### Category analogue of sup-measurability problem

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#### Abstract

A function  $F: \mathbb{R}^2 \to \mathbb{R}$  is sup-measurable if  $F_f: \mathbb{R} \to \mathbb{R}$  given by  $F_f(x) = F(x, f(x)), x \in \mathbb{R}$ , is measurable for each measurable function  $f: \mathbb{R} \to \mathbb{R}$ . It is known that under different set theoretical assumptions, including CH, there are sup-measurable non-measurable functions, as well as their category analogues. In this paper we will show that the existence of the category analogues of sup-measurable non-measurable functions is independent of ZFC. A similar result for

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the original measurable case is the subject of a work in prepartion by Rosłanowski and Shelah.

#### 1 Introduction

Our terminology is standard and follows that from [3], [4], [10], or [12]. In particular, pr:  $X \times Y \to X$  will stand for the projection onto the first coordinate. A subset A of a Polish space X is nowhere meager provided  $A \cap U$  is not meager for every non-empty open subset of X.

The ternary Cantor subset of  $\mathbb{R}$  will be identified with with its homeomorphic copy,  $2^{\omega}$ , which stands for the set of all function  $x \colon \omega \to \{0,1\}$  considered with the product topology. In particular, the basic open subsets of  $2^{\omega}$  are in the form

$$[s] \stackrel{\text{def}}{=} \{ x \in 2^{\omega} \colon s \subset f \},\$$

where  $s \in 2^{<\omega}$ . Also, since  $\mathbb{R} \setminus \mathbb{Q}$  is homeomorphic to  $2^{\omega} \setminus E$  for some countable set E (the set of all eventually constant functions in  $2^{\omega}$ ) in our more technical part of the paper we will be able replace  $\mathbb{R}$  with  $2^{\omega}$ .

The study of sup-measurable functions<sup>1</sup> comes from the theory of differential equations. More precisely it comes from the question: For which functions  $F \colon \mathbb{R}^2 \to \mathbb{R}$  does the Cauchy problem

$$y' = F(x, y), \quad y(x_0) = y_0$$
 (1)

have a (unique) a.e.-solution in the class of locally absolutely continuous functions on  $\mathbb{R}$  in the sense that  $y(x_0) = y_0$  and y'(x) = F(x, y(x)) for almost all  $x \in \mathbb{R}$ ? (For more on this motivation see [8] or [2]. Compare also [9].) It is not hard to find measurable functions which are not sup-measurable. (See [13] or [1, Cor. 1.4].) Under the continuum hypothesis CH or some weaker set-theoretical assumptions nonmeasurable sup-measurable functions were constructed in [6], [7], [1], and [8]. The independence from ZFC of the existence of such an example is the subject of a work in prepartion by Rosłanowski and Shelah.

A function  $F: \mathbb{R}^2 \to \mathbb{R}$  is a category analogue of sup-measurable function (or Baire sup-measurable) provided  $F_f: \mathbb{R} \to \mathbb{R}$  given by  $F_f(x) = F(x, f(x))$ ,  $x \in \mathbb{R}$ , has the Baire property for each function  $f: \mathbb{R} \to \mathbb{R}$  with the Baire

<sup>&</sup>lt;sup>1</sup>This is abbreviation from superposition-measurable function.

property. A Baire sup-measurable function without the Baire property has been constructed under CH in [5]. (See also [1] and [2].) The main goal of this paper is to show that the existence of such functions cannot be proved in ZFC. For this we need the following easy fact. (See [1, Prop. 1.5].)

**Proposition 1** The following conditions are equivalent.

- (i) There is a Baire sup-measurable function  $F: \mathbb{R}^2 \to \mathbb{R}$  without the Baire property.
- (ii) There is a function  $F: \mathbb{R}^2 \to \mathbb{R}$  without the Baire property such that  $F_f$  has the Baire property for every Borel function  $f: \mathbb{R} \to \mathbb{R}$ .
- (iii) There is a set  $A \subset \mathbb{R}^2$  without the Baire property such that the projection  $\operatorname{pr}(A \cap f) = \{x \in \mathbb{R} : \langle x, f(x) \rangle \in A\}$  has the Baire property for each Borel function  $f : \mathbb{R} \to \mathbb{R}$ .
- (iv) There is a Baire sup-measurable function  $F: \mathbb{R}^2 \to \{0,1\}$  without the Baire property.

The equivalence of (i) and (ii) follows from the fact that the function  $F: \mathbb{R}^2 \to \mathbb{R}$  is Baire sup-measurable if and only if  $F_f$  has a Baire property for every Borel function  $f: \mathbb{R} \to \mathbb{R}$ .<sup>2</sup>

The main theorem of the paper is the following.

Theorem 2 It is consistent with the set theory ZFC that

 $\varphi$ : for every  $A \subset 2^{\omega} \times 2^{\omega}$  for which the sets A and  $A^c = (2^{\omega} \times 2^{\omega}) \setminus A$  are nowhere meager in  $2^{\omega} \times 2^{\omega}$  there exists a homeomorphism f from  $2^{\omega}$  onto  $2^{\omega}$  such that the set  $\operatorname{pr}(A \cap f)$  does not have the Baire property in  $2^{\omega}$ .

Before proving this theorem let us notice that it implies easily the following corollary.

**Corollary 3** The existence of Baire sup-measurable function  $F: \mathbb{R}^2 \to \mathbb{R}$  without the Baire property is independent from the set theory ZFC.

<sup>&</sup>lt;sup>2</sup>It is also true that  $F: \mathbb{R}^2 \to \mathbb{R}$  is Baire sup-measurable provided  $F_f$  has the Baire property for every Baire class one function  $f: \mathbb{R} \to \mathbb{R}$ , and that  $F: \mathbb{R}^2 \to \mathbb{R}$  is sup-measurable provided  $F_f$  is measurable for every continuous function  $f: \mathbb{R} \to \mathbb{R}$ . See for example [2, lemma 1 and remark 1].

PROOF. As mentioned above under CH there exist Baire sup-measurable functions without the Baire property. So, it is enough to show that the property  $\varphi$  from Theorem 2 implies that there are no such functions.

So, take an arbitrary  $A \subset \mathbb{R}^2$  without the Baire property. By (iii) of Proposition 1 it is enough to show there exists a Borel function  $f: \mathbb{R} \to \mathbb{R}$  for which the set  $\operatorname{pr}(A \cap f)$  does not have the Baire property.

We will first show this under the additional assumption that the sets A and  $\mathbb{R}^2 \setminus A$  are nowhere meager in  $\mathbb{R}^2$ . But then the set  $A_0 = A \cap (\mathbb{R} \setminus \mathbb{Q})^2$  and its complement are nowhere meager in  $(\mathbb{R} \setminus \mathbb{Q})^2$ . Moreover, since  $\mathbb{R} \setminus \mathbb{Q}$  is homeomorphic to  $2^{\omega} \setminus E$  for some countable set E we can consider  $A_0$  as a subset of  $(2^{\omega} \setminus E)^2 \subset 2^{\omega} \times 2^{\omega}$ . Then  $A_0$  and its complement are still nowhere meager in  $2^{\omega} \times 2^{\omega}$ . Therefore, by  $\varphi$ , there exists an autohomeomorphism f of  $2^{\omega}$  such that the set  $\operatorname{pr}(A_0 \cap f) = \{x \in 2^{\omega} \setminus E : \langle x, f(x) \rangle \in A_0\}$  does not have the Baire property in  $2^{\omega}$ . Now, as before,  $f \upharpoonright (2^{\omega} \setminus E)$  can be considered as defined on  $\mathbb{R} \setminus \mathbb{Q}$ . So if  $\bar{f} : \mathbb{R} \to \mathbb{R}$  is an extension of  $f \upharpoonright (2^{\omega} \setminus E)$  (under such identification) to  $\mathbb{R}$  which is constant on  $\mathbb{Q}$ , then  $\bar{f}$  is Borel and the set  $\operatorname{pr}(A_0 \cap \bar{f})$  does not have the Baire property in  $\mathbb{R}$ .

Now, if A is an arbitrary subset of  $\mathbb{R}^2$  without the Baire property we can find non-empty open intervals U and W in  $\mathbb{R}$  such that A and  $(U \times W) \setminus A$  are nowhere meager in  $U \times W$ . Since U and W are homeomorphic with  $\mathbb{R}$  the above case implies the existence of Borel function  $f_0: U \to W$  such that  $\operatorname{pr}(A \cap f_0)$  does not have the Baire property in U. So any Borel extension  $f: \mathbb{R} \to \mathbb{R}$  of  $f_0$  works.

## 2 Reduction of the proof of Theorem 2 to the main lemma

The theorem will be proved by the method of iterated forcing, a knowledge of which is needed from this point on.

The idea of the proof is quite simple. For every nowhere meager subset A of  $2^{\omega} \times 2^{\omega}$  for which  $A^c = (2^{\omega} \times 2^{\omega}) \setminus A$  is also nowhere meager we will find a natural ccc forcing notion  $Q_A$  which adds the required homeomorphism f. Then we will start with the constructible universe V = L and iterate with finite support these notions of forcing in such a way that every nowhere meager set  $A^* \subset 2^{\omega} \times 2^{\omega}$ , with  $(2^{\omega} \times 2^{\omega}) \setminus A^*$  nowhere meager, will be taken care of by some  $Q_A$  at an appropriate step of iteration.

There are two technical problems with carrying through this idea. First is that we cannot possibly list in our iteration all nowhere meager subsets of  $2^{\omega} \times 2^{\omega}$  with nowhere meager complements since the iteration can be of length at most continuum  $\mathfrak{c}$  and there are  $2^{\mathfrak{c}}$  such sets. This problem will be solved by defining our iteration as  $P_{\omega_2} = \langle \langle P_{\alpha}, \dot{Q}_{\alpha} \rangle : \alpha < \omega_2 \rangle$  such that the generic extension V[G] of V with respect to  $P_{\omega_2}$  will satisfy  $2^{\omega} = 2^{\omega_1} = \omega_2$  and have the property that

(m) every non-Baire subset  $A^*$  of  $2^{\omega}$  contains a non-Baire subset A of cardinality  $\omega_1$ .

Thus in the iteration we will use only the forcing notions  $Q_{\alpha} = Q_A$  for the sets A of cardinality  $\omega_1$ , whose number is equal to  $\omega_2$ , the length of iteration. Condition (m) will guarantee that this will give us enough control of all nowhere meager subsets  $A^*$  of  $2^{\omega} \times 2^{\omega}$ .

The second problem is that even if at some stage  $\alpha < \omega_2$  of our iteration we will add a homeomorphism f appropriate for a given set  $A \subset 2^{\omega} \times 2^{\omega}$ , that is such that

$$V[G_{\alpha}] \models \text{"pr}(A \cap f)$$
 is not Baire in  $2^{\omega}$ ,"

where  $G_{\alpha} = G \cap P_{\alpha}$ , then in general there is no guarantee that the set  $\operatorname{pr}(A \cap f)$  will remain non-Baire in the final model V[G]. The preservation of non-Baireness of each appropriate set  $\operatorname{pr}(A \cap f)$  will be achieved by carefully crafting our iteration following a method known as the *oracle-cc* forcing iteration.

The theory of the oracle-cc forcings is described in details in [12, Ch. IV] (compare also [11, Ch. IV]) and here we will recall only the fragments that are relevant to our specific situation. In particular if

$$\Gamma \stackrel{\text{def}}{=} \{ \lambda < \omega_1 : \lambda \text{ is a limit ordinal} \}$$

then

• an  $\omega_1$ -oracle is any sequence  $\mathcal{M} = \langle M_\delta \colon \delta \in \Gamma \rangle$  where  $M_\delta$  is a countable transitive model of ZFC<sup>-</sup> that is, ZFC without the power set axiom) with a property that  $\delta + 1 \subset M_\delta$ ,  $M_\delta \models$  " $\delta$  is countable," and the set  $\{\delta \in \Gamma \colon A \cap \delta \in M_\delta\}$  is stationary in  $\omega_1$  for every  $A \subset \omega_1$ .

The existence of an  $\omega_1$ -oracle is equivalent to the diamond principle  $\diamond$ .

With each  $\omega_1$ -oracle  $\mathcal{M} = \langle M_{\delta} \colon \delta \in \Gamma \rangle$  there is associated a filter  $D_{\mathcal{M}}$  generated by the sets  $I_{\mathcal{M}}(A) = \{ \delta \in \Gamma \colon A \cap \delta \in M_{\delta} \}$  for  $A \subset \omega_1$ . It is proved in [12, Claim 1.4] that  $D_{\mathcal{M}}$  is a proper normal filter containing every closed unbounded subset of  $\Gamma$ .

We will also need the following fact which, for our purposes, can be viewed as a definition of  $\mathcal{M}$ -cc property.

Fact 4 Let P be a forcing notion of cardinality  $\leq \omega_1$ ,  $e: P \to \omega_1$  be one-toone, and  $\mathcal{M} = \langle M_{\delta} : \delta \in \Gamma \rangle$  be an  $\omega_1$ -oracle. If there exists a  $C \in D_{\mathcal{M}}$  such that for every  $\delta \in \Gamma \cap C$ 

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e^{-1}(E) is predense in P for every set E \in M_{\delta} \cap \mathcal{P}(\delta), for which e^{-1}(E) is predense in e^{-1}(\{\gamma : \gamma < \delta\}),
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then P has the  $\mathcal{M}$ -cc property.

This follows immediately from the definition of  $\mathcal{M}$ -cc property [12, Definition 1.5, p. 150].

Our proof will rely on the following main lemma.

**Lemma 5** For every  $A \subset 2^{\omega} \times 2^{\omega}$  for which A and  $A^c = (2^{\omega} \times 2^{\omega}) \setminus A$  are nowhere meager in  $2^{\omega} \times 2^{\omega}$  and for every  $\omega_1$ -oracle  $\mathcal{M}$  there exists an  $\mathcal{M}$ -cc forcing notion  $Q_A$  of cardinality  $\omega_1$  such that  $Q_A$  forces

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there exists an autohomeomorphism f of 2^{\omega} such that the sets \operatorname{pr}(f \cap A) and \operatorname{pr}(f \setminus A) are nowhere meager in 2^{\omega}.
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The proof of Lemma 5 represents the core of our argument and will be presented in the next section. In the remainder of this section we will sketch how Lemma 5 implies Theorem 2. Since this follows the standard path, as described in [12, Chapter IV], the readers familiar with this treatment may proceed directly to the next section.

First of all, to define an appropriate iteration we will treat forcings  $Q_A$  from Lemma 5 as defined on  $\omega_1$ . More precisely, in the iteration we will always replace  $Q_A$  with its order isomorphic copy  $\langle \omega_1, \leq_A \rangle$ . So, we can treat any finite support iteration  $P_{\alpha} = \langle \langle P_{\beta}, \dot{Q}_{\beta} \rangle \colon \beta < \alpha \rangle$  of  $Q_A$  forcing notions as having an absolute and fixed universe, say  $U_{\alpha} = \{g \in (\omega_1)^{\omega_2} \colon g^{-1}(\omega_1 \setminus \{0\}) \in [\alpha]^{<\omega} \}$ . This will allow us to treat the  $\Diamond_{\omega_2}$ -sequence  $\langle X_{\alpha} \colon \alpha < \omega_2 \rangle$  as a sequence of  $P_{\alpha}$ -names of subsets of  $2^{\omega} \times 2^{\omega}$ . (After appropriate coding.)

We will also need the following variant of [12, Example 2.2].

**Lemma 6** Assume that  $\diamondsuit_{\omega_1}$  holds and that  $S \subset 2^{\omega}$  is such that S and  $S^c$  are nowhere meager in  $2^{\omega}$ . Then there exists an  $\omega_1$ -oracle  $\mathcal{M}$  such that if P is an arbitrary  $\mathcal{M}$ -cc forcing then P forces that

S and  $S^c$  are nowhere meager in  $2^{\omega}$ .

PROOF. By [12, Example 2.2] for any non-empty basic open set W of  $2^{\omega}$  there are oracles  $\mathcal{M}_W^0$  and  $\mathcal{M}_W^1$  such any  $\mathcal{M}_W^0$ -cc forcing forces that  $S \cap W$  is not meager, and any  $\mathcal{M}_W^1$ -cc forcing forces that  $S^c \cap W$  is not meager. So, by [12, Claim 3.1], there is a single  $\omega_1$ -oracle  $\mathcal{M}$  which "extends" all oracles  $\mathcal{M}_W^i$ , and it clearly does the job.

Now, the iteration  $P_{\omega_2}$  is defined by choosing by induction the sequence  $\langle \langle P_{\alpha}, \dot{A}_{\alpha}, \dot{\mathcal{M}}_{\alpha}, \dot{Q}_{\alpha}, \dot{f}_{\alpha} \rangle \colon \alpha < \omega_2 \rangle$  such that for every  $\alpha < \omega_2$ 

- (a)  $P_{\alpha} = \langle \langle P_{\beta}, \dot{Q}_{\beta} \rangle : \beta < \alpha \rangle$  is a finite support iteration,
- (b)  $\dot{A}_{\alpha}$  is a  $P_{\alpha}$ -name for which  $P_{\alpha}$  forces that  $\dot{A}_{\alpha}$  and  $(\dot{A}_{\alpha})^c$  are nowhere meager subsets of  $2^{\omega} \times 2^{\omega}$ ,
- (c)  $\dot{\mathcal{M}}_{\alpha}$  is a  $P_{\alpha}$ -name for which  $P_{\alpha}$  forces that  $\dot{\mathcal{M}}_{\alpha}$  is an  $\omega_1$ -oracle and for every  $\dot{Q}$  satisfying  $\dot{\mathcal{M}}_{\alpha}$ -cc we have
  - (i) for every  $\beta < \alpha$  if  $P_{\alpha} = P_{\beta} * \dot{P}_{\beta,\alpha}$  then  $P_{\beta} \Vdash "\dot{P}_{\beta,\alpha} * \dot{Q} \text{ is } \dot{\mathcal{M}}_{\beta}\text{-cc},"$
  - (ii) if  $\alpha = \gamma + 1$  then  $\dot{Q} \Vdash \text{"pr}(\dot{f}_{\gamma} \cap \dot{A}_{\gamma}), \text{pr}(\dot{f}_{\gamma} \setminus \dot{A}_{\gamma}) \subset 2^{\omega}$  are nowhere meager in  $2^{\omega}$ ,"
- (d)  $\dot{Q}_{\alpha}$  is a  $P_{\alpha}$ -name for a forcing such that  $P_{\alpha}$  forces  $\dot{Q}_{\alpha}$  is an  $\dot{\mathcal{M}}_{\alpha}$ -cc forcing  $Q_{\dot{A}_{\alpha}}$  from Lemma 5,
- (e)  $\dot{f}_{\alpha}$  is a  $P_{\alpha+1}$ -name for which  $P_{\alpha+1}$  forces that  $\dot{f}_{\alpha}$  is a  $\dot{Q}_{\alpha}$ -name for the function f from Lemma 5.

If for some  $\alpha < \omega_2$  the sequence  $\langle \langle P_{\beta}, \dot{A}_{\beta}, \dot{\mathcal{M}}_{\beta}, \dot{Q}_{\beta}, \dot{f}_{\beta} \rangle$ :  $\beta < \alpha \rangle$  has been defined then we proceed as follows. Forcing  $P_{\alpha}$  is already determined by (a). We choose  $\dot{A}_{\alpha}$  as  $X_{\alpha}$  from the  $\diamondsuit_{\omega_2}$ -sequence if it satisfies (b) and arbitrarily, still maintaining (b), otherwise. Since steps (d) and (e) are facilitated by Lemma 5, it is enough to construct  $\dot{\mathcal{M}}_{\alpha}$  satisfying (c). For this we will consider two cases.

Case 1:  $\alpha$  is a limit ordinal.

For a moment fix a  $\beta < \alpha$  and work in  $V^{P_{\beta}}$ . Let  $\mathcal{M}_{\beta}$  and  $P_{\beta,\alpha}$  be the interpretations of  $\dot{\mathcal{M}}_{\beta}$  and  $\dot{P}_{\beta,\alpha}$ , respectively. By the inductive assumption for every  $\beta < \gamma < \alpha$  forcing  $P_{\beta,\gamma}$  is  $\mathcal{M}_{\beta}$ -cc. So, by [12, Claim 3.2],  $P_{\beta,\alpha}$  is  $\mathcal{M}_{\beta}$ -cc. Thus, by [12, Claim 3.3], in  $(V^{P_{\beta}})^{P_{\beta,\alpha}} = V^{P_{\alpha}}$  there is an  $\omega_1$ -oracle  $\mathcal{M}_{\beta}^*$  such that if Q is  $\mathcal{M}_{\beta}^*$ -cc then  $P_{\beta,\alpha} * Q$  is  $\mathcal{M}_{\beta}$ -cc.

So, in  $V^{P_{\alpha}}$ , we have  $\omega_1$ -oracles  $\mathcal{M}^*_{\beta}$  for every  $\beta < \alpha$ . Thus, by [12, Claim 3.1], in  $V^{P_{\alpha}}$  there exists an  $\omega_1$ -oracle  $\mathcal{M}_{\alpha}$  which is stronger than all  $\mathcal{M}^*_{\beta}$ 's in a sense that if Q is  $\mathcal{M}_{\alpha}$ -cc then Q is also  $\mathcal{M}^*_{\beta}$ -cc. So, there is a  $P_{\alpha}$ -name  $\dot{\mathcal{M}}_{\alpha}$  for  $\mathcal{M}_{\alpha}$  for which (c) holds.

Case 2:  $\alpha$  is a successor ordinal,  $\alpha = \gamma + 1$ . Then  $P_{\alpha} = P_{\gamma} * \dot{Q}_{\gamma}$ .

Since, by (d),  $P_{\gamma}$  forces that  $\dot{Q}_{\gamma}$  is  $\dot{\mathcal{M}}_{\gamma}$ -cc, using (c) for  $\alpha = \gamma$  we conclude that

$$P_{\beta} \Vdash "\dot{P}_{\beta,\alpha} \text{ is } \dot{\mathcal{M}}_{\beta}\text{-cc}"$$

for every  $\beta < \gamma$ . So, proceeding as in Case 1, in  $V^{P_{\alpha}}$  we can find  $\omega_1$ -oracles  $\dot{\mathcal{M}}_{\beta}^*$  such that

$$P_{\beta} \Vdash "\dot{P}_{\beta,\alpha} * \dot{Q} \text{ is } \dot{\mathcal{M}}_{\beta}\text{-cc}"$$

for every Q which is  $\dot{\mathcal{M}}_{\beta}^*$ -cc. Let  $\mathcal{M}$  be an  $\omega_1$ -oracle from Lemma 6 used with  $S = \operatorname{pr}(\dot{f}_{\gamma} \cap \dot{A}_{\gamma})$ . As above we can find, in  $V^{P_{\alpha}}$ , an  $\omega_1$ -oracle  $\mathcal{M}_{\alpha}$  which is stronger than all  $\mathcal{M}_{\beta}^*$ 's and  $\mathcal{M}$ . Then, there is a  $P_{\alpha}$ -name  $\dot{\mathcal{M}}_{\alpha}$  for which (c) holds. This finishes the construction of the iteration.

To finish the argument first note that the interpretations of  $\operatorname{pr}(\dot{f}_{\alpha} \cap \dot{A}_{\alpha})$  and  $\operatorname{pr}(\dot{f}_{\alpha} \setminus \dot{A}_{\alpha})$  in the final model V[G] remain nowhere meager in  $2^{\omega}$ . This is the case since, by (e),  $P_{\alpha+1}$  forces that

$$\operatorname{pr}(\dot{f}_{\alpha} \cap \dot{A}_{\alpha})$$
 and  $\operatorname{pr}(\dot{f}_{\alpha} \setminus \dot{A}_{\alpha})$  are nowhere meager in  $2^{\omega}$ ,

and, by (c)(i), that

every 
$$\dot{P}_{\alpha+1,\gamma}$$
 is  $\dot{\mathcal{M}}_{\alpha+1}$ -cc

while, by condition (c)(ii), every  $\dot{\mathcal{M}}_{\alpha+1}$ -cc forcing preserves nowhere meagerness of  $\operatorname{pr}(\dot{f}_{\alpha} \cap \dot{A}_{\alpha})$  and  $\operatorname{pr}(\dot{f}_{\alpha} \setminus \dot{A}_{\alpha})$ . To finish this part of the argument it is enough to note that  $P_{\alpha+1}$  forces that " $\dot{P}_{\alpha+1,\omega_2}$  is  $\dot{\mathcal{M}}_{\alpha+1}$ -cc" which follows from [12, Claim 3.2].

To complete the argument it is enough to show that each nowhere meager subset  $A^*$  of  $2^{\omega} \times 2^{\omega}$  from V[G] with nowhere meager complement contains an interpretation of some  $\dot{A}_{\alpha}$ . However,  $P_{\omega_2}$  is ccc. So, if  $\dot{A}$  is a  $P_{\omega_2}$ -name for  $A^*$  then the set

$$\left\{ \alpha \in \Gamma \colon P_{\alpha} \Vdash \dot{A} \cap V^{P_{\alpha}} \text{ is nowhere meager in } 2^{\omega} \times 2^{\omega} \right\}$$

contains a closed unbounded subset of  $\Gamma$ . Thus  $\diamondsuit_{\omega_2}$  guarantees that  $A^*$  contains an interpretation of some  $\dot{A}_{\alpha}$ .

#### 3 Proof of Lemma 5

Let  $\mathcal{K}$  be the family of all sequences  $\bar{h} = \langle h_{\xi} \colon \xi \in \Gamma \rangle$  such that each  $h_{\xi}$  is a function from a countable set  $D_{\xi} \subset 2^{\omega}$  onto  $R_{\xi} \subset 2^{\omega}$  and that

$$D_{\xi} \cap D_{\eta} = R_{\xi} \cap R_{\eta} = \emptyset$$
 for every distinct  $\xi, \eta \in \Gamma$ .

For each  $\bar{h} \in \mathcal{K}$  we will define a forcing notion  $Q_{\bar{h}}$ . Forcing  $Q_A$  satisfying Lemma 5 will be chosen as  $Q_{\bar{h}}$  for some  $\bar{h} \in \mathcal{K}$ .

So fix an  $\bar{h} \in \mathcal{K}$ . Then  $Q_{\bar{h}}$  is defined as the set of all triples  $p = \langle n, \pi, h \rangle$  for which

- (A) h is a function from a finite subset D of  $\bigcup_{\xi \in \Gamma} D_{\xi}$  into  $2^{\omega}$ ;
- (B)  $n < \omega$  and  $\pi$  is a permutation of  $2^n$ ;
- (C)  $|D \cap D_{\xi}| \leq 1$  for every  $\xi \in \Gamma$ ;
- (D) if  $x \in D \cap D_{\xi}$  then  $h(x) = h_{\xi}(x)$  and  $h(x) \upharpoonright n = \pi(x \upharpoonright n)$ .

Forcing  $Q_{\bar{h}}$  is ordered as follows. Condition  $p' = \langle n', \pi', h' \rangle$  is stronger than  $p = \langle n, \pi, h \rangle$ ,  $p' \leq p$ , provided

$$n \le n', \ h \subset h', \ \text{and} \ \pi'(s) \upharpoonright n = \pi(s \upharpoonright n) \ \text{for every } s \in 2^{n'}.$$
 (2)

Note that the second part of (D) says that for every  $x \in D$  and  $s \in 2^n$ 

$$x \in [s]$$
 if and only if  $h(x) \in [\pi(s)]$ . (3)

Also, if  $n < \omega$  we will write  $[s] \upharpoonright 2^n$  for  $\{x \upharpoonright 2^n : x \in [s]\}$ . Note that in this notation the part of (2) concerning permutations says that  $\pi'$  extends  $\pi$  in a sense that  $\pi'$  maps  $[t] \upharpoonright 2^{n'}$  onto  $[\pi(t)] \upharpoonright 2^{n'}$  for every  $t \in 2^n$ .

In what follows we will use the following basic property of  $Q_{\bar{h}}$ .

(\*) For every  $q = \langle n, \pi, h \rangle \in Q_{\bar{h}}$  and  $m < \omega$  there exist an  $n' \geq m$  and a permutation  $\pi'$  of  $2^{n'}$  such that  $q' = \langle n', \pi', h \rangle \in Q_{\bar{h}}$  and q' extends q.

The choice of such n' and  $\pi'$  is easy. First pick  $n' \geq \max\{m, n\}$  such that  $x \upharpoonright n' \neq y \upharpoonright n'$  for every different x and y from either domain D or range R = h[D] of h. This implies that for every  $t \in 2^n$  the set  $D_t = \{x \upharpoonright n' \colon x \in D \cap [t]\} \subset [t] \upharpoonright 2^{n'}$  has the same cardinality as  $D \cap [t]$  and  $H_t = \{x \upharpoonright n' \colon x \in h[D] \cap [\pi(t)]\} \subset [\pi(t)] \upharpoonright 2^{n'}$  has the same cardinality as  $h[D] \cap [\pi(t)]$ . Since, by (3), we have also  $|D \cap [t]| = |h[D] \cap [\pi(t)]|$  we see that  $|D_t| = |H_t|$ . Define  $\pi'$  on  $D_t$  by  $\pi'(x \upharpoonright n') = h(x) \upharpoonright n'$  for every  $x \in D_t$ . Then  $\pi'$  is a bijection from  $D_t$  onto  $H_t$  and this definition ensures that an appropriate part of the condition (D) for h and  $\pi'$  is satisfied. Also, if for each  $t \in 2^n$  we extend  $\pi'$  onto  $[t] \upharpoonright 2^{n'}$  as a bijection from  $([t] \upharpoonright 2^{n'}) \backslash D_t$  onto  $([\pi(t)] \upharpoonright 2^{n'}) \backslash H_t$ , then the condition (2) will be satisfied. Thus such defined  $q' = \langle n', \pi', h \rangle$  belongs to  $Q_{\bar{h}}$  and extends q.

Next note that forcing  $Q_{\bar{h}}$  has the following property, described in Fact 7, needed to prove Lemma 5. In what follows we will consider  $2^{\omega}$  with the standard distance:

$$d(r_0, r_1) = 2^{-\min\{n < \omega \colon r_0(n) \neq r_1(n)\}}$$

for different  $r_0, r_1 \in 2^{\omega}$ .

Fact 7 Let  $\bar{h} = \langle h_{\xi} : \xi \in \Gamma \rangle \in \mathcal{K}$  and  $f = \bigcup \{h : \langle n, \pi, h \rangle \in H\}$ , where H is a V-generic filter over  $Q_{\bar{h}}$ . Then f is a uniformly continuous one-to-one function from a subset D of  $2^{\omega}$  into  $2^{\omega}$ . Moreover, if for every  $\xi \in \Gamma$  the graph of  $h_{\xi}$  is dense in  $2^{\omega} \times 2^{\omega}$ , then D and f[D] are dense in  $2^{\omega}$  and f can be uniquely extended to an autohomeomorphism  $\tilde{f}$  of  $2^{\omega}$ .

PROOF. Clearly f is a one-to-one function from a subset D of  $2^{\omega}$  into  $2^{\omega}$ . To see that it is uniformly continuous choose an  $\varepsilon > 0$ . We will find  $\delta > 0$ 

such that  $r_0, r_1 \in D$  and  $d(r_0, r_1) < \delta$  imply  $d(f(r_0), f(r_1)) < \varepsilon$ . For this note that, by (\*), the set

$$S = \{ q = \langle n, \pi, h \rangle \in Q_{\bar{h}} \colon 2^{-n} < \varepsilon \}$$

is dense in  $Q_{\bar{h}}$ . So take a  $q = \langle n, \pi, h \rangle \in H \cap S$  and put  $\delta = 2^{-n}$ . We claim that this  $\delta$  works.

Indeed, take  $r_0, r_1 \in D$  such that  $d(r_0, r_1) < \delta$ . Then there exists a  $q' = \langle n', \pi', h' \rangle \in H$  stronger than q such that  $r_0$  and  $r_1$  are in the domain of h'. Therefore,  $n \leq n'$  and for j < 2

$$f(r_j) \upharpoonright n = h'(r_j) \upharpoonright n = (h'(r_j) \upharpoonright n') \upharpoonright n = \pi'(r_j \upharpoonright n') \upharpoonright n = \pi(r_j \upharpoonright n)$$

by the conditions (D) and (2). Since  $d(r_0, r_1) < \delta = 2^{-n}$  implies that  $r_0 \upharpoonright n = r_1 \upharpoonright n$  we obtain

$$f(r_0) \upharpoonright n = \pi(r_0 \upharpoonright n) = \pi(r_1 \upharpoonright n) = f(r_1) \upharpoonright n$$

that is,  $d(f(r_0), f(r_1)) \leq 2^{-n} < \varepsilon$ . So f is uniformly continuous.

Essentially the same argument (with the same values of  $\varepsilon$  and  $\delta$ ) shows that  $f^{-1}\colon f[D]\to D$  is uniformly continuous. Thus, if  $\tilde{f}$  is the unique continuous extension of f into  $\mathrm{cl}(D)$ , then  $\tilde{f}$  is a homeomorphism from  $\mathrm{cl}(D)$  onto  $\mathrm{cl}(f[D])$ .

To finish the argument assume that all functions  $h_{\xi}$  have dense graphs, take a  $t \in 2^m$  for some  $m < \omega$ , and notice that the set

$$S_t = \{q = \langle n, \pi, h \rangle \in Q_{\bar{h}} : \text{ the domain } D' \text{ of } h \text{ intersects } [t] \}$$

is dense in  $Q_{\bar{h}}$ . Indeed, if  $q = \langle n, \pi, h \rangle \in Q_{\bar{h}}$  then, by (\*), strengthening q if necessary, we can assume that  $m \leq n$ . Then, refining t if necessary, we can also assume that m = n, that is, that t is in the domain of  $\pi$ . Now, if [t] intersects the domain of h, then already q belongs to  $S_t$ . Otherwise take  $\xi \in \Gamma$  with  $D' \cap D_{\xi} = \emptyset$  and pick  $\langle x, h_{\xi}(x) \rangle \in [t] \times [\pi(t)]$ , which exists by the density of the graph of  $h_{\xi}$ . Then  $\langle n, \pi, h \cup \{\langle x, h_{\xi}(x) \rangle\} \rangle$  belongs to  $S_t$  and extends q.

This shows that  $D \cap [t] \neq \emptyset$  for every  $t \in 2^{<\omega}$ , that is, D is dense in  $2^{\omega}$ . A similar argument shows that for every  $t \in 2^{<\omega}$  the set

$$S^t = \{q = \langle n, \pi, h \rangle \in Q_{\bar{h}} \colon \text{ the range of $h$ intersects } [t] \}$$

is dense in  $Q_{\bar{h}}$ , which implies that h[D] is dense in  $2^{\omega}$ . Thus  $\tilde{f}$  is a homeomorphism from  $cl(D) = 2^{\omega}$  onto  $cl(h[D]) = 2^{\omega}$ .

Now take  $A \subset 2^{\omega} \times 2^{\omega}$  for which A and  $A^c = (2^{\omega} \times 2^{\omega}) \setminus A$  are nowhere meager in  $2^{\omega} \times 2^{\omega}$  and fix an  $\omega_1$ -oracle  $\mathcal{M} = \langle M_{\delta} : \delta \in \Gamma \rangle$ . By Fact 7 in order to prove Lemma 5 it is enough to find an  $\bar{h} = \langle h_{\xi} : \xi \in \Gamma \rangle \in \mathcal{K}$  such that

$$Q_A = Q_{\bar{h}} \text{ is } \mathcal{M}\text{-cc}$$
 (4)

and  $Q_{\bar{h}}$  forces that, in V[H],

the sets 
$$\operatorname{pr}(f \cap A)$$
 and  $\operatorname{pr}(f \setminus A)$  are nowhere meager in  $2^{\omega}$ . (5)

(In (5) function f is defined as in Fact 7.)

To define  $\bar{h}$  we will construct a sequence  $\langle \langle x_{\alpha}, y_{\alpha} \rangle \in 2^{\omega} \times 2^{\omega} \colon \alpha < \omega_1 \rangle$  aiming at  $h_{\xi} = \{ \langle x_{\xi+n}, y_{\xi+n} \rangle \colon n < \omega \}$ , where  $\xi \in \Gamma$ .

Let  $\{\langle s_n, t_n \rangle : n < \omega\}$  be an enumeration of  $2^{<\omega} \times 2^{<\omega}$  with each pair  $\langle s, t \rangle$  appearing for an odd n and for an even n. Points  $\langle x_{\xi+n}, y_{\xi+n} \rangle$  are chosen inductively in such a way that

- (i)  $\langle x_{\xi+n}, y_{\xi+n} \rangle$  is a Cohen real over  $M_{\delta}[\langle \langle x_{\alpha}, y_{\alpha} \rangle : \alpha < \xi + n \rangle]$  for every  $\delta \leq \xi$ ,  $\delta \in \Gamma$ , that is,  $\langle x_{\xi+n}, y_{\xi+n} \rangle$  is outside all meager subsets of  $2^{\omega} \times 2^{\omega}$  which are coded in  $M_{\delta}[\langle \langle x_{\alpha}, y_{\alpha} \rangle : \alpha < \xi + n \rangle]$ ;
- (ii)  $\langle x_{\xi+n}, y_{\xi+n} \rangle \in A$  if n is even, and  $\langle x_{\xi+n}, y_{\xi+n} \rangle \in A^c$  otherwise.
- (iii)  $\langle x_{\xi+n}, y_{\xi+n} \rangle \in [s_n] \times [t_n].$

The choice of  $\langle x_{\xi+n}, y_{\xi+n} \rangle$  is possible since both sets A and  $A^c$  are nowhere meager, and we consider each time only countably many meager sets. Condition (iii) guarantees that the graph of each of  $h_{\xi}$  will be dense in  $2^{\omega} \times 2^{\omega}$ . Note that if  $\Gamma \ni \delta \leq \alpha_0 < \cdots < \alpha_{k-1}$ , where  $k < \omega$ , then (by the product lemma in  $M_{\delta}$ )

$$\langle \langle x_{\alpha_i}, y_{\alpha_i} \rangle : i < k \rangle$$
 is an  $M_{\delta}$ -generic Cohen real in  $(2^{\omega} \times 2^{\omega})^k$ . (6)

For  $q = \langle n, \pi, h \rangle \in Q_{\bar{h}}$  define

$$\hat{q} = \bigcup_{\langle s,t\rangle \in \pi} [s] \times [t].$$

Clearly  $\hat{q}$  is an open subset of  $2^{\omega} \times 2^{\omega}$  and condition (2) implies that for every  $q, r \in Q_{\bar{h}}$  with  $r = \langle n', \pi', h' \rangle$ 

if 
$$q \le r$$
 then  $\hat{q} \subset \hat{r}$  and  $\hat{q} \cap ([s] \times [t]) \ne \emptyset$  for every  $\langle s, t \rangle \in \pi'$ . (7)

Also for  $\delta \in \Gamma$  let  $(Q_{\bar{h}})^{\delta} = \{\langle n, \pi, h \rangle \in Q_{\bar{h}} \colon h \subset \bigcup_{\zeta < \delta} h_{\zeta} \}$ . To prove (4) and (5) we will use also the following fact.

**Fact 8** Let  $\delta \in \Gamma$  and let  $E \in M_{\delta}$  be a predense subset of  $(Q_{\bar{h}})^{\delta}$ . Then for every  $k < \omega$  and  $p = \langle n, \pi, h \rangle \in (Q_{\bar{h}})^{\delta}$  the set

$$B_p^k = \bigcup \{(\hat{q})^k \colon \text{ } q \text{ extends } p \text{ and some } q_0 \in E\}$$
 (8)

is dense in  $(\hat{p})^k \subset (2^\omega \times 2^\omega)^k$ .

PROOF. By way of contradiction assume that  $B_p^k$  is not dense in  $(\hat{p})^k$ . Then there exist  $m < \omega$  and  $s_0, t_0, \ldots, s_{k-1}, t_{k-1} \in 2^m$  with the property that  $P = \prod_{i < k} ([s_i] \times [t_i]) \subset (\hat{p})^k$  is disjoint from  $B_p^k$ . Increasing m and refining the  $s_i$ 's and  $t_j$ 's, if necessary, we may assume that  $m \geq n$ , all  $s_i$ 's and  $t_j$ 's are different,  $\bigcup_{i < k} [s_i]$  is disjoint from the domain D of h, and  $h[D] \cap \bigcup_{i < k} [t_i] = \emptyset$ . We can also assume that  $x \upharpoonright m \neq y \upharpoonright m$  for every different x and y from D and from h[D]. Now, refining slightly the argument for (\*) we can find  $r = \langle m, \pi', h \rangle \in (Q_{\bar{h}})^{\delta}$  extending p such that  $\pi'(s_i) = t_i$  for every i < k. (Note that  $P \subset (\hat{p})^k$ .) We will obtain a contradiction with the predensity of E in  $(Q_{\bar{h}})^{\delta}$  by showing that r is incompatible with every element of E.

Indeed if q were an extension of  $r \leq p$  and an element  $q_0$  of E, then we would have  $(\hat{q})^k \subset B_p^k$ . But then, by (7) and the fact that  $\langle s_i, t_i \rangle \in \pi'$  for i < k, we would also have  $(\hat{q})^k \cap P \neq \emptyset$ , contradicting  $P \cap B_p^k = \emptyset$ . This finishes the proof of Fact 8.

Now we are ready to prove (4), that is, that  $Q_{\bar{h}}$  is  $\mathcal{M}$ -cc. So, fix a bijection  $e \colon Q_{\bar{h}} \to \omega_1$  and let

$$C = \left\{ \delta \in \Gamma \colon (Q_{\bar{h}})^{\delta} = e^{-1}(\delta) \in M_{\delta} \right\}.$$

Then  $C \in D_{\mathcal{M}}$ . (Just use a suitable nice codding or [12, Claim 1.4(4)].) Take a  $\delta \in C$  and fix an  $E \subset \delta$ ,  $E \in M_{\delta}$ , for which  $e^{-1}(E)$  is predense in  $(Q_{\bar{h}})^{\delta}$ . By Fact 4 it is enough to show that

 $e^{-1}(E)$  is predense in  $Q_{\bar{h}}$ .

Take  $p_0 = \langle n, \pi, h_0 \rangle$  from  $Q_{\bar{h}}$ , let  $h = h_0 \upharpoonright \bigcup_{\eta < \delta} D_{\eta}$  and  $h_1 = h_0 \backslash h$ , and notice that the condition  $p = \langle n, \pi, h \rangle$  belongs to  $(Q_{\bar{h}})^{\delta}$ . Assume that  $h_1 = \{\langle x_i, y_i \rangle \colon i < k\}$ . Since  $s(h_1) = \langle \langle x_i, y_i \rangle \colon i < k \rangle \in (\hat{p})^k$ ,  $B_p^k \in M_{\delta}$  (as defined from  $(Q_{\bar{h}})^{\delta} \in M_{\delta}$ ) and, by Fact 8,  $B_p^k$  is dense in  $(\hat{p})^k$  condition (6) implies that  $s(h_1) \in B_p^k$ . So there are  $q = \langle n_0, \pi_0, g \rangle \in (Q_{\bar{h}})^{\delta}$  extending p and some  $q_0 \in e^{-1}(E)$  for which  $s(h_1) \in \hat{q}^k$ . But then  $p' = \langle n_0, \pi_0, g \cup h_1 \rangle$  belongs to  $Q_{\bar{h}}$  and extends q. This finishes the proof of (4).

The proof of (5) is similar. We will prove only that  $\operatorname{pr}(f \setminus A) = \operatorname{pr}(f \cap A^c)$  is nowhere meager in  $2^{\omega}$ , the argument for  $\operatorname{pr}(f \cap A)$  being essentially the same

By way of contradiction assume that  $\operatorname{pr}(f \setminus A)$  is not nowhere meager in  $2^{\omega}$ . So there exists an  $s^* \in 2^{<\omega}$  such that  $\operatorname{pr}(f \setminus A)$  is meager in  $[s^*]$ . Let a condition  $p^* \in Q_{\bar{h}}$  and  $Q_{\bar{h}}$ -names  $\dot{U}_m$ , for  $m < \omega$ , be such

$$p^* \Vdash_{Q_{\bar{h}}}$$
each  $\dot{U}_m$  is an open dense subset of  $[s^*]$  and  $\operatorname{pr}(f \setminus A) \cap \bigcap_{m < \omega} \dot{U}_m = \emptyset$ .

For each  $m < \omega$ , since  $p^*$  forces that  $\dot{U}_m$  is an open dense subset of  $[s^*]$ , for every  $t \in 2^{<\omega}$  extending  $s^*$  there is a maximal antichain  $\langle p^m_{s,k} \colon k < \kappa^m_s \rangle$  in  $Q_{\bar{h}}$  forcing that  $\dot{U}_m \cap [t]$  contains some basic open subset [s].

Note that each of these antichains must be countable, since the forcing notion  $Q_{\bar{h}}$  is  $\mathcal{M}$ -cc and therefore ccc. Combining all these antichains we get a sequence  $\langle p_{s,k}^m \in Q_{\bar{h}} \colon m < \omega, s \in 2^{<\omega}, k < \kappa_s^m \rangle$  such that

- $\kappa_s^m \leq \omega$ ,
- $p_{s,k}^m \Vdash_{Q_{\bar{h}}} [s] \subseteq \dot{U}_m$ ,
- for every  $q \in Q_{\bar{h}}$  extending  $p^*$  and  $t \in 2^{<\omega}$  extending  $s^*$  there are  $s \in 2^{<\omega}$  and  $k < \kappa_s^m$  such that the conditions q and  $p_{s,k}^m$  are compatible and  $t \subset s$ .

Note that for sufficiently large  $\delta \in \Gamma$  we have  $p_{s,k}^m \in (Q_{\bar{h}})^{\delta}$  for all  $m < \omega$ ,  $s \in 2^{<\omega}$ , and  $k < \kappa_s^m$ .

Now, by the definition of  $\omega_1$ -oracle, the set  $B_0$  of all  $\delta \in \Gamma$  for which

$$\langle p_{s,k}^m \in Q_{\bar{h}} \colon m < \omega, s \in 2^{<\omega}, k < \kappa_s^m \rangle \in M_{\delta} \quad \text{and} \quad (Q_{\bar{h}})^{\delta} \in M_{\delta}$$

is stationary in  $\omega_1$ . (Just use a suitable nice coding, or see [12, Ch.IV, Claim 1.4(4)]). Thus, using clause (iii) of the choice of  $x_{\xi}$ 's, we may find a  $\delta \in B_0$ , an odd  $j < \omega$ , and a condition  $p_0 = \langle n_0, \pi_0, h_0 \rangle \in Q_{\bar{h}}$  such that

- $p_0 \le p^*$ ,  $s^* \subset x_{\delta+j}$ , and
- $x_{\delta+i}$  belongs to the domain of  $h_0$ .

Then  $p_0 \Vdash "x_{\delta+j} \in [s^*] \cap \operatorname{proj}(f \setminus A)"$  (remember j is odd so  $\langle x_{\delta+j}, y_{\delta+j} \rangle \in A^c$ ). We will show that

$$p_0 \Vdash x_{\delta+j} \in \bigcap_{m < \omega} \dot{U}_m,$$

which will finish the proof.

So, assume that this is not the case. Then there exist an  $i < \omega$  and a  $p_1 = \langle n, \pi, h_1 \rangle \in Q_{\bar{h}}$  stronger than  $p_0$  such that  $p_1 \Vdash "x_{\delta+j} \notin \dot{U}_i$ ." Let us define  $h = h_1 \upharpoonright \{x_\alpha : \alpha < \delta\}$  and  $h_1 \setminus h = \{\langle a_l, b_l \rangle : l < m\}$ . Notice that the condition  $p = \langle n, \pi, h \rangle$  belongs to  $(Q_{\bar{h}})^{\delta}$ . We can also assume that  $\langle x_{\delta+j}, y_{\delta+j} \rangle = \langle a_0, b_0 \rangle$ .

Now consider the set Z of all  $\langle z_0, z'_0, \dots, z_{m-1}, z'_{m-1} \rangle \in (2^{\omega} \times 2^{\omega})^m$  for which

• there exist  $s \in 2^{<\omega}$ ,  $k < \kappa_s^i$ , and  $q \in (Q_{\bar{h}})^{\delta}$  such that  $s \subset z_0$ , q extends p and  $p_{s,k}^i$ , and  $\langle z_0, z_0', \ldots, z_{m-1}, z_{m-1}' \rangle \in (\hat{q})^m$ .

Claim The set Z belongs to the model  $M_{\delta}$  and it is an open dense subset of  $(\hat{p})^m$ .

PROOF. It should be clear that Z is (coded) in  $M_{\delta}$ . (Remember the choice of  $\delta$ .) To show that it is dense in  $(\hat{p})^m$  we proceed like in the proof of Fact 8. We choose  $s_0, t_0, \ldots, s_{m-1}, t_{m-1}$  and r exactly as there. Next pick a condition  $q \in Q_{\bar{h}}$ , a sequence  $s \in 2^{<\omega}$ , and  $k < \kappa_s^m$  such that

$$s_0 \subset s$$
 and  $q$  extends  $p_{s,k}^i$  and  $r$ .

(Remember the choice of the  $p_{s,k}^i$ 's.) Clearly we can demand that  $q \in (Q_{\bar{h}})^{\delta}$ . Now note that it is possible to choose a  $\bar{z} = \langle z_0, z'_0, \dots, z_{m-1}, z'_{m-1} \rangle \in (\hat{q})^m$  such that  $s \subset z_0, s_i \subset z_i, t_i \subset z'_i$ . Then  $\bar{z} \in Z \cap \prod_{i < k} ([s_i] \times [t_i])$ .

Since Z is clearly open, this completes the proof of Claim.

Now, by (6) and the Claim above,  $\langle \langle a_l, b_l \rangle : l < m \rangle$  belongs to Z since  $\langle \langle a_l, b_l \rangle : l < m \rangle$  belongs to  $(\hat{p}_1)^m = (\hat{p})^m$ . But this means that there exist  $q = \langle n^q, \pi^q, h^q \rangle \in (Q_{\bar{h}})^{\delta}$  and  $s \in 2^{<\omega}$  such that:

- $q \leq p, q \Vdash "[s] \subseteq \dot{U}_i"$ , and
- $\langle \langle a_l, b_l \rangle : l < m \rangle \in (\hat{q})^m$ , and  $x_{\delta+j} = a_0 \in [s]$ .

But then  $p_2 = \langle n^q, \pi^q, h^q \cup \{\langle a_l, b_l \rangle : l < m \} \rangle$  belongs to  $Q_{\bar{h}}$  and extends both q and  $p_1$ . So,  $p_2$  forces that  $x_{\delta+j} = a_0 \in [s] \subseteq \dot{U}_i$ , contradicting our assumption that  $p_1 \Vdash "x_{\delta+j} \notin \dot{U}_i$ ."

This finishes the proof of (5) and of Lemma 5.

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