ON WHAT I DO NOT UNDERSTAND (AND HAVE SOMETHING TO SAY), MODEL THEORY
SH702

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Abstract. This is a non-standard paper, containing some problems, mainly in model theory, which I have, in various degrees, been interested in. Sometimes with a discussion on what I have to say; sometimes, of what makes them interesting to me, sometimes the problems are presented with a discussion of how I have tried to solve them, and sometimes with failed tries, anecdote and opinion. So the discussion is quite personal, in other words, egocentric and somewhat accidental. As we discuss many problems, history and side references are erratic, usually kept at a minimum ("See..." means: see the references there and possibly the paper itself).
The base were lectures in Rutgers Fall '97 and reflect my knowledge then. The other half, concentrating on set theory, is in print [Sh:666], but the two halves are independent. We thank A. Blass, G. Cherlin and R. Grossberg for some corrections.

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CONTENT
PART II

§1 Two cardinal theorems

During the 1960's, two cardinal theorems were popular among model theorists.

1.1 Definition. $(\lambda_1, \ldots, \lambda_n) \to_\kappa (\mu_1, \ldots, \mu_n)$ holds if whenever T is a set of κ f.o. sentences with unary predicates $P_1, \ldots, P_n$ and every finite subset of T has a model M such that $|P^{M_i}| = \lambda_i$ for $i = 1, \ldots, n$, then T has a model N such that $|P^{N_i}| = \mu_i$ for $i = 1, \ldots, n$. If κ is omitted we mean $\kappa = \aleph_0$. For notational simplicity we always assume $\lambda_1 \geq \ldots \geq \lambda_n, \mu_1 \geq \ldots \geq \mu_n$.

We shall usually speak on the case $n = 2$; we have, for general discussion, ignore the possibility of adding cardinality quantifier $(\exists^{\leq\lambda} x)$. Later the subject becomes less popular; Jensen complained "when I start to deal with gap n 2-cardinal theorems, they were the epitome of model theory and as I finished, it stopped to be of interest to model theorists".
I sympathize, though model theorists has reasonable excuses: one is that they want ZFC-provable theorems or at least semi-ZFC ones (see [Sh 666, 1.20t]) the second is that it has not been clear if there were any more.

1.2 Question: Are there more nontrivial n-cardinal ZFC theorems, or only assuming facts on cardinal arithmetic (i.e. semi ZFC ones).

Maybe I better recall the classical ones.

1.3 Theorem. *[Vaught]* $(\lambda^+, \lambda) \to (\aleph_1, \aleph_0)$.

1.4 Theorem. *[Chang]* $\mu = \mu^{<\mu} \Rightarrow (\lambda^+, \lambda) \to (\mu^+, \mu)$.

1.5 Theorem. *[Vaught]* $(\beth_\omega(\lambda), \lambda) \to (\mu_1, \mu_2)$ *when* $\mu_1 \geq \mu_2$.

But there were many independence results and positive theorems for $V = L$ (see [Ho93], [CK], [Sch85], [Sh 18]). ? CK ?

After several years of drawing a blank, I found a short and easy proof of

1.6 Claim. $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$.
In fact $(\lambda^{+\omega}, \lambda) \to (\mu_1, \mu_2)$ *if* $\mathrm{Ded}'(\mu_2) > \mu_1 \geq \mu_2$ *where*

1.7 Definition. $\mathrm{Ded}'(\mu) = \mathrm{Min}\{\lambda :$ if $\mathscr{T}$ is a tree with λ nodes and $\delta \leq \lambda$ levels, then the number of its δ-branches is $< \lambda\}$. This is essentially equal to

$$\mathrm{Ded}(\mu) = \mathrm{Min}\Big\{\lambda : \text{if } I \text{ is a linear order of cardinality } \mu \text{ then } I \text{ has } < \lambda \text{ Dedekind cuts}\Big\}.$$

See [Sh 49]. Considering the many independence proofs and natural limitations, one may ask ([CK]) ? CK ?
1.8 Question: Assume $\lambda = \lambda^{\beth_\omega(\mu)}$ and $\lambda_1 = (\lambda_1)^{<\lambda_1} \geq \mu$, do we have $(\lambda^+, \lambda, \mu) \to$

$(\lambda_1^+, \lambda_1, \mu_1)$.
Things are not commutative, if $\mu = \mu^{<\mu}$ then $(\beth_\omega(\mu^+), \mu^+, \mu) \rightarrow (\lambda, \mu^+, \mu)$ is easy and well known (a consequence of 1.4 + 1.5).
In fact, the impression this becomes set theory has some formal standing: we know that all such theorems are provably equivalent to suitable partition theorems, for formalizing this we need the following definition.

1.9 Definition. 1) Let E be an equivalence relation on $\mathscr{P}(n)$ preserving cardinality; we call such a pair (n, E) an identity. Let $\lambda \rightarrow (n, E)_\mu$ mean that if $F_\ell : [\lambda]^\ell \rightarrow \mu$ for $\ell \leq n$, then we can find $\alpha_0 < \ldots < \alpha_{n-1} < \lambda$ such that for any $u, v \in [n]^k, k \leq n$ we have:

$$uEv \Rightarrow F_k(\ldots, \alpha_\ell, \ldots)_{\ell \in u} = F_k(\ldots, \alpha_\ell, \ldots)_{\ell \in v}$$

we call (n, E) an identity of (λ, μ).
2) $\mathrm{Id}(\lambda, \mu) =: \{(n, E) : (n, E)$ is an identity of $(\lambda, \mu)\}$.

Now

1.10 Claim. *Essentially assuming $\lambda > \mu, \lambda_1 \geq \mu_1 \geq \kappa$ we have: $(\lambda, \mu) \rightarrow_\kappa (\lambda_1, \mu_1)$ iff Id$(\lambda, \mu) \supseteq$ Id(λ_1, μ_1) (see [Sh 8], [Sh:E17]).*
Fully: if $\mu_1 = \mu^{\aleph_0}$ or just $(\lambda_1, \mu_1) \rightarrow_{\aleph_0} (\lambda_1, \mu_1)$ then the equivalence holds; the implication $\Rightarrow$ holds always.

This leaves open:
1.11 Question: Prove the consistency of the existence of $\lambda \geq \mu$ such that $(\lambda, \mu) \nrightarrow (\lambda, \mu)$ (another formulation is: (λ, μ) is not $\aleph_0$-compact).

1.12 Discussion: I am sure that the statement in 1.11 is consistent. Note that all the cases we mention gives the $\aleph_0$-compactness (and a completeness theorem).

Originally the theorems quoted above were not proved in this way.

Vaught proved 1.3 by (sequence)-homogeneous models. Chang proved 1.4 by saturated models of suitable expansion of T.
Vaught 1.5 finds a consistent expansion T_1 of T which has a built-in elementary extension increasing P_1 but preserving P_2. Morley used Erdös Rado theorem to give an alternative proof. Now $(\aleph_\omega, \aleph_0) \rightarrow (2^{\aleph_0}, \aleph_0)$ was proved this way. It took me some effort to characterize the identities for the pair $(\aleph_1, \aleph_0)$, see [Sh 74], so it gives an alternate proof.

Surely Jensen's proof of his 2-cardinal theorems can be analyzed in this way, but I have not looked at this.

Now Jensen's proofs in this light, say

1.13 Theorem. *1) Fixing n, if we look at what can be Id(λ^{+n}, λ), when $V = L$, it is minimal.*
2) If $V = L$, then Id$(\mu^+, \mu), \mu$ singular, (e.g. Id$(\aleph_{\omega+1}, \aleph_\omega)$) is equal to Id$(\aleph_1, \aleph_0)$ hence is minimal.

This fits the intuition that L tends to have objects. So there are many colorings in this case.
So we can ask
1.14 Question: Fixing a pattern of cardinal arithmetic, what is the minimal possible set $\mathrm{Id}(\lambda,\mu)$ if it exists? (Minimal: varying on forcing extensions giving such patterns). As equivalent formulation is: what identities are provable? E.g. $\mu=\mu^{<\mu}, 2^\mu=\mu^{++}, \lambda=\mu^{+3}=2^{\mu^+}$.

The idea is: if we lose hope that all such pairs have the same set of identities, resolvable in ZFC, can we at least find minimal pairs. We may instead of cardinal arithmetic use e.g. "there is a kurepa tree" or whatever, but this is less appealing to me.

It is natural to ask also:
1.15 Question: Fixing a pattern of cardinal arithemtic, what is the maximal possible for set $\mathrm{Id}(\lambda,\mu)$?

A very natural case is $\lambda=\beth_n(\mu), n\geq 2$. In fact, I think it is almost sure that the following case gives it. Let $\lambda_0>\lambda_1>\ldots>\lambda_n$, with each λ_ℓ is supercompact Laver indestructible, now force by $\prod\limits_{\ell<n} P_\ell$ where P_ℓ is adding $\lambda_{\ell+1}$ Cohens subsets of λ_ℓ. I think (λ_n,λ_0) in this model has a maximal set of identities. The point is that each $\lambda_{\ell+1}$ satisfies a generalization of Halpern-Lauchli theorem (see [Sh 288, §4]).

1.16 Question: Assume GCH, μ singular limit of supercompacts. Is $\mathrm{Id}(\mu^+,\mu)$ maximal?
Jensen had found the minimal 1.2; now see [MgSh 324], there is no μ^+-tree for μ as above, so it is a natural candidate for maximality.

1.17 Question: 1) What is the maximal set of identities $\mathrm{Id}(\lambda^{+n},\lambda)$ under GCH?
2) Can we have a universe of set theory satisfying GCH $+\bigwedge\limits_{\lambda}\mathrm{Id}(\lambda^{+n},\lambda)$ maximal?
3) Similarly for $(\beth_n(\lambda),\lambda)$).
For (2),(3) we need "GCH fails everywhere (badly)", see Foreman Woodin [FW].
Generally, our knowledge on the family of forcing doing something for all cardinals seems not to be developed flexibly enough now (see [Sh 666]).

1.18 Question: 1) If λ is strongly inaccessible cardinal and λ_1 is an inaccessible non-Mahlo cardinal which has a square (or even $V=L$) and the first order ψ has a λ-like model then ψ has a λ-like model?
2) Similarly with λ being n-Mahlo, λ_1 being not $(n+1)$-Mahlo (and $V=L$) (see [Sch85]).
We know it is surely true (at least if $V=L$), but this is not a proof. The singular case is Keisler [Ke68] (and more in [Sh 18]).

* * *

We can ask
1.19 Problem: When do we have $\kappa\in\{2,\aleph_0\}$ and $\{(m_{i,1},\ldots,m_{i,n}):i<\omega\}\rightarrow_\kappa(\mu_1,\ldots,\mu_n)$ with $m_{i,\ell}<\omega$ which means:

(*) if T is first order theory of cardinality $\le \kappa$ and every finite $T' \subseteq T$ for i large enough have a model M_i such that $|P_\ell^{M_i}| = m_{i,\ell}$ for $\ell = 1,\dots,n$, then T has a model $M, |P_\ell^M| = \mu_\ell$ for $\ell = 1,\dots,n$.
We know something, see [Sh 37], [Sh 18, p.250-1]. (We ignore here the can of disjunctions; for every i large enough for some j, T' has a model $M, |P^{M_\ell}| = m_{i,j,\ell}$).

1.20 Claim. *: If $i \le m_{i,1}$ and $(m_{i,1})^i \le m_{i,2}$ then*

$$\{(m_1^i, m_2^i) : i < \omega\} \to (\aleph_0, 2^{\aleph_0}).$$

1.21 Question: For $m_{i,1}, m_{i,2}$ as in 1.20 do we have always (i.e. for every λ, provably in ZFC) $\{(m_1^i, m_2^i) : i < \omega\} \to (2^\lambda, \lambda)$? Or at least $\{(2^m, m) : m < \omega\} \to (2^\lambda, \lambda)$.

(The problem is when $\mathrm{Ded}(\lambda) < (2^\lambda)^+$). Those problems (1.19 - 1.22) are involved with problems in (finitary) Ramsey theory. Natural (and enough) to try to show consistency of (for T with Skolem functions)

$$\begin{aligned} T_{Sk} \cup \{x_\eta \ne x_\nu \ \ \& \ \ P_1(x_\eta) : \eta \in {}^\lambda 2, \eta \ne \nu \in {}^\lambda 2\} \\ \cup \{P_2(\sigma(x_{\eta_1},\dots,x_{\eta_n})) \to \sigma(x_{\eta_1},\dots,x_{\eta_n}) = \sigma(x_{\nu_1},\dots,x_{\nu_n}) : \\ n < \omega, \sigma \text{ a term and } \langle \eta_1,\dots,\eta_n\rangle \approx \langle \nu_1,\dots,\nu_n\rangle\} \end{aligned}$$

where $\langle \eta_0,\dots,\eta_{n-1}\rangle \approx \langle \nu_0,\dots,\nu_{n-1}\rangle$, for $\eta_\ell, \nu_\ell \in {}^\lambda 2$, means

(*)(a) $\eta_\ell <_{lex} \eta_k \equiv \nu_\ell <_{lex} \nu_k$, of course $<_{\ell ex}$ is lexicographic order
(b) $\ell g(\eta_{\ell_1} \cap \eta_{k_1}) < \ell g(\eta_{\ell_2} \cap \eta_{k_2}) \Leftrightarrow \ell g(\nu_{\ell_1} \cap \nu_{k_1}) < \ell g(\nu_{\ell_2} \cap \nu_{k_2})$
(c) $\eta_m(\ell g(\eta_\ell \cap \eta_k)) = \nu_m(\ell g(\nu_\ell \cap \nu_k))$.

(Main Point: level of the splitting not important, unlike the proof of the previous theorem 1.20).
This approach tells us to find more identities for the relevant finite pairs. We can, on the other hand, try to exploit that "$\mathrm{Id}(2^\lambda, \lambda)$ is smaller than suggested by the above approach" (see [Sh 430, 3.4,6.3]).

1.22 Question: Does, for $W \subseteq \omega$ infinite, $n < \omega$

$$\{(\beth_n(i), i) : i \in W\} \to (\beth_n(\lambda), \lambda)?$$

or even

$$\{\big((\beth_n(i))^i, i\big) : i \in W\} \to (\beth_{n+1}(\lambda), \lambda)?$$

Some of the theorems above have also parallel with omitting types. So considering some parallelism it is very natural to ask
1.23 Question: If $\psi \in L_{\omega_1,\omega}$ has a model of cardinality $\ge \aleph_{\omega_1}$ does it have a model of cardinality continuum? (well assuming $2^{\aleph_0} > \aleph_{\omega_1}$).

This is connected to the problem of Borel squares, a problem I had heard from Harrington about.

1.24 Definition. 1) A set $B \subseteq {}^\omega 2 \times {}^\omega 2$ contains a λ-square if for some $A \subseteq {}^\omega 2$ of cardinality λ we have $A \times A \subseteq B$ i.e. $\eta, \nu \in A \Rightarrow (\eta, \nu) \in B$.
2) A set $B \subseteq {}^\omega 2 \times {}^\omega 2$ contains a perfect square if there is a perfect set $\mathscr{P} \subseteq {}^\omega 2$ such that $\mathscr{P} \times \mathscr{P} \subseteq B$.
3) A set $B \subseteq {}^\omega 2 \times {}^\omega 2$ contains a λ-rectangle if for some $A_1, A_2 \subseteq {}^\omega 2$ of cardinality λ we have $A_1 \times A_2 \subseteq B$. We add perfect if A_1, A_2 are perfect.

The connection is (see [Sh 522]).

1.25 Claim. *Assume $MA + 2^{\aleph_0} > \aleph_{\omega_1}$, for some cardinal λ^* we have*

- $(a)_1$ *if $\psi \in L_{\omega_1,\omega}$ has a model of cardinality $\geq \lambda^*$ then it has a model of cardinality continuum*
- $(a)_2$ *for no $\lambda' < \lambda^*$ does $(a)_1$ hold*
- $(b)_1$ *if $\lambda^* < 2^{\aleph_0}$ and $\mathscr{B}$ is a Borel subset of ${}^\omega 2 \times {}^\omega 2$ and it has a λ^*-square, then $\mathscr{B}$ contains a perfect set*
- $(b)_2$ *for no $\lambda' < \lambda^*$ does $(b)_1$ hold*
- (c) *if $\lambda^* < 2^{\aleph_0}$ then λ^* is a limit cardinal of cofinality $\aleph_1$.*

In fact this λ^* essentially can be defined as $\lambda_{\aleph_1}(\aleph_0)$ where

1.26 Definition. 1) For a model M with countable vocabulary, we define

$$\mathrm{rk}_\mu : \{w \subseteq M : w \text{ finite nonempty}\} \to \text{ Ord } \cup \{\infty\}$$

(really $\mathrm{rk}_{M,\mu}$) by

$$\begin{aligned}\mathrm{rk}_\mu(w) \geq \alpha + 1 \text{ iff, for any enumeration } &\langle a_\ell : \ell < |w|\rangle \text{ of } w \\ &\text{and first order formula } \varphi(x_0, x_1, \dots, x_{n-1}) \in L_{\tau(M)} \text{ such that} \\ &M \models \varphi[a_0, a_1, \dots, a_{n-1}] \text{ we can find } \geq \mu \text{ members} \\ &a'_0 \in M \backslash \{a_0\} \text{ such that} \\ &M \models \varphi[a'_0, a_1, a_2, \dots, a_{n-1}] \text{ and } \mathrm{rk}(w \cup \{a'_0\}) \geq \alpha.\end{aligned}$$

2) $\lambda_{\mu,\alpha}(\aleph_0) =$ Min$\{\lambda :$ if M is a model of cardinality λ and countable vocabulary then $\alpha \leq$ sup$\{\mathrm{rk}_\mu(w) + 1 : w \subseteq M$ finite nonempty$\}$. We may omit μ if $\mu = 1$.

So question 1.23 can be rephrased as
<u>1.27 Question</u>: If $\lambda_{\omega_1}(\aleph_0) = \aleph_{\omega_1}$?
It is harder but we can deal similarly with rectangles and with equivalence relations (see [Sh 522] and hopefully [Sh 532]); so e.g.

<u>1.28 Question</u>: If a Borel set $\mathscr{B} \subseteq {}^\omega 2 \times {}^\omega 2$ contains an e.g. $\aleph_{\omega_1}$-rectangle (i.e. a $A_1 \times A_2, |A_1| = |A_2| = \aleph_{\omega_1}$) then does it contain a perfect rectangle?

* * *

On Hanf numbers of omitting types and relatives see Grossberg Shelah [GrSh 259]. Let $\delta_2(\lambda,\kappa)$ be the minimal ordinal δ such that if $\psi \in L_{\kappa^+,\omega}$ has a model M, $\text{otp}(M,<^M) \ge \delta, |P^M| = \lambda$, <u>then</u> ψ has a non-well ordered model N such that $N \restriction P^N \prec M \restriction P^M$.

<u>1.29 Question</u>: If $\lambda > 2^\kappa$, $\text{cf}(\kappa) \ge \aleph_0$ do we have $\delta_2(\lambda,\kappa) = (\text{cov}(\lambda,\kappa) + 2^\kappa)^+$?

<u>1.30 Question</u>: Let $\text{cf}(\kappa) > \aleph_0$; is $\delta_2(\kappa,\kappa) < (\sup\{\text{rk}_D(f) : D$ an $\aleph_1$-complete filter on $\kappa, f \in {}^\kappa\kappa\})^+$?

§2 Monadic Logic and Indiscernible

On monadic logic generally see Gurevich [G] (till '81).

We almost know how complicated the monadic theory of the real line is: of course, it is interpretable in the 2nd order theory of $2^{\aleph_0}$, while we can interpret in it the second order theory of $2^{\aleph_0}$ in V^{Cohen} (Boolean interpretation - probably the reason it (the undecidability of the monadic theory of $(\mathbb{R}, <)$) was difficult is that first order interpretation was expected; but it takes more years to see that this speaks on forcing. We cannot represent syntactically $\mathbb{N}$, but we can represent Cohen names of natural numbers), see latest version [Sh 284a].

<u>2.1 Question</u>: 1) Can we

(*a*) interpret the monadic theory of (the order) $\mathbb{R}$ in (second order theory of $2^{\aleph_0})^{V^{\text{Cohen}}}$?
<u>or</u> just show

(*b*) Turing degree(monadic theory of $\mathbb{R}$) $\leq$ Turing degree(second order theory of $2^{\aleph_0})^{V^{\text{Cohen}}}$?

There are many variants.

2.2 Definition. 1) For a logic $\mathscr{L}$, $\text{Th}_{\mathscr{L}}(M)$, the $\mathscr{L}$-theory of the structure M in the universe V is $\{\varphi : \varphi \in \mathscr{L}$ in the vocabulary of M and in V we have $M \models \varphi\}$.
2) When $\mathscr{L}$ is a logic, $\mathscr{L}(Q_t)_{t \in I}$ means we add the quantifiers Q_t, $\mathscr{L}_{\lambda,\kappa}$ means we allow (forming the formulas) take conjunctions on $< \lambda$ formulas and use a string of $< \kappa$ quantifiers. But we may use $L = L_{\omega,\omega}$ for first order, so $L_{\lambda,\kappa}, L(Q_t)$ for the expansions as above.

You may ask:
<u>2.3 Question</u>: How are the L(2nd)-th theory of $2^{\aleph_0}$ in V and in V^{Cohen} related? Of course, 2-nd stand for the quantifier on say arbitrary binary relations.
This is a different question - how many times are they equal, e.g. if $V = V_0^{\text{Cohen}}$, then they are equal.

From the point of view of monadic logic, the question I think is: can we "eliminate quantifiers" using names, and the answer "they are equal" to the second question (2.1(b)) may be accidental, in the sense that does not answer "can monadic formulas say more than the appropriate forcing statements". (They may be one definable from the other...)

We may also ask, (more specifically than in 2.3)
<u>2.4 Question</u>: Can the monadic theory of $\mathbb{R}$ be changed by adding Cohen? What if we assume $V = L$?

As indicated, the hope is a "meaningful" reduction of monadic formulas to relevant forcing statement. If we try for other direction, it is natural to try to interpret the second order theory of $2^{\aleph_0}$ in V^Q for Q another forcing, e.g. Sacks forcing.

It is reasonable to try to deal with a similar problem where the upper and lower bounds are further apart. Consider $M_\lambda = ({}^{\omega >}\lambda, \triangleleft)$ in the logic $L(Q_{pr})$, where Q_{pr} is the quantifier over pressing down unary function f, where pressing down means

"$f(x)$ is an initial segment of x".
Alternatively, ask on the monadic theory of

$$M_\lambda = ({}^{\omega>}\lambda, \triangleleft, +, \times)$$

$$\triangleleft = \{(\eta, \nu) : \eta \text{ an initial segment of } \nu \text{ both in } {}^{\omega>}\lambda\}$$

$$+ = \{(\eta, \nu, \rho) : \eta \triangleleft \rho, \nu \triangleleft \rho, \text{ all three in } {}^{\omega>}\lambda \text{ and } \ell g(\eta) + \ell g(\nu) = \ell g(\rho)\}$$

$$\begin{aligned} \times = \{(\eta, \nu, \rho) : \ & \eta, \nu, \rho \text{ belongs to} \\ & \{\eta^* \restriction n : n < \omega\} \text{ for some } \eta^* \in {}^{\omega}\lambda \text{ and} \\ & \ell g(\eta) \times \ell g(\nu) = \ell g(\rho)\} \end{aligned}$$

Now (see [Sh 205])

2.5 Theorem. *In the $L(Q_{pr})$-theory of M_λ, we can interpret the Levy$(\aleph_0, \lambda)$-Boolean valued second order theory of λ = second order theory of $\aleph_0$ in $V^{\mathrm{Levy}(\aleph_0,\lambda)}$. So the complexity of the $L(Q_{pr})$-theory of M_λ is at most that of the second order theory of λ and at least that of the second order theory of λ in $V^{\mathrm{Levy}(\aleph_0,\lambda)}$.*

(Note: this is just second order theory of $\aleph_0$ which stabilize under large cardinals). This depends on λ because in second order theory $\lambda^{V^{\mathrm{Levy}(\aleph_0,\lambda)}}$ we can e.g. interpret f.o. theory of $(L_{\lambda^+}, \in)$. So not unnatural to assume that the same is true on the $L(Q_{pr})$-theory of M_λ, this is true, e.g. if $\mathrm{Th}_{L(Q_{\mathrm{pr}})}(M_\lambda)$ is interpretable in the Levy$(\aleph_0, \lambda)$-Boolean valued second order theory of λ, that is $\aleph_0$.

<u>2.6 Problem</u>: The parallel of 2.1(b), 2.5 for $L(Q_{pr})$.

We know that the monadic theory of linear order is complicated, exactly as second order theory (so they have the same Lowenheim number). Is there a sizable class where we can have simple monadic theory?

<u>2.7 Problem</u>: Can the monadic theory of well orders be decidable? And/or has a small Lowenheim number? Even $\aleph_\omega$?
(Why "can" not "is"? Consistently monadic theory of $(\omega_2, <)$ is as complicated as you like ([GMSh 141], [LeSh 411]). Note that the statement "every stationary $S \subseteq S^2_0$ reflect" can be expressed in monadic logic on $(\omega_2, <)$, hence the theory is "set theoretically sensitive". There are theorems saying that there is a strong connection.)

There is a natural candidate for such a model of set theory, but it is not known if it works. The consequence will be that also the Lowenheim number of well ordering and the Lowenheim numbers of the class of linear orders are small.

The candidate we mention is: let $\mathbf{V}_0$ satisfies GCH, we shall force with $P_\infty = \bigcup\limits_\alpha P_\alpha$ where we use an iterated forcing $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha \text{ an ordinal } \rangle$ with full support with $\underset{\sim}{Q}_i$ defined as Q^{λ_i} in $\mathbf{V} = \mathbf{V}^{P_i}, \lambda_i =$ the ith regular uncountable cardinal in $\mathbf{V}_0$, defined as below. In universe $\mathbf{V}$ with a cardinal $\lambda = \lambda^{<\lambda}$, let Q^λ be the result of iteration of length $\lambda^+, \langle Q_{\lambda,i}, \underset{\sim}{R}_{\lambda,i} : i < \lambda^+\rangle, Q^\lambda = \bigcup_{i<\lambda^+} Q_{\lambda,i}, R_{\lambda,i}$ has cardinality λ and has an extra partial order $\leq_{pr} = \leq_{pr}^{R_{\lambda,i}}$ such that $p \leq_r q \Rightarrow p < q$ and if $\delta < \lambda^+$ is limit, $\langle p_i : i < \delta\rangle$ is $\leq_{pr}^{R_{\lambda,i}}$-increasing continuous then it has a $\leq_{pr}$-lub and for every dense open $\mathscr{I} \subseteq R^{\lambda,i}$ and $p \in R^{\lambda,i}$ there is q satisfying $p \leq_{\rm pr}^{R_{\lambda,i}} q \in \mathscr{I}$. This forcing is easy to handle and add e.g. many non reflecting stationary sets (e.g. use for regular $\lambda > \aleph_0, R = \{h : h$ is a function from some $\alpha_h < \lambda$ to $h^{-1}\{1\}$ do not reflect$\}, h_1 \leq h_2 \Leftrightarrow h_1 \subseteq h_2$ and $h_1 \leq_{pr} h_2 \Leftrightarrow h_1 = h_2 \vee (h_1 \subseteq h_2 \ \& \ h_2(\alpha_{h_1}) = 0)$.

The analysis of the monadic theory I expect uses the lemmas (and notions) of [Sh 42, §4].

* * *

Suppose we fix a first order theory T (e.g. countable), look at monadic logic on its class of models. There was much research on the monadic theory of linear orders and trees. Why? Just accident? (see Baldwin Shelah [BlSh 156]).

<u>2.8 Problem</u>: Let T be first order complete. If we cannot (f.o.) interpret second order theory in the monadic theory of model of T, <u>then</u> models of T are not much more complicated than trees.

Note: if in some model M of T expanded by unary predicates call it M^+, we can interpret a one to one function $H : A \times B \to 2$ where A, B are infinite, <u>then</u> the theory is at least as complicated as second order logic, so those are hopelessly complicated for the purpose of our present investigation. Assume not, that is

$(*)$ for any $M^+, Th_L(M^+)$ does not have the independence property.

So we feel the cut is meaningful, a dividing line. We shall return to this later (2.17) because this connects somehow to another problem also on classifying f.o. theories suggested by Grossberg and Shelah (observing $(*)$ below):

<u>2.9 Problem</u>: Investigate $\to_T$ according to properties of T, where T is a complete first order theory, where

2.10 Definition. 1) Let $\lambda \to_T (\mu)_\kappa$ mean that: if $M \models T, A \in [M]^\kappa, \bar{a}_i \in {}^kM$ for $i < \lambda$, <u>then</u> for some $Y \subseteq \lambda, |Y| = \mu$, the sequence $\langle \bar{a}_i : i \in Y\rangle$ is indiscernible over A in M.
2) Let $\lambda \to_T^{\rm loc} (\mu)_\kappa$ mean that for any finite set Δ of formulas, we get above Δ-indiscernibility.
3) We may replace T by K for a class of models, or by M if $K = \{M\}$.

2.11 Definition. T has the ω-independence property if there are $k < \omega$ and formula $\varphi_n(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_n)$ for $n < \omega$ where $\ell g(\bar{x}_i) = k$ such that for every λ and $F : [\lambda]^{<\aleph_0} \to 2$ there are $M \models T, \bar{a}_i \in {}^kM$ and $\bar{b}_n \in {}^{\ell g(\bar{y}_n)}M$ such that: $M \models \varphi_n(\bar{a}_{i_0}, \ldots, \bar{a}_{i_{n-1}}, \bar{b}_n)$ iff $F(\{i_0, \ldots, i_{n-1}\}) = 1$ (see [LwSh 560]).

So

(∗) for T with the ω-independence property $\lambda \rightarrow_T (\mu)_{\aleph_0}$ is equivalent to $\lambda \rightarrow (\mu)^{<\omega}_{\aleph_0}$ so they are maximally complicated under this test.

2.12 Problem: If T doesn't have the independence property (= have NIP), $\rightarrow_T$ is "nice" because supposedly the prototypes of the class of unstable theories having NIP is linear order, for which $\rightarrow_T$ has a nice theory (as we can go down to well ordering).

We expect a nice solution. The problem (2.9) may be partially resolved by an answer to 5.38. Though the last two problems remain open, we can use a weak answer to the last to give some information on the earlier one.

2.13 Definition. 1) Let $\lambda \rightarrow_{T,m} (\mu)_\kappa$ be defined as in 2.10 restricting ourselves to k such that $k < 1 + m$ (so for $m = \omega$ we get 2.10).
2) Let $\lambda \rightarrow^{\text{loc}}_{T,m} (\mu)_\kappa$ means that for any finite set Δ of formulas, we get above Δ-indiscernibility.

Well 2.13(2) is, of course, interesting only when the Erdös-Rado Theorem does not give the answer. Now you may ask: will it make a difference to demand $k = 1$. Surprisingly there is: it suffices to have "no $\varphi(x, y; \bar{z})$ has the order property in M" to get strong results on $\rightarrow_M$ (see [Sh 300, Ch.I,§4]). More elaborately, the surprise for me was that the condition like "no $\varphi(\bar{x}, \bar{y}; \bar{z})$ has the order property" when restricting $\ell g(\bar{x}) = \ell g(\bar{y}) = k$ but not $\ell g(\bar{z})$ has any consequences (some readers missed the point that the model was not required to be stable), even T was not required to be stable, but it is less interesting ([Sh 715, np1.11t]).

2.14 Definition. 1) We say T is $(k, r) - *$-NIP if every formula $\varphi = \varphi(\bar{x}, \bar{y}, \bar{z})$ with $\ell g(\bar{x}) = k, \ell g(\bar{y}) = r$ is $(k, r) - *$-NIP which means that: for no $\bar{a}_u, \bar{b}_\ell, \bar{c}$ for $u \subseteq \omega, \ell < \omega$ do we have $\mathfrak{C} \models \varphi[\bar{a}_u, \bar{b}_k, \bar{c}]$ iff $\ell \in u$ (so $\ell g(\bar{a}_\ell) = k, \ell g(\bar{b}_k) = r$, can be phrase by a variant of $|S^k_\varphi(A)|$ small). We may replace (ℓ, m) by a set of such pairs.
2) Similarly for other "straight" properties, see 5.15, particularly part (4), 5.16, 5.19.

Note that we have considered $\varphi(\bar{x}, \bar{y}, \bar{z})$ as the quadruple $\langle \varphi, \bar{x}, \bar{y}, \bar{z} \rangle$ with $\bar{x}\hat{\ }\bar{y}\hat{\ }\bar{z}$ a sequence with no repetitions of variables, including every variable which occurs freely in φ.
On the relationships of those properties, the independence property and the strict order property see [Sh 715].

2.15 Problem: Is there a reasonable theory for the family of $(k, m) - *$-NIP first order theories (complete) T? Or for the family of first order T without the ω-independence property? Certainly this is hopeful.
A "theory" here means say as in [Sh:c] for the class of superstable (complete first order theories) T.

2.16 Question: 1) Prove that for any $k < \omega$, for some ℓ, m (in fact, quite low) we hope that any complete first order T we have: T is $(\ell, m) - *$-non-independence,

<u>iff</u> $\lambda \rightarrow_{T,k} (\mu)_\chi$ under reasonable conditions on $\lambda, \mu, \chi, |T|$ as in the Erdös Rado Theorem (rather than large cardinals). You may use a set of (ℓ, m)'s.
2) For $k = \omega$ we similarly consider the failure of the ω-independence property. This will prove that the ω-independence property is a real dividing line for 2.9, but I have no reasonable speculations on what a theory for this property will say.

What we can get (see [Sh 197])

2.17 Theorem. *If every monadic expansion of T does not have the independence property, <u>then</u>*

$$\beth_\omega(\kappa + |T|)^+ \rightarrow_T (\kappa^+)_\kappa$$

(the property in the assumption is very strong, <u>but</u> it is reasonable in context of "why the research on monadic logic concentrates on trees + linear orders"? How is this proved? We can decompose any model to a tree sum starting by 2.17 with a large sequence of indiscernible, extend it to a decomposition, so the tree has 2 levels. However, the cardinality of the "leaves" have no apriori upper bound. But as there are many leaves such that the model is their sum we can show that the model, if it is not too little can be extended to all larger cardinalities retaining its monadic theory.

This proves that the dividing line (mentioned in 2.17 and discussed earlier) is real.

Macintyre had said that cardinals appearing in a theorem make it uninteresting (though he has moderate lately). I think inversely and find fascinating theorems showing that for the family of models of T of cardinality λ having a property is equivalent to an inside "syntactical" property of T. Also, I think it is a good way of discovering a worthwhile property of T which should be persuasive even for those who unlike myself do not see their beauty. Macintyre supposedly is even less friendly toward infinitary logics; but

<u>2.18 Thesis</u> We use infinitary logic to "drown the noise"; only from the distance you see the major outlines of the landscape clearly, so for many purposes; e.g. examining the $L_{\infty,\kappa}$-theory of models of T will give a more coherent and interesting picture whereas probably $L_{\infty,\aleph_0}$-theory gives an opaque one. It probably is not accidental that superstability was discovered looking at behaviour in cardinals like $\beth_\omega$ and not from investigating countable models.

We may consider more complicated partition relations

2.19 Definition. 1) Let $\lambda \rightarrow_T (\mu)^n_\kappa$ for first order complete T means; letting $\mathfrak{C} = \mathfrak{C}_T$, the monster model for T:

- $\boxtimes$ if $\bar{\mathbf{a}}_u \in {}^{\kappa>}\mathfrak{C}$ for $u \in [\lambda]^{\leq n}$, <u>then</u> we can find $W \in [\lambda]^\mu$ and $\bar{\mathbf{b}}_u \in {}^{\kappa>}\mathfrak{C}$ for $u \in [W]^{\leq n}$ such that:
  - (α) $\bar{\mathbf{a}}_u$ is an initial segment of $\bar{\mathbf{b}}_u$
  - (β) $v \subseteq u \in [W]^{\leq n} \Rightarrow$ Rang$(\bar{\mathbf{b}}_v) \subseteq$ Rang$(\bar{\mathbf{b}}_u)$
  - (γ) if $m < \omega$ and $i_0 < \ldots < i_{m-1}$ and $j_0 < \ldots < j_{m-1}$ from W then there is a $\mathfrak{C}$-elementary mapping f (even an automorphism of $\mathfrak{C}$) such that:

$v \in [\{0,\dots,m-1\}]^{\leq n}$ & $u_1 = \{i_\ell : \ell \in v\}$ & $u_2 = \{j_\ell : \ell \in v\}$ implies that f maps $\bar{\mathbf{b}}_{u_1}$ to $\bar{\mathbf{b}}_{u_2}$.

2) Similarly for $\rightarrow_K, \rightarrow_M$.

In [Sh:h, Ch.VI] we get that for stable T, no large cardinal is needed: cardinal bounds which is essentially $(\beth_n(\mu^-))^+$ suffice (in fact this is done in a much more general framework, and also for trees $({}^{\omega>}\lambda)$.

2.20 Problem: For first order theories T for every μ, κ how large is $\mathrm{Min}\{\lambda : \lambda \rightarrow_T (\mu)^n_\kappa\}$?

We expect a dichotomy: either suitable large cardinal are needed, so $\beth_k(\mu + \kappa + |T|)$ for $k = k_n$ large enough suffice.

* * *

Returning to classifying first order theories T by the monadic logic, the case of T stable is reasonably analyzed ([BlSh 156], [Sh 284c]), still there is a troublesome dividing line.

2.21 Problem Assume any model of T is a non-forking sum of $\langle M_\eta : \eta \in \mathbf{T}\rangle$ where $\mathbf{T} \subseteq {}^{\omega>}\lambda$ (closed under initial segments). In some cases the $\mathscr{L}$(mon)-theory is essentially exactly as complicated as that of $({}^{\omega>}\lambda, \triangleleft)$, in other cases we can interpret Q_{pr}. Can we prove the dichotomy, i.e. that always at least one of those holds.

Probably not so characteristic of me, but I asked
2.22 Question: Is the monadic-Borel theory of the real line decidable?
2) Is the monadic theory of $({}^{\omega\geq}2, \triangleleft)$ undecidable?

The meaning of monadic-Borel is that we interpret the monadic quantifier $(\exists X)\varphi$ by "there is a Borel set X" such that φ.

The choice of Borel is just a family of subsets of $\mathbb{R}$ (or ${}^{\omega\geq}2$) which is closed under reasonable operations and do not contain subsets gotten by diagonalization on the continuum. So $\mathscr{P}(\mathbb{R}) \cap L[\mathbb{R}]$ assuming AD is okay, too. If we try the $({}^{\omega\geq}2, \triangleleft)$ version, Borel determinacy + Rabin machines looks the obvious choice for trying to prove a decidability answer. For $(\mathbb{R}, <)$ it is reasonable to try to get elimination of quantifiers, i.e. an appropriate version of $UTh^n(\mathbb{R}, \bar{Q})$ should be enough ([Sh 42, §4]).

§3 Automorphisms and quantifiers

3.1 Discussion: As known for long: for first order complete theory T there are lots of models with lots of automorphisms (in the direction of saturated ones or EM ones). To build models with no nontrivial ones is hard (even in special cases - there is literature). Ehrenfeucht conjectures that the classes

$$\{\lambda : \psi \text{ has a rigid model in } \lambda, \lambda > \aleph_0\}$$

are simple (like omitting types, in particular: initial segments); "unfortunately", essentially any $\sum_2^1$ class of cardinals may occur (see [Sh 56]). So set theoretically we understand what these families of cardinals are, but model theoretically the answer is considered negative. We may try to change the question, so that we can say something interesting.

3.2 Definition. Let $\psi = \psi(\bar{R})$ be a first order sentence on the finite sequence $\bar{R}$ of predicates and function symbols (with R_0, i.e. $\{\bar{x}_0 : R_0(\bar{x}_0)\}$ being "the universe", so unambiguous and for simplicity each R_ℓ a predicate; in general $\bar{x}^0$ is not a singleton, and we may let R_1 be equality). Consider enriching first order logic by quantifiers $Q^\psi = Q^{\text{aut}}_\psi$ which means that we can apply $(Q^\psi_{\bar{\varphi}} f)$ to a formula where $\bar{\varphi} = \langle \varphi_\ell(\bar{x}^\ell, \bar{z}) : \ell < \ell g(\bar{R})\rangle, \ell g(\bar{x}^\ell) =$ arity of R_ℓ, and in the inductive definition of satisfaction $M \models (Q^\psi_{\bar{\varphi},\bar{a}} f)\vartheta$ holds when: if $\langle \varphi_i(\bar{x}^\ell, \bar{a}) : \ell < \ell g(\bar{R})\rangle$ defines in M an $\bar{R}$-model $M_{\bar{\varphi},\bar{a}}$ of ψ then there is an automorphism f of $M_{\bar{\varphi},\bar{a}}$ such that ϑ holds. So syntactically f is a variable on partial unary functions.

Note: those quantifiers ([Sh 43],[Sh:e]; really more general there, see 3.20) do not exactly fit "Lindstrom quantifiers". They can be expressed artificially by having many Lindstrom quantifiers and each Lindstrom quantifier is a case of this. But those are naturally second order quantifiers and e.g. adding two such quantifiers is more than adding the cases for each. So for a vocabulary τ in the language

$$L_{\omega,\omega}(Q^\psi)(\tau)$$

we have variables: individual variables and unary partial function variables, we can form $(Q^\psi_{\langle \psi_\ell : \ell < \ell g(\bar{R})\rangle \bar{y}})\vartheta$ if φ_ℓ, ϑ are already in $L_{\omega,\omega}(Q^\psi)(\tau)$ and satisfaction is defined as above. We may allow such quantifier to act only on models $M_{\bar{\varphi},\bar{a}}$ whose universe is $\subseteq M$ <u>or</u> to allow the set of elements of $M_{\bar{\varphi},\bar{a}}$ (equivalently $\bar{x}^0$) to be e.g. the set automorphisms of $M_{\varphi^*,\bar{b}^*}$ for any $\bar{b}^*$ satisfying say $\theta^*(\bar{y}, c)$ where ψ^*, ϑ^* are formulas in our logic of smaller depth, etc. For compactness this does not matter.

3.3 Problem: For which ψ is $L(Q^\psi)$ a compact logic?

3.4 Example: If $\bar{R} = \langle R_0\rangle, \psi = \forall x R_0(x)$, then we have quantifications on unary functions varying on permutations, so the quantifier Q^ψ gives second order logic (on nontrivial structures). So in this problem even though the models of ψ can be written as $M_0 + M_1$ or "degenerated", we get second order logic.

Note: So for this classification a sentence ψ which says "the model (of ψ) is trivial" gives a complicated logic $L_{\omega,\omega}(Q^\psi)$.

If ψ has only finite models, the logic is compact in a dull way. You may wonder if compactness holds for any sentence ψ at all, as this looks like a second order logic. However, there are interesting sentences ψ with $L(Q^\psi)$ compact:

- (a) $\psi =$ the axiom of the theory of Boolean Algebras, (i.e. conjunction of the axioms in standard axiomatization)
- (b) the axiom of the theory of ordered fields.

We expect that if the models of ψ are complicated enough, the logic will be compact. We may also have applications to the compactness: it was known

CON(there is 1-homogenous[1] atomless Boolean Algebra B
such that $\mathrm{Aut}(B)$ is not simple)

even: $\mathrm{Con}(\exists G \triangleleft \mathrm{Aut}(B)(\mathrm{Aut}(B)/G$ commutative$))$

(see [Sh 384]; it was known that $\mathrm{Aut}(B)'$ (= commutator subgroup) is simple).

So the compactness and completeness theorems show: ZFC $\vdash$ "there are such Boolean Algebras". So considering the success of the compactness and completeness theorems having such quantifier will be plausably in addition to being good by itself, also applicable.

So we are interested in:
3.5 Problem: Find more such quantifiers (homomorphisms of embeddings instead of automorphisms are welcomed, see [Sh:e] on the cases above).

The proof gives more examples but we like to have:
3.6 Problem: Characterize the ψ for which we have a compact $L(Q^\psi)$ or at least find:

- (a) general criterion
- (b) natural examples rather than those which look to the proofs one has.

We may consider also:
3.7 Problem: Characterize the strongly rigid first order theory T and the rigid ones where,

3.8 Definition. 1) First order T is called strongly rigid if: for every theory $T_1 \supseteq T$ there is a theory $T_2 \supseteq T_1$ such that the pair (T_2, T) is rigid which means that T_2 has a model M_2, such that every $f \in \mathrm{Aut}(M_2 \restriction \tau(T))$ is first order definable with parameters in M_2. We say T is super rigid if above $T_2 = T_1$. We say T is essentially rigid if (T, T) is rigid. We say (T_1, T) is rigid for $\varphi(M)$ if $\varphi(-)$ is a property of models of T_1 and $M_2 \restriction \tau(T)$ satisfies φ (e.g. $|T_1|^+$-saturated). We add "in λ" if the models is required to be of cardinality λ.
2) We add the adjective everywhere if we omit the demand "$T \subseteq T_1$" and replace $f \in \mathrm{Aut}(M_2 \restriction \tau_T)$ by f an automorphism M of T which is interpreted in M_2 by (first order) formulas with parameters (as in 3.2, of course the model M' of T has the vocabulary of T).

[1]A Boolean Algebra B is 1-homogeneous if for any $x, y \in B \backslash \{0_B, 1_B\}$, some automorphism of B map x to y

This is a way to classify T's.

Those are relatives of having rigid models. The definable automorphisms are the parallel of inner automorphisms of a group. Note that all those notions do not imply that T has a rigid model; if M is a complete, say infinite, non-abelian group (i.e. any automorphism is inner) then $\mathrm{Th}(M)$ is essentially rigid but has no rigid model. The version with $T_1 = T_2$ is the best case. If we replace T by any model of T interpretable in M_1 (as in 3.8(2)) and allow T_1 to have parallels of Skolem functions we are approximating the compactness and completeness problem discuss above. We may even let $T_1 = Th(\mathscr{H}(\chi), \in), \chi$ strong limit, and consider interpretation of T on "sets" of the model M_2 of T_2 rather than classes.

Why have we concentrated on ordered fields and Boolean Algebras?

The point is that e.g. for a dense partial order we can get a model where for every partial order definable in it, every automorphism of it as a partial order is, for a dense set of intervals, definable with parameters. (If the partial order is not dense, consider "infinite intervals"). Why "ordered field"? Only as in this case there any automorphism is determined by its action on any interval. Concerning Boolean Algebras, the underpinning point is that we consider structures $(A, B, R), R \subseteq A \times B$ which satisfy comprehension, that is:

$$(\forall y_1 \neq y_2 \in B)(\exists x \in A)(xRy_1 \equiv xRy_2)$$

and have the strong independence, that is,

$$\begin{aligned}(\forall x_1, \ldots, x_n \in A)&(\forall y_1, \ldots, y_n \in A)(\exists z)\\ &(\bigwedge_{\ell,k} x_\ell \neq y_k \Rightarrow (\bigwedge_{i=1}^{n} x_i R z \ \ \& \ \ \neg y_i R z)).\end{aligned}$$

(An abvious exampls is an atomic Boolean Algebra $B, A = \text{atoms}(B), B = B$ are okay).

For some of the readers a bell may ring. A theory T is <u>unstable</u>: iff it has the strict order property (that is some $\varphi(\bar{x}, \bar{y})$ is a partial order with infinite chains) <u>or</u> has the independence property (a relative of the strong independence property). This does not say any unstable theory will do but indicates that an unstable theory at least locally will do.

<u>Note</u>: For the theory of linear orders, for $(A, <)$, if E is a convex equivalence relation with classes A_i for $i < i^*$ and $f_i \in \text{Aut}(A, <)$ maps A_i to itself, then $\bigcup_i (f_i \restriction A_i) \in \text{Aut}(A, <)$. (We can express that informally as "models of T are, in general, decomposable"; to avoid trivialities we restrict ourselves to uncountables ones). So for T any theory of infinite linear orders, T is not strongly rigid. We need ψ (or T) to say that the model is not decomposable.

Generally,

3.9 Definition. 1) We say ψ (or T) is pseudo decomposable when: if for every n, there are a model M of ψ (or of T), M the disjoint union of the nonempty sets A_i (for $i < n$) and $f_i^1 \neq f_i^2$ from $\text{Aut}(M)$ such that

$$f^1_i \restriction (M \backslash A_i) = f^2_i \restriction (M \backslash A_i)$$

and $\bigcup\limits_{i=0}^{n-1} (f_i^{\eta(i)} \restriction A_i) \in$ Aut(M) for every $\eta \in {}^n 2$; in other words, M has a nontrivial automorphism over $M \backslash A_i$ for each i.
2) We say ψ (or T) is semi-decomposable if for every n we can find a model M of ψ and partition $\langle A_\ell : \ell < n \rangle$ of M to infinite subsets such that:

- $(*)$ for every finite set Δ_1 of formulas in $L(\tau_T)$ there is a finite set Δ_2 of formulas in $L(T)$ such that
- $(**)$ if for $\ell < n, k^\ell < \omega$ and $\bar a_\ell, \bar b_\ell \in {}^{k_\ell}(A_\ell)$ and $\bar a_\ell, \bar b_\ell$ realize the same Δ_2-type in M and $\ell = 0 \Rightarrow \bar a_\ell = \bar b_\ell$, <u>then</u> $\bar a_0 \hat{}\, \bar a_1 \hat{}\, \ldots \bar a_{n-1}, \bar b_0 \hat{}\, \bar b_1 \hat{}\, \ldots \bar b_{n-1}$ realize the same Δ_1-type in M.

3) We say almost decomposable if the function $\Delta_1 \mapsto \Delta_2$ does not depend on n.

3.10 Claim. *1) If T is pseudo decomposable, <u>then</u> we can find $T_1 \supseteq T$ such that:*

- (a) *for any model M_1 of T_1 we have: (Aut$(M_1 \restriction \tau(T))$ has cardinality $2^{\|M_1\|}$ hence some $f \in$ Aut$(M_1 \restriction \tau(T))$ is not definable in M_1 even with parameters*
- (b) *if $T = \{\psi\}$ then for models of T_1, in the logic $L(Q^\psi)$ we can interpret second order logic on M_1*
- (c) *we can embed also some product $\Pi\{G_i : i < \|M_1\|\}, G_i$ a nontrivial group.*

2) If T is semi-decomposable then T is pseudo decomposable.
3) If T is almost decomposable, <u>then</u> it is semi-decomposable and for any saturated model of cardinality λ (or just λ^+-resplended model of T), $\lambda > |T|$, we can find $\langle A_i : i < \lambda \rangle$ as in 3.10, in fact:

- (a) *if $\bar a_i, \bar b_i \in {}^{\omega >}(A_i)$ for $i < \lambda, \bar a_0 = b_0$, $tp(\bar a_i, \emptyset, M) = tp(\bar b_i, \emptyset, M)$ then $\bar a_{i_0} \hat{}\, \ldots \hat{}\, \bar a_{i_n}, \bar b_{i_0} \hat{}\, \ldots \hat{}\, \bar b_{i_n}$ realizes the same type in M for any $i_0 < \ldots < i_n < \lambda$*
- (b) *$(M, A_i)_{i \in \omega}$ is λ-saturated for $\omega \in [\lambda]^{<\lambda}$.*

For Boolean Algebras we can decompose the set of atoms, but the image of an element is not deciphered so this theory is not even pseudo decomposable.

Be careful, the statement "B is the Boolean Algebra generated by the close-open intervals of a linear order I" is not first order (this follows by the compactness so if T_1 extend the theory of Boolean algebras then it has a model with no undefinable automorphism). Now for the first problem, 3.3, the hope is that failure pseudo indecomposability is enough for compactness, it is of course necessarily by 3.10.

<u>3.11 Question(Cherlin)</u>: What occurs for vector spaces over finite fields?

Let $\mathbb{F}$ be a (fixed) finite field and let $\psi_{\mathbb{F}}$ be the conjunction of the axioms of vector spaces over the field $\mathbb{F}$ (we have binary function symbols for $x + y, x - y$, individual constant 0 and unary functions F_c for $c \in \mathbb{F}$ to denote multiplication by c). There is $T_1 \supseteq \{\psi_{\mathbb{F}}\}$ such that for models of T_1, in the logic $L(Q^{\rm aut}_{\psi_{\mathbb{F}}})$ we can interpret second order logic on M_1 (similarly for a finitely generated field).

Remark. Also for a general field this works, except that we do not have the quantifier as $\psi_{\mathbb{F}}$ is an infinite conjunction of first order formulas.
Why? Enough to have T_1 such that; for M^* a model of T_1:

(*i*) $P_1^{M^*}, P_2^{M^*}$ are disjoint subsets of the vector space
(*ii*) $P_1^{M^*} \cup P_2^{M^*}$ is an independent set in the vector space
(*iii*) F_1 is a unary function, $F_1 \restriction P_\ell^{M^*}$ is one-to-one onto M^* for $\ell = 1, 2$
(*iv*) F_2, a two-place function, is a pairing function.

How is the interpretation? For any function g from $P_1^{M^*}$ to $P_2^{M^*}$ there is an automorphism f of the vector space such that:

$$x \in P_1^M \Rightarrow f(x) = x + g(x)$$

$$x \in P_2^M \Rightarrow f(x) = x.$$

My impression is that any reasonable example will fall easily one way or the other by existing methods.

* * *

3.12 Definition. Let M be a model of $T, P \subseteq M$. We say $T' = Th(M, P)$ has the automorphic embeddability property over P if for every model (M', P') of T', every automorphism f of $M' \restriction P'$ can be extended to an automorphism of M.

3.13 Question Characterize the theories $T' = Th(M, P)$ which has the automorphic embeddability property over P.

This looks hard on us as characterization of this would probably involve $\mathscr{P}^-(n)$-diagrams as in classification over a predicate; on the case with no two cardinal models (i.e. $\|M\| > |P^M|$ assuming there is $\lambda = \lambda^{<\lambda} \geq |T|$), see [Sh 234]. The general case is, unfortunately, still in preparation ([Sh 322])); see end of §6.

* * *

There are other ways to consider quantification over automorphisms:

For a model M let $(M, \mathrm{Aut}(M))$ be the two sorted model, one sort is M, the other is the group $\mathrm{Aut}(M)$, with the application function, that is in the formulas, we allow forming $f(x)$ for x of first sort, f of second sort. We may replace $\mathrm{Aut}(M)$ by the semi-group of endomorphisms or one-to-one endomorphisms.

Now for a variety $\mathscr{V}$, the complicatedness of the first order theory of the endomorphism semi-group $\mathrm{End}(F_\lambda)$ of the free algebra with λ generators is reasonably understood (see [Sh 61]) but so far not the automorphism group in the general case though several specific cases were analyzed, (see [ShTr 605], [BTV91]).

<u>3.14 Problem</u>: For which varieties $\mathscr{V}$, letting F_α be the free algebra in $\mathscr{V}$ generated by $\{x_i : i < \alpha\}$, can we in $\mathrm{Aut}(F_\lambda)$ (first order) interpret second order theory of λ? We hope for a solution which depend "lightly" on λ (like $\mathrm{Aut}(\lambda, =)$)? We may allow quantification on elements or even use $(F_\lambda, \mathrm{Aut}(F_\lambda))$; but, of course, better is if we succeed to regain it.

The following property looks like a relevant dividing line

3.15 Definition. We say the variety $\mathscr{V}$ is Aut-decomposable if:

> if $F_{\omega 2}$ is the algebra generated freely by $\{x_i : i < \omega + \omega\}$ for $\mathscr{V}$ and $f \in \mathrm{Aut}(F_{\omega 2})$ satisfies $f(x_n) = x_n$ for $n < \omega$, <u>then</u> we have:
> f maps $\langle x_{\omega+n} : n < \omega \rangle_{F_{\omega 2}}$ to itself.

Why? For varieties $\mathscr{V}$ with this property we can repeat the analysis of $\mathrm{Aut}(\lambda, =)$ which is the group of permutations of λ; though first order interpretation of elements has to be reconsidered. But this is not needed in generalizing the "upper bound", the equivalences. That is for proving, say $\mathscr{V}$ with countable vocabulary for simplicity, that $\mathrm{Th}(\mathrm{Aut}(F_{\aleph_\alpha}))$ depend "lightly" on α; i.e. if for $\ell = 1, 2, \alpha_\ell = \delta_\ell + \gamma_\ell, \gamma_\ell < ((2^{\aleph_0})^+)^\omega$ (ordinal exponentiations) and $\mathrm{Min}\{\mathrm{cf}(\alpha), (2^{\aleph_0})^+\} = \mathrm{Min}\{\mathrm{cf}(\alpha_2), (2^{\aleph_0})^+\}, \gamma_1 = \gamma_2$ then $\mathrm{Th}(\mathrm{Aut}(F_{\aleph_1})) = \mathrm{Th}(\mathrm{Aut}(F_{\aleph_{\alpha_2}}))$. On the other hand, if it fails an automorphism of F_λ code a complicated subset of λ.

* * *

We may look at questions on "are there logics with specified properties?"
An old problem (see [BF]):
<u>3.16 Question</u>: Is there an $\aleph_1$-compact extension of $L(Q)$ which has interpolation (Craig)?

I prefer
<u>3.17 Question</u>: Is there in addition to first order logic a compact logic which has interpolation?

Barwise prefers to look at definability properties of logics (e.g. characterizing $L_{\infty,\omega}$) but my taste goes to:

<u>3.18 Problem</u>: Find (nontrivial) implications between properties of logics.
See for example [Mw85], [Sh 199]; interpolation and Beth theorems are, under reasonable assumption, equivalent; and amalgamation essentially implies compactness. After great popularity in the seventies, the interest has gone down, a contributing factor may have been the impression that there are mainly counterexamples. This seems to me too early to despair.
However, Vannanen's book [Va9x] should appear.

<u>3.19 Discussion</u>: So we are interested in enriching first order logic by additional quantifiers preserving compactness and getting interpolation.

A natural play is to allow second order variables X but restrict the existential quantifier to cases when the relation P (or function) satisfies some first order sentence ψ with some specific old R as a parameter (e.g. P is f, an automorphism of a model of ψ, a case discussed above). Another way ([Sh 43]) is to replace "exist" by "the family of those satisfying it belongs to a family $\mathbb{D}(M)$ of such relations over M". An example introduced in [Sh 43] is the case of a unary predicate, with $\mathbb{D}(M)$ being the club filter on $[|M|]^{\aleph_0}$; or equivalently for the strength of the logic, the family of stationary subsets of $[|M|]^{\aleph_0}$. Those quantifiers are $(aaP), (stP)$, respectively. This logic has many properties like $L(Q)$, see [BKM78], some like second order, [ShKf 150], [Sh 199].

Now interpolation holds for the pair of logic[2] $(L(Q^{\rm cf}_{\le\aleph_0}), L(aa))$ which means: if φ_ℓ is a sentence in $L(Q^{cf}_{\le\aleph_0})(\tau_\ell)$ for $\ell = 1,2$ and $\vdash \varphi_1 \to \varphi_2$ then for some $\psi \in L(aa)(\tau_1 \cap \tau_2)$ we have $\vdash \varphi_1 \to \psi$ and $\vdash \psi \to \varphi_2$. Also the Beth closure of $L(Q^{\rm cf}_{\le 2^{\aleph_0}})$ is compact so there is a compact logic which satisfies the "implicit definability implies the explicit definability"; moreover, is reasonably natural (at least in my eyes). Seems near the mark but not in it. Consider (see [HoSh 271]) the following logic: let $\aleph_0 < \kappa < \lambda$ and κ, λ are compact cardinals, and expand first order logic by all the connectives of the form $\bigwedge\limits_D \varphi_i$ where D is a κ-complete ultrafilter on some $\theta \in [\kappa, \lambda)$, meaning naturally $\{i < \theta : \varphi_i\} \in D$. It has interpolation but not full compactness (only μ-compact for $\mu < \kappa$).

* * *

More formally and fully

3.20 Definition. 1) Assume

(a) $\psi(\bar R, S)$ be a sentence (usually in first order) in the vocabulary $\bar R$, i.e. $\bar R$ is a list with no repetitions of predicates and function symbols, S an additional predicate, each have a given arity (for notational simplicity R_0 is a unary predicate for "the universe" each R_ℓ is a predicate)

(b) $\mathbb{D}$ is a function, its domain is $K_{\bar R} = \{M : M$ a model, the vocabulary of M is $\{R_i : 1 + i < \ell g(\bar R)\}, R^M_0 = |M|\}, \mathbb{D}(M)$ is a family of subsets of $\{N : N$ is an expansion of M by $S^N\}$ and, of course, if f is an isomorphism from M_1 onto M_2 then f maps $\mathbb{D}(M_1)$ onto $\mathbb{D}(M_2)$.

The quantifier $Q_{\bar\varphi,\mathbb{D}}$, syntactically acts as $(Q_{\bar\varphi,\mathbb{D}}S)\vartheta$ where S is a variable on n-place relations, $\bar\varphi = \langle \varphi_i(\bar x_i, \bar z_i) : i < \ell g(\bar R)\rangle$ and $\ell g(\bar x_i) =$ arity$(R_i)\rangle$ and $M_{\bar\varphi,\bar a} = M_{\bar\varphi,\bar a}[\mathfrak{A}]$ is defined as in Definition 3.2 and

$$\begin{aligned}\mathfrak{A} \models &(Q_{\bar\varphi,\mathbb{D}}S)\vartheta(S,\bar a) \text{ iff}\\ &\{S : S \text{ an } n\text{-place relation on } \{x : \varphi_0(x,\bar z)\}\\ &\text{ and } (M_{\bar\varphi,\bar a}[\mathfrak{A}], S) \models \vartheta(S,\bar a)\}\\ &\text{ belongs to } \mathbb{D}(M_{\bar\varphi,\bar a}[\mathfrak{A}]).\end{aligned}$$

[2] $Q^{\rm cf}_{\le\aleph_0}$ tells us the cofinality of a linear order is $\le \aleph_0$

2) Similarly when for defining $M_{\bar{\varphi},\bar{a}}[\mathfrak{A}]$ we replace equality by an equivalence relation R_1.

A variant of 3.17 is

3.21 Question: Is there a reasonably defined such quantifier Q such that $L(Q)$ is compact and has interpolation? or at least has the Beth property?

§4 Relatives of the main gap

A main gap theorem here means, for a family of classes of models, that for each class K <u>either</u> we have a complete set of invariants for models of K (presently, which are basically just sets) <u>or</u> it has quite complicated models (see below after 4.11).

This seems obviously a worthwhile dichotomy, if it occurs indeed, and have been approached as a dichotomy on the number of models (but see below).

We know for a countable first order, T complete for simplicity, that $I(\aleph_\alpha, T) =: \{M/\cong : M \models T \ \ \& \ \ \|M\| = \aleph_\alpha\}$ behave nicely (either $I(\aleph_\alpha, T) = 2^{\aleph_\alpha}$ for every $\alpha > 0$ or $< \beth_{\omega_1}(|\alpha| + \aleph_0)$ for every α). But many relatives of this question are open.

I thought a priori on several of them that they will be easier, but have worked more on the case of models so the earlier solution in [Sh:c] does not prove this thought wrong; still this a priori opinion is not necessarily true.

<u>4.1 Question</u> 1) Prove the main gap for the class of $\aleph_1$-saturated models.
2) Prove the main gap for the class of $\aleph_0$-saturated models.

Now 4.1(1) have looked a priori relatively not hard, in fact the work in [Sh:c] seems to solve it "except" for lack of regular types, so in the decomposition theorem we are lacking how to exhaust the model.

Another direction is:

<u>4.2 Problem</u>: Let T be countable stable complete first order theory. Show that if $\neg(*)$, <u>then</u> T has otop (or dop; for otp we allow types over countable sets), where

(∗) if $M_0 \prec M_\ell \prec \mathfrak{C}$ for $\ell = 1, 2$ and $M_1 \underset{M_0}{\bigcup} M_2$, (i.e. $\mathrm{tp}_*(M_2, M_2)$ does not fork over M_0) <u>then</u> there is a prime model (even $F_{\aleph_0}$-prime) over $M_1 \cup M_2$.

<u>Note</u>: For superstable this is true (this was the main last piece for the main gap, see [Sh:c, Ch.XII]).

<u>4.3 Discussion</u>: Our problem is that the proof there uses induction on ranks, and generally stable theories have less well understood theory of types (not enough regular types exist), just as in 4.1. However, if we assume T superstable without DOP, then every regular type is either trivial (= the dependence relation is) or of depth zero ([Sh:c, Ch.X,§7]). There is some parallel theorem for stable theories without DOP, it may be helpful.

Maybe relevant is the theory of types for stable T in [Sh:c, Ch.V,§5], [Sh 429] and Hernandez [He92] which proved that if $\mathbf{I}_0, \mathbf{I}_1$ are indiscernible not orthogonal then for some indiscernible $\mathbf{J}, \mathbf{I}_\ell \leq_s \mathbf{J}$ ([Sh:c, Ch.V,§1]), but in spite of early expectation this has not been enough to solve 4.1(1).

Where could 4.2 help? For the theories which are "low" for the main gap, a model is characterized up to isomorphism by its $L_{\infty,\aleph_1}$(dimensional quantifier) theory. But we may look at logics allowing e.g. a sequence of quantifiers with countable length (even ω_1), as investigated by the Finnish school. We know that for unstable theories, and for stable theories with DOP we have the nonstructure, see Hyttinen

Shelah [HySh 676]. It seems that 4.2 would complete a piece in finding another dividing line here. Some stable, unsuperstable theories become low. Essentially, the hope is that <u>either</u> every model of T can be coded by trees with at most δ^* levels, δ fixed, even countable or $\leq \omega_1$ <u>or</u> we have the order property (even independence property) in a stronger logic (in NOPOT or NDOP holds). However, 4.2 is not enough, we need also a decomposition theorem.

<u>4.4 Question</u>: If T is countable stable with NDOP and NOPOT and $(*)$ of 4.2 holds, does the decomposition theorem hold at least for shallow T?
Interpretation of groups may be relevant, particularly non-isolated types, because non-orthogonal, weakly orthogonal types tend to involve groups.
Note that here the existnece of $\mathbf{F}^\ell_{\aleph_0}$-primary model on $N \cup \{a\}$ included inside a given $M \supseteq N \cup \{a\}$ is not assured.

Another problem is
<u>4.5 Problem</u>: Prove the main gap for $K_T =: \cap\{M_1 \restriction T : M_1 \text{ a model of } T_1, T \subseteq T_1, |T_1| \leq 2^{\aleph_0}\}$.
Note that if $(\forall \lambda < 2^{\aleph_0}) 2^\lambda = 2^{\aleph_0}$, then we can find one T_1 which suffices (as by Robinson lemma we have "amalgamation" for theories, so there is a universal (oven "saturated") T_1, i.e. if $T_1' \supseteq T$ is complete, $|T_1'| \leq 2^{\aleph_0}$ by changing names of predicates not in $\tau(T)$ we can embed T_1' into T_2 over T.
Like all these problems, possibly a large part of the work is already done, but though a priori I thought this was easier, it is not necessarily true. The natural hidden order property is by $\exists^{\geq \lambda} x \varphi(x,y,z)$ (cardinality quantifiers) (maybe on the number of equivalence classes or dimension for Δ-indiscernible sets, Δ finite), we hope there will not be a need to consider several cardinality quantifiers simultaneously. If M is a model of T which looks like $(A, 0, P, E)$, $A = \omega \cup \{(n,m,\ell) : \ell < k(n,m)\}$, $P = \omega$, F_1, F_2 unary functions, $F_\ell(n_1,n_2,k) = n_\ell$ for $\ell = 1,2$, $E = \{((n_1,m_1,k_1),(n_2,m_1,k_2)) : n_1 = n_2, m_1 = m_2 \text{ and } k_1,k_2 < k(n_1,m_1)\}$ and the function $k(n,m)$ random enough, T has a hidden order property, that is, the formula $\varphi(x,y) =: (\exists z)(F_1(z) = x \;\;\&\;\; F_2(z) = y \;\;\&\;\; (\exists^{\geq \aleph_1} z')(z' E z))$.
We phrase it appropriately (and there are fewer divisions).

The very low parts of the hierarchy have been analyzed, i.e. the bottom part: categorical

$$\aleph_\alpha > 2^{\aleph_0} \Rightarrow I(\aleph_\alpha, K_T) = 1 \text{ or } I(\aleph_\alpha, K_T) \geq 2^{|\alpha|}.$$

For the main gap, we can assume T is superstable and we should analyze for $M \in K_T$, which we know is $\aleph_\varepsilon$-saturated and it is natural to analyze the different dimensions.

<u>Note</u>: If T is a theory of one equivalence relation E saying there is an equivalence class with n elements, for infinitely many n, it is not in the lowest class, but still we understand it. For the T above, if for every n we have $\aleph_0$ classes with n elements then $K_T = \{M : \text{there are } \|M\| \text{ classes of cardinality } \|M\|, \text{ for each } x \in M, \text{ has } \|M\|\text{-classes with } (x/E)\text{-elements and } \forall n \exists x(|X/E| = n)\}$.
In the first case (i.e. $I(\aleph_\alpha, K_T) = 1$) every model is saturated. We expect that if $I(\aleph_\alpha, K_T) = 2^\alpha$, then for $M \in K_T$, there is an equivalence relation between indiscernible sets but on the set of equivalence classes, there is no further structure

(well, maybe unary "predicates") such that no two equivalence classes have the same dimension.

In general, the theme taken for granted is:
4.6 Thesis: If K is a reasonable[3] class of models then:

> the behaviour of the function $I(\aleph_\alpha, K)$ is uniform, the "same" for all relevant cardinals.

Have not really looked. The expectation is that after this level, we'll have 2^{2^α}, (and also $2^{|\alpha|^{\aleph_0}}, 2^{|\alpha|^{2^{\aleph_0}}}$, etc), and $\beth_n(|\alpha|)$, (or $\beth_n(|\alpha|^{\aleph_0}), \beth_n(|\alpha|^{2^{\aleph_0}})$) and $\beth_\zeta(|\alpha|)$ for each $\zeta \in [\omega, \omega_1)$ and then after our ω_1 steps we have $\aleph_\alpha \mapsto 2^{\aleph_\alpha}$.

The main[4] question left in [Sh:c] is
4.7 Problem: Prove the main gap for uncountable T.

The problem in proving is the lack of primary models, particularly over non-forking triples of models. Maybe more interpretation of groups will help in solving this. Maybe replacing "primary models" by prime models, and isolated types by unavoidable ones may help.

(Recall that B is primary over A if $B = A \cup \{a_i : i < \alpha\}$ and $\text{tp}(a_i, A \cup \{a_j : j < i\})$ is isolated for $i < \alpha$.)
Isolated types have been great (for $\aleph_0$-categoricity, no T with exactly two countable models, Morley theorem), but for an uncountable theory they are not sufficient, the lack of them does not witness much. Still there can be prime models.

Maybe we should look at derived non-elementary classes, where we look for hidden order and if there is none we get nicer properties. Maybe even define such classes inductively on $\alpha < \omega_1$ (or even $D(x = x, L, \infty)$, but carry enough connection to the original T to be able to finish soon (and carry enough to continue, see [Sh:h], [Sh 600]).

It may be reasonable to start with analyzing unidimensional T (concerning 4.7).

4.8 Thesis: All such problems have a "good" solution, (unlike Ehrenheuft Conjecture, see [Sh 54], see §3).

The audience asked
4.9 Question: Can a theory T be "nice" in spite of having many models, maybe still models of T can be understood by invariants.

4.10 Answer: "Nice" certainly yes (see §5 as you may choose to consider say linear order as reasonable invariants and so ask for which first order theories such invariant suffice). But not true, if you define a generalized cardinal invariant as follows (for simplicity $|T| = \aleph_0$).

Depth zero: cardinal invariant is a cardinal

Depth $\alpha + 1$: cardinal invariant are sets of sequences of length $\leq 2^{\aleph_0}$ of cardinal invariants of depth α or a cardinal invariant of depth α

depth δ for δ limit: depth α for some $\alpha < \delta$

---

[3]but see on rigidity!

[4]why I have been feeling so? As for almost all this book, countability plays a minor role

4.11 Claim. *If models of T of cardinality $\aleph_\alpha$ are characterized up to isomorphism by generalized cardinal of depth $\le \gamma_T$, then $I(\aleph_\alpha, T) \le \beth_{\gamma_T+1}(2^{\aleph_0})$ (see [Sh 200]).*

Really, the main gap for countable complete T's is a division to three cases. If T is in the upper case, model of T codes stationary sets; if T is in the lower case, a model can be described by a tree with $\le \omega$ levels and depth $\le \gamma_T < \omega_1$; and if T is in the middle case, a model can be described by a tree with $\le \omega$ levels, but can have depth an arbitrarily large ordinal. The first case is T unsuperstable or is NDOP or NDTOP, the second case are the deep theories (which are superstable, NDOP, NOTOP) and the third are the rest.

A theological question is which of those two dividing lines is the more striking dividing line. Probably between the upper case and the rest. Clearly the fact that from the isomorphism type of a model of T we can naturally compute a stationary set modulo a club (see 4.13 below), getting any such set, say that the class of models of T is very complicated, whereas a tree with ω levels seems reasonably understood though their number (up to isomorphism) is large. We can look at it in another way: if we "understood" the isomorphism types of M, forcing notions "which do not do much damage" (including preserving inequality of cardinality of the relevant sets), preserve non isomorphism of models if T is in the lower or middle case. E.g. if $\lambda = \operatorname{cf}(\lambda) > |T| = \aleph_0$ and $\mathbb{P}$ is a forcing notion not adding ω-sequences to λ preserving cardinalities $\le \lambda$ then $\mathbb{P}$ preserves non isomorphism of models of T of cardinality $\le \lambda$ iff T is in the lower of middle case. It seems a very weak demand of a complete set of invariants to be preserved by such a change in the universe. This is the intended meaning of the word (main) gap here, though to say that the isomorphism types of models of T are all "simple", "well understood" is open to variations, here the "good, well understood" case is very good, and the "bad" are so bad, that it is an evidence to this dividing line to be a major natural division (on c.c.c. forcing - see 4.18 below). E.g. we may above require the forcing to add no $(< \lambda)$-sequences getting the same division.

The audience asked

4.12 Question: Can we assign stationary sets as invariants?

4.13 Answer: In restricted classes of models it works but the question is what the connection should be between the model and the stationary set. That is, generally, there are enough stationary sets to code models in cardinality λ, so we have to say $M, S(M)/\mathscr{D}_\lambda$ (or $\mathscr{D}_{\le\kappa}(\lambda)$ or whatever) should be nicely connected. Hence this remains vague. Note that if we aim not at a complete set of invariants but as an evidence for nonstructure, then we can. That is, for any T in the upper case we can naturally assign a stationary subset of λ modulo $\mathscr{D}_\lambda$ as an invariant to models of T of cardinality $\lambda = \operatorname{cf}(\lambda)$ such that any stationary subset of λ (or of $\{\delta < \lambda : \operatorname{cf}(\delta) = \aleph_0\}$) appears. E.g. let T be unsuperstable. If say M has universe $\lambda = \operatorname{cf}(\lambda) > \aleph_1$, use $\{\delta < \lambda : M \restriction \delta \prec M$ and for every $\bar b \in {}^{\omega>}M$ every countable subtype of $\operatorname{tp}(b, M_\delta, M)$ is realized in $M_\delta\}$.

[Why? Let $\lambda = \operatorname{cf}(\lambda) > |T| + \mu$, where $\mu = \operatorname{cf}(\mu) > \aleph_0$. Let Φ be proper template for $K^{\rm tr}_\omega$, $\mathrm{EM}_{\tau(T)}(I, \Phi)$ a model of T, witnessing unsuperstability, let I be a linear order of the form $\lambda + J, J$ isomorphic to the inverse of μ and for $\delta \in S^* = \{\delta < \lambda : \operatorname{cf}(\delta) = \aleph_0\}$ let η_δ be an increasing ω-sequence of ordinals with limit δ. Now for $S \subseteq S^*$ let $I_S = {}^{\omega>}(\lambda + J) \cup \{\eta_\delta : \delta \in S\} \cup \{\eta : \eta \in {}^\omega(\lambda + J)$ is eventually zero$\}$. We can check that the invariant of $\mathrm{EM}(I_S, \Phi)$ is $S/\underset{\sim}{D}_\lambda$.]

However, this is not an invariant which characterizes up to isomorphism. The cases of NDOP, NOTOP are in face easier (can use the end of [Sh:e, Ch.III,§3]).

Classifying will not die as
4.14 Thesis: In any reasonable classification (in the present sense) there are examples of the "complicated" class which are actually well understood so should be prototypes of another class which is analyzable.

Hodges in his thesis had asked about
4.15 Question: When does a first order theory have a $\prec$-minimal model in λ? What can be $\mathrm{PrSp}(T)$?

4.16 Definition. 1) M is a $\prec$-minimal model in λ if it can be elementary embeddable into any other model of N of $\mathrm{Th}(M)$ of cardinality λ.
2) $\mathrm{PrSp}(T) = \{\lambda : M \text{ has a } \prec\text{-minimal model of } T\}$.

We may consider

4.17 Definition. $\mathrm{PrSp}'(T) = \{(\lambda,\mu) : M$ a model of T of cardinality μ which is $(\lambda,\prec)$-embeddable$\}$ where M is $(\lambda,\prec)$-embeddable if it is embeddable into every model of $\mathrm{Th}(M)$ of cardinality λ.

Hodges gave some examples of $\mathrm{PrSp}(T)$ and then I add a few others. Hodges showed that if T is the theory of infinite atomic Boolean Algebra, then $\mathrm{PrSp}(T) = \{\lambda : \lambda \text{ is strong limit}\}$. Also if I is a linear order with no monotonic sequence of elements of length $\lambda, \lambda = \mathrm{cf}(\lambda)$, then in $EM(I,\Phi)$ there is no formula defining when restricted to some set, a well order of length λ.

My old remarks and a theorem of Hrushovski that "unidimensional stable theory is superstable" gives
Fact: Assume T is stable, $\mathrm{cf}(\lambda) > |T|$; if $\lambda \in PrSp(T)$ then T is unidimensional (hence superstable) and $\mathrm{cf}(\mu) > |T| \Rightarrow \mu \in PrSp(T)$.

[why?
Case 1: T not unidimensional.
As in [Sh:c, Ch.V,§2].

Case 2: T is unidimensional, superstable. As in [Sh:c, Ch.IX,§2].

* * *

As discussed above we know that for complicated theories (say unstable or unsuperstable or ones with DOP or OTOP), models can code stationary sets hence "isomorphism types are very set theoretic sensitive". E.g. changing by forcing shooting a club of $\lambda = \mathrm{cf}(\lambda)$ disjoints to some S. It is natural to consider: "nice forcing notion can make non-isomorphic models to isomorphic".

4.18 Problem: For which first order countable T is there a c.c.c. forcing notion making non-isomorphic isomorphic.

Now (see [BLSh 464]) for unsuperstable T the answer is no and so with superstable with DOP or OTOP.

By [LwSh 518], there is T among the remaining with the answer no, after preliminary c.c.c. forcing. Laskowski and me agree there is a serious unsatisfactory point in the paper, but do not agree on its identity. He thinks it is the preliminary c.c.c. forcing. However, I think that as anyhow we deal with forcing this is minor, but the restriction on T major (whereas he think not).

Newelski told me that for T superstable countable, $p \in S^m(A)$ with uncountably many stationarization ([Sh:c, Ch.III]), he considered the meagre ideal on the set of stationarizations of p (in connection with Vaught conjecture for superstable T). Subsequently in [LwSh 560] we used the null ideal on this space for proving in the proof (could have used the meagre). He asked

4.19 Question: Are there T, A, p as above such that the ideal of null sets and of meagre sets are different?

* * *

We can measure the number of models in other ways.

4.20 Definition. : Let $I'_\kappa(\lambda, K) = \{M/\equiv_{\infty,\kappa}: M \in K, \|M\| = \lambda\}$ where K is a class of models of a fixed vocabulary, $\tau(K)$ $\equiv_{\infty,\kappa}$ is the equivalence relation of having the same $L_{\infty,\kappa}$- theory. If $\kappa = \aleph_0$ we may omit it, if $K = \{M : M \models T\}$ we may write T.

Starting my Ph.D. studies, I note (concentrating on $\kappa = \aleph_0$ but Rabin was not enthusiastic)

4.21 Theorem. *If K is elementary (or defined by $\psi \in L_{\lambda^+,\kappa}$), $\lambda = \lambda^{<\kappa} \geq |\tau(K)|$ and $I_\kappa(\lambda, K) \leq \lambda$, then $\lambda \leq \mu \Rightarrow I_\kappa(\mu, K) \leq I_\kappa(\lambda, K)$.*
(Later I understand that this is easy by Levy absoluteness; see [Sh 11] and see Nadel's thesis).

So
4.22 Question: For first order T, what can

$$\operatorname{Min}\{\lambda : I_\kappa(\lambda, T) \leq \lambda = \lambda^\kappa\}$$

be when it is $< \infty$, i.e. can you give a better bound than the Hanf number of $L_{(\tau(T)+\aleph_0)^+,\omega}$ (well ordering) $L_{\infty,\omega}$.

Lately, Laskowski and me investigate what can be the supremum of the $L_{\infty,\kappa}$ Karp height for models of T, so a theory is considered complicated if this is not bounded; this is closely related, see [LwSh 560]. The point is that while case $\kappa = \aleph_0$ is opaque the cases of many bigger κ is at least at present, not a dead end, supporting 2.18.

§5 Unstable first order theory

The major theme of classification theory has been for me, since [Sh 1]:
5.1 Meta Problem: Find worthwhile dividing lines on the family of (complete first order) theories.
A dividing line is not just a good, interesting property, it is one for which we have something to say on both sides; so for some problems naturally a solution goes by working on each side separately.

Of course phrased as "find dividing lines among the possible mathematical theories" this is too general and too vague to lead to mathematical theorems. But it is quite natural to restrict ourselves to the family of classes of models of first order theories (complete, and even countable).

Almost by definition, a dividing line is an interesting property (though not inversely: the class of non groups among $(A, F), F$ an associative two-place function or non 0-minimal first order theories are not so remarkable), but it is remarkable that, for our contexts, there are some. I have changed the name of [Sh:a] from "stability and the number of nonisomorphic models" to "classification theory"in order to stress its aim - finding meaningful dividing lines.

We believe good test problems are needed and, of course, problems on the number of non-isomorphic models were inherently interesting and serve well. But they could not serve for unstable theories. We shall see below how some problems succeed or fails in this role, but sometime we do not know of a good candidate. I have considered at various times $\lambda \mapsto \sup\{|S(A)|^+ : |A| \leq \lambda\}$, Keisler's order (i.e. saturation of ultraproducts), SP (see below) and later $\triangleleft^*$ ([Sh 500]) and the existence of universal models. Sometime getting a full ZFC answer (on which I work hard in [Sh:c]) seems too much so decide that it is reasonable to content myself with:
5.2 Half ZFC or Poor Man ZFC Answer: The result on the lower half of a dividing line will be ZFC (or semi-ZFC, i.e. depending on cardinal arithmetic in relevant cardinals), whereas in the complicated, upside we allow consistency results (in semi-ZFC: you may distinguish between cases to high consistency strength and those really consistent you may argue to add diamond, etc.).

This may help, as getting a too fine division is not our aim. Also if we are more interested in the dividing lines themselves, consistency results should be enough. This is even more relevant in classifying non-elementary classes and in classification over a predicate.
Note that if we look at "having complicated phenomena" as barrier to positive theorems, clearly a consistency result suffices.

* * *

5.3 Discussion: I find it particularly nice if the property have some equivalent definitions by "outside notions" and "by inside notions", some got for dealing with the "down side", some with "the upside". To clarify consider the example of stability; unstable theories are characterized by the order property (inside property for the upside, helpful in proving the class of models of an unstable T is complicated), stable theories are characterized by having finite local ranks ($R^n(p, \varphi, 2) < \omega$) (inside property for the downside; helpful in developing stability theory, showing we can in some senses understand the class of models of a stable T), instability is characterized by "for every λ for some $A, |S(A)| > \lambda \geq |A|$" (a weak outside property for the upside), stability by "for every $\lambda, |A| \leq \lambda \Rightarrow |S(A)| \leq \lambda^{|T|}$ (a weak outside property for the downside); late coming outside property characterizing "unstable T" is, "has many κ-resplended models of cardinality λ" where $\lambda = \lambda^{<\kappa} > 2^{|T|}$, (outside property for the upside), "stable T has exactly one $|T|^+$-resplended model of cardinality λ when $\lambda = \lambda^{|T|}$" (outside property for downside; see [Sh:e, Ch.V]).

* * *

Considering unstable theories, we knew they have the independence property or the strict order property, but not necessarily both, so the simplest prototypes of unstable theories are the T_{ord}, the theory of dense linear order and the theory T_{rg} of random graph. We have earlier in §2 discussed $(k,m)-*$-NIP and it is natural to ask on the inter-relations of them, the strict order property and the independence property, see [Sh 715].

For the neighborhood of T_{rg}, the problem I had chosen as a test problem was
5.4 Problem: Classify first order theories by

$$SP(T) = \big\{(\lambda,\kappa) : \text{every model } M \text{ of } T \text{ of cardinality } \lambda \text{ has an elementary extension } N \text{ of cardinality } \lambda \text{ which is } \kappa\text{-saturated}\big\}$$

or, for simplicity

$$SP'(T) = \{(\lambda,\kappa) \in SP(T) : \lambda^{2^{|T|}} = \lambda > 2^\kappa \text{ and } \kappa = \operatorname{cf}(\kappa) > 2^{|T|}\}.$$

[Why "for every M", not just there is M? Because then, letting $T = Th(M), M = M_1 + M_2, T_\ell = Th(M_\ell)$ and T_2 trivial (e.g. $\mathrm{Th}(\omega,=)$, that is, having infinite models, all relations empty) we easily can check that $SP'(T)$ is maximal; that is, equal to $= \{(\lambda,\kappa) : \lambda = \lambda^{2^{|T|}}, \lambda > 2^\kappa, \kappa = \operatorname{cf}(\kappa) > 2^{|T|}\}$) and the intended intuition is to say that T, T_1 has the same complicatedness.

Now [Sh:93] give a semi-ZFC answer to the question on for which T is $\mathrm{SP}'(T)$ is minimal (i.e. are maximally complicated under this criterion).

5.5 Theorem. *If T is not simple, then $SP'(T)$ is minimal (that is, is equal to $\{(\lambda,\kappa) : \lambda = \lambda^\kappa, \kappa = cf(\kappa) > 2^{|T|}\}$).*

The other directions, if T is not simple (hence having the tree property) then $\mathrm{SP}(T)$ is minimal, holds by [Sh:c]. For this [Sh:93] began the generalization [Sh:a, Ch.II,III] to simple theories, I suggested to some to continue but only lately Hrushovski[5]

[5]his preprint has not appeared (and, unlike the others, will not), it investigates the generalization of "geometric stability theory" and group interpretations for theories minimal for $D(-,L,\infty)$. He has some theories of fields and investigating finite models with few types in mind

and then Kim, Pillay Laskowski, Buechler, Morgan, Shami and others use and investigate parallels of [Sh:c] to simple theories; for surveys see [GIL97x], [KiPi]. ? GIL97x ?

We expect that

5.6 Conjecture: There is a finer division of simple theories to $\omega + 1$ families, by the properties Pr_n such that if T has the Pr_n-th property for every n then $SP'(T)$ is minimal, two theories of the same family (i.e. satisfying Pr_n but not $\mathrm{Pr}_{n+1}(T)$ and let $n = n(T)$) essentially have the same $\mathrm{SP}'(T)$, but if two have different $n(T)$ then consistently there is a cardinal separating them (in the $\mathrm{SP}(T') \setminus \mathrm{SP}(T'')$ if $n(T'') < n(T')$); this should be the relatively easy part. A prototype of a counter-example to the n-th property, $k \geq 3$ is the model completion of T_k, where T_k say: $(x_1, \ldots, x_n)$ is symmetric irreflexive, $R(x_1, \ldots, x_n) \to P(x), xSy \to P(x) \ \& \ \neg P(y)$ and $\neg(\exists x, \ldots, x_n y)(\neg P(y) \ \& \ R(x_1, \ldots, x_n) \ \& \ \bigwedge\limits_{\ell=1}^{n} x_\ell S y)$.

The intention is that $\mathrm{Pr}_n(T)$ is a syntactical property which implies:

> $\boxtimes$ if M is a model of $T, \mu = \mu^{|T|} \leq \|M\| \leq 2^\mu$ and $M^* \prec M, \|M^*\| \leq |T|, p^* \in S(M^*), \Phi =: \{\varphi(x, \bar a) : \bar\in M$ and $\{\varphi(x, \bar a)\} \cup p^*$ does not fork over $M^*\}$, then can be represented as $\bigcup\limits_{i<\mu} \Gamma_i$ and $\varphi_1, \ldots, \varphi_n \in \Gamma_i \Rightarrow p^* \cup \{\varphi_1, \ldots, \varphi_n\}$ does not fork over M^*. (For $n = 2$, [Sh:93, 7.8] is a version). We may use [Sh 234].

What is $SP(T_{rg})$?

For any μ let $\log(\mu) = \ \mathrm{Min}\{\lambda : 2^\lambda \geq \mu\}$ if, now if $\mu \geq (\log(\mu))^{<\kappa}$ then $(\lambda, \kappa) \in SP(T_{rg})$ (using [EK]). If $(\log(\mu))^{<\kappa} > \mu$ this is conected to SCH. By [GiSh 597] the answer is independent. Note that T_{rg} is minimal among simple theories in the sense that $SP(T)$ is maximal among unstable theories.

It is not a priori clear that the answer is so coherent, there may be a myriad of properties with many independent results; I have not tried this direction. This will not help us much in classification. Here I am not sure if the "armies of god" will prevail. In other words, I am not sure it is a good test problem any more.

* * *

As said above any unstable theory has the independence property or strict order property. So among unstable theories the theory of random graphs and the theory of linear order are in some sense the simplest. So we can expect to have a theory of some family of first order theories for which linear order is a prototype (as discussed earlier for theory of random graph). Best, of course, is if we can have something for all T without the independence property (see after 2.9). It was encouraging ([Sh:c, Ch.III,§7]).

5.7 Theorem. *(T first order without the independence property).*
If $\kappa > |T|$ is regular $A \subseteq \mathfrak{C}$ then we can find a κ-saturated $M \prec \mathfrak{C}$ such that $A \subseteq M$ and M is in some sense constructible over $A : |M| = A \cup \{a_i : i < \alpha\}$ and $tp(a_i, A \cup \{a_j : j < i\})$ does not split over some $B_i \subseteq A \cup \{a_j : j < i\}$ which has cardinality $< \kappa$.

For long there was no reasonable candidate for test question: (the results on $\lambda \mapsto \sup\{|S(A)| : |A| \le \lambda\}$ were satisfactory but do not lead to something). Now [LwSh 560] start to classify by the $L_{\infty,\kappa}$-Karp height; note that some superstable theories are maximal there.

We may look for a parallel of [Sh:93], e.g.

5.8 Definition. Assume T without the independence property and $\lambda = \lambda^{<\lambda} + |T| < \mu, M \in EC(\mu, T), N \prec_{L_{|T|^+,|T|^+}} M, \|N\| = 2^{|T|}, p^* \in S(N)$ and

$$\begin{aligned}\mathscr{P}_\ell = \mathscr{P}_\ell(p^*, M) =: \Big\{ p \in S(M) :& p \text{ in some sense does not fork over } p^* \\ & \text{which means that } p^* \subseteq p \text{ and :} \\ & (a) \quad \text{if } \ell = 1 \text{ letting } P_\varphi = P_{\varphi(x,\bar y)} = \{\bar c \in {}^{\ell g(\bar y)}M : \varphi(x,\bar a) \in p\} \\ & \qquad \text{we have } (N, P_\varphi)_\varphi \prec_{L_{|T|^+,|T|^+}} (M, P_\varphi)_\varphi \\ & (b) \quad \text{if } \ell = 2, \text{ then for every } \varphi(x,\bar a) \in p \text{ and for } A \subseteq N, |A| \le |T| \\ & \qquad \text{there is } \varphi(x,\bar a') \in p^* \text{ such that } \bar a, \bar a' \\ & \qquad \text{realizes the same type over } A \text{ inside } M \Big\}\end{aligned}$$

$$\mathscr{F}_\ell(p, M) = \cup\{p : p \in \mathscr{P}_\ell(p^*, M)\}.$$

We can make $\mathscr{F}_\ell(p, M)$ to a Boolean Algebra (as in the later parts of [Sh:93]).

<u>5.9 Question</u>: Can you force this Boolean Algebra by a λ^+-c.c. λ-complete forcing notion to be "simple" in some sense? best: union of λ subalgebra which are interval Boolean Algebra.

Probably too much to hope for but the direction may be reasonable, see more 5.36 - 5.40 and [Sh 715].

* * *

Not having the strict order property look to me a priori very promising dividing line, however, the test problems which look promising lead to smaller classes (see below on [Sh 500, §2]). This includes

5.10 Definition. 1) The universality spectrum of T is

$$\begin{aligned} UvSp(T) = \{\lambda :& T \text{ has a universal model, i.e. every other model of} \\ & T \text{ of cardinality } \lambda \text{ can be elementarily embedded into it}\}\end{aligned}$$

2) The pairs-Universality spectrum of T is

$$\mathrm{UvpSp}(T) = \{(\lambda, \mu) : \lambda \leq \mu \text{ and there } M \in EC(\mu, T) \text{ into which every } N \in EC(\lambda, T) \text{ can be elementarily embedded}\}$$

(we can look at the size of a universal family; cov sheds light on the connections, see [Sh 457], [DjSh 614]).

Under GCH for $\lambda > |T|$ the answer is known, so we can look only for weak solutions involving consistency, "semi-ZFC solutions" as suggested in 5.2 above.
Now the theory of universal graphs consistently has large universal spectrum even for $\lambda < 2^{<\lambda}$ ([Sh 175a]). So once we know ([KjSh 409]) that the theory of linear order has few (e.g. $2^\lambda > \lambda^{++} \Rightarrow \lambda^{++} \notin \mathrm{UvSp}(T_{\mathrm{ord}})$), and that this applies to any T with the strict order property, it raises hope that this is a good test problem for that property.

Alas, it may be good but not for the strict order property as ([Sh 500, §2]) NSOP_4 suffices where (see [Sh 500], [DjSh 692]):

5.11 Definition. 1) T has the SOP (the strong order property) if some type $p(\bar{x}, \bar{y})$ defined in $\mathfrak{C}_T$ a partial order with arbitrarily long ($< \bar{\kappa}$) chains).
2) T has SOP_n (the strong n-order property, $n \geq 3$) if for some formula $\varphi(\bar{x}, \bar{y})$:

- (a) there is an infinite indiscernible sequence ordered by φ
- (b) we cannot find $m \leq n$ and $\bar{a}_0, \dots, \bar{a}_{m-1}, \bar{a}_m = \bar{a}_0$ such that $\varphi(\bar{a}_\ell, \bar{a}_{\ell+1})$ for $\ell < m$.

3) T has SOP_2 if some $\varphi(\bar{x}, \bar{y})$ has it which means that we can find in $\mathfrak{C}_T, \bar{a}_\eta \in {}^{\ell g(\bar{y})}(\mathfrak{C}_T)$ for $\eta \in {}^{\omega >}2$ such that:

- (a) if $\eta\char`^\langle \ell \rangle \trianglelefteq \eta_\ell \in {}^{\omega >}2$ for $\ell = 0, 1$ then $\{\varphi(\bar{x}, \bar{a}_{\eta_1}), \varphi(\bar{x}, \bar{a}_{\eta_0})\}$ is inconsistent
- (b) if $\eta \in {}^\omega 2$ then $\{\varphi(\bar{x}, \bar{a}_{\eta \restriction n} : n < \omega\}$ is inconsistent.

4) T has SOP_1 is defined as in (3) only in clause (a) we demand $\eta_0 = \eta\char`^\langle 0 \rangle$.

5.12 Problem: 1) Develop a theory for NSOP T's.
2) Develop a theory for NSOP_n T's.
3) Find additional evidence of complicatedness to the SOP_n's (and SOP).
Earlier I thought that the most promising is the case $n = 3$, a prototypical theory seems T_{feq} ([Sh 457]), but now we know that $n = 2$ is a real dividing line ([DjSh 692]). However, we have $\mathrm{SOP}_n \Rightarrow \mathrm{SOP}_{n+1}$ and for $n \geq 3$ the inverse implication fails, but for $n = 1, 2$?

Now [Sh 457, §1] indicates another direction, see Dzamonja Shelah [DjSh 710]; there for theory T with tree coding we prove some non-existence of universal models.

5.13 Definition. 1) The formula $\varphi(\bar x, \bar y, \bar z)$ is tree coding in T, if for every (equivalently some) $\lambda \geq \kappa \geq \aleph_0$ we can find in $\mathfrak{C}_T, \bar c_\nu(\nu \in {}^\kappa\lambda), \bar b_\eta(\eta \in {}^{\kappa>}\lambda)$,
$\bar a_\alpha(\alpha < \kappa)$ such that:

(a) $\models \varphi[\bar c_\nu, \bar b_\eta, \bar a_\alpha]$ if $\eta = \nu \restriction \alpha \ \ \& \ \ \nu \in {}^\kappa\lambda$

(b) if $\alpha < \kappa$ and $\nu, \rho \in {}^\kappa\lambda$ and $\nu \restriction \alpha \neq \rho \restriction \alpha$ then $\varphi(\bar c_\nu, \bar y, \bar a_\alpha), \varphi(\bar c_\rho, \bar y, \bar a_\alpha)$ are contradictory.

2) T has tree coding if some $\varphi(\bar x, \bar y, \bar z)$ has.

5.14 Problem: Develop the theory of T's without tree coding (and further non-structure theorems for those with).
Clearly in some sense the dividing line stable/unstable is simpler than superstable/unsuperstable not to mention NDOP/DOP, etc. The following definitions tries explicate this. The point being that many properties are properties of a formula $\varphi(\bar x, \bar y)$ in T.

5.15 Definition. Fix T and $\mathfrak{C} = \mathfrak{C}_T$.
1) For first order formula $\varphi = \varphi(\bar x, \bar y)$ and $\bar a_0, \dots, \bar a_{n-1} \in {}^{\ell g(\bar y)}\mathfrak{C}$, let $\mathbf{u}_\varphi(\bar a_0, \dots, \bar a_{n-1}) = \{u \subseteq n :\models (\exists \bar x) \bigwedge\limits_{\ell < n} \varphi(\bar x, \bar a_\ell)^{\text{cf}(i \in u)}\}$ where $\varphi^{\text{true}} = \psi, \psi^{\text{false}} = \neg\psi$.
2) For first order formula $\varphi = \varphi(\bar x, \bar y)$ let

$$\begin{aligned}\Gamma_\varphi = \{(n, \mathbf{u}) :& \text{for some } n < \omega \text{ and } \bar a_0, \dots, \bar a_{n-1} \in {}^{\ell g(\bar y)}\mathfrak{C};\\ & \text{we have } \mathbf{u}_\varphi(\bar a_0, \dots, \bar a_{n-1}) = \mathbf{u}\}.\end{aligned}$$

3) We let $\Gamma_T = \{\Gamma_\varphi : \varphi = \varphi(\bar x, \bar y) \in L(T)\}$. A division of first order theories is straightly defined if: for some Γ it is the family of T such that $\Gamma \in \Gamma_T$ the first order T are divided to those T's that $\Gamma \in \Gamma_T$ (the up sets) and those T's that $\Gamma \notin \Gamma_T$ (the down side).
4) Let $\Gamma_{T,\varphi(\bar x, \bar y, \bar z)} = \{\Gamma_{\varphi(\bar x, \bar y, \bar c)} : \bar c \in {}^{\ell g(z)}\mathfrak{C}\}$.

5.16 Definition. 1) For Γ as above we say: T is Γ-high if $\Gamma \in \Gamma_T$ and Γ-low otherwise.
2) We say that a class $\mathfrak{T}$ of complete first order theories is straight if the truth value of $T \in \mathfrak{T}$ is determined by Γ_T.

A variant which seems to capture the main point is:

5.17 Definition. 1) Let $\Gamma^* = \{(n, F_1, F_2) : n < \omega, F_1, F_2$ are disjoint families of sets of partial functions from $\{0, \dots, n-1\}$ to $\{\text{true false}\}\}$.
2) We say that $\langle \bar a_0, \dots, \bar a_{n-1}\rangle$ does φ-realizes $(n, F_1, F_2) \in \Gamma^*$ if $f \in F_1 \Rightarrow \mathfrak{C} \models (\exists \bar x) \bigwedge\limits_{\ell \in \text{ Dom}(f)} \varphi(\bar x, \bar a_\ell)^{f(\ell)}$ and $f \in F_2 \Rightarrow \mathfrak{C} \models \neg(\exists \bar x) \bigwedge\limits_{\ell \in \text{ Dom}(f)} \varphi(\bar x, \bar a_\ell)^{f(\ell)}$. We can apply this to $\varphi(\bar x, \bar y, \bar c), \bar c$ from $\mathfrak{C}_T$.
3) For $\Gamma \subseteq \Gamma^*$ we say that $\varphi(\bar x, \bar y)$ has the weak Γ-property (in T) if any $(n, F_1, F_2) \in$

Γ is φ-realized by some $\langle \bar{a}_0, \dots, \bar{a}_{n-1} \rangle$. We say that $\varphi(\bar{x}, \bar{y})$ has the strong Γ-property if for $(n, F_1, F_2) \in \Gamma^*$ we have $(n, F_1, F_2) \in \Gamma$ iff (n, F_1, F_2) is φ-realized by some $\langle \bar{a}_0, \dots, \bar{a}_{n-1} \rangle$. We say $\varphi(\bar{x}, \bar{y}, \bar{z})$ has such a property if this holds for some $\varphi(\bar{x}, \bar{y}, \bar{c})$.
4) T has the weak/strong Γ-property if some $\varphi(\bar{x}, \bar{y}, \bar{z})$ has it. T has the weak/strong pure Γ-property if some $\varphi(\bar{x}, \bar{y})$ has it.
5) We say that a class (or property) $\mathfrak{T}$ of complete first order theories is weakly/strong simply high straight if for some $\Gamma \subseteq \Gamma^*$ we have: $T \in \mathfrak{T}$ if T has the weak/strong Γ-property in T. The class $\mathfrak{T}$ is weakly/strongly simply low straight if it is the compliment of a simply high straight one.
6) Omitting the "weak" and "strong" we shall mean weak(ly).
7) $\mathbf{\Gamma}^{**} = \{\Gamma \subseteq \Gamma^*$: for some T, some $\varphi(\bar{x}, \bar{y})$ has the Γ-property$\}$.

5.18 Fact: 1) For any $\Gamma \subseteq \Gamma^*$, the truth of "$T$ has the weak Γ-property" is determined by Γ_T.
2) Allowing in Definition 5.16(10, (2), 5.17 for the weak versions, formulas $\varphi(\bar{x}, \bar{y}, \bar{c})$, does not make a difference for having the Γ-property.

5.19 Observation: The following properties can be represented as "T has the weak Γ-property".
1) T is unstable.
2) T has the independence property.
3) T has the strict order property.
4) T has the tree property (equivalently, is not simple).
5) T has NSOP_n (the n-strong order property)(where $n \geq 3$).
6) T has the NSOP_2.
7) T has the NSOP_1.

Proof. Only 5) is not immediate.
It suffices to show

$(*)$ T has NSOP_n iff for some $\varphi = \varphi(\bar{x}, \bar{y})$ we have

$(*)_\varphi$ there is an indiscernible set $\langle \bar{a}_\ell : \ell < \omega \rangle$ such that

(a) for every $0 = i_0 < \dots < i_n = \omega, \{\varphi(\bar{x}, \bar{a}_m)^{\text{if}(\ell \text{ even})} : \ell < n, m \in [i_\ell, i_{\ell+1})\}$ is consistent

(b) for no $i_0 < \dots < i_n$ do we have $\models (\exists \bar{x}) \bigwedge\limits_{\ell \leq n} \varphi(\bar{x}, \bar{a}_{i_\ell})^{\text{if}(\ell \text{ even})}$

$\square_{5.19}$

In this context, there are naturally the most complex theories:

5.20 Definition. 1) We say that $\varphi(\bar{x}, \bar{y})$ straightly maximal (in T) if Γ_φ is maximal.

2) We say $\varphi(\bar{x}, \bar{y}, \bar{z})$ is strongly straightly maximal (in T) if $\Gamma_{T, \varphi(\bar{x}, \bar{y}, \bar{z})}$ is maximal. 3) Call T straightly maximal if some $\varphi(\bar{x}, \bar{y})$ is.
4) Call T strongly straightly maximal if some $\varphi(\bar{x}, \bar{y}, \bar{z})$ is.

An example is true arithmetic, i.e. $Th(\omega, 0, 1, +, \times)$
5.21 Problem: 1) Develop a theory

- (a) for non-straightly maximal T
- (b) for non-strongly straightly maximal.

2) Find natural nonstructure theorem, i.e. witness for having complicated models

- (a) for straightly maximal T's
- (b) for strongly straightly maximal T's.

* * *

Now 5.21 seems quite persuasive to me, but I have to say I do not know of a test problem, nor what should we expect of a good theory for the nonmaximal theory. Note that this scheme does not include the $(k,n) - *$-NIP where "arity" is important.

We can easily adapt the definitions to include it, but the present version is not necessarily a drawback - the present version does not discriminate elements from seven-tuples, etc., and
5.22 Thesis: It is certainly reasonable to map the continents and oceans before we look at hills and lakes (if we can, of course).

Now superstability does not fit this scheme, too, again it is a finer distinction; yet, we write down this version.

5.23 Definition. Fix T and $\mathfrak{C} = \mathfrak{C}_T$ and α an ordinal.
1) Let $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i, \bar{c}_i) : i < \alpha \rangle$ be a sequence with $\bar{c}_i$ from $\mathfrak{C}_T$.
For $n < \omega$ and $\bar{a}_{i,\ell} \in {}^{(\ell g(\bar{y}_i))}\mathfrak{C}$ let

$$\mathbf{u}_{\bar{\varphi}}(\langle \bar{a}_{i,\ell} : i < \alpha, 0 < n \rangle) = \Big\{ u \subseteq \alpha \times n : \text{the type } \{\varphi(\bar{x}, \bar{a}_{i,\ell})^{\text{if}((i,\ell) \in u)} : i < \alpha, \ell < n\} \text{ is consistent} \Big\}.$$

2) For a sequence $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i, \bar{c}_i) : i < \alpha \rangle$ as above let

$$\Gamma_{\bar{\varphi}} = \Big\{ (n, \mathbf{u}) : \text{for some } n \text{ and } \bar{a}_{i,\ell} \in {}^{(\ell g(\bar{y}_i)}\mathfrak{C} \text{ (for } i < \alpha, \ell < n) \text{ we have } \mathbf{u}_{\bar{\varphi}}(\langle \bar{a}_{i,\ell} : i < \alpha, \ell < n \rangle) = \mathbf{u} \Big\}.$$

3) Let $\Gamma^\alpha_T = \{\Gamma_{\bar{\varphi}} : \bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i) : i < \alpha \rangle\}$.
4) Let $\Gamma^*_2 = \{(n, F_1, F_2) : n < \omega$ and F_1, F_2 are disjoint families of partial finite functions from $\alpha \times n$ to {true,false}}.
5) We say that $\langle \bar{a}_{i,\ell} : i < \alpha, \ell < n \rangle$ does $\bar{\varphi}$-realizes $(n, F_1, F_2) \in \Gamma^*_\alpha$ if is as above and:

$(*)_1$ $f \in F_1 \Rightarrow \mathfrak{C} \models (\exists \bar{x})[\bigwedge_{(i,\ell) \in \mathrm{Dom}(f)} \varphi_i(\bar{x}, \bar{a}_{i,\ell})^{f(i,\ell)}]$

$(*)_2$ $f \in F_2 \Rightarrow \mathfrak{C} \models \neg(\exists \bar{x})[\bigwedge_{(i,\ell) \in \mathrm{Dom}(f)} \varphi_i(\bar{x}, \bar{a}_{i,\ell})^{f(i,\ell)}]$.

6) For $\Gamma \subseteq \Gamma^*$ we say that $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i, \bar{c}_i) \rangle$ has the Γ-property in T if every $(n, F_1, F_1) \in \Gamma$ is $\bar{\varphi}$-realized for T.
7) We say that T has the Γ-property for $\Gamma \subseteq \Gamma^*_\alpha$ if some $\bar{\varphi} = \langle \varphi_i(\bar{x}, \bar{y}_i) : i < \alpha \rangle$ has the Γ-property.

What about DOP, OTOP, etc?

5.24 Definition. For a logic $\mathscr{L}$, $(T, \mathscr{L})$ has any of the properties defined above if we allow the formulas φ to be in L but as $\mathscr{L}$ possibly fail compactness we should like large case so:

- (a) $(T, \mathscr{L})$ has the order property if for some $\varphi(\bar{x}, \bar{y}) \in \mathscr{L}$ with $\gamma =: \ell g(\bar{x}) = \ell g(\bar{y})$, for every linear order I there are a model M of T and $\bar{a}_t \in {}^\gamma M$ for $t \in I$ such that, for any $t, s \in I$ we have $M \models \varphi[\bar{a}_t, \bar{a}_s]$ iff $t <_I s$
- (b) $(T, \mathscr{L})$ has the independence property if for some $\varphi(\bar{x}, \bar{y}) \in \mathscr{L}$, for every λ there are a model M of $T, \bar{a}_\gamma \in {}^{\ell g(\bar{y})}M$ for $\gamma < \lambda, \bar{a}_u \in {}^{\ell g(\bar{x})}M$ for $u \subseteq \lambda$ such that $M \models \varphi[\bar{a}_u, \bar{b}_\gamma]$ iff $\gamma \in u$.

But if we look at NDOP or NOTOP, (for superstable T, in the standard definition) we do not fully use $\mathscr{L} = L_{|T|^T, |T|^*}$ or $\mathscr{L} = L_{|T|^+, \aleph_0}, \mathscr{L} = L_{|T|^+, \aleph_0}(Q^{\dim \geq \aleph_1})$ (if we use finite sequences, sufficient for superstable T, see [Sh:c, Ch.XII]) or $\mathscr{L} = L_{|T|^+, |T|^+}$, we rather use formulas of specific form. But the order property and independence property becomes equivalent, and main gap tend to show equivalence of such versions.

* * *

The universality spectrum raises many problems both set theoretic and model theoretic. For the set theoretic side, we still do not know enough on $\mathrm{UvSp}(T_{\mathrm{ord}})$ and also the universe UvSp for the theory of graphs (see [DjSh 659]).

5.25 Problem: Is it consistent that for some $\lambda, \lambda, \mu, \lambda < \mu < 2^\lambda$ and $\mu \in \mathrm{UvSp}(T)$ for every countable T? (equivalently true arithmetic).

Still theories with SOP_4 look essentially maximal (as the results on linear orders hold for them)
5.26 Problem: Does every NSOP_4 theory T have consistently a non-trivial universality spectrum?

5.27 Thesis: The way, (a good way), a reasonable way to develop the theory of NSOP_4 and/or NSOP_4 first order theories is

- $(*)$ start by asking which first order theories fall under the persuasion of [Sh 457, §4]?

So we know that if T has SOP_4 then not, whereas if T is simple, then yes. Lastly, for some T which has SOP_3 & NSOP_4, the answer is yes.

The nice scenario is if those will be exactly SOP_4. If this succeeds, this will be very good for investigating universality spectrum. It may give something on the theory of NSOP_4. Maybe, a right parallel of non-forking. If it fails, it still gives important information on universality. May give information on NSOP_3.

5.28 Discussion: One may pose the question: is universality just a tool toward classifying?

Answer: In some sense, yes.

But, I believe the right way to classify is to choose a worthwhile relevant test problem (like number of non-isomorphic models). So it is true that in a sense the classification is higher, real aim but still the universality spectrum and classifying $I(\lambda, T)$ are very important. Reason for optimism concerning the universality spectrum is: the positive and negative answers (guessing clubs) and [Sh 457, §4] seems to speak on the same thing.

5.29 Question: CON(in λ^+ there is universal linear order & $2^\lambda > \lambda^+$ & $\lambda = \lambda^{>\lambda} > \aleph_0$). If this fails, we can look at the examples in [Sh 500, §2] (existentially complete directed graph with no ($\le k$)-cycle).

Of course: we would like to ask for which first order theories the proof in [Sh 457], [DjSh 614] will work?
For PA (piano arithmetic)? Conceivably for PA we can prove that: there is no universal in more cardinalities than the obvious ones ($\lambda = 2^{<\lambda}$, where $\lambda > |T|$ for simplicity) or can try there all theories. If we fail for linear order but succeed for some other T's, it should be very illuminating, maybe revealing new dividing lines.

I have not looked at
5.30 Question: Does all simple unstable countable theories have the same universality spectrum? Or, do they have many possible spectrums?

The natural way: look at forcing for graphs and think of a non-trivial simple theory such that if in the beginning we force many models of it in χ, there would not be co-habitation.

If we discover too fine a distinction, it will not be so exciting to investigate.

Even so, a Major question is
5.31 Problem: Find the maximal class for UnSp, that is a dividing line in the sense that they behave like linear order (at present).
If for all first order T we have the consistency hoped for linear order, but many such theories behave differently and there is no alternate proofs for "there is no universal" in ZFC (+ cardinal arithmetic), then finer distinction among such theories look not inviting.

My feeling: the dividing line of the proof in [Sh 500, §2] is a major dividing line, the one for universality.

To get semi-ZFC distinction
5.32 Question: Generalize [Sh 457] to λ^{++}.

Clearly for NSOP_4 theories and probably more this fails; i.e. we get some notion but the property required in [Sh 457] fails; but this may provide a theory of types to NSOP_4 theory (or a new dividing line).

Of course, we may like to know more on simple theories
5.33 Problem: For which theories the consistency results on graphs ([Sh 175], [Sh 175a], [DjSh 659]?) can be generalized?

Even for graphs (but probably not hard):
5.34 Problem: Can we in [Sh 175a] get the consistency for all regular cardinals in the intervals? also for the singulars?

* * *

5.35 Discussion: In the spectrum from in the one end finding the bare outlines, finding some order in the total chaos, to the other end, perfectly understanding on what we know not little, I prefer the first. So though I was (and am still) sure that there is much more to be said on superstable/stable theories (in fact, this essentially follows from the belief that it is an important dividing line) not to say on theories of finite Morley rank, and on simple theories, I am more excited from starting new frameworks.

Of course, I believe that such general theorems of f.o. theories will have meaningful application for specific theories (though I do not agree with A. Robinson that this is the aim of model theory or a needed justification; but I agree it is a worthwhile one), in fact, such applicability is highly suggestive from belief in the meaningfulness of the dividing line (if the theory is serious). Well, some may argue that has not simple theories proved to be the only one with reasonable non-forking (by Kim and Pillay [KiPi])? Yes, but this had been done for stable, too, and maybe trying to generalize is not the only way to find an understanding of such theories. For example, probably the theory of NSOP_3 theories will replace elements by formulas, and we shall have to make parallel replacement moving from NSOP_n to NSOP_{n+1}. E.g. consider: for a formula $\varphi(\bar{x},\bar{a})$ in M_1 and $M_0 \prec M_1$ and type $p \in S(M_1)$ to which $\varphi(x,\bar{a})$ belongs, as in 5.8 $q \subseteq p \restriction M_0, |q| \le |T|, \varphi(x,\bar{a})$ reflect nicely in M_0. However, in some sense having proved the main gap for countable f.o. theory, I feel my task (on first order theories) was done, just like [Sh 460] in cardinal arithmetic.

* * *

In linear order, if $\langle a_t : t \in J\rangle$ is indiscernible ($\equiv$ monotonic) over $A, t_0 < s < t_1$ and $\{t_0, s, t_1\} \subseteq J$, then $\mathrm{tp}(a_s, \{a_{t_0}, a_{t_1}\}) \vdash \mathrm{tp}(a_s, A)$.

5.36 Question: Can we prove a similar phenomena for NIP theories?

This cannot be literally true as for stable theories it is false. Probably we should "divide" the works between stable like parts and the above idea.

On the other hand putting together intervals of length $|T|$ and adding we can find $\langle b'_t : t \in I\rangle$ such that $\bar{b}_t \subseteq \bar{b}'_t$ and for $t_0 < t_1 < t_2$, $\mathrm{tp}(\bar{b}_{t_1}, \bar{b}'_{t_0}{}^\frown \bar{b}'_{t_2}) \vdash \mathrm{tp}(\bar{b}_{t_1}, \cup\{\bar{b}_s : s \notin (t_0,t_2)\}$.

In some sense, a model of a stable theory M can be represented by a well ordering and unary functions:

5.37 Fact: If $\mathrm{Th}(M)$ is stable and $|M| = \{a_\alpha : \alpha < \alpha^*\}$, we can find $f_{\varphi,\ell} : \alpha^* \to \alpha^*$ satisfying $f_{\varphi,\ell}(\alpha) <$ Max$\{2, \alpha\}$ (for $\varphi = \varphi(x, \bar{y}) \in L(\tau_T), \ell < n_\varphi < \omega$) such that $\mathrm{tp}(\langle a_{\alpha_1}, \ldots, a_{\alpha_n}\rangle, \emptyset, M)$ can be reconstructed from equalities between composition of $f_{\varphi,\ell}$ (the point being that $\mathrm{tp}_\varphi(a_\alpha, \{a_\beta : \beta < \alpha\})$ is definable by some $\psi(\bar{y}, \bar{c}), \bar{c} \subseteq \{a_\beta : \beta < \alpha\}$).

5.38 Problem: 1) For NIP theories, does something parallel hold with equalities replace by some ($\leq |T|$) linear orderings of α^*?
2) Find parallel theories for other properties of T.

5.39 Problem: Investigate first order T which are NIP (i.e. without the independence property).

5.40 Question: For T with NIP:
1) If $A \subseteq B \subseteq \mathfrak{C}_T, p \in S(A)$, does there exist $q \in S(B)$ extending p which does not fork over A?
2) Do ordered groups play here a role similar to groups for stable theories?

5.41 Question: For (complete) T with the independence property, $T_1 \supseteq T$, and θ and for simplicity λ a successor of regular $> 2^\theta$, are there θ-resplened models M_1 of T_1 with $M_1 \restriction \tau_T$ has large $L_{\infty,\lambda}$-Karp height?

§6 Classifying non-elementary classes

I see this as the major problem of model theory. Cherlin presses me to expand on this point; now in '69 Morley and Keisler told me that model theory of first order logic is essentially done and the future is the development of model theory of infinitary logics (particularly fragments of $L_{\omega_1,\omega}$). By the eighties it was clearly not the case and attention was withdrawn from infinitary logic (and generalized quantifiers, etc.) back to first order logic. Now, of course, it is better to prove theorems in a wider context, also we may recall that algebraists are not restricting their attention to elementary classes; but wider context may have a heavy price in content, it is not clear that there interesting theory left at all. As the theory for the family of first order theories has widened and deepened this attention was justified. But, of course, it would be wonderful if we have at all a classification theory for nonelementary classes. Just generalizing with changes here and there is not so exciting, but clearly, if there is a theory at all, there are in it many dividing lines of different character; the danger it is the other direction: having too weak theory.

Of course , this is phrased too generally, e.g. I feel classes defined by $\psi \in L_{\aleph_1,\aleph_1}$ are probably hopeless (we can easily code behaviour which are very set theoretically sensitive). So "non elementary" should be restricted to a reasonable class, and there are choices. The first case I considered was $(K_D, \prec)$ where

6.1 Definition. Let T be a first order complete theory, $D \subseteq D(T) = \cup\{D(M) : M$ a model of $T\}$ where $D(M) = \{\text{tp}(\bar{a}, \emptyset, M) : \bar{a} \in {}^m M, m < \omega\}$, (so D codes T, well when $D \neq \emptyset$). Let
1) $K_D = \{M : M$ a model of T (so $\tau(M) = \tau(T)$) and moreover $D(M) \subseteq D\}$ (and $\prec$ is the usual being elementary submodel order).
2) M is λ-sequence-homogeneous (or just λ-homogeneous) <u>if</u> for every elementary map f of M (i.e. f one to one from $\text{Dom}(f) \subseteq M$ to $\text{Rang}(f) \subseteq M$ and f preserve first order formulas) of cardinality $< \lambda$ and $a \in M$ there is an elementary map f' of M satisfying $f \subseteq f'$ & $a \in \text{Dom}(f')$.
3) M is (λ, D)-homogeneous <u>if</u> M is λ-homogeneous and $D(M) = D$.

The reason for considering K_D was that "(λ, D)-homogeneous" was similar to "λ-saturated". The older notion of model homogenous had not looked managable to me (see 6.2(1),(2) below).

6.2 Definition. 1) M is λ-model-homogeneous if: for every isomorphism f from $M_1 \prec M$ onto $M_2 \prec M, M_1 \prec N_1 \prec M, \|N_1\| < \lambda$ there is $N_2, M_2 \prec N_2 \prec M$ and an isomorphism f' from N_1 onto N_2 extending f.
2) M is model homogenous if it is $\|M\|$-model homogeneous.
3) $\mathscr{D}_\kappa(M) = \{N/\cong : N \prec M, \|N\| \leq \kappa\}$.
4) M is $(\lambda, \mathscr{D})$-homogeneous if M is λ-model homogeneous and $\mathscr{D}_{|\tau(\mathscr{D})|+\aleph_0}(M) = \mathscr{D}$.
5) $K_{\mathscr{D}} = \{M : \mathscr{D}_{|\tau(\mathscr{D})|+\aleph_0}(M) \subseteq \mathscr{D}\}$.

Still we do not know the answer to
<u>6.3 Question</u>: 1) Is there a "reasonable" upper bound to

$$\mu^*_\kappa = \Big\{\text{Min}\{\lambda : \text{there is no } (\lambda, D)\text{-homogeneous model of cardinality } \lambda\} : \text{ for some complete first order theory } T \text{ of cardinality } \kappa, D \subseteq D(T)\Big\}.$$

2) Similarly for $(\lambda, \mathscr{D})$-homogeneity.
I think that it is known (by the Kazachstan school, under GCH) that $\mu^*_{\aleph_0} \geq \aleph_\omega$.

But more central for me is
6.4 Problem: 1) How much of the theory on stable theories can be generalized to $(K_D, \prec)$ for stable D?
2) Similarly for superstable; where

6.5 Definition. 1) K_D is stable if (for every λ there is a (λ, D)-homogeneous model of cardinality $\geq \lambda$, and) for arbitrarily large λ, K_D is stable in λ which means $A \subseteq M \in K_D, |A| = \lambda \Rightarrow S(A, M) = \{\text{tp}(a, A, M) : a \in M\}$ has cardinality $\leq \lambda$.
2) K_D is superstable if the stability holds for every large enough λ.
Investigation of $(K_D, \prec)$ have been carried, see the introduction of [HySh 676]. There is little on $(\lambda, \mathscr{D})$-homogeneity (see [Sh 237c], [Sh 300]). The interest is mainly in D such that for every λ there is (λ, D)-homogeneous model of cardinality λ, but anyhow in definition 6.5, it suffices to deal with "small λ", the rest follows.

6.6 Problem: 1) Prove the main gap for

$$I(\lambda, K_D) = \{M/\cong : M \in K_D, \|M\| = \lambda\}.$$

2) Prove the main gap for

$$I(\lambda, \{M \in K_D : M \text{ is } (\kappa, D)\text{-homogeneous}\}) = |\{M/\cong : M \in K_D \text{ has cardinality } \lambda \text{ and is } (\kappa, D)\text{-homogeneous}\}|.$$

Certainly for first order classes I considered as the main case version (1) (note: when $D = D(T)$ we get back the elementary classes as special cases). However, here the interest started with (D, μ)-homogeneous model so probably part (2) is more natural. However, the problem has not been resolved even for countable first order $T, D = D(T)$; see [HySh 676, §0] on what was done.

What is lost in this context compared with the first order one? Formulas are not so interesting any more, except as part of a complete type. There is a remnant of compactness: there is $a \in \mathfrak{C}_D$ realizing a type $p \in S(A)$ iff for every finite B the type $p \restriction B$ is realized. Also the Hanf numbers of omitting types is helpful and (D, κ)-homogeneous is quite parallel to κ-saturated; large parts of stability theory for such models has been generalized to this context and much more is still to be done. Note that it should not all be parallel to the first order case, first there are new aspects (like λ-goodness), also some early work was done first in this context (the stability spectrum $\kappa(D)$ and $\lambda(D)$) and lastly, something like $\mathfrak{C}^{eq}$ may be better here in some respects.

Another direction has been universal class, where a class K of τ-models closed under isomorphism is a universal class when $M \in K$ iff every finitely generated submodel belongs to K (see [Sh 300], [Sh:h]). This context is incomparable with first order; a universal class is certainly not necessarily first order, and also the inverse implication fails. Now there may be long sequences on which a quantifier free formula defines order, in which case we have a strong nonstructure. Otherwise we can define being a submodel $M \le N$, axiomatize the setting and start developing the parallel of [Sh:c], with types being defined by chasing arrows rather than as a set of formulas, starting with the parallel of the theorem "saturated $\equiv$ homogeneous universal", and having some new dividing lines, getting regular types, etc. The idea was that assuming some possible reasons for strong nonstructure does not hold, we can define a stronger notion of submodel $<_1$ (like $\prec_{\Sigma_1}$) and prove that $\mathfrak{K}^+ = (\mathfrak{K}, <_1)$ is inside our setting. We think that after enough such strengthening, the intersection is similar enough to the first order case to prove the main gap, but this was not done.

<u>6.7 Question</u>: Does the main gap (of course with depth possibly quite large) hold for universal classes?

Note that though first order formulas does not play a role, types, dimension of indiscernible sets, prime models, orthogonality and regularity does. Also we believe that the idea of changing inductively the context will be helpful (as it is in [Sh 600]).

We may rather look at classes defined say by $\psi \in L_{\omega_1,\omega}$, here it is harder to begin.

Note that generally in this section I have thought that we should expect not just the situation in cardinals $\lambda \le |T|$ to be different than in "large enough λ" (as was the case for first order) but say $\lambda <$ relevant Hanf number of $L_{\omega_1,\omega}$, so the small cardinal should have different behaviour. The theory is not totally empty as we can prove some things:

6.8 Theorem. *Assume $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for $n < \omega$.*
1) If $\psi \in L_{\omega_1,\omega}$ have "few" models in $\aleph_1, \dots, \aleph_n$ (essentially $I(\aleph_m, \psi) < 2^{\aleph_m}$) but has an uncountable model <u>then</u> ψ has a model in $\aleph_{n+1}$.
2) If $\psi \in L_{\omega_1,\omega}$ have few models in $\aleph_1, \aleph_2, \dots, \aleph_n, \dots (n < \omega)$ but has an uncountable model, <u>then</u> ψ has models on all cardinalities.
3) If $\psi \in L_{\omega_1,\omega}$ is categorical in $\aleph_1, \aleph_2, \dots, \aleph_n, \dots, (n < \omega)$, <u>then</u> ψ is categorical in every $\lambda > \aleph_0$; in fact under the assumption of part (2), ψ is excellent, and for excellent classes categoricity is one $\lambda > \aleph_0$ suffice here (essentially [Sh 87a], [Sh 87b] when "few" is strengthened a little, see more in [Sh 600], more on excellent class [GrHa89]).

We do not know:
<u>6.9 Problem</u>: If $\psi \in L_{\omega_1,\omega}$ (or even $\psi \in L_{\kappa^+,\omega}$) is categorical in one $\lambda \ge \beth_{\omega_1}$ (or $\lambda \ge \beth_{(2^\kappa)^+}$), <u>then</u> ψ is categorical in every such λ?

Some wonder why "$\lambda \geq \beth_{\omega_1}$"? Now $\lambda \geq \aleph_\omega$ is necessarily as by [HaSh 323], ψ may categorical in $\aleph_0, \dots, \aleph_n$, but not in λ if $2^{\aleph_n} < 2^\lambda$ (or so).

Others wonder why such modest question, isn't the main gap better? Of course it is, but I think it is more reasonable first to resolve the categoricity. But are "a class of models of $\psi \in L_{\kappa^+,\omega}$" the best context? Thinking of putting [Sh 87a] + [Sh 87b] together with results on $L_{\omega_1,\omega}(Q)$ in [Sh 48], I consider ([Sh 88]) abstract elementary classes. I have preferred this context, certainly the widest I think has any chance at all.
In [Sh 87a], [Sh 87b], [Sh 88] it is proved:

- $(*)_2$ catgoricity (of $\psi \in L_{\omega_1,\omega}(Q)$) in $\aleph_1$ implies the existence of a model of ψ of cardinality $\aleph_2$;
- $(*)_3$ if $n > 0, 2^{\aleph_0} < 2^{\aleph_1} < \ldots < 2^{\aleph_n}, \psi \in L_{\omega_1,\omega}$ and $1 \leq I(\aleph_\ell, \psi) < \mu_{\rm wd}(\aleph_\ell)$ for $1 \leq \ell \leq n$, then ψ has a model of cardinality $\aleph_{n+1}$
- $(*)_4$ if $2^{\aleph_0} < 2^{\aleph_1} < \ldots, \psi \in L_{\omega_1,\omega}$ and $1 \leq (\aleph_\ell, \psi) < \mu_{\rm wd}(\aleph_\ell)$ for $\ell = 1, 2, \ldots$, then ψ has a model in every infinite cardinal and is categorical in one $\lambda > \aleph_0$ iff it is categorical in every $\lambda > \aleph_0$.

Now the problems were:
6.10 Problem: Prove $(*)_3, (*)_4$ in the context of an abstract elementary class $\mathfrak{K}$ which is $PC_{\aleph_0}$.

6.11 Problem: Parallel results in ZFC; e.g. prove $(*)_3$ when $n = 1, 2^{\aleph_0} = 2^{\aleph_1}$. By [Sh 88, §6] there are classes categorical in $\aleph_1$ if MA, but not so if $2^{\aleph_0} < 2^{\aleph_1}$ so really there is here a different model theory involved.

6.12 Problem: Construct examples; e.g. $\mathfrak{K}$ (or $\psi \in L_{\omega_1,\omega}$), categorical in $\aleph_0, \aleph_1, \dots, \aleph_n$ but not in $\aleph_{n+1}$ (see [HaSh 323]).

6.13 Problem: If $\mathfrak{K}$ is λ-a.e.c. (abstract elementary class), and is categorical in λ and λ^+, does it necessarily have a model in λ^{++}? assuming $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$? In [Sh 576] we solve a somewhat weaker version of 6.13.

It is reasonable to be willing to assume large cardinal, if we can develop some interesting theory. In [MaSh 285] a version of Los Conjecture for $T \subseteq L_{\kappa,\omega}, \kappa$ compact cardinal was proved (starting for large enough successor).
6.14 Question: 1) If $T \subseteq L_{\kappa^+,\omega}$ is categorical in one limit $\lambda > \beth_{(2^{\kappa+|T|})^+}$, then T is categorical in every $\lambda \geq \beth_{(2^{\kappa+|T|})^+}$.
2) Similarly for $\mathfrak{K}$ a κ-a.e.c. with amalgamation.
3) Similarly for $\mathfrak{K}$ as κ-a.e.c.
Note: that for (2) there are some results ([Sh 394]).

Moreover
6.15 Problem: 1) Develop classification (or at least stability) theory for $T \subseteq L_{\kappa,\omega}$ at least if κ is compact, or even just measurable.
In Kolman Shelah [KlSh 362], [Sh 472] the parallel (downward part) is proved for κ-measurable.

Several cases lead to
6.16 Problem: Classify Φ proper for linear order (more accurately $(\Phi,\tau),\tau\subseteq\tau(\Phi)$) according to the function $I(\lambda,K_{\Phi,\tau})$ where

$$K_{\Phi,\tau}=\{EM_\tau(I,\Phi): I \text{ a linear order of cardinality } \lambda\}.$$

Probably as a first step we should consider generic $I\subseteq({}^\lambda 2,<_{lex})$ of cardinality λ (and then try to work in ZFC). Maybe it is reasonable to restrict ourselves to a dense family of Φ's, see [Sh 394].

6.17 Problem: More interesting classes to serve as index models. We have considered linear orders, trees with $\kappa+1$ levels, ordered graphs (see [Sh:e, Ch.III,end of §2], [LwSh 560]).

If $T\subseteq L_{\kappa,\omega},\kappa$ compact have compactness for $L_{\kappa,\kappa}$-types and can prove (under categoricity or a failure of a nonstructure assumption) that $\prec_{L_{\kappa,\omega}}=\prec_{L_{\kappa,\kappa}}$. But when we consider e.g. κ-a.e.c. with amalgamation, we may have a formal description of a type $p\in S(M)$ having $p\restriction N$ wherever $N\le_{\mathfrak{K}} M$ has small cardinality, neither knowing it there a $\le_{\mathfrak{K}}$-extension of M in which it is realized; not knowing it is unique. Remember the type was defined by chasing $\le_{\mathfrak{K}}$-embedding.

In [Sh 576] we consider whether we can do anything without any remnant of compactness (i.e. without E.M.-models, no large cardinals, no omitting type theorems) with some success. This is continued in [Sh 600], where we look at an abstract version of superstability (proved to occur in "nature" relying on earlier work.

* * *

There may be, however, limitations. First order logic was characterized e.g. by Lowenheim Skolem to $\aleph_1+$ compactness, now those are the first step, and we may well have the parallel of the theory without having the basic properties (Lowenheim Skolem and compactness).

6.18 Problem Can we characterize what part of stability theory are actually peculiar to first order?

We may consider generalizing the definitions and theorems on simple theories (see §5 particularly 5.4, 5.5). Now the context which seems less hostile is (D,λ)-homogeneous one (see the beginning of the section).

6.19 Definition. Assume D is a finite diagram.
1) Let $\kappa_{\theta,\sigma}(D)$ be the first regular (for simplicity) cardinal κ such that there is no increasing continuous sequence $\langle A_i:i\le\kappa\rangle$ of D-sets each of cardinality $<\kappa+\theta$ and $p\in S^m_D(A_\kappa)$ such that for every $i<\kappa,p\restriction A_{i+1}$ does (θ,σ)-divide over A_i (see below).
2) We say that $p\in S_D(B)$ does (θ,σ)-divide over A if:
($A\subseteq B$ are D-sets and in some D-set $C\supseteq B$)
there are $\bar b\in{}^{\theta>}B$ and sequence $\langle\bar b_t:t\in I\rangle$ in which
$\bar b$ appears, as $\bar b_{t^*},|I|=\sigma,C=B\cup\bigcup\limits_t\bar b_t$ and there are no
D-set $C_1\supseteq C$ and $\bar d\in{}^m(C_1)$ such that:

(∗) letting $\gamma=\ell g(\bar b)$ and $q(x,\bar y)\in S^{1+\gamma}(B)$ be $\{\varphi(x,\bar y,\bar c):\bar c\subseteq B,\varphi(x,\bar b,\bar c)\in p\}$ we have: $\bar d$ realizes $q(x,\bar b_t)$ for at least two $t\in I$.

3) If we omit σ from (1) we mean $\beth((2^{\theta+\kappa+(\tau(D))})^+)$ and in (2) we mean $\beth((2^{|B|+(\tau(D))})^+)$.

The value of $|I|$ is to allow us to use the Hanf number for omitting type, no point to increase further.

Of course

6.20 Claim. *1) If $p \in S_D(B)$ does θ-divide over A, <u>then</u> p does θ-divide$^+$ over A which means that we can choose $\langle \bar{b}_t : t \in I \rangle$ to be an indiscernible sequence.*
2) If $p \in S_D(B)$ does θ-divide$^+$ over A then for every σ, we have $p \in S_D(B)$ does (θ, σ)-divide over A.
3) If $\kappa < \kappa_{\theta,\sigma}(D), \mu = \mu^{<\kappa} > \kappa + \theta, D$ is μ-good, <u>then</u> we can find a D-set $A, |A| = \mu$, and $p_i \in S_D(A_i), A_i \subseteq A, |A_i| = \kappa + \theta$ for $i < \mu^\kappa$ such that for $i \neq j, p_i, p_j$ are contradictory, i.e. no $p \in S_D(A_i \cup A_j)$ extend p_i and p_j.
4) If $\kappa_{\theta,\sigma}(D)$ is $\geq \beth((2^{\theta+\sigma+|\tau(D)|})^+)$ then it is ∞.
5) If $\kappa_\theta(D)$ is $\geq \beth((2^{\theta+|\tau(D)|})^+)$ then it is ∞.

In Definition 6.19(2), we can demand, in $(*)$ instead two, a fix $n < \omega$, we do real change. If we ask $\mu \geq \aleph_0$, the theorem on Hanf numbers are no longer helpful, but weakened forms of the statement 6.20(3) holds.
We now may generalize the test problem from [Sh:93].

6.21 Theorem. *Assume the axiom $(Ax)_\mu$ of [Sh 80], $2^\mu > \lambda > \mu, \lambda^{<\mu} = \lambda, \mu^{<\mu} = \mu$.*

If D is a good finite diagram and $\kappa_{\mu,\mu^+}(D) \leq \mu$ and A is a D-set of cardinality $\leq \lambda$ <u>then</u> we can find a (D, μ)-homogeneous model M into which A can be embedded.

However
<u>6.22 Question</u>: Is $\langle \kappa_{\theta,\sigma}(D) : \theta, \sigma \rangle$ characterized by few invariants? Mainly, is $\langle \kappa_\theta(D) : \theta \rangle$ constant for θ large enough and $\langle \kappa_{\theta,\theta^+}(D) : \theta \rangle$.

* * *

This may be connected to the $\mathscr{P}^{(-)}(n)$-diagram theme. Looking at the proof of Morley's theorem, it struck me as a phenomenal good luck that categoricity could be gotten from a global property (saturation) rather than by painstakingly analyzing the models. A model M of cardinality λ, with vocabulary of cardinality μ, can be represented by on an increasing continuous elementary chain $\langle M_i : i < \lambda \rangle$ with $M = \bigcup\limits_{i<\lambda} M_i, M_i$ of cardinality $|i| + \mu$. Now for each i, we have to analyze M_{i+1} over M_i, so we represent the model $\langle M_{i+1}, M_i \rangle$ as an increasing continuous elementary chain $\langle (M_{i+1}, j, M_{i,j}) : j < \|M_{i+1}\| \rangle, \|M_{i+1,j}\| = \|M_{i,j}\| = |j| + \mu$ and now our problem is to construct $M_{i+1,j+1}$ over $M_{i,j}, M_{i+1,j}, M_{i,j+1}$, so we have to represent $(M_{i+1,j+1}, M_{i+1}, j, M_{i,j+1}, M_{i,j})$ by an increasing continuous elementary chain. After n such stages we have a $\mathscr{P}(n)$-diaigram $\langle M_u : u \in \mathscr{P}(n) \rangle$, for $n = 0$ this is just M, i.e. $M_\emptyset = M$, and for $\bar{M} = \langle M_u : u \in \mathscr{P}(n+1) \rangle$ letting $\bar{M}^- =: \langle M_u : u \in \mathscr{P}(n) \rangle$ and $\bar{M}^+ =: \langle M_{u \cup \{n-1\}} : u \in \mathscr{P}(n) \rangle$, both are $\mathscr{P}(n)$-diagrams and $\bar{M}^- \prec \bar{M}^+$. We can say $\bar{M}$ is a $(\lambda', \mathscr{P}(n))$-diagram if in addition $\|M_u\| = \lambda'$ for $u \in \mathscr{P}(n)$.

So to understand a model M in λ, for each $n < \omega$ and $\lambda' \in [\mu, \lambda)$ for each $(\lambda', \mathscr{P}(n))$-diagram $\langle M_n : u \in \mathscr{P}(n)\rangle$ we have to understand M_n over $\bar{M}^* = \langle M_u : u \in \mathscr{P}^-(n)\rangle$ where $\mathscr{P}^-(n) = \mathscr{P}(n)\backslash\{n\}, \bar{M}^*$ is called a $(\lambda', \mathscr{P}^-(n))$-diagram. So for categoricity, "understand" means in particular that it is essentially unique up to isomorphism (the "essentially" hint that we may have "time up to λ" to "correct" some things). What have we gained? Just naturally we can prove statements by induction on λ': a statement on $\mathscr{P}^{(-)}(n)$-diagrams for all n simultaneously (or for $\lambda = \mu^{+n}$, prove for $(\mu^{+m}, \mathscr{P}(n-m))$!) The gain is that the statement for $(\lambda', \mathscr{P}(n))$ for $\lambda' > \mu$ naturally used $\lambda'' \in [\mu, \lambda')$ and $\mathscr{P}(n+1)$.

To prove existence of a model in λ, we similarly prove by induction on $\lambda' \in [\mu, \lambda)$ that a $(\lambda', \mathscr{P}^-(n))$-diagram can be completed to a $(\lambda', \mathscr{P}(n))$-diagram.

Of course, we expect more conditions, complicating our induction.

<u>6.23 Thesis</u>: For complicated problems (on say all cardinals) we expect we need such a $\mathscr{P}^-(n)$ analysis.

This scheme was used in [Sh 87b] mentioned above, and also [SgSh 217], [Sh:c, Ch.XII], [Sh 234]. Returning to simple finite diagrams, for proving goodness from good behaviour in small cardinals, etc., this seems reasonable. This also applies to the hopeful Pr_n for 5.6.

§7 Finite model theory
0-1 Laws

Many were interested but hope is faint.
<u>7.1 Problem</u>: Find a logic with 0-1 law (or at least convergence or at least with very weak 0-1 law) from which finite combinatorialist can draw conclusion, novel for them.

But see [Fri99]. We know that say for the random model $(n, <, \mathscr{R}), \mathscr{R}$ a random 2-place relation, the 0-1 law and even convergence fails ([CHSh 245]) but the very weak 0-1 law holds ([Sh 551], a continuation with accurate estimates Boppana Spencer [BoSp]). However, this positive result goes through without telling us what first order formulas can define (in any random enough such model).

<u>7.2 Question</u>: Find the model theoretic content of the very weak 0-1 laws for $(n, <, \mathscr{R})$ and $(n, \mathscr{F}), \mathscr{F}$ a random 2-place function.
We hope for a very weak "elimination of quantifiers", saying hopefully one which gives: first order formulas can say much on "small set", but little on the majority.

Let $G_{n,p}$ be the random graph with set of vertices $[n] = \{1, \dots, n\}$ and edge probability p. It seems to me natural
<u>7.3 Problem</u>: 1) Characterize the sequences $\langle p_n : n < \omega \rangle$ of probabilities (that is reals in the interval $[0, 1)$) such that for every first order sentence ψ in the language of graphs we have

<u>Possibility a</u>: $(0 - 1$ law):
$\langle \text{Prob}(G_{n,p_n} \models \psi) : n < \omega \rangle$ converge to zero or converge to 1.

<u>Possibility b</u>: (convergence):
$\langle \text{Prob}(G_{n,p_n} \models \psi) : n < \omega \rangle$ converge.

<u>Possibility c</u>: (very weak $0 - 1$ law):
$\langle \text{Prob}(G_{n+1,p_{n+1}} \models \psi) - \text{Prob}(G_{n,p_n} \models \psi) : n < \omega \rangle$ converge to zero.

2) Like part (1) replacing G_{n,p_n} by the $G_{n,\bar{p}}$, the random graph with set of vertices $\{1, \dots, n\}$ and the probability of $\{i, j\}$ being an edge is $p_{(i-j)}$ (see [LuSh 435]).
3) Other cases (say random model on $\{1, \dots, n\}$ with vocabulary τ).
A solution for 7.3(2) case should be in [Sh 581].

In the cases of $0 - 1$ laws considered we usually get a dichotomy; say $\mathscr{M}_n$ is the n-random structure, say on $\{1, \dots, n\}$; the dichotomy has the form: either (a) or (b) where

- (a) we have a complex case, i.e. we can define in $\mathscr{M}_n$ (if n large enough $\mathscr{M}_n$ random enough) an initial segment of arithmetic of size $k_{\mathscr{M}_n}$, say of order of magnitude $\sim \log(n)$ or at least $\log_*(n)$ (or weakly complex: $\underset{\sim}{k}_n$ going to infinity or at least for some $\varepsilon > 0$ for every $k^*, \varepsilon <$ lim sup $\text{Prob}(k_{\underset{\sim}{M}_n} \geq k^*)$)
- (b) we have a simple case, so $0 - 1$ law (or at least convergence) (see [Sh 550]). So

7.4 Problem: 1) Prove for reasonable classes of 0-1 contexts $\langle \mathscr{M}_n : n < \omega \rangle$ such dichotomies.
2) Investigate the family of $\langle \mathscr{M}_n : n < \omega \rangle$ which are nice (in the direction of having $0 - 1$ laws), like closure under relevant operations.
Concerning part (2), see [Sh 550], [Sh 637].

7.5 Problem: In §2 we discuss investigating reasonable partial orders among generalized quantifiers. Make a parallel investigating on finite models.

See [Sh 639] which try to do for the finite what [Sh 171] do to a large extent for the infinite case.

§8 More on finite partition theorems

See discussion in [Sh 666, §8].
8.1 Question: What is the order of magnitude of the Hales-Jewitt numbers, $HJ(n,c)$ (see Definition 8.2(3) below).

8.2 Definition. 1) Let Λ be a finite nonempty alphabet, we define $f^{10}_\Lambda(m,c)$ where $m,c \in \mathbb{N}, |\Lambda|$ divide m, as the first $k \le \omega$ divisible by $|\Lambda|$ such that:

- $(*)$ if d is a c-colouring of ${}^{[k]}\Lambda$, i.e. a function from $\{\eta : \eta$ a function from $[k] = \{1,\dots,k\}$ into a set with $\le c$ members$\}$, then we can find $\langle M_\ell : \ell < m\rangle$ and η^* such that:
  - (a) $M_\ell \subseteq [k], \ell \ne m \Rightarrow M_\ell \cap M_m = \emptyset$ and η^* is a function from $M \backslash \bigcup\limits_\ell M_\ell$ into Λ
  - (b) $\|M_\ell\| = \|M_0\| > 0$ for $\ell < m$
  - (c) for $\nu_1,\nu_2 \in S = \{\eta : \eta \in {}^{[k]}\Lambda, \eta^* \subseteq \eta$, and each $\eta \restriction M_\ell$ is constant$\}$ we have $d(\nu_1) = d(\nu_2)$ provided that for every $\alpha \in \Lambda$ for $i \in \{1,2\}$ we have
    $|\{\ell < m : \nu_i \restriction M_\ell$ is constantly $\alpha\}| = m/|\Lambda|$
  - (d) if $\alpha,\beta \in \Lambda$ and $\nu \in S$ then
    $|\{a \in M\backslash \bigcup\limits_{\ell<m} M_\ell : \eta^*(a) = \alpha\}| = |\{a \in M\backslash \bigcup\limits_{\ell<m} M_\ell : \eta^*(a) = \beta\}|$.

2) Now $f^9_\Lambda(m,c)$ is defined similarly without clause (d).
3) $HJ_\Lambda(m,c)$ is defined similarly omitting (d), and replacing (b), (c) by:

$(b)'$ $M_\ell \ne \emptyset$

$(c)'$ $d \restriction S$ is constant.

Lastly let $HJ(n,c) = HJ_{[n]}(1,c)$.
4) Let $f^8_\Lambda(m,c)$ be defined as in part (2), replacing clause (b) by (b)$'$ from part (3).
5) We define $f^{10,*}_\Lambda(m,c)$ as in part (1) replacing clause (c) by

$(c)^+$ for $\nu_1,\nu_2 \in S = \{\eta : \eta \in {}^{[k]}\Lambda, \eta^* \subseteq \eta$ and each $\eta \restriction M_\ell$ is constant$\}$ we have $d(\nu_1) = d(\nu_2)$ provided that for every $\alpha \in \Lambda$ we have $|\{\ell < m : \nu_1 \restriction M_\ell$ is constantly $\alpha\}| = |\{\ell < m : \nu_2 \restriction M_\ell$ is constantly $\alpha\}|$.

6) We define $f^{9,*}_\Lambda(m,c)$ as we have defined $f^{10,*}_\Lambda(m,c)$ omitting clause (d).
7) We define $f^{8,*}_\Lambda(m,c)$ as we have defined $f^{10,*}_\Lambda(m,c)$ omitting clause (d) and replacing clause (b) by clause (b)$'$ from part (3).

Remark. So $HJ_\Lambda(m,c)$ is the Hales-Jewett number for alphabet Λ, getting m-dimensional subspace.

8.3 Fact: 1) $f^{10}_\Lambda(m,c) \ge f^9_\Lambda(m,c) \ge f^8_\Lambda(m,c)$.
2) $f^{10,*}_\Lambda(m,c) \ge f^{9,*}_\Lambda(m,c) \ge f^{8,*}_\Lambda(m,c)$.

3) $f^{10}_\Lambda(m,c) \le f^{10,*}_\Lambda(m,c)$ and $f^9_\Lambda(m,c) \le f^{9,*}_\Lambda(m,c)$ and $f^8_\Lambda(m,c) \le f^{8,*}_\Lambda(m,c)$.
4) $f^{8,*}_\Lambda(m,c) \le HJ_\Lambda(m,c)$.

Proof. Read the definitions.

We can deal similarly with the density (like Szemeredi theorem) version of those functions.

May those numbers be helpful for HJ-number? First complimentarily to 8.3, clearly

8.4 Claim. *$HJ_\Lambda(m,c) \le f^{8,*}_\Lambda(m^*,c)$ if m^* satisfies:*

> $\boxtimes$ *assume that $Par = \{\bar\ell : \bar\ell = \langle \ell_\alpha : \alpha \in \Lambda\rangle, \ell_\alpha \in [0,m^*)$ and $\Sigma\{\ell_\alpha : \alpha \in \Lambda\} = m^*\}$ and d is a c-colouring of Par; <u>then</u> we can find $\bar\ell^\alpha \in Par$ for $\alpha \in \Lambda$ such that $d \restriction \{\bar\ell^\alpha : \alpha \in \Lambda\}$ is constant and for some $\ell^* > 0$ and $\langle \ell^*_\alpha : \alpha \in \Lambda\rangle$ we have for any distinct $\alpha,\beta \in \Lambda : \ell^\alpha_\beta = \ell^*_\beta, \ell^\alpha_\alpha = \ell^*_\alpha + \ell^*$.*

Remark. We can choose $\alpha^* \in \Lambda$ let $\Lambda^* = \Lambda\backslash\{\alpha^*\}$ and restrict ourselves to $\mathrm{Par}' = \{\bar\ell \in \mathrm{Par} : \alpha \in \Lambda^* \Rightarrow \ell_\alpha \le m^{**} =: m^*/|\Lambda|\}$ and let $\mathrm{Par}'' = \{\bar\ell \restriction \Lambda^* : \bar\ell \in \mathrm{Par}'\}$, now $\mathrm{Par}'' = {}^\Lambda[0,m^{**})$, and $\bar\ell \mapsto \bar\ell \restriction \Lambda^*$ is a one-to-one map from Par' onto Par''. So clearly it suffices to find a d-monocromatic $\{\bar\ell^*\} \cup \{\bar\ell^\alpha : \alpha \in \Lambda^*\} \subseteq \mathrm{Par}''$ and $m > 0$ such that $\ell^\alpha_\beta = \ell^*_\beta$ if $\beta \ne \alpha \in \Lambda^*, \ell^\alpha_\beta = \ell^*_\alpha + m$ if $\beta = \alpha \in \Lambda^*$. Now this holds by $\boxtimes$ which is a case of the $|\Lambda^*|$-dimensional of v.d.W. $\square_{8.4}$

8.5 Claim. *1) $f^{10}_\Lambda(m,c) \le m \times HJ(|\Lambda|^m,c)$ so f^{10}_Λ is not far from the Hales Jewett numbers.*
2) $f^{9,*}_\Gamma(m,c) \le m \times HJ(|\Gamma|^m,c)$.

Proof. 1) Let M_k be $\{0,\dots,k-1\}$.

Let Λ_1 be the set of function π from $\{0,\dots,m-1\}$ to Λ such that $\alpha \in \Lambda \Rightarrow |\pi^{-1}\{\alpha\}| = m/|\Lambda|$.

Let $n_1 = |\Lambda_1|$ so $n_1 \le |\Lambda|^m$ and $k_1 = HJ(|\Lambda|^m,c)$ and $k = m \times k_1$.

Let d be a c-colouring of $V = {}^\Lambda(M_k)$. Let $V_1 = {}^{\Lambda_1}(M_{k_1})$ and we define a function g from V_1 onto V as follows:

for $\eta \in V_1$, we have to define $\langle g(\eta)(a) : a \in M_k\rangle, g(\eta)(a) \in \Lambda$, now for $a \in \{0,\dots,k-1\}$ noting $m[a/m] \le a < m[a/m]+m$ we define $g(\eta)(a) = (\eta([a/m]))([a/m]-m[a/m])$.

We define a c-colouring d_1 of V_1: $d_1(\eta) = d(g(\eta))$. So there is nonempty $N \subseteq M$ and ρ^*_1 a function from $M\backslash N$ into Λ_1 such that $d_1 \restriction \{\rho \in {}^{\Lambda_1}(M_{k_1}) : \rho^*_1 \subseteq \rho$ and $\rho \restriction N$ is constant$\}$ is constant. Let for $\ell < m, N_\ell = \{a : [a/m] \in N$ and $[a/m]-m[a/m] = \ell\}$ and $\rho^* \in {}^\Lambda(M_k\backslash \bigcup\limits_{\ell<m} N_\ell)$ be such that $\rho^* \subseteq \rho \in {}^\Lambda(M_k) \Rightarrow \rho^*_1 \subseteq g(\rho)$. Now check.

2) Similar proof. $\square_{8.5}$

<u>8.6 Question</u>: 1) Can we give better bounds to $f^\ell_\Lambda(m,c)$ than through HJ for $\ell = 8,9,10$?
2) What is the order of magnitude of f^8, f^9, f^{10}?
3) What about f^{11} (see 8.11 below) and $f^{10,*}, f^{9,*}, f^{8,*}$?

8.7 Definition. 1) For a set A let

- (a) $\mathrm{seq}_\ell(A) = \{\eta : \eta$ is a sequence of length ℓ with no repetitions, of elements of $A\}$
- (b) $\mathrm{seq}(A) = \bigcup\limits_{\ell \geq 1} \mathrm{seq}_\ell(A)$
- (c) $\mathrm{seq}_{m,\ell}(A) = \{\bar\eta : \bar\eta = \langle \eta_i : i < m\rangle, \eta_i \in \mathrm{seq}_\ell(A)$ and $i_1 < i_2 \Rightarrow \mathrm{Rang}(\eta_{i_1}) \cap \mathrm{Rang}(\eta_{i_2}) = \emptyset$
- (d) $\mathrm{seq}^*_m(A) = \cup\{\mathrm{seq}_{m,\ell}(A) : \ell \geq 1\}$
- (e) $\mathrm{seq}^*(A) = \cup\{\mathrm{seq}^*_m(A) : m \geq 1\}$.

2) For $\bar\eta \in \mathrm{seq}^*(A)$ let
$\mathrm{son}(\bar\eta) = \{\nu : \nu \in \mathrm{seq}(A)$ and ν is a concatenation of some members of $\{\eta_\ell : \ell < \ell g(\bar\eta)\}$, in any order$\}$
$\mathrm{legson}(\bar\eta) = \{\nu \in \mathrm{son}(\bar\eta) : \mathrm{Rang}(\nu) = \bigcup\limits_i \mathrm{Rang}(\eta_i)\}$,
$\mathrm{dis}(\bar\eta) = \{\bar\nu \in \mathrm{seq}^*(A) :$ each ν_i is from $\mathrm{son}(\bar\eta)\}$,
$\mathrm{leg\ dis}(\bar\eta) = \{\bar\nu \in \mathrm{dis}(\bar\eta) : \bigcup\limits_i \mathrm{Rang}(\nu_i) = \bigcup\limits_i \mathrm{Rang}(\eta_i)\}$.
3) $f^{12}(m,c)$ is the first k such that $k = \omega$ or $k < \omega$ and for every c-colouring d of $\mathrm{seq}([0,k))$ there is $\bar\eta \in \mathrm{seq}^*_m(A)$ such that the set $\mathrm{son}(\bar\eta)$ is d-monocromatic.
There are other variants.

<u>8.8 Question</u>: Is $f^{12}(m,c)$ finite?

8.9 Definition. 1) For groups G,H and subset Y of H and cardinal κ let $G \rightarrow (Y,H)_\kappa$ means that for any κ-colouring d of G (i.e. d is a function from G into a set of cardinality $\leq \kappa$) there is an embedding h of H into G such that $d \restriction \{h(y) : y \in Y\}$ is constant.
2) $G \rightarrow (Y,H)_{\kappa,\theta}$ is defined similarly but $d \restriction \{h(y) : y \in Y\}$ has range with $< \theta$ members.
3) If $Y = H$ we may omit it.

<u>8.10 Question</u>: 1) Investigate $G \rightarrow (Y,H)_c$ for finite groups.
2) Assume H is a finite permutation group, Y is one conjugacy class (say permutation of order two) and c finite, does $G \rightarrow (Y,H)_c$ exist? (This is connected to 8.7, just interpret $\eta \in \mathrm{seq}(A)$ of even legnth $2n$ with the permutation of A permuting $\eta(i)$ with $\eta(n+1)$ for $i < n$.
3) Similarly when we colour subgroups of G.

Similar problems to 8.6 are

8.11 Definition. 1) (See [Sh 679] and the notation there). Let $f^1_\Lambda(m,c)$ is the first k such that $k = \omega$ (i.e. infinity) or

$(*)_k$ k is divisible by $|\Lambda_{\mathrm{id}}|$ and letting $M = M^\tau_k$, we have: for every c-colouring d of $\mathrm{Space}_{\bar\Lambda}(M)$, we can find an m-dimensional subspace S such that

- (a) letting $\langle M_\ell : \ell < m\rangle$ be a witness for S (see Definition [Sh 679, 1.7t(5)] the dimension of $M_\ell, |P^{M_\ell}|$, is the same for $\ell = 0,\dots,m-1$

(b) let $K = K^\tau_{[0,m)}, N = c\ell_M(\bigcup_{\ell<m} M_\ell)$ and f as in (c) of Definition [Sh 679, 1.7t(5)]; if $\rho_1, \rho_2 \in \mathrm{Space}_{\bar\Lambda}(K)$ and $\nu_1, \nu_2 \in S$ are such that $b \in N \Rightarrow \nu_1(b) = \rho_1(\hat f(b)) \quad \& \quad \nu_2(b) = \rho_2(\hat f(b))$ and there is an automorphism of K mapping ρ_1 to ρ_2 then $d(\nu_1) = d(\nu_2)$.

2) If $\tau = \{\mathrm{id}\}, \Lambda = \Lambda_{\mathrm{id}}$ we write $f^{11}_\Lambda(m,c)$ and above (b) means:

$(b)'$ for $\nu_1, \nu_2 \in S$ we have $d(\nu_1) = d(\nu_2)$ if $\nu_1 \restriction (M \backslash \bigcup_{\ell<m} M_\ell) = \nu_2 \restriction (M \backslash \bigcup_{\ell<m} M_\ell)$ and for every $\alpha \in \Gamma$ we have

$$|\{\ell < m : \nu_1 \restriction M_\ell \text{ is constantly } \alpha\}| = |\{\ell < m : \nu_2 \restriction M_\ell \text{ is constantly } \alpha\}|.$$

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