable theories.

**Claim 1.19** Suppose that  $T$  is a countable theory that is  $\triangleleft_{\lambda^*}^*$ -maximal. Then it is maximal in the Keisler order  $\triangleleft_\lambda$ .

**Proof of the Claim.** Suppose otherwise and let  $T_1$  be a theory that is  $\triangleleft^*$ -maximal but not maximal in the Keisler order  $\triangleleft_\lambda$ . In particular we have  $T_{\text{ord}} \triangleleft_\lambda^* T_1$ , so there is a  $\lambda$ -relevant  $(T_{\text{ord}}, T_1)$ -superior triple  $(T, \bar{\varphi}_0, \bar{\varphi}_1)$ -exemplifying this. By Observation 1.4(0) we may assume that the interpretation  $\bar{\varphi}_1$  is trivial, so  $T_1 \subseteq T$ - for simplicity.

Since  $T$  is not maximal in the Keisler order  $\triangleleft_\lambda$ , by [Sh c] 4.2 (1) there is a regular ultrafilter  $\mathcal{D}$  which is not good and a model  $M$  of  $T$  such that  $M^\lambda/\mathcal{D}$  is nevertheless  $\lambda^+$ -compact. We can extend  $M$  to a model  $N$  of  $T$  and consider  $N^* = N^\lambda/\mathcal{D}$ . This is a model of  $T$  and by the Extension Theorem for ultrafilters we have that  $[N^*]^{\bar{\varphi}_1} = M^\lambda/\mathcal{D}$ , so it is  $\lambda^+$ -compact and hence it is  $\lambda^+$ -saturated. Again by the Extension Theorem we have that  $[N^*]^{\bar{\varphi}_1} = (N^{\bar{\varphi}_1})^\lambda/\mathcal{D}$ . Now on the one hand we have by the  $\triangleleft_{\lambda^*}^*$ -maximality of

$T_1$  that  $(N^{\varphi_1})^\lambda/\mathcal{D}$  must be  $\lambda^+$ -saturated, hence  $\lambda^+$ -compact. But on the other hand  $(N^{\varphi_1})^\lambda/\mathcal{D}$  cannot be  $\lambda^+$ -compact because  $\mathcal{D}$  is not a good ultrafilter and  $T_{\text{ord}}$  is maximal in the Keisler order, contradicting [Sh c] 4.2 (1).  $\star_{1.19}$

## 2 On the properties $\text{SOP}_2$ and $\text{SOP}_1$

In his paper [Sh 500], S. Shelah investigated a hierarchy of properties unstable theories without strong order property may have. This hierarchy is named  $\text{NSOP}_n$  for  $3 \leq n < \omega$ , where the acronym NSOP stands for “not strong order property”. The negation of  $\text{NSOP}_n$  is denoted by  $\text{SOP}_n$ . It was shown in [Sh 500] that  $\text{SOP}_{n+1} \implies \text{SOP}_n$ , that the implication is strict and that  $\text{SOP}_3$  theories are not simple. In this section we investigate two further notions, which with the intention of furthering the above hierarchy, we name  $\text{SOP}_2$  and  $\text{SOP}_1$ . The original definitions of  $\text{SOP}_n$  for  $n \geq 3$  do not immediately lend themselves to extending the hierarchy for  $n = 1, 2$ , but the properties we define nevertheless fulfill that role. In section 3, a connection between this hierarchy and  $\triangleleft_\lambda^*$ -maximality will be established.

Recall from [Sh 500] one of the equivalent definitions of  $\text{SOP}_3$ . (The equivalence is established in Claim 2.19 of [Sh 500]).

**Definition 2.1** (1) A (complete) theory  $T$  has  $\text{SOP}_3$  iff there is an indiscernible sequence  $\langle \bar{a}_i : i < \omega \rangle$  and formulae  $\varphi(\bar{x}, \bar{y})$ ,  $\psi(\bar{x}, \bar{y})$  such that

- (a)  $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$  is contradictory,
- (b) for some sequence  $\langle \bar{b}_j : j < \omega \rangle$  we have

$$i \leq j \implies \models \varphi[\bar{b}_j, \bar{a}_i] \text{ and } i > j \implies \models \psi[\bar{b}_j, \bar{a}_i],$$

- (c) for  $i < j$ , the set  $\{\varphi(\bar{x}, \bar{a}_j), \psi(\bar{x}, \bar{a}_i)\}$  is contradictory.

(2)  $\text{NSOP}_3$  stands for the negation of  $\text{SOP}_3$ .

**Definition 2.2** (1)  $T$  has  $\text{SOP}_2$  if there is a formula  $\varphi(\bar{x}, \bar{y})$  which exemplifies this property in  $\mathfrak{C} = \mathfrak{C}_T$ , which means:

There are  $\bar{a}_\eta \in \mathfrak{C}$  for  $\eta \in \omega^{>2}$  such that



- (a) for every  $\rho \in {}^\omega 2$ , the set  $\{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright n}) : n < \omega\}$  is consistent,
  - (b) if  $\eta, \nu \in {}^{\omega > 2}$  are incomparable,  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$  is inconsistent.
- (2)  $T$  has  $\text{SOP}_1$  if there is a formula  $\varphi(\bar{x}, \bar{y})$  which exemplifies this in  $\mathfrak{C}$ , which means:

There are  $\bar{a}_\eta \in \mathfrak{C}$ , for  $\eta \in {}^{\omega > 2}$  such that:

- (a) for  $\rho \in {}^\omega 2$  the set  $\{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright n}) : n < \omega\}$  is consistent.
  - (b) if  $\nu \frown \langle 0 \rangle \leq \eta \in {}^{\omega > 2}$ , then  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu \frown \langle 1 \rangle})\}$  is inconsistent.
- (3)  $\text{NSOP}_2$  and  $\text{NSOP}_1$  are the negations of  $\text{SOP}_2$  and  $\text{SOP}_1$  respectively.

The following Claim establishes the relative position of the properties introduced in Definition 2.2 within the (N)SOP hierarchy.

**Claim 2.3** For any complete first order theory  $T$ , we have

$$\text{SOP}_3 \implies \text{SOP}_2 \implies \text{SOP}_1.$$

**Proof of the Claim.** Suppose that  $T$  is  $\text{SOP}_3$ , as exemplified by  $\langle \bar{a}_i : i < \omega \rangle$ ,  $\langle \bar{b}_j : j < \omega \rangle$  and formulae  $\varphi(\bar{x}, \bar{y})$  and  $\psi(\bar{x}, \bar{y})$  (see Definition 2.1), and we shall show that  $T$  satisfies  $\text{SOP}_2$ . We define

$$\vartheta(\bar{x}, \bar{y}^0 \frown \bar{y}^1) \equiv \varphi(\bar{x}, \bar{y}^0) \wedge \psi(\bar{x}, \bar{y}^1), \text{ where } \text{lg}(\bar{y}^0) = \text{lg}(\bar{y}^1).$$

Let us first prove the consistency of

$$\Gamma \stackrel{\text{def}}{=} T \cup \{ \neg(\exists \bar{x}) [\vartheta(\bar{x}, \bar{y}_\eta) \wedge \vartheta(\bar{x}, \bar{y}_\nu)] : \eta \perp \nu \text{ in } {}^{\omega > 2} \} \cup \bigcup_{n < \omega} \{ (\exists \bar{x}) [\bigwedge_{k \leq n} \vartheta(\bar{x}, \bar{y}_{\eta \upharpoonright k})] : \eta \in {}^{n 2} \}.$$

Suppose for contradiction that  $\Gamma$  is not consistent, then for some  $n < \omega$ , the following set is inconsistent:

$$\Gamma' \stackrel{\text{def}}{=} T \cup \{ \neg(\exists \bar{x}) [\vartheta(\bar{x}, \bar{y}_\eta) \wedge \vartheta(\bar{x}, \bar{y}_\nu)] : \eta, \nu \text{ incomparable in } {}^{n \geq 2} \} \cup \{ (\exists \bar{x}) [\bigwedge_{k \leq n} \vartheta(\bar{x}, \bar{y}_{\eta \upharpoonright k})] : \eta \in {}^{n 2} \}.$$

Fix such  $n$ . We pick ordinals  $\alpha_\eta, \beta_\eta < \omega$  for  $\eta \in {}^{n \geq 2}$  so that

$$(i) \nu \triangleleft \eta \implies \alpha_\nu < \alpha_\eta < \alpha_\eta + 1 < \beta_\eta < \beta_\nu,$$

$$(ii) \beta_{\eta \smallfrown \langle 0 \rangle} < \alpha_{\eta \smallfrown \langle 1 \rangle}.$$

For  $\eta \in {}^{n \geq 2}$  let  $\bar{a}_\eta^* \stackrel{\text{def}}{=} \bar{a}_{\alpha_\eta} \smallfrown \bar{a}_{\beta_\eta}$ . We show that  $\mathfrak{C}$  and  $\{\bar{a}_\eta^* : \eta \in {}^{n \geq 2}\}$  exemplify that  $\Gamma'$  is consistent. So, if  $\eta \in {}^{n \geq 2}$  then we have  $\bigwedge_{k \leq n} \vartheta[\bar{b}_{\alpha_\eta+1}, \bar{a}_{\eta \upharpoonright k}^*]$  as for every  $k \leq n$  we have  $\alpha_{\eta \upharpoonright k} < \alpha_\eta + 1$ , so  $\varphi[\bar{b}_{\alpha_\eta+1}, \bar{a}_{\alpha_{\eta \upharpoonright k}}]$  holds, but also for all  $k \leq n$ , as  $\eta \upharpoonright k \trianglelefteq \eta$ , we have  $\beta_{\eta \upharpoonright k} > \alpha_\eta + 1$ , so  $\psi[\bar{b}_{\alpha_\eta+1}, \bar{a}_{\beta_{\eta \upharpoonright k}}]$  holds. Hence  $(\exists \bar{x})[\bigwedge_{k \leq n} \vartheta(\bar{x}, \bar{a}_{\eta \upharpoonright k}^*)]$ . On the other hand, if  $\nu \smallfrown \langle l \rangle \trianglelefteq \eta_l$  for  $l < 2$ , then  $\{\vartheta(\bar{x}, \bar{a}_{\eta_0}^*), \vartheta(\bar{x}, \bar{a}_{\eta_1}^*)\}$  is contradictory as the conjunction implies  $\psi(\bar{x}, \bar{a}_{\beta_{\eta_0}}) \wedge \varphi(\bar{x}, \bar{a}_{\alpha_{\eta_1}})$ , which is contradictory by  $\beta_{\eta_0} < \alpha_{\eta_1}$  and (c) of Definition 2.1. This shows that  $\Gamma'$  is consistent, hence we have also shown that  $\Gamma$  is consistent.

Having shown that  $\Gamma$  is consistent, we can find witnesses  $\{\bar{a}_\eta^* : \eta \in {}^{\omega > 2}\}$  in  $\mathfrak{C}$  realising  $\Gamma$ . Now we just need to show that  $\{\vartheta(\bar{x}, \bar{a}_{\eta \upharpoonright n}^*) : n < \omega\}$  is consistent for every  $\eta \in {}^\omega 2$ . This follows by the compactness theorem and the definition of  $\Gamma$ . Hence we have shown that  $\text{SOP}_3 \implies \text{SOP}_2$ .

The second part of the claim is obvious (and the witnesses for  $\text{SOP}_2$  can be used for  $\text{SOP}_1$  as well).  $\star_{2.3}$

**Question 2.4** Are the implications from Claim 2.3 reversible?

**Claim 2.5** If  $T$  satisfies  $\text{SOP}_1$ , then  $T$  is not simple. In fact, if  $\varphi(\bar{x}, \bar{y})$  exemplifies  $\text{SOP}_1$  of  $T$ , then the same formula exemplifies that  $T$  has the tree property.

**Proof of the Claim.** Let  $\varphi(\bar{x}, \bar{y})$  and  $\{\bar{a}_\eta : \eta \in {}^{\omega > 2}\}$  exemplify  $\text{SOP}_1$ . Then

$$\Gamma_\eta \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{a}_{\eta \smallfrown \langle 0 \rangle_n \smallfrown \langle 1 \rangle}) : n < \omega\}$$

for  $\eta \in {}^{\omega > 2}$  consists of pairwise contradictory formulae. (Here  $\langle 0 \rangle_n$  denotes a sequence consisting of  $n$  zeroes.) For  $n < \omega$  and  $\nu \in {}^{n\omega}$  let

$$\rho_\nu \stackrel{\text{def}}{=} \langle 0 \rangle_{\nu \langle 0 \rangle + 1} \smallfrown \langle 1 \rangle \smallfrown \langle 0 \rangle_{\nu \langle 1 \rangle + 1} \cdots \smallfrown \langle 0 \rangle_{\nu \langle n-1 \rangle + 1} \smallfrown \langle 1 \rangle,$$

so  $\rho_\nu \in {}^{\omega > 2}$  and  $\nu \trianglelefteq \eta \implies \rho_\nu \trianglelefteq \rho_\eta$ . For  $\nu \in {}^{n\omega}$  let  $\bar{b}_\nu = \bar{a}_{\rho_\nu}$ . We observe first that  $\{\varphi(\bar{x}, \bar{b}_{\nu \smallfrown \langle k \rangle}) : k < \omega\}$  is a set of pairwise contradictory formulae,

for  $\nu \in {}^n\omega$ ; namely, if  $k_0 \neq k_1$ , then  $\varphi(\bar{x}, \bar{b}_{\nu \smallfrown \langle k_l \rangle})$  for  $l < 2$  are two different elements of  $\Gamma_{\rho_\nu}$ . On the other hand,  $\{\varphi(\bar{x}, \bar{b}_{\nu \smallfrown n}) : n < \omega\}$  is consistent for every  $\nu \in {}^\omega\omega$ . Hence  $\varphi(\bar{x}, \bar{y})$  and  $\{\bar{b}_\nu : \nu \in {}^{\omega>} \omega\}$  exemplify that  $T$  has the tree property, and so  $T$  is not simple. ★<sub>2.5</sub>

This ends the discussion of the properties of  $\text{SOP}_1$  and  $\text{SOP}_2$  that are directly relevant to the main thesis of the paper—the reader only interested in the connection with the order  $\triangleleft^*$  can now turn directly to §3. The rest of this section however contains some further syntactic developments of these properties which are of interest if one wishes to understand the type theory induced by them. The indiscernibility results we have here were recently used by Shelah and Usvyatsov [ShUs 844] to define a rank function on  $\text{NSOP}_1$  theories (see Theorem 2.22)).

The definition of when a theory has  $\text{SOP}_1$  can be made in another equivalent fashion.

**Definition 2.6** Let  $\varphi(\bar{x}, \bar{y})$  be a formula of  $\mathcal{L}(T)$ . We say  $\varphi(\bar{x}, \bar{y})$  has  $\text{SOP}'_1$  iff there is  $\langle \bar{a}_\eta : \eta \in {}^{\omega>} 2 \rangle$  in  $\mathfrak{C}_T$  such that

- (a)  $\{\varphi(\bar{x}, \bar{a}_{\rho \smallfrown n})^{\rho(n)} : n < \omega\}$  is consistent for every  $\rho \in {}^\omega 2$ , where we use the notation

$$\varphi^l = \begin{cases} \varphi & \text{if } l = 1, \\ \neg\varphi & \text{if } l = 0 \end{cases}$$

for  $l < 2$ .

- (b) If  $\nu \smallfrown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega>} 2$ , then  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$  is inconsistent.

We say that  $T$  has property  $\text{SOP}'_1$  iff some formula of  $\mathcal{L}(T)$  has it.

**Claim 2.7** (1) If  $\varphi(\bar{x}, \bar{y})$  exemplifies  $\text{SOP}_1$  of  $T$  then  $\varphi(\bar{x}, \bar{y})$  (hence  $T$ ) has property  $\text{SOP}'_1$ .

- (2) If  $T$  has property  $\text{SOP}'_1$  then  $T$  has  $\text{SOP}_1$ .

**Proof of the Claim.** (1) Let  $\{\bar{a}_\eta : \eta \in {}^{\omega>} 2\}$  and  $\varphi(\bar{x}, \bar{y})$  exemplify that  $T$  has  $\text{SOP}_1$ . For  $\eta \in {}^{\omega>} 2$  we define  $\bar{b}_\eta \stackrel{\text{def}}{=} \bar{a}_{\eta \smallfrown \langle 1 \rangle}$ . We shall show that  $\varphi(\bar{x}, \bar{y})$  and  $\{\bar{b}_\eta : \eta \in {}^{\omega>} 2\}$  exemplify  $\text{SOP}'_1$ .

Given  $\hat{\eta} \in {}^\omega 2$ . Let  $\bar{c}$  exemplify that item (a) from Definition 2.2(2) holds for  $\hat{\eta}$ . Given  $n < \omega$ , we consider  $\varphi[\bar{c}, \bar{b}_{\hat{\eta} \upharpoonright n}]^{\hat{\eta}(n)}$ . If  $\hat{\eta}(n) = 1$ , then, as  $\bar{b}_{\hat{\eta} \upharpoonright n} = \bar{a}_{\hat{\eta} \upharpoonright n \smallfrown \langle 1 \rangle} = \bar{a}_{\hat{\eta} \upharpoonright (n+1)}$ , we have that  $\varphi[\bar{c}, \bar{b}_{\hat{\eta} \upharpoonright n}]^{\hat{\eta}(n)} = \varphi[\bar{c}, \bar{a}_{\hat{\eta} \upharpoonright (n+1)}]$  holds. If  $\hat{\eta}(n) = 0$ , then

$$(\hat{\eta} \upharpoonright n) \smallfrown \langle 0 \rangle = \hat{\eta} \upharpoonright (n+1).$$

As  $\varphi[\bar{c}, \bar{a}_{\hat{\eta} \upharpoonright (n+1)}]$  holds, by (b) of Definition 2.2(2), we have that  $\varphi[\bar{c}, \bar{a}_{\hat{\eta} \upharpoonright n \smallfrown \langle 1 \rangle}]$  cannot hold, showing again that,  $\varphi[\bar{c}, \bar{b}_{\hat{\eta} \upharpoonright n}]^{\hat{\eta}(n)} = \neg \varphi[\bar{c}, \bar{a}_{\hat{\eta} \upharpoonright n \smallfrown \langle 1 \rangle}]$  holds. This shows that  $\{\varphi(\bar{x}, \bar{b}_{\hat{\eta} \upharpoonright n})^{\hat{\eta}(n)} : n < \omega\}$  is consistent, as exemplified by  $\bar{c}$ .

Suppose  $\nu \smallfrown \langle 0 \rangle \sqsubseteq \eta \in {}^{\omega > 2}$  and that  $\varphi[\bar{d}, \bar{b}_\eta] \wedge \varphi[\bar{d}, \bar{b}_\nu]$  holds. So both  $\varphi[\bar{d}, \bar{a}_{\eta \smallfrown \langle 1 \rangle}]$  and  $\varphi[\bar{d}, \bar{a}_{\nu \smallfrown \langle 1 \rangle}]$  hold. On the other hand, as  $\nu \smallfrown \langle 0 \rangle \sqsubseteq \eta$ , clearly  $\nu \smallfrown \langle 0 \rangle \sqsubseteq \eta \smallfrown \langle 1 \rangle$ , and so (b) of Definition 2.2(2) implies that  $\{\varphi(\bar{x}, \bar{a}_{\eta \smallfrown \langle 1 \rangle}), \varphi(\bar{x}, \bar{a}_{\nu \smallfrown \langle 1 \rangle})\}$  is contradictory, a contradiction. Hence the set  $\{\varphi(\bar{x}, \bar{b}_\eta), \varphi(\bar{x}, \bar{b}_\nu)\}$  is contradictory

(2) Define first for  $\eta \in {}^{\omega \geq 2}$  an element  $\rho_\eta \in {}^{\omega \geq 2}$  by letting

$$\rho_\eta(3k) = \eta(k),$$

$$\rho_\eta(3k+1) = 0,$$

$$\rho_\eta(3k+2) = 1,$$

and if  $\text{lg}(\eta) = m < \omega$ , then  $\text{lg}(\rho_\eta) = 3m$ . Note that for  $\eta \in {}^\omega 2$  and  $k < \omega$  we have  $\rho_{\eta \upharpoonright k} = \rho_\eta \upharpoonright (3k)$ .

Let  $\varphi(\bar{x}, \bar{y})$  and  $\{\bar{a}_\eta : \eta \in {}^{\omega > 2}\}$  exemplify property  $\text{SOP}'_1$ . We pick  $c_0 \neq c_1$  and define for  $\eta \in {}^{\omega > 2}$

$$\bar{b}_{\eta \smallfrown \langle 1 \rangle} \stackrel{\text{def}}{=} \bar{a}_{\rho_\eta} \smallfrown \bar{a}_{\rho_\eta \smallfrown \langle 1 \rangle} \smallfrown \langle c_0, c_1 \rangle,$$

$$\bar{b}_{\eta \smallfrown \langle 0 \rangle} \stackrel{\text{def}}{=} \bar{a}_{\rho_\eta \smallfrown \langle 0, 0 \rangle} \smallfrown \bar{a}_{\rho_\eta} \smallfrown \langle c_0, c_1 \rangle,$$

$$\bar{b}_{\langle \rangle} \stackrel{\text{def}}{=} \langle c_0 \rangle_{2n+2},$$

where  $\langle c \rangle_k$  stands for the sequence of  $k$  entries, each of which is  $c$ , and  $n = \text{lg}(\bar{y})$  in  $\varphi(\bar{x}, \bar{y})$ . We define

$$\begin{aligned} \psi(\bar{x}, \bar{z}) \equiv \psi(\bar{x}, \bar{z}^0 \smallfrown \bar{z}^1 \smallfrown w^0 \smallfrown w^1) \equiv \\ [w^0 = w^1] \vee [\varphi(\bar{x}, \bar{z}^0) \wedge \neg \varphi(\bar{x}, \bar{z}^1)], \end{aligned}$$

where  $\bar{z} = \bar{z}^0 \smallfrown \bar{z}^1 \smallfrown \langle w^0, w^1 \rangle$  and  $lg(\bar{z}^0) = lg(\bar{z}^1) = lg(\bar{y})$ . We now claim that  $\psi(\bar{x}, \bar{z})$  and  $\{\bar{b}_\eta : \eta \in {}^\omega > 2\}$  exemplify that  $SOP_1$  holds for  $T$ . Before we start checking this, note that for  $\eta \in {}^\omega > 2$  we have:

- <sub>1</sub>  $\psi(\bar{d}, \bar{b}_{\langle \rangle})$  holds for any  $\bar{d}$ ,
- <sub>2</sub>  $\psi(\bar{d}, \bar{b}_{\eta \smallfrown \langle 0 \rangle})$  holds iff  $\varphi(\bar{d}, \bar{a}_{\rho_\eta \smallfrown \langle 0, 0 \rangle}) \wedge \neg\varphi(\bar{d}, \bar{a}_{\rho_\eta})$  holds,
- <sub>3</sub>  $\psi(\bar{d}, \bar{b}_{\eta \smallfrown \langle 1 \rangle})$  holds iff  $\neg\varphi(\bar{d}, \bar{a}_{\rho_\eta \smallfrown \langle 1 \rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\rho_\eta})$  holds.

Let us verify 2.2(2)(a), so let  $\eta \in {}^\omega 2$ . Pick  $\bar{c}$  such that  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright n}]^{\rho_\eta(n)}$  holds for all  $n < \omega$ . We claim that

$$\psi[\bar{c}, \bar{b}_{\eta \upharpoonright n}] \text{ holds for all } n. \quad (*)$$

The proof is by a case analysis of  $n$ .

If  $\underline{n = 0}$ , this is trivially true. If  $\underline{n = k + 1}$  and  $\underline{\eta(k) = 0}$ , then we need to verify that  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k \smallfrown \langle 0, 0 \rangle}]$  holds and  $\neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k}]$  holds. We have

$$\rho_\eta \upharpoonright k \smallfrown \langle 0, 0 \rangle = \rho_\eta \upharpoonright (3k + 2),$$

and  $\rho_\eta(3k + 2) = 1$ . Hence  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k \smallfrown \langle 0, 0 \rangle}]$  holds by the choice of  $\bar{c}$ . On the other hand, we have  $\rho_\eta \upharpoonright k = \rho_\eta \upharpoonright (3k)$ , and  $\rho_\eta(3k) = \eta(k) = 0$ , hence  $\neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k}]$  holds.

If  $\underline{n = k + 1}$  and  $\underline{\eta(k) = 1}$ , then we need to verify that  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k}]$  holds while  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k) \smallfrown \langle 1 \rangle}]$  does not. As  $\rho_\eta \upharpoonright k = \rho_\eta \upharpoonright (3k)$ , and  $\rho_\eta(3k) = \eta(k) = 1$ , we have that  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k}]$  holds. Note that  $\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k+2)}]$  holds as  $\rho_\eta(3k+2) = 1$ . We also have  $(\rho_\eta \upharpoonright (3k + 1)) \smallfrown \langle 0 \rangle \leq \rho_\eta \upharpoonright (3k + 2)$ . Hence  $\neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k+1)}]$  by part (b) in Definition 2.6. But

$$\neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k+1)}] \equiv \neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright (3k) \smallfrown \langle 1 \rangle}] \equiv \neg\varphi[\bar{c}, \bar{a}_{\rho_\eta \upharpoonright k \smallfrown \langle 1 \rangle}]$$

holds, so we are done proving (\*).

Let us now verify 2.2(2)(b). So suppose  $\nu \smallfrown \langle 0 \rangle \leq \eta$  and consider  $\{\psi(\bar{x}, \bar{b}_{\nu \smallfrown \langle 1 \rangle}), \psi(\bar{x}, \bar{b}_\eta)\}$ . Let  $\sigma$  and  $l$  be such that  $\eta = \sigma \smallfrown \langle l \rangle$ .

Case 1.  $\nu = \sigma$ .

Hence  $l = 0$ . So  $\psi(\bar{x}, \bar{b}_\eta) \implies \neg\varphi(\bar{x}, \bar{a}_{\rho_\nu})$  and  $\psi(\bar{x}, \bar{b}_{\nu \smallfrown \langle 1 \rangle}) \implies \varphi(\bar{x}, \bar{a}_{\rho_\nu})$ , by •<sub>2</sub> and •<sub>3</sub> respectively, showing that  $\{\psi(\bar{x}, \bar{b}_\eta), \psi(\bar{x}, \bar{b}_{\nu \smallfrown \langle 1 \rangle})\}$  is inconsistent.

Case 2.  $\nu \triangleleft \sigma$  and  $l = 0$ .

Hence  $\nu \frown \langle 0 \rangle \trianglelefteq \sigma$ . Clearly  $\rho_\nu \frown \langle 0 \rangle \trianglelefteq \rho_\sigma \frown \langle 0, 0 \rangle$ , as

$$\rho_\sigma(\text{lg}(\rho_\nu)) = \sigma(\text{lg}(\nu)) = 0.$$

We have  $\psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle}) \implies \varphi(\bar{x}, \bar{a}_{\rho_\nu})$  by  $\bullet_3$  and  $\psi(\bar{x}, \bar{b}_\eta) = \psi(\bar{x}, \bar{b}_\sigma \frown \langle 0 \rangle)$  implies  $\varphi(\bar{x}, \bar{a}_{\rho_\sigma \frown \langle 0, 0 \rangle})$  by  $\bullet_2$ , while the two formulae being implied are contradictory, by (b) in the definition of  $\text{SOP}'_1$ .

Case 3.  $\nu \triangleleft \sigma$  and  $l = 1$ .

Observe that  $\psi(\bar{x}, \bar{b}_\eta) \implies \varphi(\bar{x}, \bar{a}_{\rho_\sigma})$  by  $\bullet_3$  and  $\psi(\bar{x}, \bar{b}_{\nu \frown \langle 1 \rangle}) \implies \varphi(\bar{x}, \bar{a}_{\rho_\nu})$ . As above, using  $\nu \frown \langle 0 \rangle \trianglelefteq \sigma$ , we show that the set  $\{\varphi(\bar{x}, \bar{a}_{\rho_\nu}), \varphi(\bar{x}, \bar{a}_{\rho_\sigma})\}$  is inconsistent.  $\star_{2.7}$

**Conclusion 2.8**  $T$  has  $\text{SOP}_1$  iff  $T$  has property  $\text{SOP}'_1$  from Claim 2.7.

**Question 2.9** Is the conclusion of 2.8 true when the theory  $T$  is replaced by a formula  $\varphi$ ?

*Start changes*

It turns out that witnesses to being  $\text{SOP}_1$  can be chosen to be highly indiscernible.

**Definition 2.10** (1) Given an ordinal  $\alpha$  and sequences  $\bar{\eta}_l = \langle \eta_0^l, \eta_1^l, \dots, \eta_{n_l}^l \rangle$  for  $l = 0, 1$  of members of  ${}^{\alpha > 2}$ , we say that  $\bar{\eta}_0 \approx_1 \bar{\eta}_1$  iff

(a)  $n_0 = n_1$ ,

(b) the truth values of

- $\eta_k^l = \langle \rangle$ ,
- $\eta_{k_1}^l \cap \eta_{k_2}^l \trianglelefteq \eta_{k_3}^l \cap \eta_{k_4}^l$

do not depend on  $l$ ,

(c)  $\eta_{k_1}^l \not\trianglelefteq \eta_{k_2}^l \implies \eta_{k_1}^0 (\text{lg}(\eta_{k_1}^0 \cap \eta_{k_2}^0)) = \eta_{k_1}^1 (\text{lg}(\eta_{k_1}^1 \cap \eta_{k_2}^1))$

for  $k_1, k_2, k_3, k_4 \leq n_0$ .

(2) We say that the sequence  $\langle \bar{a}_\eta : \eta \in {}^\alpha 2 \rangle$  of elements of  $\mathfrak{C}$  (for an ordinal  $\alpha$ ) is *1-fully binary tree indiscernible (1-fbti)* iff whenever  $\bar{\eta}_0 \approx_1 \bar{\eta}_1$  are sequences of elements of  ${}^\alpha 2$ , then

$$\bar{a}_{\bar{\eta}_0} \stackrel{\text{def}}{=} \bar{a}_{\eta_0^0} \frown \dots \frown \bar{a}_{\eta_0^{n_0}}$$

and the similarly defined  $\bar{a}_{\bar{\eta}_1}$ , realise the same type in  $\mathfrak{C}$ .

(3) Suppose that  $\delta$  is a limit ordinal  $> 0$ . Define  $h^* = h_\delta^* : {}^{\delta>2} \rightarrow {}^{\delta>2}$  by letting for  $\eta \in {}^{\delta>2}$

- $lg(h^*(\eta)) = 2lg(\eta) + 1$ ,
- $i < lg(h^*(\eta)) \implies h^*(\eta)(2i) = 0, h^*(\eta)(2i + 1) = \eta(i)$ ,
- $h^*(\eta)(2lg(\eta)) = 1$ .

For  $n < \omega$  and  $\bar{\eta} \in {}^n({}^{\delta>2})$  we define  $h^*(\bar{\eta}) = \langle h^*(\eta_l) : l < n \rangle$ .

We say  $\bar{\eta} \approx_2 \bar{\nu}$  iff  $h^*(\bar{\eta}) \approx_1 h^*(\bar{\nu})$ . We define 2-fbti like 1-fbti but using  $\approx_2$  in place of  $\approx_1$ .

**Observation 2.10 A** The following can be easily checked:

- (1) Let  $\bar{\eta}, \bar{\nu} \in {}^n({}^{\alpha>2})$  and let  $\bar{\eta}'$  and  $\bar{\nu}'$  be the closures of  $\bar{\eta}, \bar{\nu}$ , respectively, under intersections. Then  $\bar{\eta} \approx_1 \bar{\nu}$  iff  $\bar{\eta}' \approx_1 \bar{\nu}'$ .
- (2) If  $\langle \bar{a}_{\bar{\eta}} : \bar{\eta} \in {}^{\delta>2} \rangle$  is 1-fbti then  $\langle \bar{a}_{h^*(\bar{\eta})} : \bar{\eta} \in {}^{\delta>2} \rangle$  is 2-fbti.
- (3)  $h^*(\eta)$  is never  $\langle, \rangle$  and  $h^*(\eta_0)$  is never a strict initial segment of  $h^*(\eta_1)$ .

**Claim 2.11** If  $t \in \{1, 2\}$  and  $\langle \bar{b}_\eta : \eta \in {}^\omega 2 \rangle$  are of given constant length, and  $\delta \geq \omega$  is a (limit for  $t = 2$ ) ordinal, then we can find  $\langle \bar{a}_\eta : \eta \in {}^{\delta>2} \rangle$  such that

- (a)  $\langle \bar{a}_\eta : \eta \in {}^{\delta>2} \rangle$  is  $t$ -fbti,
- (b) if  $\bar{\eta} = \langle \eta_m : m < n \rangle$ , where each  $\eta_m \in {}^{\delta>2}$ , is given, then we can find  $\nu_m \in {}^\omega 2 (m < n)$  such that with  $\bar{\nu} \stackrel{\text{def}}{=} \langle \nu_m : m < n \rangle$ , we have  $\bar{\nu} \approx_t \bar{\eta}$  and sequences  $\bar{a}_{\bar{\eta}}$  and  $\bar{b}_{\bar{\nu}}$  realise the same type in  $\mathfrak{C}$ .

**Proof of the Claim.**<sup>2</sup> Let us first deal with  $t = 1$ . By Observation 2.10 A (1) above, we may reduce to checking clause (b) only for tuples  $\bar{b}_\eta$  where  $\eta$  is closed under intersections. By Compactness Theorem it suffices to assume  $\delta = \omega$ . The proof goes through a series of steps through which we obtain increasing degrees of indiscernibility. We shall need some auxiliary definitions. Let  $\alpha$  be an infinite ordinal.

**Definition 2.12** (1) Given  $\bar{\eta} = \langle \eta_0, \dots, \eta_{k-1} \rangle$ , a sequence of elements of  ${}^{\alpha>2}$ , and an ordinal  $\gamma$ . We define  $\bar{\eta}' = \text{cl}_\gamma(\bar{\eta})$  as follows:

$$\bar{\eta}' = \langle \langle \rangle, \eta_0, \eta_0 \upharpoonright \gamma, \eta_1, \eta_1 \upharpoonright \gamma, \eta_0 \cap \eta_1, \eta_2, \eta_2 \upharpoonright \gamma, \eta_0 \cap \eta_2, \eta_1 \cap \eta_2 \dots \rangle.$$

We also define  $u_\gamma[\bar{\eta}] = \{\eta_l \in \bar{\eta} : \text{lg}(\eta_l) > \gamma\}$ .

(2) We say that  $\bar{\eta} \approx_{\gamma,n} \bar{\nu}$  iff  $\bar{\eta}' \stackrel{\text{def}}{=} \text{cl}_\gamma(\bar{\eta})$  and  $\bar{\nu}' \stackrel{\text{def}}{=} \text{cl}_\gamma(\bar{\nu})$  satisfy

- (i)  $\bar{\eta}' = \langle \eta'_l : l < m \rangle$  and  $\bar{\nu}' = \langle \nu'_l : l < m \rangle$  are both in  ${}^m(\alpha>2)$  for some  $m$ ,
- (ii) for  $l < m$  we have  $\eta'_l \in {}^{\gamma \geq 2} \iff \nu'_l \in {}^{\gamma \geq 2}$ , and for such  $l$  we have  $\eta'_l = \nu'_l$ ,
- (iii)  $n \geq |u_\gamma[\bar{\eta}]|$ ,
- (iv)  $\eta'_l, \eta'_k \in u_\gamma[\bar{\eta}] \implies [\text{lg}(\eta'_l) < \text{lg}(\eta'_k) \iff \text{lg}(\nu'_l) < \text{lg}(\nu'_k)]$ ,
- (v)  $\eta'_{l_1} \trianglelefteq \eta'_{l_2} \iff \nu'_{l_1} \trianglelefteq \nu'_{l_2}$ , and the same holds for the equality,
- (vi) if  $\eta'_{l_1}$  is not an initial segment of  $\eta'_{l_2}$ , then  $\eta'_{l_1}(\text{lg}(\eta'_{l_1} \cap \eta'_{l_2})) = \nu'_{l_1}(\text{lg}(\nu'_{l_1} \cap \nu'_{l_2}))$ .

(3)  $\langle \bar{a}_\eta : \eta \in {}^{\alpha>2} \rangle$  is  $(\gamma, n)$ -indiscernible iff for every  $k$ , for every  $\bar{\eta}, \bar{\nu} \in {}^k(\alpha>2)$  with  $\bar{\eta} \approx_{\gamma,n} \bar{\nu}$ , the tuples  $\bar{a}_{\bar{\eta}}$  and  $\bar{a}_{\bar{\nu}}$  realise the same type.

(4)  $(\leq \gamma, n)$ -indiscernibility is the conjunction of  $(\beta, n)$ -indiscernibility for all  $\beta \leq \gamma$ .

(5) We say that  $\langle \bar{a}_\eta : \eta \in {}^{\alpha>2} \rangle$  is 0-*fbti* iff it is  $(\gamma, n)$ -indiscernible for all  $\gamma$  and  $n$ .

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<sup>2</sup>Note that the definition of  $\approx_1, \approx_2$  has changed from the one given in the published version of this paper, but the following proof is basically the same as the one there.



**Note 2.12 A** (1)  $\text{cl}_0(\bar{\eta})$  is simply the closure of  $\bar{\eta}$  under intersections, joined with  $\langle \rangle$  in appropriate places.

(2)  $\bar{\eta} \approx_{\gamma,n} \bar{\nu}$  iff  $\text{cl}_\gamma(\bar{\eta}) \approx_{\gamma,n} \text{cl}_\gamma(\bar{\nu})$ .

**Subclaim 2.13** If  $\bar{a}_\eta \in {}^k\mathfrak{C}$  for  $\eta \in {}^{\omega>}2$  are tuples of constant length and closed under intersections, then

for any  $\alpha \geq \omega$  we can find  $\bar{a}' = \langle \bar{a}'_\eta : \eta \in {}^{\alpha>}2 \rangle$  such that

(x)  $\bar{a}'$  is 0-fbti,

(xx) for every  $m$  and a finite set  $\Delta$  of formulae, we can find  $h : {}^{m \geq 2} \rightarrow {}^{\omega > 2}$  such that

( $\alpha$ )  $\langle \bar{a}'_\eta : \eta \in {}^{m \geq 2} \rangle$  and  $\langle \bar{a}_{h(\eta)} : \eta \in {}^{m \geq 2} \rangle$  realise the same  $\Delta$ -type,

( $\beta$ )  $h$  satisfies  $h(\eta) \hat{\langle} l \rangle \leq h(\eta \hat{\langle} l \rangle)$  for  $\eta \in {}^{m > 2}$  and  $l < 2$ , and

$$\text{lg}(\eta_1) = \text{lg}(\eta_2) \implies \text{lg}(h(\eta_1)) = \text{lg}(h(\eta_2)).$$

**Proof of the Subclaim.** By Compactness Theorem it suffices to work with  $\alpha = \omega$ .

Let  $(*)_{\gamma,n}$  be the conjunction of the statement  $(x)_{\gamma,n}$  given by

$\bar{a}'$  is  $(\leq \gamma, n)$ -indiscernible,

and  $(xx)$  above. We prove by induction on  $n$  and then  $\gamma$  that for any  $\gamma \leq \omega$  we can find  $\bar{a}'$  for which  $(*)_{\gamma,n}$  holds.

$n = 0$ . We use  $\bar{a}'_\eta = \bar{a}_\eta$ .

$n + 1$ . By induction on  $\gamma \leq \omega$ , we prove that there is  $\bar{a}'$  for which  $(*)_{\gamma,n+1} + (*)_{\omega,n} + (xx)$  holds.

$\gamma < \omega$ .

Without loss of generality, the sequence  $\langle \bar{a}_\eta : \eta \in {}^{\omega > 2} \rangle$  is  $(\leq \omega, n)$ -indiscernible, as  $(xx)$  as a relation between  $\langle \bar{a}_\eta : \eta \in {}^{\omega > 2} \rangle$  and  $\langle \bar{a}'_\eta : \eta \in {}^{\omega > 2} \rangle$  is transitive. Suppose we are given  $\bar{\eta}^*, \bar{\nu}^*$  satisfying  $\bar{\eta}^* \approx_{\gamma,n+1} \bar{\nu}^*$ . By Note 2.1.2 A, we may assume  $\bar{\eta}^*, \bar{\nu}^*$  to be the same as their  $\text{cl}_\gamma$  closures and the same will hold for any  $\bar{\eta}, \bar{\nu}$  that we mention in this context.

If  $|u_\gamma[\bar{\eta}^*]| \leq n$ , the conclusion follows by the assumptions. We shall assume  $|u_\gamma[\bar{\eta}^*]| > n$ , so  $|u_\gamma[\bar{\eta}^*]| = n+1$ . Moreover, if  $\min(u_\gamma[\bar{\eta}^*]) = \min(u_\gamma[\bar{\nu}^*])$  and for any  $l$  with  $lg(\eta_l^*) = \min(u_\gamma[\bar{\eta}^*])$  we have  $\eta_l^* = \nu_l^*$ , then using  $(x)_{\min(u_\gamma[\bar{\eta}^*]),n}$ , we get that  $\bar{a}_{\bar{\eta}^*}$  and  $\bar{a}_{\bar{\nu}^*}$  realise the same type. By the same argument, fixing a finite set  $\Delta$  of formulae, for every  $\bar{\eta}$ , we get that the  $\text{tp}_\Delta(\bar{a}_{\bar{\eta}})$  depends just on the

$$\bar{\eta}/ \approx_{\gamma,n+1} \stackrel{\text{def}}{=} \Upsilon \text{ and } \{\eta_l : l < lg(\bar{\eta})\} \cap {}^{\min(u_\gamma[\bar{\eta}])}2 = \{\eta_l : l \in v^\Upsilon\}$$

for some  $v^\Upsilon \subseteq lg(\bar{\eta})$ . Let us define  $F_{\Upsilon,\Delta}^0$  by  $F_{\Upsilon,\Delta}^0(\langle \eta_l : l \in v^\Upsilon \rangle) = \text{tp}_\Delta(\bar{a}_{\bar{\eta}})$ . By the closure properties of  $\bar{\eta}$  and the definition of  $\approx_{\gamma,n+1}$ , we get that for  $l_1 \neq l_2 \in v^\Upsilon$  the truth value of  $\eta_{l_1} \upharpoonright (\gamma+1) = \eta_{l_2} \upharpoonright (\gamma+1)$  depends only on  $\Upsilon$ . We can hence replace  $v^\Upsilon$  by a set  $v_*^\Upsilon \subseteq v^\Upsilon$  such that  $\langle \eta_l : l \in v_*^\Upsilon \rangle$  are the representatives under the equality of the restrictions to  $\gamma+1$ .

As we have fixed  $\Delta$ , there is a finite set  $A$  of  $\Upsilon$ s that can be used as representatives for the values of  $F_{\Upsilon,\Delta}^0$ . Let  $r$  be the size of the range of  $F_{\Upsilon,\Delta}^0$ . Let  $k^* = 2^{\gamma+1}$  (so finite) and let  $\{\mu_k^* : k < k^*\}$  list  ${}^{\gamma+1}2$ . We define a function  $F_{\Upsilon,\Delta}$  on  $k^*$  ( $\omega > 2$ ) by letting

$$F_{\Upsilon,\Delta}(x_0, \dots, x_k, \dots)_{k < k^*} \stackrel{\text{def}}{=} F_{\Upsilon,\Delta}^0(\langle \eta_l : l \in v_*^\Upsilon \rangle),$$

$$\text{where } \eta_l \upharpoonright (\gamma+1) = \mu_k^* \implies \eta_l = \mu_k^* \frown x_k.$$

Define a function  $F$  with arity  $k^*$  so that  $F((\dots, x_k, \dots)_{k < k^*})$  is defined iff for some  $m < \omega$  we have  $\{x_k : k < k^*\} \subseteq {}^m 2$  and then

$$F((\dots, x_k, \dots)_{k < k^*}) = \langle F_{\Upsilon,\Delta}((\dots, x_k, \dots)_{k < k^*}) : \Upsilon \in A \rangle.$$

Therefore  $F$  is a function from  $\bigcup_{m < \omega} \prod_{k < k^*} \text{lev}_m(\omega > 2)$  into a set of size  $r$ . We recall the following definition and restatement of the Halpern-Lauchli theorem [HaLa], due to Laver and Pincus and presented in [PiHa].

**Definition 2.13 A** (1) A tree  $S$  is *strongly embedded* in a tree  $T$  if there is a strictly increasing embedding  $f^*$  of  $S$  as a suborder of  $T$  such that

- any nonmaximal node in  $f(S)$  has the same number of immediate successors in  $T$  and in  $f(S)$ , and

- all nodes on any common level of  $S$  are mapped by  $f$  to a common level of  $T$ .

(2) A nonempty subtree of  ${}^{\omega}>\omega$  is *well-behaved* if it is finitely branching and has no maximal nodes (hence it has  $\omega$  levels).

**Halpern-Lauchli theorem** Let  $r, d < \omega$ . Suppose that  $\langle T_i : i < d \rangle$  are well-behaved trees and that  $c$  is a colouring of  $\bigcup_{n < \omega} \prod_{i < d} \text{lev}_n(T_i)$  into  $r$  colours. Then there are  $f^*$ ,  $\langle S_i : i < d \rangle$  and  $\langle h_i : i < d \rangle$  such that

- $f^* : \omega \rightarrow \omega$  is a strictly increasing function,
- each  $S_i$  is a well-behaved tree,
- $h_i$  is a strong embedding of  $S_i \rightarrow T_i$ ,
- for each  $n < \omega$  and  $i < d$ , the common height in  $T_i$  of elements of  $h_i \text{``lev}_n(S_i)$  is  $f^*(n)$ , and
- $\bigcup_{n \in u} \prod_{i < d} h_i \text{``lev}_n(S_i)$  is  $c$ -monochromatic.

Moreover, in the case that all  $T_i$  are the same tree, we can assume that all  $h_i$  are contained in a common function  $h$ .

Therefore we can apply the Halpern-Lauchli theorem to  $F$ . We get a sequence  $\langle S_k : k < k^* \rangle$  of well-behaved trees exemplify the conclusion of the Halpern-Lauchli theorem with  $h_k = h \upharpoonright S_k$  and  $f^*(n) = \text{ht}[h \text{``lev}_n(S_k)]$ . Since the only well-behaved subtree of  ${}^{\omega}>2$  is  ${}^{\omega}>2$  itself, we can conclude that there is  $h : {}^{\omega}>2 \rightarrow {}^{\omega}>2$  such that

- $h \upharpoonright {}^{\gamma \geq 2}$  is the identity,
- $\text{lg}(h(\eta))$  depends just on  $\text{lg}(\eta)$  (not on  $\eta$ ),
- $h(\eta) \smallfrown \langle l \rangle \triangleleft h(\eta \smallfrown \langle l \rangle)$  for  $l = 0, 1$ ,
- for some  $c$  we have that for all  $m < \omega$

$$\{\eta_k : k < k^*\} \subseteq {}^m 2 \implies F((h(\eta_0), h(\eta_1), \dots, h(\eta_k), \dots)_{k < k^*}) = c.$$

Let  $\bar{a}'_\eta$  for  $\eta \in \omega^{>2}$  be defined to be  $\bar{a}_\eta$  if  $\eta \in \gamma^{>2}$ , and otherwise  $\bar{a}_{h(\nu)}$  for the unique  $\nu$  such that  $\eta \upharpoonright (\gamma + 1) = \mu_k^*$  and  $\eta = \mu_k^* \frown \nu$ .

We have obtained the desired conclusion, but localized to  $\Delta$ . The induction step ends by an application of the compactness theorem.

$\gamma = \omega$  This is vacuously true.

Having carried the induction, the conclusion of the Subclaim follows from  $\bigwedge_n (*)_{0,n}$ .  $\star_{2.13}$

Now we go *back to the proof of the Claim*. Given  $\langle \bar{b}_\eta : \eta \in \omega^{>2} \rangle$  as in the assumptions, by the Subclaim we can assume that they are 0-fbti. We choose by induction on  $n$  a function  $h_n : n^{\geq 2} \rightarrow \omega^{>2}$  as follows. Let  $h_0(\langle \rangle) = \langle \rangle$ . If  $h_n$  is defined, let

$$k_n \stackrel{\text{def}}{=} \max\{\text{lg}(h_n(\eta)) + 1 : \eta \in n^{\geq 2}\}$$

and let

$$h_{n+1}(\langle \rangle) = \langle \rangle, \quad h_{n+1}(\langle 1 \rangle \hat{\ } \nu) = \langle 1 \rangle \hat{\ } h_n(\nu), \quad h_{n+1}(\langle 0 \rangle \hat{\ } \nu) = \langle 0, \dots, 0 \rangle \hat{\ } h_n(\nu),$$

where the sequence of 0s in the last part of the definition has length  $k_n$ . The point of the definition of  $h_n$  is that if  $\bar{\eta}^l = \langle \eta_0^l, \dots, \eta_{n_l}^l \rangle$  for  $l = 0, 1$  are given and  $n^* = \text{lg}(\text{cl}_0(\bar{\eta}^0))$ , then

$$\bar{\eta}^0 \approx_1 \bar{\eta}^1 \implies \langle h_{n^*}(\eta_0^0), \dots, h_{n^*}(\eta_{n_0}^0) \rangle \approx_{0,n^*} \langle h_{n^*}(\eta_0^1), \dots, h_{n^*}(\eta_{n_1}^1) \rangle.$$

To check this, we verify the six relevant items of the definition of  $\approx_{0,n^*}$ .

- (i) Follows because  $n_0 = n_1$  by the definition of  $\approx_1$ .
- (ii) If  $h_{n^*}(\eta_i^0) \cap h_{n^*}(\eta_j^0) = \langle \rangle$  then  $\eta_i^0 \cap \eta_j^0 = \langle \rangle$  so  $\eta_i^1 \cap \eta_j^1 = \langle \rangle$  by the definition of  $\approx_1$ , and hence  $h_{n^*}(\eta_i^1) \cap h_{n^*}(\eta_j^1) = \langle \rangle$ . The opposite implication holds by symmetry.
- (iii) Follows by the definition of  $n^*$ .

(iv) Suppose

$$0 < \lg(h_{n^*}(\eta_i^0) \cap h_{n^*}(\eta_j^0)) < \lg(h_{n^*}(\eta_k^0) \cap h_{n^*}(\eta_s^0)).$$

Let  $m \leq n^*$  be the first such that

$$0 < \lg(h_{n^*}(\eta_i^0 \upharpoonright m) \cap h_{n^*}(\eta_j^0 \upharpoonright m)) < \lg(h_{n^*}(\eta_k^0 \upharpoonright m) \cap h_{n^*}(\eta_s^0 \upharpoonright m)).$$

Clearly,  $m > 0$ . To simplify the notation, let us assume that  $m = n^*$ . Let  $\eta_t^0 = \langle l_t \rangle \frown \nu_t^0$  for  $t \in \{i, j, k, s\}$  and for some  $l_t \in \{0, 1\}$  depending on  $t$ . The situation we describe can happen iff  $l_i = l_j = 1$  and  $l_k = l_s = 0$ , by the definition of  $h_n$ . By the definition of  $\approx_1$  this can be recognised by the  $\approx_1$ -type of  $\bar{\eta}^0$ .

(v), (vi) Follow because the corresponding properties are preserved by  $h_n$ .

Fix an  $n < \omega$  and define  $\bar{a}_\eta = \bar{b}_{h_n(\eta)}$  for  $\eta \in {}^{n \geq 2}$ . By the above argument it follows that  $\langle \bar{a}_\eta : \eta \in {}^{n \geq 2} \rangle$  are 1-fbti. As  $n$  was arbitrary, we can finish by compactness.

For  $t = 2$ , we use exactly the same proof.  $\star_{2.11}$

The following Theorem 2.15 will finally tell us that witnesses for  $\text{SOP}_1$  can be chosen with a certain degree of indiscernability. We need to introduce a new notion of indiscernability:

**Definition 2.14** (1) Given an ordinal  $\alpha$  and sequences  $\bar{\eta}_l = \langle \eta_0^l, \eta_1^l, \dots, \eta_{n_l}^l \rangle$  for  $l = 0, 1$  of members of  ${}^{\alpha > 2}$ , we say that  $\bar{\eta}_0 \approx_3 \bar{\eta}_1$  iff

(a)  $n_0 = n_1$ ,

(b) the truth values of

- $\eta_k^l = \langle \rangle$ ,
- $\eta_{k_1}^l \cap \eta_{k_2}^l \trianglelefteq \eta_{k_3}^l \cap \eta_{k_4}^l$

do not depend on  $l$ ,

(c)  $\eta_{k_1}^l \not\trianglelefteq \eta_{k_2}^l \implies \eta_{k_1}^0 (\lg(\eta_{k_1}^0 \cap \eta_{k_2}^0)) = \eta_{k_1}^1 (\lg(\eta_{k_1}^1 \cap \eta_{k_2}^1))$ ,

(d)  $\eta_{k_1}^l \not\leq \eta_{k_2}^l \implies \eta_{k_1}^0 = (\eta_{k_1}^0 \cap \eta_{k_2}^0) \frown \langle 1 \rangle$  iff  $\eta_{k_1}^1 = (\eta_{k_1}^1 \cap \eta_{k_2}^1) \frown \langle 1 \rangle$ .

for  $k_1, k_2, k_3, k_4 \leq n_0$ .

(2) We say that the sequence  $\langle \bar{a}_\eta : \eta \in {}^{\alpha>}2 \rangle$  of elements of  $\mathfrak{C}$  (for an ordinal  $\alpha$ ) is *3-fully binary tree indiscernible (3-fbti)* iff whenever  $\bar{\eta}_0 \approx_1 \bar{\eta}_1$  are sequences of elements of  ${}^{\alpha>}2$ , then

$$\bar{a}_{\bar{\eta}_0} \stackrel{\text{def}}{=} \bar{a}_{\eta_0^0} \frown \dots \frown \bar{a}_{\eta_{n_0}^0}$$

and the similarly defined  $\bar{a}_{\bar{\eta}_1}$ , realise the same type in  $\mathfrak{C}$ .

**Theorem 2.15** Suppose that  $T$  has  $\text{SOP}_1$  as witnessed by  $\varphi(\bar{x}, \bar{y})$  and a sequence  $\bar{a} = \langle \bar{a}_\eta : \eta \in {}^{\omega>}2 \rangle$ . Then there is  $\bar{d} = \langle \bar{d}_\eta : \eta \in {}^{\omega>}2 \rangle$  exemplifying that  $\varphi(\bar{x}, \bar{y})$  has  $\text{SOP}_1$  and  $\langle \bar{d}_\eta : \eta \in {}^{\omega>}2 \setminus \{\langle \rangle\} \rangle$  is 3-fbti.

**Proof.** Let  $k^* = \text{lg}(\bar{y})$ . First define  $\bar{b}_\eta$  for  $\eta \in {}^{\omega>}2$  by  $\bar{b}_\eta = \bar{a}_{\eta \frown \langle 0 \rangle} \frown \bar{a}_{\eta \frown \langle 1 \rangle}$ . Let for any  $\bar{z}_0, \bar{z}_1$  of length  $k^*$  and  $l \in \{0, 1\}$ ,  $\psi_l(\bar{x}, \bar{z}_0 \frown \bar{z}_1) \equiv \varphi(\bar{x}, \bar{z}_l)$ . Now we use Claim 2.11 applied to  $\langle \bar{b}_\eta : \eta \in {}^{\omega>}2 \rangle$ . Therefore we can find  $\bar{c} = \langle \bar{c}_\eta : \eta \in {}^{\omega>}2 \rangle$  such that

(a)  $\bar{c}$  is 1-fbti,

(b) for any finite  $n$  and  $\bar{\eta} \in {}^n({}^{\omega>}2)$  there is  $\bar{\nu} \in {}^n({}^{\omega>}2)$  such that  $\bar{\nu} \approx_1 \bar{\eta}$  and  $\bar{b}_\eta$  and  $\bar{c}_\nu$  realise the same type in  $\mathfrak{C}$ .

Let  $\bar{d}_\eta$  for  $\eta \in {}^{k^*}({}^{\omega>}2)$  be defined by induction on the length of  $\eta$  so that  $\bar{d}_{\eta \frown \langle 0 \rangle} \frown \bar{d}_{\eta \frown \langle 1 \rangle} = \bar{c}_\eta$  and  $\bar{d}_{\langle \rangle} = \bar{c}_{\langle \rangle}$ . This is possible by the choice of  $\bar{b}$  and  $\bar{c}$ .

**Claim 2.16** If  $\nu \frown \langle 0 \rangle \leq \eta$  then  $\varphi(\bar{x}, \bar{d}_\eta)$  and  $\varphi(\bar{x}, \bar{d}_{\nu \frown \langle 1 \rangle})$  are incompatible.

**Proof of the Claim.** Let  $\eta = \rho \frown \langle l \rangle$  for some  $l \in \{0, 1\}$ . Consider  $\{\psi_l(\bar{x}, \bar{c}_\rho), \psi_1(\bar{x}, \bar{c}_\nu)\}$ , we claim that this set is inconsistent. We know that

$$\psi_l(\bar{x}, \bar{c}_\rho) \equiv \varphi(\bar{x}, \bar{d}_{\rho \frown \langle l \rangle}) \equiv \varphi(\bar{x}, \bar{d}_\eta), \quad \psi_1(\bar{x}, \bar{c}_\nu) \equiv \varphi(\bar{x}, \bar{d}_{\nu \frown \langle 1 \rangle}).$$

By the 1-fbti property of  $\bar{c}$  and the choice of  $\bar{c}$  with respect to  $\bar{b}$  it suffices to check that  $\{\psi_l(\bar{x}, \bar{b}_\rho), \psi_1(\bar{x}, \bar{b}_\nu)\}$  is inconsistent. This means that  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu \frown \langle 1 \rangle})\}$  is inconsistent, which is true by the choice of  $\bar{a}$ .  $\star_{2.16}$

**Claim 2.17** For any  $\rho \in {}^\omega 2$ ,  $\{\varphi(\bar{x}, d_{\rho \upharpoonright n}^-) : n < \omega\}$  is consistent.

**Proof of the Claim.** It suffices to show that for any

$$\langle \rangle \triangleleft \eta_0 \triangleleft \eta_1 \triangleleft \dots \triangleleft \eta_k$$

the set  $\{\varphi(\bar{x}, \bar{d}_{\eta_{l+1} \upharpoonright \lg(\eta_l) \frown \eta_{l+1} \upharpoonright \lg(\eta_l)}) : l > k\} \cup \{\varphi(\bar{x}, \bar{d}_{\langle \rangle})\}$  is consistent. This means  $\{\psi_{\eta_{l+1} \upharpoonright \lg(\eta_l)}(\bar{x}, \bar{c}_{\eta_{l+1} \upharpoonright \lg(\eta_l)}) : l < k\} \cup \{\varphi(\bar{x}, \bar{c}_{\langle \rangle})\}$  is consistent. By the choice of  $\bar{b}$  and  $\bar{c}$  this is to say  $\{\psi_{\eta_{l+1} \upharpoonright \lg(\eta_l)}(\bar{x}, \bar{b}_{\eta_{l+1} \upharpoonright \lg(\eta_l)}) : l < k\} \cup \{\varphi(\bar{x}, \bar{a}_{\langle \rangle})\}$  or  $\{\varphi(\bar{x}, \bar{a}_{\eta_{l+1} \upharpoonright \lg(\eta_l)}) : l < k\} \cup \{\varphi(\bar{x}, \bar{a}_{\langle \rangle})\}$  is consistent, but this is true by the choice of  $\bar{a}$ .  $\star_{2.17}$

**Claim 2.18**  $\langle d_\eta : \eta \in {}^\omega 2 \setminus \{0\} \rangle$  is 3-fbti.

**Proof of the Claim.** Suppose that  $\bar{\eta}_0 \approx_3 \bar{\eta}_1$  and consider  $\bar{d}_{\bar{\eta}_0}$  and  $\bar{d}_{\bar{\eta}_1}$ . For each  $\eta_l^k$  let  $\nu_k^l$  be such that  $\eta_k^l = \nu_k^l \frown \langle m_k^l \rangle$  for some  $m_k^l \in \{0, 1\}$  and let  $\bar{\nu}_0, \bar{\nu}_1$  be defined from  $\nu_k^l (l \in \{0, 1\}, k < \lg(\bar{\eta}_0))$ . Then  $\bar{\eta}_0 \approx_3 \bar{\eta}_1 \implies \bar{\nu}_0 \approx_3 \bar{\nu}_1$ , hence  $\bar{c}_{\bar{\nu}_0}$  and  $\bar{c}_{\bar{\nu}_1}$  realise the same type, which implies that  $\bar{d}_{\bar{\eta}_0}$  and  $\bar{d}_{\bar{\eta}_1}$  do.

$\star_{2.18}$

$\star_{2.15}$

*End changes.*

As we mentioned before, it would be really interesting to know if  $\text{SOP}_2$  and  $\text{SOP}_1$  are equivalent. A step towards understanding this question is provided by the next claim which shows that in the case of theories which are  $\text{SOP}_1$  and  $\text{NSOP}_2$ , the witnesses to being  $\text{SOP}_1$  can be chosen to be particularly nice. *note a change here to 3-fbti from the old version*

**Claim 2.19** Suppose that  $\varphi(\bar{x}, \bar{y})$  satisfies  $\text{SOP}_1$ , but for no  $n$  does the formula  $\varphi_n(\bar{x}, \bar{y}_0, \dots, \bar{y}_{n-1}) \equiv \bigwedge_{k < n} \varphi(\bar{x}, \bar{y}_k)$  satisfy  $\text{SOP}_2$ . Then there are witnesses  $\langle \bar{a}_\eta : \eta \in {}^{\omega > 2} \rangle$  for  $\varphi(\bar{x}, \bar{y})$  satisfying  $\text{SOP}_1$  which in addition satisfy:

(c) if  $X \subseteq {}^{\omega > 2}$ , and there are no  $\eta, \nu \in X$  such that  $\eta \frown \langle 0 \rangle \trianglelefteq \nu$ , then  $\{\varphi(\bar{x}, \bar{a}_\eta) : \eta \in X\}$  is consistent.

(d)  $\langle \bar{a}_\eta : \eta \in {}^{\omega > 2} \rangle$  is 3-fbti.

(In particular, such a formula and witnesses can be found for any theory satisfying  $\text{SOP}_1$  and  $\text{NSOP}_2$ .)

**Proof of the Claim.** We shall be using the following colouring theorem, for which we could not find a specific reference and so we include a proof of it.

**Lemma 2.20** Suppose  $\text{cf}(\kappa) = \kappa$  and we colour  ${}^{\kappa}>2$  by  $\theta < \kappa$  colours. Then there is an embedding  $h : {}^{\omega}>2 \rightarrow {}^{\kappa}>2$  such that  $h(\eta)\langle l \rangle \trianglelefteq h(\hat{\eta}\langle l \rangle)$  and  $\text{Rang}(h)$  is monochromatic.

**Proof of the Lemma.** Let  $c$  be a colouring as in the assumptions and let  $\{a_i : i < \theta\}$  list  $\text{Rang}(c)$ . We claim that there is  $\nu^* \in {}^{\kappa}>2$  and  $j < \theta$  such that for every  $\nu \in {}^{\kappa}>2$  satisfying  $\nu^* \trianglelefteq \nu$  there is  $\rho \in {}^{\kappa}>2$  with  $\nu \trianglelefteq \rho$  and  $c(\rho) = j$ . For otherwise, we can choose by induction on  $i \leq \theta$  a member  $\eta_i \in {}^{\kappa}>2$  with  $i < j \implies \eta_i \trianglelefteq \eta_j$  such that for no  $\rho \in {}^{\kappa}2$  do we have  $\eta_{i+1} \trianglelefteq \rho$  and  $c(\rho) = i$ , using  $\theta < \text{cf}(\kappa)$ . As  $\theta < \kappa$ , we obtain a contradiction.

Having found such  $\nu^*, j$  we define  $h(\eta)$  for  $\eta \in {}^n2$  by induction on  $n < \omega$ . For  $n = 0$  we choose  $h(\langle \rangle)$  to satisfy  $\nu^* \trianglelefteq h(\langle \rangle)$  and  $c(h(\langle \rangle)) = j$ , which is possible by the choice of  $\nu^*$  and  $j$ . For  $n + 1$ , for any  $\eta \in {}^{n+1}2$  we choose for  $l = 0, 1$  a member  $h(\eta \smallfrown \langle l \rangle)$  of  ${}^{\kappa}>2$  which is above  $h(\eta) \smallfrown \langle l \rangle$  and on which  $c$  is  $j$ , which again is possible by the choice of  $\nu^*$  and  $j$ . ★<sub>2.20</sub>

Let  $\varphi(\bar{x}, \bar{y})$  be a  $\text{SOP}_1$  formula which is not  $\text{SOP}_2$ , and moreover assume that for no  $n$  does the formula  $\varphi_n$  defined as above satisfy  $\text{SOP}_2$ . By Theorem 2.15, we can find witnesses  $\langle \bar{a}_\eta : \eta \in {}^{\omega}>2 \rangle$  which are 3-fbti. By the compactness theorem, we can assume that we have a 1-fbti sequence  $\langle \bar{a}_\eta : \eta \in {}^{\omega_1}>2 \rangle$  with the properties corresponding to (a) and (b) of Definition 2.2(2), namely

(a) for every  $\eta \in {}^{\omega_1}2$ , the set  $\{\varphi(\bar{x}, \bar{a}_{\eta \upharpoonright \alpha}) : \alpha < \omega_1\}$  is consistent,

(b) if  $\nu \smallfrown \langle 0 \rangle \trianglelefteq \eta \in {}^{\omega_1}>2$ , then  $\{\varphi(\bar{x}, \bar{a}_{\nu \smallfrown \langle 1 \rangle}), \varphi(\bar{x}, \bar{a}_\eta)\}$  is inconsistent.

We shall now attempt to choose  $\nu_\eta$  and  $w_\eta$  for  $\eta \in {}^{\omega_1}>2$ , by induction on  $l\text{g}(\eta) = \alpha < \omega_1$  so that:

(i)  $\nu_\eta \in {}^{\omega_1}>2$ ,



- (ii)  $\beta < \alpha \implies \nu_{\eta \upharpoonright \beta} \triangleleft \nu_\eta$ ,
- (iii)  $\beta < \alpha \implies \nu_\eta(\text{lg}(\nu_{\eta \upharpoonright \beta})) = \eta(\beta)$ ,
- (iv)  $w_\eta \subseteq \omega_1^{>2}$  is finite and  $\nu \in w_\eta \implies \text{lg}(\nu) < \text{lg}(\nu_\eta)$ ,
- (v) if  $\text{lg}(\eta)$  is a limit ordinal  $> 0$ , then  $w_\eta = \emptyset$ ,
- (vi) if  $\eta \in \beta 2$  and  $l < 2$ , then  $w_{\eta \frown \langle l \rangle} \subseteq \{\rho \in \omega_1^{>2} : \nu_\eta \frown \langle l \rangle \trianglelefteq \rho\}$  and  $\max\{\text{lg}(\rho) : \rho \in w_{\eta \frown \langle l \rangle}\} < \text{lg}(\nu_{\eta \frown \langle l \rangle})$ ,
- (vii) for each  $\eta$  there is  $\rho^* = \rho_\eta^*$  such that
  - ( $\alpha$ )  $\nu_\eta \triangleleft \rho^* \in \omega_1 2$ ,
  - ( $\beta$ )  $|\{\alpha < \omega_1 : \rho^*(\alpha) = 1\}| = \aleph_1$ ,
  - ( $\gamma$ ) letting

$$p_\eta(\bar{x}) \stackrel{\text{def}}{=} \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_{\eta \upharpoonright \gamma} \text{ for some } \gamma \leq \text{lg}(\eta)\},$$

we have that for all large enough  $\beta^*$ , the set

$$p_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\rho^* \upharpoonright \beta}) : \beta > \beta^* \wedge \rho^*(\beta) = 1\}$$

is consistent,

- (viii)  $p_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta \frown \langle 0 \rangle} \cup w_{\eta \frown \langle 1 \rangle}\}$  is inconsistent.

Before proceeding, we make several remarks about this definition. Firstly, requirements (vii) and (viii) taken together imply that for each  $\eta \in \omega_1^{>2}$  we have that  $w_{\eta \frown \langle 0 \rangle} \cup w_{\eta \frown \langle 1 \rangle} \neq \emptyset$ . Secondly, the definition of  $w_{\eta \frown \langle l \rangle}$  for  $l \in \{0, 1\}$  implies that

$$\bigwedge_{l=0,1} \rho_l \in w_{\eta \frown \langle l \rangle} \implies \rho_0 \perp \rho_1.$$

Thirdly, in (vii), any  $\rho^*$  which satisfies that  $\nu_\eta \triangleleft \rho^*$  and  $|\{\gamma : \rho^*(\gamma) = 1\}| = \aleph_1$  can be chosen as  $\rho_\eta^*$ , by indiscernibility.

Now let us assume that a choice as above is possible, and we have made it. Hence for each  $\eta \in \omega_1^{>2}$  there is a finite  $q_\eta \subseteq p_\eta$  such that

$$q_\eta(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta \frown \langle 0 \rangle} \cup w_{\eta \frown \langle 1 \rangle}\} \quad (*)$$

is inconsistent. Notice that there are  $q$  and  $\eta^* \in {}^{\omega_1}2$  such that

$$(\forall \eta_1)[\eta^* \trianglelefteq \eta_1 \in {}^{\omega_1}2 \implies (\exists \eta_2 \in {}^{\omega_1}2)(\eta_1 \trianglelefteq \eta_2 \wedge q_{\eta_2} = q)].$$

Namely, otherwise, we would have the following: each  $p_\eta$  is countable, hence for every  $\eta$  there is  $g(\eta)$  with  $\eta \triangleleft g(\eta) \in {}^{\omega_1}2$  and

$$g(\eta) \trianglelefteq \eta_1 \implies q_{\eta_1} \not\subseteq p_\eta.$$

Let  $\eta_0 \stackrel{\text{def}}{=} \langle \rangle$ , and for  $n < \omega$  let  $\eta_{n+1} = g(\eta_n)$ . Let  $\eta \stackrel{\text{def}}{=} \cup_{n < \omega} \eta_n$ , hence  $p_\eta = \cup_{n < \omega} p_{\eta_n}$  (as  $w_\eta = \emptyset$ ), and so  $q_\eta \subseteq p_{\eta_n}$  for some  $n$ , a contradiction.

Having found such  $q, \eta^*$ , by renaming and using Lemma 2.20, we can assume that  $\eta^* \stackrel{\text{def}}{=} \langle \rangle$  and that for all  $\eta \in {}^\omega 2$  we have  $q_\eta = p_\langle \rangle = q$  (as  $\eta \trianglelefteq \nu \implies p_\eta \subseteq p_\nu$ ). For  $\eta \in {}^{\omega > 2}$  let  $\bar{\tau}_\eta$  list  $w_\eta$ . Without loss of generality, by thinning and renaming, we have that for all  $\eta_1, \eta_2$ ,

$$\langle \nu_{\eta_1} \rangle \frown \bar{\tau}_{\eta_1 \frown \langle 0 \rangle} \frown \bar{\tau}_{\eta_1 \frown \langle 1 \rangle} \approx_1 \langle \nu_{\eta_2} \rangle \frown \bar{\tau}_{\eta_2 \frown \langle 0 \rangle} \frown \bar{\tau}_{\eta_2 \frown \langle 1 \rangle}.$$

Similarly to the proof of Claim 2.7, we can define a formula  $\psi(\bar{x}, \bar{y})$  and  $\{\bar{b}_\eta : \eta \in {}^{\omega > 2}\}$  such that

$$\psi(\bar{x}, \bar{b}_\eta) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_\eta\}.$$

We claim that  $\psi(\bar{x}, \bar{y})$  and  $\{\bar{b}_\eta : \eta \in {}^{\omega > 2}\}$  exemplify  $\text{SOP}_2$  of  $T$ , which is then a contradiction (noting that  $\psi$  is a formula of the form  $\varphi_n$  for some  $n$ , where  $\varphi_n$  was defined in the statement of the Claim). We check the two properties from Definition 2.2(1).

To see (a), let  $\eta \in {}^\omega 2$  be given. We have that  $p_\eta$  is consistent, and  $q \subseteq p_\eta$ . For  $n < \omega$ , we have

$$\psi(\bar{x}, \bar{b}_{\eta \upharpoonright n}) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta \upharpoonright n}\}.$$

As this is a conjunction of a set of formulae each of which is from  $p_\eta$ , we have that  $\{\psi(\bar{x}, \bar{b}_{\eta \upharpoonright n}) : n < \omega\}$  is consistent. To check (b), suppose  $\eta \perp \nu \in {}^{\omega > 2}$ . Let  $n$  be such that  $\eta \upharpoonright n = \nu \upharpoonright n$  but  $\eta(n) \neq \nu(n)$ . Hence

$$\psi(\bar{x}, \bar{b}_\eta) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta \upharpoonright n \frown \eta(n)}\}$$

and

$$\psi(\bar{x}, \bar{b}_\nu) \equiv \bigwedge q \wedge \bigwedge \{\varphi(\bar{x}, \bar{a}_\rho) : \rho \in w_{\eta \upharpoonright n \smallfrown \nu(n)}\},$$

so taken together, the two are contradictory by the choice of  $q$ .

We conclude that the choice of  $\nu_\eta$  and  $w_\eta$  cannot be carried throughout  $\eta \in {}^{\omega_1}2$ . So, there is  $\alpha < \omega_1$  and  $\eta \in {}^\alpha 2$  such that  $\nu_\eta, w_{\eta \smallfrown \langle l \rangle}, \nu_{\eta \smallfrown \langle l \rangle}$  for  $l < 2$  cannot be chosen, and  $\alpha$  is the first ordinal for which there is such  $\eta$ . Let  $\nu_\eta^0 \in {}^{\omega_1}2 \triangleright \cup_{\beta < \alpha} \nu_{\eta \upharpoonright \beta} \smallfrown \langle \eta(\alpha - 1) \rangle$  if the latter part is defined, otherwise let  $\nu_\eta^0 \triangleright \cup_{\beta < \alpha} \nu_{\eta \upharpoonright \beta}$ . This choice of  $\nu_\eta = \rho$  for any  $\rho \supseteq \nu_\eta^0$  with  $\rho \in {}^{\omega_1}2$  satisfies items (i)-(iii) above. We conclude that  $w_{\eta \smallfrown \langle l \rangle}$  for  $l < 2$  using any  $\rho \supseteq \nu_\eta^0$  with  $\rho \in {}^{\omega_1}2$  for  $\nu_\eta$  could not have been chosen, and examine why this is so. Note that  $p_\eta$  is already defined. Let

$$\Theta \stackrel{\text{def}}{=} \left\{ \begin{array}{l} (\rho, \gamma, w) : \nu_\eta^0 \triangleleft \rho \in {}^{\omega_1}2, \\ \text{lg}(\nu_\eta^0) \leq \gamma < \omega_1, \\ (\exists^{\aleph_1} \beta < \omega_1)(\rho(\beta) = 1), \\ w \subseteq \{\Upsilon \in {}^{\omega_1}2 : \rho \upharpoonright \gamma \trianglelefteq \Upsilon\} \text{ is finite and} \\ \text{for some } \beta_\rho < \omega_1 \text{ the set} \\ p_\eta \cup \{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright \beta}) : \rho(\beta) = 1 \ \& \ \beta \in [\beta_\rho, \omega_1)\} \cup \\ \cup \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w\} \\ \text{is consistent} \end{array} \right\}.$$

We make several **observations**:

(0) If  $(\rho, \gamma, w) \in \Theta$  and  $w \subseteq w'$  with  $w'$  finite and  $w' \setminus w$  is contained in  $\{\rho \upharpoonright \beta : \beta_\rho \leq \beta \wedge \rho(\beta) = 1\}$ , then  $(\rho, \gamma, w') \in \Theta$ .

[This is obvious.]

(1) If  $(\rho_l, \gamma, w_l) \in \Theta$  and for some  $\sigma \in {}^{\omega_1}2$  with  $\nu_\eta^0 \trianglelefteq \sigma$  we have  $\sigma \smallfrown \langle l \rangle \triangleleft \rho_l \upharpoonright \gamma$  for  $l < 2$ , while  $\rho_0$  and  $\rho_1$  are eventually equal, then  $(\rho_l, \text{lg}(\sigma), w_0 \cup w_1) \in \Theta$ .

[Why? We have  $w_l \subseteq \{\Upsilon \in {}^{\omega_1}2 : \rho_l \upharpoonright \gamma \trianglelefteq \Upsilon\}$  is finite, so clearly  $w_0 \cup w_1 \subseteq \{\Upsilon \in {}^{\omega_1}2 : \sigma \trianglelefteq \Upsilon\}$  is finite. By the assumption, we have that for some  $\beta_l < \omega_1$  for  $l < 2$

$$p_\eta \cup \{\varphi(\bar{x}, \bar{a}_{\rho_l \upharpoonright \beta}) : \beta > \beta_l \wedge \rho_l(\beta) = 1\} \cup \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_l\}$$

is consistent. Suppose that (1) is not true with  $l = 0$  and let  $\beta^* \geq \max\{\beta_0, \beta_1\}$  be such that  $\beta^* < \omega_1$  and for  $\beta > \beta^*$  the equality  $\rho_0(\beta) = \rho_1(\beta)$  holds. Hence

we have that

$$p_\eta \cup \{\varphi(\bar{x}, \bar{a}_{\rho_0 \upharpoonright \beta}) : \beta > \beta^* \wedge \rho_0(\beta) = 1\} \cup \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_0 \cup w_1\}$$

is inconsistent. By increasing  $w_0$  if necessary, (0) implies that

$$p_\eta \cup \{\varphi(\bar{x}, \bar{a}_\Upsilon) : \Upsilon \in w_0 \cup w_1\}$$

is inconsistent. Let  $\nu_\eta \stackrel{\text{def}}{=} \sigma$ , for  $l < 2$  let  $w_{\eta \smallfrown \langle l \rangle} = w_l$ , and let  $\nu_{\eta \smallfrown \langle l \rangle} \stackrel{\text{def}}{=} \rho_l \upharpoonright \beta_l^*$  for a large enough  $\beta_l^*$  so that  $\beta^* < \beta_l^*$  and  $\max(\{lg(\Upsilon) : \Upsilon \in w_{\eta \smallfrown \langle l \rangle}\}) < \beta_l^*$ . This choice shows that we could have chosen  $\nu_\eta, w_{\eta \smallfrown \langle l \rangle}$  as required, contradicting the choice of  $\eta$ .]

(2) If  $\nu_\eta^0 \triangleleft \rho \in {}^{\omega_1}2$  for some  $\rho$  such that there are  $\aleph_1$  many  $\beta < \omega_1$  with  $\rho(\beta) = 1$ , and  $lg(\nu_\eta^0) \leq \gamma < \omega_1$ , then  $(\rho, \gamma, \emptyset) \in \Theta$ .

[Why? By the choice of  $p_\eta$  and the remark about the freedom in the choice of  $\rho^*$  that we made earlier.]

Now we use the choice of  $\eta$  to define witnesses to  $T$  being  $\text{SOP}_1$  which also satisfy the requirements of the Claim. For  $\tau \in {}^{\omega > 2}$ , let  $\bar{b}_\tau \stackrel{\text{def}}{=} \bar{a}_{\nu_\eta^0 \smallfrown \tau}$ . Let us check the required properties. Properties (a),(b) and (d) follow from the choice of  $\{\bar{a}_\sigma : \sigma \in {}^{\omega_1 > 2}\}$ . Let  $X^* \subseteq {}^{\omega > 2}$  be such that there are no  $\sigma, \nu \in X^*$  with  $\sigma \smallfrown \langle 0 \rangle \trianglelefteq \nu$ , we need to show that  $\{\varphi(\bar{x}, \bar{b}_\tau) : \tau \in X^*\}$  is consistent. It suffices to show the same holds when  $X^*$  replaced by an arbitrary finite  $X \subseteq X^*$ . Fix such an  $X$ . Clearly, it suffices to show that for some  $\rho, \gamma$ , letting  $w = \{\nu_\eta^0 \smallfrown \tau : \tau \in X\}$ , we have  $(\rho, \gamma, w) \in \Theta$ .

Let  $\rho^* \in {}^{\omega_1}2$  be such that  $\nu_\eta^0 \triangleleft \rho^*$  and  $\rho^*(\beta) = 1$  for  $\aleph_1$  many  $\beta$ . By induction on  $n \stackrel{\text{def}}{=} |X|$  we show:

there is  $\rho \in {}^{\omega_1}2$  such that for some  $\gamma \geq \max\{lg(\sigma) : \sigma \in w\}$ , we have  $(\rho, \gamma, w) \in \Theta$  and  $\beta > \gamma \implies \rho(\beta) = \rho^*(\beta)$ , while  $\rho(\gamma) = 1$ .

$n = 0$ . Follows by observation (2) above.

$n = 1$ . Let  $X = \{\tau\}$  and  $\gamma = lg(\tau) + lg(\nu_\eta^0)$ . Let  $\rho \in {}^{\omega_1}2$  be such that  $\rho \upharpoonright \gamma = \nu_\eta^0 \smallfrown \tau$ ,  $\rho(\gamma) = 1$  and  $\beta > \gamma \implies \rho(\beta) = \rho^*(\beta)$ . By observation (2) above, we have that  $(\rho, \gamma, \emptyset) \in \Theta$ . Then, by observation (0), we have  $(\rho, \gamma, w) \in \Theta$ .

$n = k + 1 \geq 2$ . *Case 1.*  $w$  is linearly ordered by  $\triangleleft$ .

Let  $\tau \in w$  be of maximal length, so clearly  $\sigma \in w \setminus \{\tau\} \implies \sigma \frown \langle 1 \rangle \sqsubseteq \tau$ . Let  $\rho \in {}^{\omega_1}2$  be such that  $\tau \frown \langle 1 \rangle \triangleleft \rho$  and  $\beta > \text{lg}(\tau)$ , while  $\rho(\beta) = \rho^*(\beta)$ . Now continue as in the case  $n = 1$ .

*Case 2.* Not Case 1.

Let  $\sigma \in {}^{\omega_1}2$  be  $\triangleleft$ -maximal such that  $(\forall \tau)(\tau \in w \implies \sigma \sqsubseteq \tau)$ . This is well defined, as  $w \neq \emptyset$  is finite. Let  $w_l \stackrel{\text{def}}{=} \{\tau \in w : \sigma \frown \langle l \rangle \sqsubseteq \tau\}$ , so  $w_0 \cap w_1 = \emptyset$  but neither of  $w_0, w_1$  is empty. Now we have that  $\sigma \notin w$ , as otherwise we could choose  $\tau \in w_0$  such that  $\sigma \frown \langle 0 \rangle \sqsubseteq \tau$ , obtaining an easy contradiction with our assumptions on  $X$ . Hence  $w = w_0 \cup w_1$ . We can now use observation (1) and the inductive hypothesis. ★<sub>2.19</sub>

To complete this discussion of the syntactic properties (N)SOP<sub>1,2</sub> we shall quote a result from [ShUs 844] in which the understanding of SOP'<sub>1</sub> and the witnesses for SOP<sub>1</sub> developed here was used to show that NSOP<sub>1</sub> theories admit a rank function.

**Definition 2.21** Given (partial) types  $p(\bar{x}), q(\bar{y})$  and a formula  $\varphi(\bar{x}, \bar{y})$ . By induction on  $n < \omega$  we define when

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y})) \geq n.$$

$n = 0$ . This happens iff both  $p(\bar{x})$  and  $q(\bar{y})$  are consistent.

$n + 1$ . The rank is  $\geq n + 1$  iff for some  $\bar{c}$  realising  $q(\bar{y})$  both

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c})\}, q(\bar{y})) \geq n$$

and

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c}))\}) \geq n.$$

If the rank is  $\geq n$  for all  $n$  then we say it is infinite, otherwise we say it is finite.

**Theorem 2.22** (Shelah-Usvyatsov [ShUs 844]) A theory  $T$  is NSOP<sub>1</sub> iff

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(\bar{x} = \bar{x}, \bar{y} = \bar{y}) < \infty$$

for every formula  $\varphi(\bar{x}, \bar{y})$ .

### 3 $\triangleleft^*$ -maximality revisited

In this section we come back to our main thesis, which is that properties  $SOP_2$  and the maximality in the  $\triangleleft^*$ -order are closely connected.

Our main proof will use two auxiliary notions. The first is the order  $\triangleleft_{\lambda}^{**}$ , which is a version of the  $\triangleleft_{\lambda}^*$ -order.

**Definition 3.1** (1) For (complete first order theories)  $T_1, T_2$  and a regular cardinal  $\lambda > |T_1|, |T_2|$ , let  $T_1 \triangleleft_{\lambda}^{**} T_2$  mean:

There is a  $\lambda$ -relevant  $(T_1, T_2)$ -superior  $(T^*, \bar{\varphi}, \bar{\psi})$  (see Definition 1.2) such that  $T^*$  has Skolem functions and if  $T^{**} \supseteq T^*$  is complete with  $|T^{**}| < \lambda$  then

( $\oplus$ ) there is a model  $M$  of  $T^{**}$  of size  $\lambda$  and an  $M^{\bar{\psi}}$ -type  $p$  omitted by  $M$  such that for every elementary extension  $N$  of  $M$  of size  $\lambda$  which omits  $p$  and a type  $q$  (in one variable) over  $N^{\bar{\varphi}}$ , there is an elementary extension of  $N$  of size  $\lambda$  which realises  $q$  and omits  $p$ .

(2) Let  $T_1 \triangleleft^{**} T_2$  mean that  $T_1 \triangleleft_{\lambda}^{**} T_2$  holds for all large enough regular  $\lambda$ .

(3)  $T_1$  is said to be  $\triangleleft_{\lambda}^{**}$ -maximal iff there is no  $T_2$  such that  $T_1 \triangleleft_{\lambda}^{**} T_2$ . Similarly for  $\triangleleft^{**}$ .

The connection between this notion and  $\triangleleft^*$  is given by the following claim:

**Claim 3.2** Suppose that  $T_1, T_2$  are theories and  $\lambda > |T_1|, |T_2|$  satisfies  $2^{\lambda} = \lambda^+$ . Then

$$T_1 \triangleleft_{\lambda^+}^* T_2 \implies \neg(T_2 \triangleleft_{\lambda}^{**} T_1).$$

**Proof.** This statement is just a reformulation of the beginning of the proof of Theorem 1.17. In other words, let  $(T, \bar{\varphi}_1, \bar{\varphi}_2)$  show that  $T_1 \triangleleft_{\lambda^+}^* T_2$ . This means that  $|T| < \lambda^+$  but since  $\lambda^{<\lambda} = \lambda$  and  $\lambda > |T_1|, |T_2|$  we may assume that  $|T^*| < \lambda$ . Namely since there is a consistent theory  $T \supseteq \bar{\varphi}_1 \cup \bar{\varphi}_2$  in which  $\bar{\varphi}_i$  interprets  $T_i$ , and each  $T_i$  has size  $< \lambda$ , there is a consistent theory  $T'$  of size  $< \lambda$  which does the same. Without loss of generality  $T' \subseteq T$ . In particular  $|\tau(T')| < \lambda$  so by extending  $T'$  to a complete subtheory of  $T$  and renaming we may assume  $T'$  is complete. Any model  $M$  of  $T$  has a reduct  $N$  that is a model of  $T'$  and that satisfies  $M^{\bar{\varphi}} = N^{\bar{\varphi}}$  and similarly for  $\bar{\psi}$ .

Hence  $(T', \bar{\varphi}, \bar{\psi})$  is a  $\lambda$ -relevant  $(T_1, T_2)$ -superior that exemplifies  $T_1 \triangleleft_{\lambda^+}^* T_2$ , so by renaming we may assume  $|T| < \lambda$ .

Suppose for contradiction that  $T_2 \triangleleft_{\lambda}^{**} T_1$  and let  $(T^*, \bar{\varphi}, \bar{\psi})$  exemplify this. Without loss of generality,  $\bar{\varphi}_1 = \bar{\psi}$  and  $\bar{\varphi}_2 = \bar{\varphi}$  and the common vocabulary of  $T$  and  $T^*$  is  $\tau(\bar{\varphi}_1) \cup \tau(\bar{\varphi}_2)$ . Hence  $T^{**} = T \cup T^*$  is consistent by Robinson Consistency Criterion. Without loss of generality  $T^{**}$  is complete. Hence let  $M$  be a model of  $T^{**}$  of size  $\lambda$  and  $p$  be a  $M^{\bar{\psi}}$  type omitted by  $M$  exemplifying the definition of  $\triangleleft_{\lambda}^{**}$ . Using the assumption  $2^\lambda = \lambda^+$  we can build by induction an elementary extension  $N$  of  $M$  with  $|N| = \lambda^+$ , with  $N$  omitting  $p$  and being  $\bar{\varphi}$ -saturated. This is a contradiction with the choice of  $T$ .  $\star_{3.2}$

**Corollary 3.3** Suppose that for all large enough regular  $\lambda$  we have  $2^\lambda = \lambda^+$ . Then any  $\triangleleft^*$ -maximal theory is also  $\triangleleft^{**}$ -maximal.

**Proof.** Suppose otherwise and let  $T$  exemplify this. Hence for every  $\kappa$  there is regular  $\lambda > \kappa$  such that  $T$  is not  $\triangleleft^{**}$ -maximal and  $2^\lambda = \lambda^+$ . Hence  $T$  is not  $\triangleleft_{\lambda^+}^*$ -maximal by Claim 3.2, a contradiction.  $\star_{3.3}$

The next notion we need is a syntactic property.

**Definition 3.4** Let  $T$  be a theory.

(1) For a formula  $\sigma(x, \bar{y})$  we say that  $\sigma(x, \bar{y})$  has  $\text{SOP}_2''$  iff for some [by compactness equivalently all] regular  $\lambda > |T|$  there is a sequence

$$\langle \bar{e}_{\bar{\eta}} : \bar{\eta} = \langle \eta_0, \dots, \eta_{n^*-1} \rangle, \eta_0 \triangleleft \eta_1 \triangleleft \dots \triangleleft \eta_{n^*-1} \in {}^\lambda \lambda \text{ and } \text{lg}(\eta_i) \text{ a successor} \rangle$$

such that

( $\alpha$ ) for each  $\eta \in {}^\lambda \lambda$ , the set

$$\left\{ \begin{array}{l} \sigma(x, \bar{e}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \upharpoonright (\alpha_0 + 1), \eta \upharpoonright (\alpha_1 + 1), \dots, \eta \upharpoonright (\alpha_{n^*-1} + 1) \rangle \\ \text{and } \alpha_0 < \alpha_1 < \dots < \alpha_{n^*-1} < \lambda \end{array} \right\}$$

is consistent

( $\beta$ ) for every large enough  $m$ , if  $g : {}^{n^* \geq m} \rightarrow {}^{\lambda >} \lambda$  satisfies

$$\rho \triangleleft \nu \implies g(\rho) \triangleleft g(\nu)$$

and

$$\rho \in {}^{n^* \geq m} \implies \text{lg}(g(\rho)) \text{ is a successor,}$$

while for  $l < n^* - 1$

$$(g(\rho)) \smallfrown \langle l \rangle \leq g(\rho \smallfrown \langle l \rangle),$$

then

$$\{\sigma(x, \bar{e}_{\langle g(\rho|1), g(\rho|2), \dots, g(\rho) \rangle}) : \rho \in {}^{n^*} m\}$$

is inconsistent. Here  $n^* = \text{lg}(\bar{y})$  in  $\sigma(x, \bar{y})$ .

(2)  $T$  is said to have  $\text{SOP}_2''$  iff some  $\sigma(x, \bar{y})$  exemplifies it.

Our theorem 3.6 is phrased in terms of  $\text{SOP}_2''$ . Answering a question from an earlier version of this paper Shelah and Usvyatsov proved in [ShUs 844] the following Theorem 3.5, which then can be used together with theorem 3.6 to prove Corollary 3.9 which states that  $\triangleleft^*$ -maximality implies  $\text{SOP}_2$ .

**Theorem 3.5** (Shelah-Usvyatsov [ShUs 844]) For any theory  $T$ ,  $T$  has  $\text{SOP}_2$  iff it has  $\text{SOP}_2''$ .

**Main Theorem 3.6** For any theory  $T$  and regular cardinal  $\lambda > |T|$ , if  $T$  is  $\triangleleft_\lambda^{**}$ -maximal then  $T$  has  $\text{SOP}_2''$ .

**Proof.** Let  $T$  be a given theory and let  $\lambda = \text{cf}(\lambda) > |T|$ . We shall assume that  $T$  is  $\triangleleft_\lambda^{**}$ -maximal and prove that  $T$  has  $\text{SOP}_2''$ . To make the reading of the proof easier we shall break it into stages.

**Stage A.** Let  $T_{\text{tree}}^n \stackrel{\text{def}}{=} \text{Th}({}^{n \geq 2}, <_{\text{tr}})$  for  $n < \omega$ , where  $<_{\text{tr}}$  stands for the relation of “being an initial segment of”, and let  $T_{\text{tree}} \stackrel{\text{def}}{=} \lim \langle T_{\text{tree}}^n : n < \omega \rangle$ , that is to say the set of all  $\psi$  which are in  $T_{\text{tree}}^n$  for all large enough  $n$ . In order to use our assumptions at a later point, let us fix a theory  $T^*$  which is a  $\lambda$ -relevant  $(T_{\text{tree}}, T)$ -superior with Skolem functions (such a  $T^*$  is easily



seen to exist), and let  $\bar{\varphi}, \bar{\psi}$  be the interpretations of  $T_{\text{tree}}$  and  $T$  in  $T^*$ , respectively. We can without loss of generality, by renaming if necessary, assume that  $\mathcal{L}(T) \subseteq \mathcal{L}(T^*)$ , so the interpretation  $\bar{\psi}$  is trivial.

As  $|T|, |T^*| < \lambda$ , we can find  $A \subseteq \lambda$  which codes  $T$  and  $T^*$ . Working in  $\mathbf{L}[A]$ , we shall define a model  $M$  of  $T^*$  of size  $\lambda$  as follows. Let

$$\begin{aligned} \Gamma \stackrel{\text{def}}{=} & T^* \cup \{\varphi_{=(x_\eta, x_\nu)} : \eta \in {}^{\lambda}>\lambda\} \\ & \cup \{x_\eta <_\varphi x_\nu : \eta \triangleleft \nu \in {}^{\lambda}>\lambda\} \\ & \{ \neg(x_\eta <_\varphi x_\nu) : \neg(\eta \triangleleft \nu) \text{ for } \eta, \nu \in {}^{\lambda}>\lambda \}. \end{aligned}$$

By a compactness argument and the fact that  $\bar{\varphi}$  interprets  $T_{\text{tree}}$  in  $T^*$ , we see that  $\Gamma$  is consistent. Let  $M$  be a model of  $\Gamma$  of size  $\lambda = \lambda^{<\lambda}$  (as we are in  $\mathbf{L}[A]$ ). For  $\eta \in {}^{\lambda}>\lambda$  let  $a_\eta$  be the realisation of  $x_\eta$  in  $M$ . For  $\eta \in {}^\lambda\lambda$ , let

$$p_\eta(x) \stackrel{\text{def}}{=} \{a_{\eta \upharpoonright \alpha} <_\varphi x : \alpha < \lambda\}$$

By the choice of  $M$  and the compactness argument it follows that each  $p_\eta$  is a (consistent) type. Note that for  $\eta_0 \neq \eta_1 \in {}^\lambda\lambda$ , types  $p_{\eta_0}$  and  $p_{\eta_1}$  are contradictory. Let

$$p'_\eta(x) = \{a <_\varphi x : \text{for some } \alpha < \lambda, a <_\varphi a_{\eta \upharpoonright \alpha}\}.$$

By the axioms of  $T_{\text{tree}}$ , we have that  $p_\eta$  and  $p'_\eta$  are equivalent. Now we observe that by the size of  $M$  there is  $\eta^* \in {}^\lambda\lambda$  such that the type  $p'_{\eta^*}$  is omitted in  $M$ , and  $p'_{\eta^*}$  is not definable in  $M$ , i.e. for no formula  $\vartheta(y, \bar{z})$  and  $\bar{c} \subseteq M$  do we have: for  $a \in M$ , the following are equivalent:  $[a <_\varphi x] \in p'_{\eta^*}$  and  $M \models \vartheta[a, \bar{c}]$ . Let  $p \stackrel{\text{def}}{=} p'_{\eta^*}$  for such a fixed  $\eta^*$ . For  $\alpha < \lambda$ , let  $a_\alpha \stackrel{\text{def}}{=} a_{\eta^* \upharpoonright \alpha}$ . We now go back to  $V$  and make an observation about  $M$ .

**Subclaim 3.7**  $T_{\text{tree}}$  satisfies the following property:

for any formula  $\vartheta(x, \bar{y})$  we have that  $T_{\text{tree}} \vdash \sigma = \sigma(\vartheta)$ , where

$$\begin{aligned} \sigma \equiv & (\forall \bar{y}) [ ((\forall x_1, x_2) \vartheta(x_1, \bar{y}) \wedge \vartheta(x_2, \bar{y})) \implies x_1 \leq_{\text{tr}} x_2 \vee x_2 \leq_{\text{tr}} x_1 ] \\ & \implies (\exists z) (\forall x) (\vartheta(x, \bar{y}) \implies x \leq_{\text{tr}} z) ]. \end{aligned}$$

**Proof of the Subclaim.** Let  $\vartheta(x, \bar{y})$  be given. By the definition of  $T_{\text{tree}}$  we only need to show that  $T_{\text{tree}}^n \vdash \sigma$  for all large enough  $n$ , which is obvious as for every  $n$  the tree  $n \geq 2$  has the top level.  $\star_{3.7}$

Hence the interpretation  $\bar{\varphi}$  of  $T_{\text{tree}}$  in  $T^*$  satisfies the same statement claimed about  $T_{\text{tree}}$ . We conclude:

⊗ if  $M \prec N$  and  $p$  is not realised in  $N$ , then there is no  $\vartheta(x, \bar{c})$  with  $\bar{c} \subseteq N$  such that  $\vartheta(a_{\eta^* \upharpoonright \alpha}, \bar{c})$  for all  $\alpha < \lambda$  holds and every two elements of  $N$  satisfying  $\vartheta(x, \bar{c})$  are  $<_{\varphi}$ -comparable.

**Stage B.** We shall choose a filtration  $\bar{M} = \langle M_i : i < \lambda \rangle$  of  $M$ , and an increasing sequence  $\langle \alpha_i : i < \lambda \rangle$ , requiring:

- (a)  $M_i \prec M$  and  $M_i$  are  $\prec$ -increasing continuous of size  $< \lambda$ , with  $M$  being the  $\bigcup_{i < \lambda} M_i$ ,
- (b)  $a_{\alpha_i} \in M_{i+1} \setminus M_i$ .

We may note that the branch induced by  $\{a_{\alpha_i} : i < \lambda\}$  is the same as the one induced by  $\{a_{\alpha} : \alpha < \lambda\}$ . Hence  $p$  is realised in any model in which  $p'(x) \stackrel{\text{def}}{=} \{a_{\alpha_i} <_{\varphi} x : i < \lambda\}$  is realised (or even the similarly defined type using any unbounded subset of  $\{\alpha_i : i < \lambda\}$ ). Hence, by renaming, without loss of generality we have  $\alpha_i = i$  for all  $i < \lambda$ .

**Stage C.** At this point we shall use the  $\triangleleft_{\lambda}^{**}$ -maximality of  $T$ , which implies that it is not true that  $T \triangleleft_{\lambda}^{**} T_{\text{tree}}$ . In particular, our  $T^*$ ,  $M$  and  $p$  do not exemplify this, hence there is  $N$  with  $M \prec N$  and  $\|N\| = \lambda$ , such that  $N$  omits  $p$ , but for some  $N^{[\bar{\psi}]}$ -type  $q$  over  $N$ , whenever  $N \prec N^+$  and  $N^+$  realises  $q$ , also  $N^+$  realises  $p$ . By ⊗, the branch induced by  $\{a_{\eta^* \upharpoonright \alpha} : \alpha < \lambda\}$  is not definable in  $N$ , so without loss of generality  $N = M$ . We can also assume that  $q$  is a complete type over  $M^{[\bar{\psi}]}$ . Let us now use the choice of  $q$  to define for each club  $E$  of  $\lambda$  a family of formulae associated with it, and to show that each of these families is inconsistent. We use the abbreviation c.d. for “the complete diagram of”.

For any club  $E$  of  $\lambda$  we define

$$\Gamma_E \stackrel{\text{def}}{=} \text{c. d.}(M) \cup q(x) \cup \{\neg(a_i <_{\varphi} \tau(x, \bar{b})) : i \in E, \tau \text{ a term of } T^*, \bar{b} \subseteq M_i\}.$$

Clearly, for any club  $E$ , if  $\Gamma_E$  is consistent then there is a model  $N$  in which  $\Gamma_E$  is realised. Identifying any  $b \in M$  with its interpretation in  $N$  and letting  $a^*$  be the interpretation of  $x$  from  $\Gamma_E$ , we can assume that  $N$  is an elementary

extension of  $M$  in which  $q$  is realised by  $a^*$ . As  $T^*$  has Skolem functions, we have  $M \prec N$ . Let  $N_1$  be the submodel of  $N$  with universe

$$A^* \stackrel{\text{def}}{=} M \cup \bigcup_{i \in E} \{\tau(a^*, \bar{b}) : \bar{b} \subseteq M_i \text{ and } \tau \text{ a term of } T^*\}.$$

Note that the size of  $N_1$  is  $\lambda$ . Clearly,  $N_1$  is closed under the functions of  $T^*$ , so  $M \subseteq N_1 \subseteq N$ . As  $T^*$  has Skolem functions, we get that  $M \prec N_1 \prec N$ . By the third part of the definition of  $\Gamma_E$ ,  $p$  is omitted in  $N_1$ . This is in contradiction with our assumptions, as  $a^* \in N_1$  realises  $q(x)$ .

Hence we can conclude

for every club  $E$  of  $\lambda$ , the set  $\Gamma_E$  is inconsistent.

**Stage D.** Now we start our search for a formula that exemplifies that  $T$  has  $\text{SOP}_2''$ . In the following definitions, we shall use the expression “an almost branch” or the abbreviation a.b. to stand for a set linearly ordered by  $<_\varphi$  (but not necessarily closed under  $<_\varphi$ -initial segments and not necessarily unbounded). Let

$$\Theta_{T^*}^0 \stackrel{\text{def}}{=} \left\{ \begin{array}{l} \vartheta(x, y, \bar{z}) : \text{there is } l = l_\vartheta < \omega \text{ such that} \\ \text{for every } M^* \models T^*, a \in M^*, \bar{c} \subseteq M^*, \text{ the set} \\ \vartheta(a, y, \bar{c})^{M^*} \text{ is the union of } \leq l \text{ a.b. in } M^{*[\bar{c}]} \end{array} \right\},$$

and let  $\Theta_{T^*}$  be the set of all  $\vartheta(x, \bar{y}, \bar{z})$  of the form  $\bigvee_{j < n} \vartheta_j(x, y_j, \bar{z}_j)$  for some  $\vartheta_0, \dots, \vartheta_{n-1} \in \Theta_{T^*}^0$  (where  $\bar{y} = \langle y_j : j < n \rangle$  and  $\bar{z} = \widehat{\langle \bar{z}_j : j < n \rangle}$ ). The formulae in  $\Theta_{T^*}$  will be called candidates. For every candidate

$$\vartheta(x, \bar{y}, \bar{z}) \equiv \bigvee_{j < n} \vartheta_j(x, y_j, \bar{z}_j)$$

and a  $\bar{\psi}$ -formula  $\sigma(x, t)$ , we consider the following game  $\mathfrak{D}_{n, \sigma, \vartheta}$  (whose definition also depends on our fixed  $p, q$  and  $\bar{M}$ ), played by two players  $\exists$  and  $\forall$ . The game starts by  $\exists$  playing  $\bar{b}^0$  from  ${}^{lg(\bar{z}_0)}M$ , then  $\forall$  playing  $\alpha_0 < \lambda$ . After that  $\exists$  chooses  $\beta_0 \in (\alpha_0, \lambda)$  and  $\bar{b}^1 \in {}^{lg(\bar{z}_1)}M$  such that  $\bar{b}^0 \in {}^{lg(\bar{z}_0)}M_{\beta_0}$ , after which  $\forall$  chooses  $\alpha_1 < \lambda$  etc., finishing by  $\exists$  choosing  $\bar{b}^{n-1} \in {}^{lg(\bar{z}_{n-1})}M$  and  $\forall$

choosing  $\alpha_{n-1}$ , while  $\exists$  chooses  $\beta_{n-1} \in (\alpha_{n-1}, \lambda)$  such that  $\bar{b}^{n-1} \in {}^{lg(\bar{z}_{n-1})}M_{\beta_{n-1}}$ . Player  $\exists$  wins this game iff for some  $\bar{e} \in {}^{lg(\bar{t})}M$  we have

$$\sigma(x, \bar{e}) \in q \text{ and } M \models (\forall x)[\sigma(x, \bar{e}) \implies \vartheta(x, \langle a_{\beta_0}, \dots, a_{\beta_{n-1}} \rangle, \widehat{\langle b^k \rangle}_{k < n})]. \quad (\otimes_1)$$

(Note: the constants  $a_{\beta_k}$  are from the set  $\{a_i : i < \lambda\}$  we chose above.) Observe that every sequence  $\langle \alpha_0, \dots, \alpha_{n-1} \rangle \in {}^n\lambda$  is an admissible sequence of moves for  $\forall$ .

We shall show that for some  $n \geq 1$  and  $\sigma, \vartheta$ , player  $\exists$  has a winning strategy in the game  $\mathcal{D}_{n, \sigma, \vartheta}$ , where  $\vartheta = \bigvee_{j < n} \vartheta_j$  as above. As these are determined games, it suffices to show that for some  $n \geq 1$  and  $\sigma, \vartheta$ , player  $\forall$  does not have a winning strategy. Suppose that this is not the case, arguing in  $(\mathcal{H}(\chi), \in, <^*_\chi, \bar{M}, p, q)$ , where  $\chi$  is large enough and  $<^*_\chi$  is a fixed well ordering of  $\mathcal{H}(\chi)$ . Fix for a moment  $(n, \sigma, \vartheta)$ . Player  $\forall$  has a winning strategy in  $\mathcal{D}_{n, \sigma, \vartheta}$ , which, replacing the ordinals  $\alpha_l$  by constants  $a_{\alpha_l}$ , can be represented by a sequence of functions  $G^l_{n, \sigma, \vartheta}$  for  $l < n$  (in  $(\mathcal{H}(\chi), \in, <^*_\chi, \bar{M}, p, q)$ ), where for  $l < n$ , if the play up to time  $l$  has been  $\bar{b}_0, \alpha_0, \beta_0, \dots, \alpha_{l-1}, \beta_{l-1}, \bar{b}^l$ , then  $G^l_{n, \sigma, \vartheta}$  applied to this play is  $a_{\alpha_l}$  for the  $\alpha_l$  in the choice of player  $\forall$ . We shall assume that these functions are the  $<^*$ -first which can act in this manner. Using this and elementarity, we notice that for every  $n, \sigma, \vartheta$  the values of  $G^l_{n, \sigma, \vartheta}$  take place in  $M$ , and that

$$E_0 \stackrel{\text{def}}{=} \{ \delta < \lambda : (\forall \sigma, \vartheta)(\forall n)(\forall l < n)[M \cap \text{Skolem}_{(\mathcal{H}(\chi), \in, \bar{M}, G^l_{n, \sigma, \vartheta})}(M_\delta) = M_\delta] \}$$

is a club of  $\lambda$  (as  $|T^*|, ||M_i|| < \lambda$  for all  $i$  and  $\bar{M}$  is increasing continuous). Let  $E \stackrel{\text{def}}{=} \text{acc}(E_0)$ . Consider now the set  $\Gamma_E$ . It is contradictory, so there is a finite subset of it which is contradictory. Hence for some  $n_0, n_1, n_2 < \omega$  and formulae  $\varrho_l(\bar{z}_l)$  ( $l < n_0$ ) from the c.d.( $M$ ), formulae  $\sigma_k(x, \bar{e}_k)$  ( $k < n_1$ )  $\in q(x)$ , ordinals  $\delta_0 < \dots < \delta_{n_2-1} \in E$ , a sequence  $\langle \bar{b}_{j,l} : j < n_2, l < l_j \rangle$  with  $\bar{b}_{j,l} \subseteq M_{\delta_j}$  and terms  $\langle \tau_{j,l} : j < n_2, l < l_j \rangle$  of  $T^*$ , the following is inconsistent:

$$\bigwedge_{l < n_0} \varrho_l(\bar{z}_l) \wedge \bigwedge_{k < n_1} \sigma_k(x, \bar{e}_k) \wedge \bigwedge_{j < n_2, l < l_j} \neg (a_{\delta_j} <_\varphi \tau_{j,l}(x, \bar{b}_{j,l})).$$

As  $\varrho_l$  come from the c.d.( $M$ ) and  $q(x)$  is a complete type over  $M^{\bar{\psi}}$ , we may assume that  $n_0 = 1$  and  $n_1 = 1$ . Note that we must have  $n_2 \geq 1$  and that

there is no loss of generality in assuming that  $\bar{b}_{j,l} = \bar{b}_j$  for all  $l < l_j$  for  $j < n$ . We shall omit the subscript 0 from  $\varrho, \sigma, \bar{e}$ . Let  $n = n_2$  and let us define  $\vartheta_j(x, y_j, \bar{z}_j)$  for  $j < n$  by

$$\vartheta_j(x, y_j, \bar{z}_j) \equiv \bigvee_{l < l_j} y_j <_{\varphi} \tau_{j,l}(x, \bar{z}_j),$$

and let  $\vartheta = \bigvee_{j < n} \vartheta_j$ . Note that for each  $j$  we have that  $\vartheta_j \in \Theta_{T^*}^0$ , as  $<_{\varphi}$  is a tree order. Hence  $\vartheta$  is a candidate,  $\sigma(x, \bar{e}) \in q(x)$ , and since  $M \models \varrho[\bar{d}]$  for some  $\bar{d}$  we have

$$M \models (\forall x)[\sigma(x, \bar{e}) \implies \bigvee_{j < n} \vartheta_j(x, a_{\delta_j}, \bar{b}_j)]. \quad (*)$$

Now we consider the following play of  $\mathcal{D}_{n,\sigma,\vartheta}$ . Let  $\exists$  choose  $\bar{b}_0$ . Recall that  $\bar{b}_0 \subseteq M_{\delta_0}$ . The strategy  $G_{n,\sigma,\vartheta}^0$  of  $\forall$  yields an ordinal  $\alpha_0$ . By the choice of  $E_0$  we have  $\alpha_0 < \delta_0$  and  $\bar{b}_0 \in M_{\delta_0}$ , so we can let  $\exists$  choose  $\beta_0 = \delta_0$ . Let  $\exists$  choose  $\bar{b}_1$  and then let  $\forall$  choose  $\alpha_1$  according to the strategy, etc. At the end of the play, player  $\forall$  should have won (as he/she used the supposed winning strategy), but clearly (\*) implies that  $\exists$  won, a contradiction.

**Stage E.** We conclude that (for our  $\lambda, \bar{M}, p, q$ ), for some  $\sigma, \vartheta$  and  $n \geq 1$  the player  $\exists$  has a winning strategy in the game  $\mathcal{D}_{n,\sigma,\vartheta}$ , call it  $St$ . Let us fix  $n = n^*, \sigma, \vartheta$ , and  $St$  and use them to get  $SOP_2''$ .

For any  $\bar{\alpha} = \langle \alpha_0, \dots, \alpha_{n-1} \rangle \in {}^n\lambda$ , we can let  $\langle \bar{b}^{\bar{\alpha}|k}, \beta^{\bar{\alpha}|(k+1)} : k < n \rangle$  be the sequence of moves that  $\exists$  plays by following the winning strategy  $St$  in a play in which  $\forall$  plays  $\bar{\alpha}$ , as the dependence is as marked. Let  $E$  be a club of  $\lambda$  such that if  $k \leq n$  and  $\alpha_0 < \dots < \alpha_{k-1} < \delta \in E$ , then  $\bar{b}^{\langle \alpha_0, \dots, \alpha_{k-1} \rangle} \in {}^{lg(\bar{z}_j)}M_{\delta}$ . (Such a club can be found by a method similar to the one used in Stage D). Renaming the  $M_i$  and  $a_i$ 's, we can without loss of generality assume that  $E = \lambda$ . For  $\bar{\alpha} \in {}^n\lambda$  let  $\bar{e}^{\bar{\alpha}}$  be such that:

$$M \models \forall x[\sigma(x, \bar{e}^{\bar{\alpha}}) \implies \bigvee_{j < n} \vartheta_j(x, a_{\beta^{\bar{\alpha}|(j+1)}}, \bar{b}_j^{\bar{\alpha}|(j+1)})].$$

Notice that  $\sigma$  is a formula in the language of  $T$ . We shall show that  $\sigma$ , together with a conveniently chosen sequence of  $\bar{e}_{\bar{\eta}}$ 's, exemplifies  $SOP_2''$ . The proof now proceeds similarly to the proof of Main Claim 1.13. Namely

**Lemma 3.8** There are sequences

$$\langle N_\eta : \eta \in {}^{\lambda>}\lambda \rangle, \langle h_\eta : \eta \in {}^{\lambda>}\lambda \rangle$$

such that

- (i)  $h_\eta$  is an elementary embedding of  $M_{lg(\eta)}$  into  $\mathfrak{C}_{T^*}$  with range  $N_\eta$ ,
- (ii)  $\nu \trianglelefteq \eta \implies h_\nu \subseteq h_\eta$ ,
- (iii) for  $\alpha \neq \beta < \lambda$  and  $\eta \in {}^{\lambda>}\lambda$  we have

$$h_{\eta \smallfrown \langle \alpha \rangle}(a_{lg(\eta)}) \perp_\varphi h_{\eta \smallfrown \langle \beta \rangle}(a_{lg(\eta)}),$$

- (iv)  $N_{\eta_0} \cap N_{\eta_1} = N_{\eta_0 \cap \eta_1}$  for all  $\eta_0, \eta_1$ .

**Proof of the Lemma.** This Lemma has the same proof as that of Main Claim 1.13 Stage B. In the notation of that proof, ignore  $b_{\delta_i}$ . When defining  $\Gamma$  use

$$\Gamma = \cup_{\alpha < \lambda} \Gamma_0^\alpha \cup \cup_{\alpha < \lambda} \Gamma_3^\alpha \cup \Gamma_4 \cup \Gamma_2^+,$$

where  $\Gamma_2^+ = \{x_0^\alpha \perp_\varphi x_0^\beta : \alpha \neq \beta < \lambda\}$  and  $\Gamma_0^\alpha, \Gamma_3^\alpha$  and  $\Gamma_4$  are defined as in the proof of Main Claim 1.13, allowing for the replacement of  ${}^{\lambda>}2$  by  ${}^{\lambda>}\lambda$  by using  $\{\bar{x}^\alpha : \alpha < \lambda\}$  in place of  $\{\bar{x}^0, \bar{x}^1\}$ . Assumptions on  $\Gamma_0^\alpha, \Gamma_2^+$  and  $\Gamma_3^\alpha$  are analogous to the ones we made in that proof. Fact 1.16 still holds, except that we drop the last set from the definition of  $r(\bar{x})$ . The rest of the proof is the same, recalling that the branch induced by  $\{a_i : i < \lambda\}$  is undefinable in  $M$ .  $\star_{3.8}$

**Stage F.** For  $\eta \in {}^\lambda\lambda$ , let  $h_\eta \stackrel{\text{def}}{=} \cup_{\alpha < \lambda} h_{\eta \upharpoonright \alpha}$ . Let  $q_\eta \stackrel{\text{def}}{=} h_\eta(q)$ , hence each  $q_\eta$  is a consistent type. For  $\bar{\eta} = \langle \eta_0, \dots, \eta_{n-1} \rangle$  and  $\eta_0 \triangleleft \dots \triangleleft \eta_{n-1}$  with  $lg(\eta_i) = \alpha_i + 1$ , let  $\bar{e}_{\bar{\eta}} \stackrel{\text{def}}{=} h_{\eta_{n-1}}(\bar{e}^{\langle \alpha_0, \dots, \alpha_{n-1} \rangle})$ .

Suppose now that  $\eta \in {}^\lambda\lambda$  is given, and consider the set

$$\{\sigma(x, \bar{e}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \upharpoonright (\alpha_0 + 1), \dots, \eta \upharpoonright (\alpha_{n-1} + 1) \rangle \text{ for some } \alpha_0 < \dots < \alpha_{n-1} < \lambda\}.$$

This set is a subset of  $q_\eta$ , and is hence consistent. This proves property  $(\alpha)$  from the definition of  $\text{SOP}_2''$ . For  $(\beta)$ , let  $m$  be large enough and  $g : {}^{n \geq m} \rightarrow {}^{\lambda>}\lambda$

be as in the statement of  $(\beta)$ . For  $\rho \in {}^n m$  let  $\bar{e}_{g_\rho} \stackrel{\text{def}}{=} \bar{e}_{\langle g(\rho \upharpoonright 1), \dots, g(\rho) \rangle}$  (note that this is always defined). We shall now show that the set

$$\{\sigma(x, \bar{e}_{g_\rho}) : \rho \in {}^n m\}$$

is inconsistent. Suppose otherwise, so let  $d \in \mathfrak{C}_{T^*}$  realise it. For each  $\rho \in {}^n m$ , let  $\eta_\rho \in {}^\lambda \lambda \supseteq g(\rho)$  and let  $\bar{\alpha}^\rho \stackrel{\text{def}}{=} \langle \alpha_0^\rho, \dots, \alpha_{n-1}^\rho \rangle$  satisfy  $\text{lg}(g(\rho \upharpoonright k)) = \alpha_k^\rho + 1$  for  $k \leq n$ , so for each  $k < n$  we have  $g(\rho \upharpoonright (k+1)) = \eta_\rho \upharpoonright (\alpha_k^\rho + 1)$ . Now we have that for each  $\rho \in {}^n m$

- (i)  $\sigma(x, \bar{e}_{g_\rho}) \equiv \sigma(x, h_{\eta_\rho \upharpoonright (\alpha_{n-1}^\rho + 1)}(\bar{e}^{\bar{\alpha}^\rho})) \in q_{\eta_\rho} \upharpoonright \sigma_{\eta_\rho}(x)$
- (ii)  $N_{\eta_\rho} \models (\forall x)[\sigma(x, \bar{e}_{g_\rho}) \implies \vartheta(x, \langle h_{\eta_\rho}(a_{\beta \bar{\alpha}^\rho \upharpoonright 1}), \dots, h_{\eta_\rho}(a_{\beta \bar{\alpha}^\rho}) \rangle, \widehat{h_{\eta_\rho}(\bar{b}^{\bar{\alpha}^\rho_j})})]$   
(hence the same holds in  $\mathfrak{C}_{T^*}$ ),
- (iii)

$$\vartheta(x, \langle h_{\eta_\rho}(a_{\beta \bar{\alpha}^\rho \upharpoonright 1}), \dots, h_{\eta_\rho}(a_{\beta \bar{\alpha}^\rho}) \rangle, \widehat{h_{\eta_\rho}(\bar{b}^{\bar{\alpha}^\rho_j})}) \implies \bigvee_{j < n} \vartheta_j(x, h_{\eta_\rho}(a_{\beta \bar{\alpha}^\rho \upharpoonright (j+1)}), h_{\eta_\rho}(\bar{b}_j^{\bar{\alpha}^\rho \upharpoonright (j+1)}))$$

for our  $\vartheta_0, \dots, \vartheta_{n-1}$ .

For each  $\rho \in {}^n m$  let  $j(\rho) < n$  be the first such that

$$\vartheta_j(d, h_{\eta_\rho}(a_{\beta \bar{\alpha}^\rho \upharpoonright (j+1)}), h_{\eta_\rho}(\bar{b}_j^{\bar{\alpha}^\rho \upharpoonright (j+1)}))$$

holds. Let  $l^* = \max\{l_0^\vartheta, \dots, l_{n-1}^\vartheta\}$ .

As  $m$  is large enough, there are  $\rho_0, \dots, \rho_{l^*} \in {}^n m$  such that  $j(\rho_s) = j^*$  for all  $s \in \{0, \dots, l^*\}$ , while  $\rho_s \upharpoonright j^*$  is fixed and  $\rho_s(j^*) \neq \rho_t(j^*)$  for  $s \neq t \leq l^*$ . (We use that there is a full  $l^*+1 \geq n$  subtree  $t^*$  of  ${}^{n \geq m}$  such that for all  $\rho \in t^* \cap {}^n m$  we have  $j(\rho) = j^*$ . Choose  $\rho_s$  belonging to  $t^*$  and splitting at the level  $j^*$ ). In particular,  $\alpha_0^{\rho_s} = \alpha_0, \dots, \alpha_{j^*-1}^{\rho_s} = \alpha_{j^*-1}$  is fixed, and so is  $h_{\eta_{\rho_s}} \upharpoonright M_{\alpha_{j^*-1}^*+1}$ , but

$$g(\rho_s) \upharpoonright (\alpha_{j^*-1} + 2) \text{ for } s \leq l^* \text{ are incomparable in } {}^\lambda \lambda. \quad (**)$$

Let  $\bar{\alpha} \stackrel{\text{def}}{=} \bar{\alpha}^{\rho_0}$ .

For each  $\rho \in {}^n m$  and  $k < n$  we have that  $\bar{b}^{\bar{\alpha}^{\rho \uparrow (k+1)}} \in M_{\alpha_{k+1}^\rho}$  (by the choice of  $E$ ), so in particular  $\bar{b}^{\bar{\alpha}^{\rho \uparrow j^*}} \in M_{\alpha_{j^*-1+1}^\rho}$ , and hence  $h_{\eta_{\rho_s}}(\bar{b}^{\bar{\alpha}^{\rho \uparrow j^*}})$  is a fixed  $\bar{b}^*$ . By the choice of  $d$  and definitions of  $j^*, l^*$  and  $\Theta_{T^*}$ , there are  $s \neq t < l_{\vartheta_{j^*}} \leq l^*$  such that  $h_{\eta_{\rho_s}}(a_{\beta^{\bar{\alpha}^{\rho_s \uparrow (j^*+1)}}})$  and  $h_{\eta_{\rho_t}}(a_{\beta^{\bar{\alpha}^{\rho_t \uparrow (j^*+1)}}})$  are on the same almost branch. Now note that for all  $\rho$  we have

$$a_{\beta^{\bar{\alpha}^{\rho \uparrow (j^*+1)}}} \in M_{\beta^{\bar{\alpha}^{\rho \uparrow (j^*+1)}}_{+1}} \setminus M_{\beta^{\bar{\alpha}^{\rho \uparrow (j^*+1)}}}$$

and  $\beta^{\bar{\alpha}^{\rho \uparrow (j^*+1)}} > \alpha_{j^*}^\rho$ . Hence  $h_{\eta_{\rho_s}}(a_{\beta^{\bar{\alpha}^{\rho_s \uparrow (j^*+1)}}})$  and  $h_{\eta_{\rho_t}}(a_{\beta^{\bar{\alpha}^{\rho_t \uparrow (j^*+1)}}})$  are incomparable, by property (iii) in Lemma 3.8, a contradiction. This shows  $(\beta)$  from the definition of  $\text{SOP}_2''$ , so finishing the proof.  $\star_{3.6}$

Putting this together with Corollary 3.3 and Shelah-Usvyatsov theorem 3.5 above we get the following corollary 3.9.

**Corollary 3.9** (1) Suppose that  $T$  is a theory that is  $\triangleleft^*$ -maximal in some universe of set theory in which  $2^\lambda = \lambda^+$  holds for all large enough regular  $\lambda$ . Then  $T$  has  $\text{SOP}_2$ .

(2) Suppose that  $T$  is a theory that is  $\triangleleft_{\lambda^+}^*$ -maximal in some universe of set theory in which  $\lambda$  is regular and  $2^\lambda = \lambda^+$ . Then  $T$  has  $\text{SOP}_2$ .

**Proof.** (1) Let  $W$  be a universe of set theory in which  $2^\lambda = \lambda^+$  holds for all large enough regular  $\lambda$  and in which  $T$  is  $\triangleleft^*$ -maximal. Hence by Corollary 3.3  $T$  is  $\triangleleft^{**}$ -maximal in  $W$  and hence by Main Theorem 3.6 in  $W$  it satisfies  $\text{SOP}_2''$ . By Shelah-Usvyatsov Theorem 3.5 above  $T$  satisfies  $\text{SOP}_2$  in  $W$ . An application of the Compactness Theorem shows that satisfying  $\text{SOP}_2$  is absolute, hence  $T$  satisfies  $\text{SOP}_2$  in  $V$ .

(2) This follows similarly, but more directly, from Main Theorem 3.6 and the Shelah-Usvyatsov Theorem 3.5.  $\star_{3.9}$

This section hence provides us with the proof of one side of our thesis that  $\text{SOP}_2$  and  $\triangleleft^*$ -maximality are closely connected. Recall that Shelah proved in [Sh 500] that  $\text{SOP}_3$  implies  $\triangleleft^*$ -maximality. So an important open question (provided that  $\text{SOP}_3$  are not actually equivalent, which we still do not know) is



**Question 3.10** Does  $SOP_2$  imply  $\triangleleft^*$ -maximality?

In a partial answer to this question posed in an earlier version of the paper Shelan and Usvyatsov in Theorem 3.12 of [ShUs 844] provided a local positive answer to this question, where by “local” we mean that they proved that any theory with  $SOP_2$  is  $\triangleleft^*$  above  $T_{\text{tree}}$  when only types localised by a certain formula are considered (see Definition 1.3).

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