SUPERATOMIC BOOLEAN ALGEBRAS: MAXIMAL RIGIDITY

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Abstract. We prove that for any superatomic Boolean Algebra of cardinality $> \beth_4$ there is an automorphism moving uncountably many atoms. Similarly for larger cardinals. Any of those results are essentially best possible

Annotated Content

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§4 On Independence

[We show e.g. that $\beth_4(\sigma)$ consistently can be improved.]

§0 Introduction

We show that any superatomic Boolean Algebra has an automorphism moving uncountably many atoms if it is large enough, really $> \beth_4$; similarly replacing $\aleph_0$ by θ; (an automorphism moves an atom if its image is not itself). We then show that those results are essentially best possible. Recall that for many other natural classes of Boolean Algebras behave differently; there are arbitrarily large members with few automorphisms (and even endomorphisms). Of course, we can express those results in topological terms. See [M] and [M1] on Boolean Algebra.
Rubin and Koppleberg [RuKo01] have proved: if $\diamondsuit_{\lambda^+} + 2^{\lambda^+} = \lambda^{++}$, then there is a superatomic Boolean Algebra $\mathbb{B}$ of cardinality λ^{++} with λ atoms and exactly λ^+ automorphisms answering a question 80 of Monk [M2], i.e. in a preliminary version asking for a consistent example.
By [Sh 641, §1], provably in ZFC, there is a superatomic Boolean Algebra $\mathbb{B}$ such that $|\mathrm{Aut}(\mathbb{B})| < |\mathrm{End}(\mathbb{B})|$ answering question 96 of Monk [M2, p.291].
By [Sh 641, §2], provably in ZFC, there is a superatomic Boolean Algebra $\mathbb{B}$ such that $|\mathrm{Aut}(\mathbb{B})| < |\mathbb{B}|$, answering Problem 80 of [M2, p.291]. In fact, if μ is strong limit, $\mu > \mathrm{cf}(\mu) = \aleph_0$ and $\lambda = \mathrm{Min}\{\lambda : 2^\lambda > 2^\mu\}$, then there is a Boolean Algebra $\mathbb{B}$ with 2^μ atoms, 2^λ elements and every automorphism of $\mathbb{B}$ moves $< \mu$ atoms so $|\mathrm{Aut}(\mathbb{B})| \le 2^\mu < 2^\lambda$.

Notation

0.1 Definition. 1) For a Boolean Algebra $\mathbb{B}$ its operation are denoted by $x \cap y, x \cup y, x - y, -x$ and $0_{\mathbb{B}}$ is its zero. Let us define the ideal $\mathrm{id}_\alpha(\mathbb{B})$ by induction:

$\mathrm{id}_0(\mathbb{B}) = \{0\}$

$\mathrm{id}_\beta(\mathbb{B}) = \{x_1 \cup \ldots \cup x_n : n < \omega$ and $x_\ell \in \mathbb{B}$ for $\ell = 1, \ldots, n$ such that for some $\alpha < \beta$ and for each $\ell \in \{1, \ldots, n\}$ the element $x_\ell/\mathrm{id}_\alpha(B)$ is an atom of $\mathbb{B}/\mathrm{id}_\alpha(B)$ or $x_\ell \in \mathrm{id}_\alpha(B)\}$.

Hence for limit δ we have

$$\mathrm{id}_\delta(\mathbb{B}) = \bigcup_{\beta<\delta} \mathrm{id}_\beta(\mathbb{B}).$$

Let $\mathrm{id}_\infty(\mathbb{B}) = \bigcup_\alpha \mathrm{id}_\alpha(\mathbb{B})$.

2) For $x \in \mathrm{id}_\infty(\mathbb{B})$ let $\mathrm{rk}(x, \mathbb{B}) = \mathrm{Min}\{\alpha : x \in \mathrm{id}_{\alpha+1}(\mathbb{B})\}$.
3) $\mathbb{B}$ is superatomic if $\mathbb{B} = \mathrm{id}_\infty(\mathbb{B})$ and $\mathrm{rk}(\mathbb{B})$ is the ordinal α such that $\mathbb{B}/\mathrm{id}_\alpha(\mathbb{B})$ is a finite Boolean Algebra (so $\mathbb{B} = \mathrm{id}_{\alpha+1}(\mathbb{B})$).
4) For a Boolean Algebra $\mathbb{B}$ and $x \in \mathbb{B}$ let $\mathbb{B} \restriction x$ be $\mathbb{B}$ restricted to $\{y \in \mathbb{B} : y \leq_{\mathbb{B}} x\}$, it is a Boolean Algebra.
5) Define by induction on $n = 1, 2, \ldots$:

$$\beth_1(< \theta) = 2^{<\theta} = \sum_{\kappa < \theta} 2^\kappa$$

$$\beth_{n+1}(< \theta) = 2^{\beth_n(<\theta)}$$

0.2 Observation. If $\mathbb{B}$ is a superatomic and D_n is an ultrafilter of $\mathbb{B}$ for $n < \omega$ <u>then</u> for some infinite $u \subseteq w$ the sequence $\langle D_n : n \in u \rangle$ covering to some ultrafilter of D of $\mathbb{B}$, i.e. for every $x \in \mathbb{B}$ for all but finitely many $n \in u$ we have $x \in D_n \leftrightarrow x \in D$.

Proof. Among the pairs $\{(x, \alpha) : x \in \mathbb{B}, \mathrm{rk}(x, \mathbb{B}) = \alpha$ and $(\exists^\infty n) x \in D_n\}$ choose one (x, α) with x minimal. Without loss of generality $x/\mathrm{id}_\alpha(\mathbb{B})$ is an atom let $u = \{n < w : x \in D_n\}$ and check that $D := \{y \in \mathbb{B} : \mathrm{rk}(y \cap x, \mathbb{B}) = \alpha\}$ is as required.

§1 Superatomic Boolean Algebras have nontrivial automorphisms

1.1 Main Theorem. *Assume*

- (a) $\mathbb{B}$ *is a superatomic Boolean Algebra with no automorphism moving* $\geq \theta$ *atoms; that is, if* π *is an automorphism of* $\mathbb{B}$ *then* $\{x : x \in$ *atom*$(\mathbb{B})$ *and* $\pi(x) \neq x\}$ *is a set of cardinality* $< \theta$
- (b) θ *regular uncountable.*

<u>*Then*</u> $|\mathbb{B}| \leq \beth_4(< \theta)$, *so if* $\theta = \sigma^+$ *then* $|\mathbb{B}| \leq \beth_4(\sigma)$.

Remark. If $|\mathbb{B}|$ is close to $\beth_4(< \theta)$, the proof says much on the structure of $\mathbb{B}$.

Proof. Let $\mathbb{B}$ be the Boolean algebra satisfying clause (a) and let μ be the number of atoms of $\mathbb{B}$. Without loss of generality

$\boxtimes_1$ $\mathbb{B}$ is a Boolean Algebra of subsets of μ and its atoms are the singletons $\{\alpha\}, \alpha < \mu$; so, e.g. $\mathbb{B} \models a - b = c$ iff $a \backslash b = c$.

Let $I =: [\mu]^{<\theta} \cap \mathbb{B} = \{x \in \mathbb{B} : |x| < \theta\}$, clearly I is an ideal of $\mathbb{B}$ and let $Y =: \{x : x \in \mathbb{B}$ and x/I is an atom of $\mathbb{B}/I\}$.
We shall prove (after some preliminary things) that:

$\boxtimes_2$ if $x \in Y$ then $|x| \le \beth_2(<\theta)$, i.e. $2^{(2^{<\theta})}$.

We shall say that a set $a \subseteq \mu$ is $\mathbb{B}$-autonomous if $(\forall y \in I)(y \cap a \in \mathbb{B})$; in this case we let $\mathbb{B} \restriction a = \mathbb{B} \cap \mathscr{P}(a)$; this notation is compatible with 0.1(4).
Clearly

$\otimes_1$ the family of $\mathbb{B}$-autonomous subsets of μ is a Boolean ring and even a Boolean algebra of subsets of μ (i.e. closed under $a \cap b, a \cup b, a \backslash b$) and include I and even $\mathbb{B}$

$\otimes_2$ for a $\mathbb{B}$-autonomous set $a, \mathbb{B} \restriction a = \{x \in \mathbb{B} : x \subseteq a\}$ is a Boolean ring of subsets of a which include $\{\{\alpha\} : \alpha \in a\}$.

Also

$\otimes_3$ if a_0, a_1 are $\mathbb{B}$-autonomous subsets of $\mu, x \in Y, a_0 \subseteq x, a_1 \subseteq x$ and $\mathbb{B} \restriction a_0 \cong \mathbb{B} \restriction a_1$ over $\mathbb{B} \restriction (a_1 \cap a_2) = \mathbb{B} \cap \mathscr{P}(a_1 \cap a_2)$, then there is an automorphism h of $\mathbb{B}$ such that h maps a_0 to a_1, a_1 to a_0 and $\alpha \in \mu \backslash a_0 \backslash a_1 \Rightarrow h(\{\alpha\}) = \{\alpha\}$.

[Why? Let g be an isomorphism from $\mathbb{B} \restriction a_0$ onto $\mathbb{B} \restriction a_1$ over $\mathbb{B} \restriction (a_0 \cap a_1)$; now we define a permutation h of atom$(\mathbb{B}) = \{\{\alpha\} : \alpha < \mu\}$; let $\alpha \in a_0 \Rightarrow h(\{\alpha\}) = g(\{\alpha\}), h(g(\{\alpha\})) = \{\alpha\}$ and $\alpha \in \mu \backslash a_0 \backslash a_1 \Rightarrow h(\{\alpha\}) = \{\alpha\}$, by the demands on g clearly h is a well defined permutation of atom$(\mathbb{B})$. Now h can be naturally extended to an automorphism $\hat{h}$ of $\mathscr{P}(\mu)$ as a Boolean Algebra, it is of order two. We have to check that $\hat{h}$ maps $\mathbb{B}$ onto itself; even into itself suffice (because of "order two"). Clearly $\hat{h}(x) = x$ and $\hat{h} \restriction (\mathbb{B} \restriction (\mu \backslash x))$ is the identity. So it is enough to check that: $\hat{h} \restriction (\mathbb{B} \restriction x)$ is an automorphism of $\mathbb{B} \restriction x$. But $I \cap (\mathbb{B} \restriction x)$ is a maximal ideal of the Boolean Algebra $\mathbb{B} \restriction x$ (as $x \in Y$) hence it is enough to check that $\hat{h}$ maps $I \cap (\mathbb{B} \restriction x)$ into itself. As $b \in I \cap (\mathbb{B} \restriction x) \Rightarrow b = (b \backslash a_0 \backslash a_1) \cup (b \cap a_0 \cap a_1) \cup (b \cap a_0 \backslash a_1) \cup (b \cap a_1 \backslash a_0)$, and all four are in I; clearly it is enough to check the following statements: $b \in I$ & $b \subseteq x \backslash a_0 \backslash a_1 \Rightarrow h(b) \in I$, and $\ell < 2$ & $b \in I$ & $b \subseteq x \cap a_\ell \backslash a_{1-\ell} \Rightarrow \hat{h}(b) \in I$ and lastly $b \in I$ & $b \subseteq a_0 \cap a_1 \Rightarrow \hat{h}(b) \in I$. The second implication holds by the choice of g, the first as $\hat{h}(b) = b$ in this case and the last one as $h \restriction \{\{\alpha\} : \alpha \in a_0 \cap a_1\}$ is the identity so again $\hat{h}(b) = b$.]

$\otimes_4$ if $b \subseteq \mu, |b| \le 2^{<\theta}$ then for some $\mathbb{B}$-autonomous set c we have $b \subseteq c \subseteq \mu, |c| \le 2^{<\theta}$.
[Why? Find c satisfying $b \subseteq c \subseteq \mu, |c| \le 2^{<\theta}$ such that $(\forall y \in [c]^{<\theta})[(\exists z)(y \subseteq$

$z \in I) \to (\exists z \subseteq c)(y \subseteq z \in I)]$, just close θ times recalling θ is regular. Now if $y \in I$ then $|y| < \theta$ hence $y \cap c \in [c]^{<\theta}$ so there is z such that $y \cap c \subseteq z \in I$ & $z \subseteq c$; hence $y \cap c = y \cap z \in I$. This proves that c is $\mathbb{B}$-autonomous as required.]

Now we return to the promised $\boxtimes_2$.

Proof of $\boxtimes_2$. Toward contradiction assume $x \in Y$ and $|x| > \beth_2(<\theta)$; let $\alpha_i \in x$ for $i < (\beth_2(<\theta))^+$ be pairwise distinct, let a_i be a $\mathbb{B}$-autonomous set of cardinality $\leq 2^{<\theta}$ such that $\{\alpha_{i+\varepsilon} : \varepsilon < 2^{<\theta}\} \subseteq a_i$ (exists[1] by $\otimes_4$), and without loss of generality $a_i \subseteq x$ (just use $a_i \cap x$, it is as required by $\otimes_1$). For some club C of $(\beth_2(<\theta))^+$, we have $i < j \in C \Rightarrow a_i \cap \{\alpha_{j+\varepsilon} : \varepsilon < 2^{<\theta}\} = \emptyset$ hence $i < j \in C \Rightarrow |a_j \backslash a_i| \geq 2^{<\theta}$. Now $I \cap \mathscr{P}(a_i)$ has cardinality $\leq |a_i|^{<\theta} \leq 2^{<\theta}$ (as θ is regular) hence $\mathbb{B} \restriction a_i$ has cardinality $\leq 2^{<\theta}$. It follows that there are a stationary $S \subseteq \{\delta < (\beth_2(<\theta))^+ : \mathrm{cf}(\delta) = (2^{<\theta})^+\}$ and a^* such that $i \in S$ & $j \in S$ & $i \neq j \Rightarrow a_i \cap a_j = a^*$ (the Δ-system lemma). Also as $a_j \subseteq X \in Y$ and $|a_j| = 2^{<\theta}$ and $|\mathbb{B}_i \restriction a_i| = 2^{<\theta}$ the number of isomorphism types of $(\mathbb{B} \restriction a_i, \{\alpha\})_{\alpha \in a^*}$ is at most $\leq \beth_2(<\theta)$ hence for some $i < j$ from $C \cap S$ we have $\mathbb{B} \restriction a_i \cong B \restriction a_j$ over $\mathbb{B} \restriction a^*$, but $|a_j \backslash a_i| \geq 2^{<\theta} \geq \theta$ hence by $\otimes_3$ there is an automorphism h of $\mathbb{B}$ which moves $\geq 2^{<\theta}$ atoms, contradiction.
Next

$\boxtimes_3$ $|Y/I| \leq \beth_3(<\theta)$.
[Why? If not, we can find $x_i \in Y$ for $i < (\beth_3(<\theta))^+$ such that $i \neq j \Rightarrow x_i/I \neq x_j/I$. As $|x_i| \leq \beth_2(<\theta)$ by $\boxtimes_2$, by the Δ-system lemma for some unbounded $A \subseteq (\beth_3(<\theta))^+$ the set $\langle x_i : i \in A\rangle$ is a Δ-system hence without loss of generality $\langle x_i : i \in A\rangle$ are pairwise disjoint (by substruction; not really needed just clearer). As $\mathbb{B} \restriction x_i$ is a Boolean Algebra of cardinality $\leq \beth_2(<\theta)$ (as $I \cap \mathscr{P}(x_i)$ is a maximal ideal of $\mathbb{B} \restriction x_i$ and $I \cap \mathscr{P}(x_i) \subseteq [x_i]^{<\theta}$ and $|x_i| \leq \beth_2(<\theta)$ by $\boxtimes_2$) there are at most $\beth_3(<\theta)$ isomorphism types of $\mathbb{B} \restriction x_i$. So for some $i \neq j$ in A we have $\mathbb{B} \restriction x_i \cong \mathbb{B} \restriction x_j$, so as in the proof of $\otimes_3$ there is an automorphism h of $\mathbb{B}$ mapping x_i to x_j, x_j to x_i and $h \restriction (\mathbb{B} \restriction (1_{\mathbb{B}} - x_i - x_j))$ is the identity hence h moves $\geq |x_i \backslash x_j| \geq \theta$ atoms because $x_i \neq x_j \bmod I$.]

Choose a set $\{x_\alpha : \alpha < \alpha^* \leq \beth_3(<\theta)\}$ of representatives of Y/I and let $x^* = \bigcup\limits_{\alpha<\alpha^*} x_\alpha$, so $x^* \subseteq \mu, |x^*| \leq \beth_3(<\theta)$.
Define $J = \{a \in \mathbb{B} : a \cap x^* = \emptyset\}$.

$\boxtimes_4$ $J \subseteq I$.
[Why? If not, there is $x \in J \backslash I$ such that x/I is an atom of $\mathbb{B}/I$ so $x/I \in$

[1] we can use also $\{a_{i+\varepsilon} : \varepsilon < \theta\}$

$\{x_\alpha/I : \alpha < \alpha^*\}$, so for some $\alpha, x/I = x_\alpha/I$ hence $|x \backslash x_\alpha| < \theta$ hence $|x \cap x_\alpha| \geq \theta$ hence $x \cap x^* \neq \emptyset$ hence $x \notin J$, a contradiction.]

Define an equivalence relation $\mathscr{E}$ on $\mathbb{B} : y_1 \mathscr{E} y_2$ iff $y_1 \cap x^* = y_2 \cap x^*$. Clearly $\mathscr{E}$ has $\leq 2^{|x^*|}$ equivalence classes and $2^{|x^*|} \leq \beth_4(< \theta)$; also $y_1 \mathscr{E} y_2 \rightarrow y_1 \backslash y_2 \in J$, in fact $y_1 \mathscr{E} y_2 \leftrightarrow (y_1 \Delta y_2 \in J)$ (see J's definition). Choose a set of representatives $\{y_\gamma : \gamma < \gamma^*\}$ for $\mathscr{E}$ so $|\gamma^*| \leq \beth_4(< \theta)$ and let $\mathbb{B}^*$ be the subalgebra of $\mathbb{B}$ which $\{y_\gamma : \gamma < \gamma^*\}$ generates. So $|\mathbb{B}^*| \leq \beth_4(< \theta)$ and, being superatomic, the number of ultrafilters of $\mathbb{B}^*$ is also $\leq \beth_4(< \theta)$. Next $\mathbb{B}$ is generated by $J \cup \mathbb{B}^*$ as for $y \in \mathbb{B}$ there is γ such that $y \mathscr{E} y_\gamma$ and $y_\gamma \in \mathbb{B}^*, y - y_\gamma \in J, y_\gamma - y \in J$ hence $y \in \langle J \cup \mathbb{B}^* \rangle$. For D an ultrafilter of $\mathbb{B}^*$ let $Z_D = \{\alpha < \mu : (\forall y \in \mathbb{B}^*)(\alpha \in y \leftrightarrow y \in D)\}$.
Clearly

$\boxtimes_5$ for every $\alpha \in \mu \backslash x^*$ there is a unique ultrafilter $D = D[\alpha]$ on $\mathbb{B}^*$ such that $\alpha \in Z_D$ (and the number of such ultrafilters is $\leq \beth_4(< \theta)$).

Now

$\boxtimes_6$ $\mu \leq \beth_4(< \theta)$.
[Why? Assume that not. By $\otimes_4$ for each $i < \mu$ we can find a $\mathbb{B}$-autonomous a_i such that $|a_i| \leq 2^{<\theta}$ and $[i, i+2^{<\theta}) \subseteq a_i$; let $a_i = \{\beta_{i,\varepsilon} : \varepsilon < \varepsilon_i\}$ with $\beta_{i,\varepsilon}$ increasing with ε. Clearly for some unbounded $A \subseteq (\beth_4(< \theta))^+$ for all $i \in A$ the following does not depend on $i : \varepsilon_i$ and $D[\beta_{i,\varepsilon}]$ for $\varepsilon < \varepsilon_i$ (use $\boxtimes_5$), and $\{u \in [\varepsilon_i]^{<\theta} : \{\beta_{i,\varepsilon} : \varepsilon \in u\} \in I\}$, and for $\zeta < 2^{<\theta}, \varepsilon = \varepsilon(i, \zeta) =$ the unique ε such that $\beta_{i,\varepsilon} = i + \zeta$ and without loss of generality for $j < i$ in $A, a_j \cap [i, i + 2^{<\theta}) = \emptyset$. By the Δ-system lemma without loss of generality for some a^* we have: for $i < j$ in $A, a_i \cap a_j = a^*$. So by $\otimes_1$ the set a^* is $\mathbb{B}$-autonomous and also $a_i \backslash a^*$ is so we can use $a_i \backslash a^*$, so without loss of generality for $i \neq j$ in $A, a_i \cap a_j = \emptyset$ and as $|x^*| \leq \beth_4(< \theta)$ clearly without loss of generality $i \in A \Rightarrow a_i \cap x^* = \emptyset$. So for $i \neq j$ in A there is a permutation g of order two of μ interchanging a_i, a_j, that is $g(\beta_{i,\varepsilon}) = \beta_{j,\varepsilon}, g(\beta_{j,\varepsilon}) = \beta_{i,\varepsilon}$ and $g(\{\beta\}) = \beta$ for $\beta \in \mu \backslash a_i \backslash a_j$. Clearly g can be extended to an automorphism $\hat{g}$ of $\mathscr{P}(\mu)$ and $\hat{g} \restriction \mathbb{B}^*$ is the identity (the proof is like that proof of $\otimes_3$ using "$\mathbb{B}$ is generated by $J \cup \mathbb{B}^*$" and "$D[\beta_{i,\varepsilon}]$ does not depend on i. So we get a contradiction.]

So $|J| \leq |[\mu]^{<\theta}| = \mu^{<\theta} \leq (\beth_4(< \theta))^{<\theta} = \beth_4(< \theta)$ and $|\mathbb{B}^*| \leq |\mathbb{B}/\mathscr{E}| \leq \beth_4(< \theta)$ so as $\mathbb{B}$ is generated by $J \cup \mathbb{B}^*$ together we get the desired conclusion. $\square_{1.1}$

1.2 Discussion. 0) We can strengthen the conclusion of **?** to:
$\longrightarrow$ scite{dg.1} undefined

$\oplus$ one of the following occurs (where I is as in the proof)
 (a) there is $a \in \mathbb{B} \backslash I$ such that $a \cap a = 0$

(b) there is an ideal $J \subseteq I$ containing $2^{<\theta}$ pairwise disjoint elements such that $a, b \in J \Rightarrow a \cap f(b) = 0_{\mathbb{B}}$.

1) We can weaken the assumption "$\mathbb{B}$ is superatomic" to "$\mathbb{B}/I^1_{<\theta}[\mathbb{B}]$ is superatomic", where:

$(*)_1$ for a Boolean Algebra $\mathbb{B}$ and infinite cardinal θ we define $I^1_{<\theta}[\mathbb{B}] = \{x \in \mathbb{B} : \mathbb{B} \restriction x$ has (algebraic) density $< \theta\}$ (see a little in [Sh 397, §1]). For $\mathbb{B}$ superatomic this is the I in the proof of 1.1 on such Boolean Algebras.

[We can choose a maximal set Z of pairwise disjoint elements of $\{x \in \mathbb{B} : x \neq 0_{\mathbb{B}}$ and $\pi(\mathbb{B} \restriction x) < \theta\}$, now without loss of generality $\mathbb{B}$ is a Boolean subalgebra of $\mathscr{P}(\mu)$ such that $x \in Z \Rightarrow x \in [\mu]^{<\theta}$, and continue as in the proof of 1.1.]
2) What if we just assume "$\mathbb{B}/I_{<\theta}[\mathbb{B}]$ is atomic"? One point in the proof may fail: the number of ultrafilters of $\mathbb{B}^*$ is not necessarily $\leq |\mathbb{B}^*| \leq \beth_4(<\theta)$ but is $\leq 2^{|\mathbb{B}^*|} \leq 2^{2^{|Y|}} \leq \beth_5(<\theta)$, so we should replace $\beth_4(<\theta)$ by $\beth_5(<\theta)$ in the conclusion in parts (1),(2).
3) We may in parts (1),(2) replace "$\pi(\mathbb{B} \restriction x)$, algebraic density, is $< \theta$" by "$d(\mathbb{B} \restriction x)$, i.e. $\mathbb{B} \restriction x$ has topological density $< \theta$" (recalling that any Boolean Algebra $\mathbb{B}'$ can be embedded into a Boolean subalgebra of $\mathscr{P}(d(\mathbb{B}'))$; but the bound is seemingly bigger.

So we use $I^2_{<\theta}[\mathfrak{B}] = \{x \in B : d(\mathbb{B} \restriction x) < \theta\}$. Note $I^1_{<\theta}[\mathbb{B}] \subseteq I^2_{<\theta}[\mathbb{B}]$.
4) In both parts (1),(3) and part (2) we have to make easy changes to adapt the proof of 1.1. Let $k = 1, \mu_1 = 2^{<\theta}$ for part (1),(2) and $\mu_1 = ?, \beth_3(<\theta), k = 2$ for part (3). We try to indicate some changes and we redefine I as $I^k_{<\theta}[\mathbb{B}]$

$\boxtimes'_1$ without loss of generality $\mathbb{B} \subseteq \mathscr{P}(\mu)$ and $a \in I = I^k_{<\theta}[\mathbb{B}] \Leftrightarrow |a| < \theta$.

[Why? Let Z be a maximal set of pairwise $\mathbb{B}$-disjoint members of $I^k_{<\theta}[\mathbb{B}] \backslash \{0_{\mathbb{B}}\}$. For each $z \in I^k_{<\theta}[\mathbb{B}]$ let $\mathscr{D}_z$ be a dense subset of ultrafilter $(z, \mathbb{B}) = \{D : D$ an ultrafilter of $\mathbb{B}$ be such that $z \in D\}$ of cardinality $< \theta$. Let $\mu = \cup\{\mathscr{D}_z : z \in Z\}$ and let $\bar{D} = \langle D_\alpha : \alpha < \mu\rangle$ list this set. There is a natural mapping $h = h_{\bar{D}}$ from $\mathbb{B}$ to $\mathscr{P}(\mu) : h(a) = \{\alpha < \mu : a \in D_\alpha\}$.]

Easily

$(*)_1$ h embeds $\mathbb{B}$ into $\mathscr{P}(\mu)$.

[Why? Trivially h is a homomorphism. If $c \in \mathbb{B}\backslash\{0\}$ then for some $a \in Z$ we have $a \cap c > 0_{\mathbb{B}}$ hence for some $\alpha < D_\alpha \in \mathscr{D}_a$ we have $a \in D$, let $D = D_\alpha, \alpha < \mu$ so $\alpha \in a$. So the kernel of α is $\{0_{\mathbb{B}}\}$, so we are done.]

$(*)_2$ h maps $I^k_{<\theta}[\mathbb{B}]$ into $[\mu]^{<\theta}$.

[Why? Let $b \in I^k_{<\theta}[\mathbb{B}]$, let $Z_b = \{a \in Z : b \cap a > 0\}$, it is a subset of Z, now for each $D \in \mathscr{D}_b$ we have $Z_b \cap D| \leq 1$ in fact $|Z \cap D| \leq 1$. So if $|Z_b| \geq \theta$ then $|Z_b| > |\mathscr{D}_b|$ so for some $c \in Z_b$ we have $(\forall D \in \mathscr{D}_b)(c \notin D)$, but this contradicts the choice of Z_b. So $|Z_b| < \theta$, so $\cup\{D : b \in D$ and $D \in \mathscr{D}_c$ for some $c \in Z_b\}$ has cardinality $\sum\limits_{c \in Z_b} |\mathscr{D}_c| < \theta$ and is a subset of $h(b)$.]

$(*)_3$ h maps $I^k_{<\theta}[\mathbb{B}]$ onto $[\mu]^{<\theta} \cap$ Rang(B) (?).

[Why? Doubtful.]

So without loss of generality

$\oplus$ h is the identity.

So the rest is easier.

Now

$\oplus$ if we assume $\mathbb{B}/I^k_{<\theta}[\mathbb{B}]$ is superatomic.
Otherwise we have just

$\oplus$ $\mathbb{B} \subseteq \mathscr{P}(\mu), I$ an ideal of $\mathbb{B} \subseteq [\mu]^{<\theta}$ and $\mathbb{B}/I$ is atomic.

So the assumption toward contradiction is

$\oplus$ $|\mathbb{B}| > \beth_5(< \theta)$ and $\neg(a), \neg(b)$ where
- (a) there is an automorphism f of $\mathbb{B}$ such that for some $c \in \mathbb{B} \backslash I, f(c0 \cap c = 0_{\mathbb{B}}$
- (b) there is a permutation π of μ inducing an automorphism of $\mathbb{B}$ such that for some $X \subseteq \mu$ of cardinality $\leq 2^\theta$, the union of $I \cap \mathscr{P}(x)$ such that $\pi(X) \cap X = \emptyset$.

We add

- $\otimes'_0$ if $b \in I$ then $|\mathbb{B} \restriction b| \leq 2$ for some $\sigma < \theta$
- $\otimes''_0$ if $x \in [\mu] \leq 2^{<\theta}$ then $|X| \leq \theta_k$ (so for $k = 2$ let θ_k be the bound
- $\otimes'_1$ we say that X is $\mathbb{B}$-autonomous when X is a sub-Boolean ring of I and $(\forall a \in I)(\exists b \in X)[b \leq_{\mathbb{B}} a \wedge (\forall c \in X)(a \cap cleb)]$
- $\otimes'_3$ if $X_1, X_2 \subseteq I$ are $\mathbb{B}$-autonomous, $x \in Y$ and $X_1 \cup Y_2 \subseteq \mathbb{B} \restriction x$ and X_1, X_2 are isomorphic over $X_1 \cap X_2$ then there is an automorphism of $\mathbb{B}$ over $X_1 \cap X_2$ mapping X_1 onto X_2
- $\otimes'_4$ if $X \subseteq I, |X| \leq \mu_1$ <u>then</u> there is a $\mathbb{B}$-autonomous $X' \subseteq I$ of cardinality $\leq \mu_1$ such that $X \subseteq X'$.

[Why? If $k = 1$ we can find X' of cardinality $\le 2^{<\theta}$, if there is $b' \in I$ above ever member of $\mathscr{U}$, <u>then</u> there is such $b' \in X'$; now check as there. FILL.]

Theorem. *The pair $\mathbb{B}, I)$ satisfies $\odot$ if the Boolean Algebra $\mathbb{B}$ and ideal I satisfies: if $\boxtimes$ below holds when:*

$\odot$ (a) *$\mathbb{B}$ has cardinality $\le \beth_5(<\theta), |I| \le \beth_4(<\theta)$*

(a)* *if in ? is strengthened to $\mathbb{B} \restriction b$ has algebraic density $< \theta$ then $|\mathbb{B}| \le \beth_4(<\theta), |I| \le ?$*

(b) *add on s a $(\lhd\theta)$, ... see end of §3*

$\odot$ (a) *$\mathbb{B}$ is a Boolean algebra*

(b) *I is an ideal of $\mathbb{B}$*

(c) *if $b \in I \backslash \{0_{\mathbb{B}}\}$ <u>then</u> $d(\mathbb{B} \restriction b)$, the topological density is $< \theta$*

(d) *$\mathbb{B}/I$ is an atomic Boolean algebra*

(e) *for no $b \in \mathbb{B} \backslash I$ does has an automorphism π such that $\pi(b) \cap b = 0_{\mathbb{B}}$*

(f) *for no ideal $J \subseteq I$ of cardinality $2^{<\theta}$ with $2^{<\theta}$ pairwise disjoint non-zero members does $\mathbb{B}$ has an automorphism π such that $b, c \in J \Rightarrow b \cap \pi(c) = 0_{\mathbb{B}}$.*

1.3 Discussion. 1) We can adapt 2.1 from §2 below to the case of 1.2(2), i.e. show that $\beth_5(<\theta)$ cannot be improved in general. E.g. let $\langle d_\zeta : \zeta < \lambda = 2^\mu \rangle$ be an independent family of subsets of μ (so any finite Boolean combination of them is infinite) and let $\mathbb{B}^*$ be the Boolean subalgebra of $\mathscr{P}(\mu)$ generated by $\{d_\alpha : \alpha < \lambda = 2^\mu\} \cup \{\{i\} : i < \mu\}$. We let $\lambda' = 2^\lambda$, let $\{c^*_\gamma : \gamma < \lambda'\}$ be an independent family of subsets of λ and let $X^* = \bigcup\limits_{\alpha<\mu} X_\alpha \cup \{x^*_\gamma : \gamma < \lambda'\}$. We ignore $\mathscr{A}'$ (and omit clause (k) of the assumption) and among the generators of $\mathbb{B}$, clause (i), (ii) remains and

$$(iii)' \ c_\zeta = \{x \in X \text{: for some } \alpha \in d_\zeta \text{ we have } x \in X_\alpha\} \cup \{x^*_\gamma : \zeta \in c^*_\gamma, \gamma \in [\mu, \lambda')\}.$$

2) We may consider replacing automorphism by monomorphisms. The problem is only in the proof of 2.1, "f maps J_1 into J_1" does not seem to follow.

§2 Constructing counterexamples

We would like to show that the bound $\beth_4(<\theta)$ from 1.1 is essentially best possible. The construction (in 2.1) is closely related to the proof in §1, but we need various assumptions. So in particular κ here corresponds to $\sup\{|\mathbb{B} \restriction a| : a \in Y\} \le \beth_2(<\theta)$ there, μ here corresponds to $|Y| \le \beth_3(<\theta)$ there, λ' here corresponds to $|\text{atom}(\mathbb{B})| \le \beth_4(<\theta)$ there. We shall deal with them later.

2.1 Lemma. *Assume*

(a) $\theta \le \kappa \le \mu \le \lambda' \le \lambda$ *and* $\theta = \text{cf}(\theta) \ge \aleph_1$

(b) *there is an* $\mathscr{A} \subseteq [\mu]^{\aleph_0}$ *almost disjoint (i.e.* $A \ne B \in \mathscr{A} \Rightarrow |A \cap B| < \aleph_0$*) such that* $(\forall A \in [\mu]^\theta)(\exists B \in \mathscr{A})(B \subseteq^* A)$ *and* $|\mathscr{A}| = \mu$

(c) $\bar{\mathbb{B}} = \langle \mathbb{B}_\alpha : \alpha < \mu \rangle$

(d) $\mathbb{B}_\alpha$ *is a superatomic Boolean Algebra with* $\le \kappa$ *atoms such that any automorphism of* $\mathbb{B}_\alpha$ *moves* $< \theta$ *atoms and* $|\mathbb{B}_\alpha| \le \lambda$*; moreover if* $c_1, c_2 \in I_\alpha$ *(see below) and* f *is an isomorphism from* $\mathbb{B}_\alpha \restriction (1 - c_1)$ *onto* $\mathbb{B}_\alpha \restriction (1 - c_2)$ *then* $\theta > |\{x \in \text{atom}(\mathbb{B}_\alpha) : x \le_{\mathbb{B}_\alpha} c_1$ *or* $f(x) \ne x\}|$

(e) $I_\alpha = \{b \in \mathbb{B}_\alpha : |\{x \in \text{atom}(\mathbb{B}_\alpha) : x \le b\}| < \theta\}$ *is a maximal ideal of* $\mathbb{B}_\alpha$

(f) *there is an infinite set* $\{a^\alpha_n : n < \omega\}$ *of pairwise distinct atoms of* $\mathbb{B}_\alpha$ *such that for every* $a \in I_\alpha$ *the set* $\{n < \omega : a^\alpha_n \le a\}$ *is finite*

(g) *if* $\alpha \ne \beta$ *<u>then</u> for no* $a_\alpha \in I_\alpha, a_\beta \in I_\beta$ *do we have* $\mathbb{B}_\alpha \restriction (1_{\mathbb{B}_\alpha} - a_\alpha) \cong \mathbb{B}_\beta \restriction (1_{\mathbb{B}_\beta} - a_\beta)$

(h) $\mathbb{B}^*$ *is a superatomic Boolean Algebra*

(i) $\mathbb{B}^*$ *has* μ *atoms*

(j) $\mathbb{B}^*$ *has* λ *elements*[2]

(k) *if* $\lambda' > \mu$ *<u>then</u> we have* $\chi, \mathscr{A}', I^*$ *satisfying:*

  (α) $\mathscr{A}' \subseteq [\lambda']^{\aleph_0}$ *is a MAD family of cardinality* χ

  (β) I^* *is an ideal of* $\mathbb{B}^*$ *containing* $\text{id}_1(\mathbb{B}^*)$*, included in* $id_{\text{rk}(\mathbb{B})}(\mathbb{B}^*)$ *suhc that the Boolean algebra* $\mathbb{B}^*/I^*$ *is isomorphic to* $\{a \subseteq \chi : a$ *is finite or co-finite*$\}$*; so* $\chi \le |\mathbb{B}^*| = \lambda$ *follows*

(gamma) *if* π *is a partial*[3] *permutation of* λ'*, Dom*$(\pi) = \lambda' \backslash Z_1$*, Rang*$(\pi) = \lambda' \backslash Z_2$ *where* $Z = Z_1 \cup Z_2 \in [\lambda']^{<\theta}$ *satisfying* $A \in \mathscr{A}' \Rightarrow |(A \Delta \pi''(A)) \backslash Z| < \aleph_0$*, <u>then</u>* π *has support* $\{\alpha < \lambda' : \pi(\alpha) \ne \alpha\}$ *of cardinality* $< \theta$*.*

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[2] if there is a tree $\mathscr{T}$ with $\le \mu$ nodes and $\ge \lambda$ branches (= maximal linearly ordered subsets) then such $\mathbb{B}^*$ exists

[3] i.e. π is one to one such that $\text{Dom}(\pi) \subseteq \lambda$ and $\text{Rang}(\pi) \subseteq \lambda$

<u>Then</u> *we can find* $\mathbf{B}$ *such that:*

- (α) $\mathbf{B}$ *is a superatomic Boolean Algebra*
- (β) $\mathbf{B}$ *has* λ' *atoms and* λ *elements*
- (γ) *every automorphism* g *of* $\mathbf{B}$ *moves* $< \theta$ *atoms; i.e.* $|\{x \in \text{ atom}(\mathbf{B}) : g(x) \neq x\}| < \theta$.

Proof. Without loss of generality $\mathbb{B}^*$ is a Boolean Algebra of subsets of $\{w_1, \alpha : \alpha < \mu\}$ with $\{\omega_1 \alpha\} : \alpha < \mu\}$ being the atoms of $\mathbb{B}^*$. If $\lambda' = \mu$ let $\mathscr{A}' = \emptyset, \chi = 0, I^* = \mathbb{B}^*$.

Without loss of generality $\mathbb{B}_\alpha$ is a subalgebra of $\mathscr{P}(X_\alpha)$ and the set of atoms of $\mathbb{B}_\alpha$ is $\{\{x\} : x \in X_\alpha\}$. Without loss of generality $\alpha \neq \beta \Rightarrow X_\alpha \cap X_\beta = \emptyset$ and let $X = \cup\{X_\alpha : \alpha < \mu\}$.

If $\lambda' = \mu$ let $Y^* = \emptyset$ and if $\lambda' > \mu$, let $Y^* \subseteq \mathbb{B}^*$ be such that $|Y^*| = \chi$ and $\{y/I^* : y \in Y^*\}$ is the set of atoms of $\mathbb{B}^*/I^*$ with no repetitions; without loss of generality

$\boxtimes_0$ for every $y \in Y^*$ for some $\alpha, y/\text{id}_\alpha(\mathbb{B}^*)$ is an atom of $\mathbb{B}^*/\text{id}_\alpha(\mathbb{B}^*)$ and $(\forall z)[z \leq_{\mathbb{B}^*} y \rightarrow (z \in \text{ id}_\alpha(\mathbb{B}^*) \equiv z \in I^*)]$.

[Why is this possible? For each $y \in \mathbb{B}^* \backslash I^*$ let $\alpha = \alpha(y) =:$ Min$\{\text{rk}_{B^*}(y - x) : x \in I^*\}$ and choose x^0_y exemplifying it, so $(y - x^0_y)/\text{id}_\alpha(\mathbb{B}^*)$ is the union of finitely many atoms of $\mathbb{B}^*/\text{id}_\alpha(\mathbb{B}^*)$, say $y_1/\text{id}_\alpha(\mathbb{B}^*), \ldots, y_n/\text{id}_\alpha(\mathbb{B}^*)$ where $n \geq 1$ and without loss of generality $y_\ell \leq_{\mathbb{B}^*} y$. So $\{y_1, \ldots, y_n\}$ cannot be all in I^* and there cannot be two $y_\ell \in \mathbb{B}^* \backslash I^*$, so there is a unique $\ell = \ell(*)$ such that $y_\ell \notin I^*$, let $x^*_y = (1 - y_{\ell(*)}) \cup x^0_y$. Now $\{y - x^*_y : y \in Y^*\}$ is as required.]

Let Y^+ be such that $Y^+ \subseteq \mathbb{B}^*, \langle y/ \text{ id}_{\text{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*) : y \in Y^+\rangle$ list with no repetitions $\{y/ \text{ id}_{\text{rk}(y,\mathbb{B}^*)} : y/\text{id}_{\text{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*)$ an atom of $\mathbb{B}^*/\text{id}_{\text{rk}(y,\mathbb{B}^*)}(\mathbb{B}^*)\}$ and $Y = \{y \in Y^+ : \text{rk}(B^*) > \text{ rk}(y, \mathbb{B}^*) > 0\}$.

Without loss of generality $Y^{\max} = \{y \in Y^+ : \text{rk}(y, \mathbb{B}) = \text{ rk}(\mathbb{B})$ is a partition of $1_\mathbb{B}$. For $y \in Y^+$ let D_y be the ultrafilter on $\mathbb{B}^*$ generated by $\{y\} \cup \{1 - x : x \in \mathbb{B}^*, \text{rk}(x, \mathbb{B}^*) < \text{ rk}(y, \mathbb{B}^*)\}$ for each $y \in Y$. Without loss of generality $Y^* \subseteq Y$ and for some $Y^{\max} \in Y^{\max}$ we have $y \in Y^+ \Rightarrow y < y^{\max}$. Also as $\mathbb{B}^*/I^*$ is isomorphic to the Boolean Algebra of finite, co-finite subsets of $\chi, y \in Y \Rightarrow \text{ rk}(y, \mathbb{B}) < \text{ rk}(\mathbb{B})$ and clause $(k)(\beta)$ of the assumption of 2.1 clearly $y \in Y \backslash Y^* \Rightarrow \{y' \in Y^* : y' - y \in \text{ id}_{\text{rk}(y',\mathbb{B}^*)}(\mathbb{B}^*)\}$ is finite so without loss of generality is empty for $y \in Y \backslash Y^*$ (singleton for $y \in Y^*$, of course), note that if $\lambda' > \mu$ then Y^* is of cardinality $|\mathscr{A}'|$ and without loss of generality $|Y \backslash Y^*| = \lambda$.

Let g be a one-to-one function from μ onto X and for $A \in \mathscr{A}$ (from clause (b)) let $\{\gamma_{A,k} : k < \omega\}$ list A without repetition. Let $g^* : \mu \rightarrow \mu$ be $g^*(\gamma) =$ the unique $\alpha < \mu$ such that $g(\gamma) \in X_\alpha$.

For $\beta < \mu$ let $\mathbf{i}(\beta)$ be the unique $i < \omega_1$ such that $(\exists\alpha)(\omega_1\alpha \le \beta = \omega_1\alpha + i)$. For $A \in \mathscr{A}$ we define $\mathbf{i}(A) =$ Min$\{i < \omega_1 : \mathbf{i}(g^*(\gamma)) < i$ for every $\gamma \in u_A\}$

$\boxtimes_1$ we have $\langle \alpha[A] : A \in \mathscr{A}, u_A$ well defined$\rangle$ such that:

- (i) $\alpha[A] < \mu$
- (ii) $\alpha(A) \in \{w - 1\beta + \mathbf{i} : \beta < \mu$ and $\mathbf{i}(A) \le i < \omega_1\}$
- (iii)) $\alpha[A_1] = \alpha[A_2] \Rightarrow A_1 = A_2$
- (iv)) $\{\alpha[A_1] : A \in \mathscr{A}$, Rang$(g^* \restriction A)$ infinite$\}$ is equal to $\{\omega_1\alpha + i : \alpha < \mu, i < i < \omega_1\}$.

Now by induction on $i < \omega_1$ we choose y_α when $\mathbf{i}(\alpha) < i$ and u_A, y_A when $\mathbf{i}(A) \le i$ such that:

$\boxtimes$ (a) $y_\alpha \in Y$ or z_α is an atom of $\mathbb{B}^*$

- (b) if Rang$(g^* \restriction A)$ is finite then $u_A = \emptyset$
- (c) if Rang$(g^* \restriction A)$ is infinite then
  - (α) u_A is an infinite subset of A
  - (β) $g^* \restriction u_A$ is one to one
  - (γ) $z_A \in Y$
  - (δ) $\langle D_{g_\beta} : \beta \in u_A\rangle$ converge to $D_{g_{\alpha[A]}}$, which means that for every $x \in \mathbb{B}$ for all but finitely many $\beta \in u_A$ we have $x \in D_{z_{\alpha[A]}} \Leftrightarrow x \in D_{y_\beta}$.

This is easy by Observation 0.2.

For $\alpha < \mu$ let a_α be $\{g(\gamma) : \gamma \in u_A\}$ if $A \in \mathscr{A}, \alpha[A] = \alpha$ and $\emptyset$ if there is no such A so if $u_A = \emptyset$, i.e. Rang$(g \restriction A)$ is finite then $a_\alpha = \emptyset$. Toward defining our Boolean Algebra let $\{x^*_\gamma : \gamma \in [\mu, \lambda')\}$ be pairwise distinct elements not in X. Let $\mathscr{A}'' = \{\{\mu + i : i \in A\} : A \in \mathscr{A}'\}$ so it is a maximal almost disjoint family of countable subsets of $[\mu, \lambda')$, as in clause (k) of the assumption so if $\mu = \lambda'$ then $\mathscr{A}'' = \emptyset = \mathscr{A}', \lambda' - \mu = 0, (\lambda' - \mu)^{\aleph_0} = 0 = |Y^*|$ and if $\lambda' > \mu$ then $|\mathscr{A}''| = |\mathscr{A}'| = \chi = |Y^*|$ so let $\langle d_y : y \in Y^*\rangle$ list $\mathscr{A}''$ with no repetitions.

Now we define our Boolean Algebra $\mathbf{B}$. It is the Boolean Algebra of subsets of $X^* = X \cup \{x^*_\gamma : \gamma \in [\mu, \lambda')\}$ generated by the following (recall that a_α may be empty)(recall that $X = \cup\{X_\alpha : \alpha < \mu\}$):

$\boxtimes_2(i)$ the sets $\{a \in \mathbb{B}_\alpha : |a| < \theta\} \cup \{a \cup a_\alpha : a \in \mathbb{B}_\alpha, |a| \ge \theta\}$ when $\alpha < \mu$

(ii) $\{x^*_\gamma\}$ for $\gamma \in [\mu, \lambda')$

(*iii*) the sets c_y (for $y \in Y$) where

$$c_y =: \big\{x \in X : \text{for some } \alpha < \mu \text{ we have } x \in X_\alpha \ \& \ y \in D_{z_\alpha}\} \cup \{x^*_\gamma : \gamma \in [\mu, \lambda') \text{ and } y \in Y^* \text{ and } \gamma \in d_y\big\}.$$

Clearly

$\otimes_0$ $\mathbf{B}$ is a subalgebra of $\mathscr{P}(X^*)$, including all the singletons hence is atomic; has λ' atoms and λ elements.

[Why? The least trivial is $x \in X = \bigcup_{\alpha<\mu} X_\alpha \Rightarrow \{x\} \in \mathbf{B}$, but if $x \in X_\alpha$, then $\{x\}$ is an atom of $\mathbb{B}_\alpha$ hence belongs to $\mathbf{B}$.]

Note that

$\otimes_1(i)$ $X_\alpha \cap a_\beta$ has at most one element for $\alpha, \beta < \mu$

(*ii*) $X_\alpha \cap X_\beta$ is (except when $\alpha = \beta$),

(*iii*) $a_\alpha \cap a_\beta$ is finite (when $\alpha \neq \beta$) by $\mathscr{A}$ being MAD

(*iv*) $(X_\alpha \cup a_\alpha) \cap (X_\beta \cup a_\beta)$ is finite for $\alpha \neq \beta < \mu$ which holds by clauses (i) + (ii) + (iii)

(*v*) if $\alpha < \mu$ and $y \in Y$, then the set $(X_\alpha \cup a_\alpha)\backslash c_y$ is finite or the set $(X_\alpha \cup a_\alpha) \cap c_y$ is finite.

[Why? Recalling $\mathbb{B}^*$ is a subalgebra of $\mathscr{P}(\mu)$ and the definition of c_y clearly $c_y \cap X_\alpha \in \{X_\alpha, \emptyset\}$. Also $X_\alpha \subseteq c_y$; so if $a_\alpha = \emptyset$ we are done. So assume $\alpha = \alpha[A]$ so u_A is infinite and it suffices to prove that for all but finitely many $\beta \in a_\alpha$ we have $\beta \in c_y \leftrightarrow X_\alpha \subseteq c_y$. But $a_\alpha = \{g(\gamma) : \gamma \in u_A\}$ so this means: for all but finitely many $\gamma \in u_A$ we have $g(\gamma) \in c_y \leftrightarrow X_\alpha \subseteq C_y$. But the definition of c_y and g^* this means: for all but finitely many $\gamma \in u_A$ we have g^* this means: for all but finitely many $\gamma \in u_A$ we have $g^*(\gamma) \in y \leftrightarrow y \in D_{z_{g^*(\gamma)}} \leftrightarrow y \in D_{z_\alpha}$ but $z_\alpha = z_A$ and $\langle D_{z_\gamma} : \gamma \in u_A\rangle$ converge to $D_{z_{\alpha[A]}}$ so we are done.]

$\otimes_2$ for $\alpha < \mu$, we have $a \in \mathbb{B}_\alpha \ \& \ |a| < \theta \Rightarrow a \in \mathbf{B} \ \& \ \mathbf{B} \restriction a = \mathbb{B}_\alpha \restriction a$ but $a \in \mathbb{B}_\alpha \Rightarrow \mathbb{B}_\alpha \restriction a$ is superatomic so $\{a \in \mathbb{B}_\alpha : |a| < \theta\} \subseteq \mathrm{id}_\infty(\mathbf{B})$.
[Why? For the first implication we should check that every one of the generators of $\mathbf{B}$ listed in $\boxtimes_2(i),(ii),(iii)$ above satisfies: its intersection with a belong to $\mathbb{B}_\alpha \restriction a$. For $\boxtimes_2(ii)$ this is trivial, for $\boxtimes_2(i)$ use $\otimes_1(i)-(iv)$ and for $\boxtimes_2(iii)$ use $\otimes_1(v)$. The rest follows.]

$\otimes_3$ for $\alpha < \mu$ the set $I^+_\alpha =: \{a \in \mathbf{B} : a \subseteq X_\alpha \cup a_\alpha \text{ and } |a| < \theta\}$ satisfies

(*i*) it is equal to $\{a \cup b : a \in \mathbb{B}_\alpha \ \& \ |a| < \theta \text{ and } b \subseteq a_\alpha \text{ is finite}\}$

(*ii*) it is a maximal ideal of $\mathbf{B} \restriction (X_\alpha \cup a_\alpha)$

[Why? Easy. The main point concerns $(X_\alpha \cup a_\alpha) \cap (X_\beta \cup a_\beta)$ satisfying clause (i) when it has cardinality $< \theta$ this holds by $\otimes_1(iv)$ and $(X_\alpha \cup a_\alpha) \cap c_y$ has cardinality $< \theta$ or $(X_\alpha \cup a_\alpha) \backslash c_y$ has cardinality $< \theta$ which holds by $\otimes_1(v)$.]

$\otimes_4$ $\alpha < \mu \Rightarrow X_\alpha \cup a_\alpha \in \mathrm{id}_\infty(\mathbf{B})$
[Why? First $X_\alpha \cup a_\alpha \in \mathbf{B}$ by clause (i) of $\boxtimes_2$ above, second if $X_\alpha \cup a_\alpha \notin \mathrm{id}_\infty(\mathbf{B})$ then by $\otimes_2$ for some ordinal ζ we have $[a \in \mathbb{B}_\alpha \ \& \ |a| < \theta \Rightarrow a \in \mathrm{id}_\zeta(\mathbf{B})]$, hence by $\otimes_3$ above $(X_\alpha \cup a_\alpha)$ is an atom of $\mathbf{B}/\mathrm{id}_\zeta(\mathbf{B})$ for ζ large enough, hence $X_\alpha \cup a_\alpha$ belong to $\mathrm{id}_{\zeta+1}(\mathbf{B})$, contradiction.]

$\otimes_5$ for $\alpha < \mu, \mathbf{B} \restriction (X_\alpha \cup a_\alpha) \cong \mathbb{B}_\alpha$ hence if $\alpha < \beta < \mu$ then for no c_α such that $c_\alpha \in \mathbf{B}_\alpha, c_\alpha \leq_{\mathbf{B}} X_\alpha \cup a_\alpha, |c_\alpha| < \theta$ and $c_\beta \in \mathbf{B}, c_\beta \leq_{\mathbf{B}} X_\beta \cup a_\beta, |c_\beta| < \theta$ do we have $\mathbf{B} \restriction (X_\alpha \cup a_\alpha \backslash c_\alpha) \cong \mathbf{B} \restriction (X_\beta \cup a_\beta \backslash c_\beta)$.
[Why? By clauses (f) + (e) of the assumption, the first phrase holds. The "hence" follows by clause (g) of the assumption.]

Let J_1 be the ideal of $\mathbf{B}$ generated by $\bigcup_{\alpha<\mu} I^+_\alpha \cup \{x^*_\gamma : \gamma \in [\mu, \lambda')\}$. (We will see that J_1 is I of the analysis in 1.1, positive part, i.e. $J_1 = [\lambda']^{<\theta} \cap B$).

Let J_2 be the ideal of $\mathbf{B}$ generated by $J_1 \cup \{X_\alpha \cup a_\alpha : \alpha < \mu\}$. Let J^+_ℓ be the ideal of the Boolean Algebra $\mathscr{P}(\mu)$ is generated by J_ℓ

$\otimes_6(i)$ $J_1 \subseteq \mathrm{id}_\infty(\mathbf{B})$
(ii) $J_1 \subseteq [X^*]^{<\theta}$ is a (proper) ideal
(iii) $J_1 \subseteq J_2 \subseteq \mathrm{id}_\infty(\mathbb{B})$ and J_2/J_1 is the ideal of $\mathbf{B}/J_1$ generated by its atoms, i.e. $\mathrm{id}_1(\mathbf{B}/J_1)$ where the atoms are $(X_\alpha \cup a_\alpha)/J_1$.

[Why? For clause (i), note that $\mathrm{id}_\infty(\mathbf{B})$ is an ideal of $\mathbf{B}$ and the generators of J_1 are in it by $\otimes_4$ (for $X_\alpha \cup a_\alpha$ that is for the members of I^+_α) and by the $\{x^*_\gamma\}$ being atomic (for $\gamma \in [\mu, \lambda')$). Clause (ii) is obvious. For clause (iii) follows by $J_1 \subseteq J_2 \subseteq \mathbb{B}$ holds by the choice of J. By $\otimes_3$ each $(X_\alpha \cup a_\alpha)/J_1$ is an atom of $\mathbf{B}/\mathbf{J}_1$. But are there more atoms? if not then by the definition of a $\mathbf{B}$ as generated by ..., we can find $n_1 \leq n_2 < \omega$ and $y_0, \ldots, y_{n_2-1} \in Y \cup Y^{\max}$ such that $c = \bigcap_{\ell=0}^{n_1-1} c_{y_\ell} - \bigcup_{\ell=n_1}^{n_k-1} c_{y_\ell}$ satisfies c_y/J_1 is an atom of $\mathbf{B}/J_1$. Let $y = \bigcap_{\ell=0}^{m-1} c_\ell - \bigcup_{\ell=n_1}^{n_2-1} c_{y_\ell} \in \mathbb{B}$.

Case 1: $y \in \mathrm{id}_1(\mathbb{B})$.

Say $y = \{\alpha_i : \ell < n\} \in [\mu]^{<\aleph_0}$ such that $\mathbf{i}(\alpha_\ell) = 0$ for $\ell < n$. Let $\beta \in \mu \backslash \{\alpha_\ell : \ell < \mu\}$, what is $c \cap X_\beta$? We can prove that it is empty by induction on $\mathbf{i}(\beta)$. Similarly $c \cap S = \emptyset$, so necessarily $c \in J_2$ as required.

Case 2: $y \in \mathrm{id}_1(\mathbb{B})$.

Then we can find distinct $\beta_n < \mu$ with $\mathbf{i}(\beta_n) = 0$ for $n < \omega$ such that $n < \omega \Rightarrow \beta_n \in y$. Then $\cup\{X_{\beta_n} : n < \omega\} \subseteq C$ hence $c \notin J_2$. So we are done.]

We shall prove that

$\otimes_7$ $\mathbf{B}/J_2$ is isomorphic to a homomorphic image of $\mathbb{B}^*$.

Toward proving $\otimes_7$ let $S = \{x^*_\gamma : \gamma \in [\mu, \lambda')\}$ and define a function h as follows: its domain is $\{c_y : y \in Y \cup Y^{\max}\}$ and $h(c_y) = y$ for $y \in Y \cup Y^{\max}$, so h is into $\mathbb{B}^*$.

Now

$(*)$ if $n_1 \le n < \omega, m_1 \le m < \omega, y_0, \dots, y_{n-1} \in Y \cup Y^{\max}$ is with no repetitions, then[4]:

in $\mathbf{B}, \tau_1 =: \bigcap\limits_{\ell < n_1} c_{y_\ell} - \bigcup\limits_{\ell = n_1}^{n-1} c_{y_\ell}$ belongs to J_2 if

in $\mathbb{B}^*, \tau_2 =: \bigcap\limits_{\ell < n_1} y_\ell - \bigcup\limits_{\ell = n_1}^{n-1} y \in \mathrm{id}_1(\mathbb{B})$.

[Why? First, assume that the second statement holds (so $\tau_2 \subseteq \{\alpha_\ell : \ell < m\} \in [\mu]^{<\aleph_0}$ then by the choice of the c_y's trivially $\tau'_1 =: \bigcap\limits_{\ell < n_1} (c_{y_\ell} \backslash S) - \bigcup\limits_{\ell = n_1}^{n-1} (c_{y_\ell} \backslash S) = \bigcup\{X_\beta : y_0, \dots, y_{n_1 - 1} \in D_{z_\beta}$ but $y_{n_1}, \dots, y_{n_2 - 1} \notin D_{z_\beta}\} = \cup\{X_\beta : \emptyset = \tau_2 \in D_{z_\beta}\} \subseteq \cup\{X_{\alpha_\ell} \cup a_{\alpha_\ell} : \ell < m\}$ but $(\tau'_1 \Delta \tau_1) \subseteq S \cup \bigcup\limits_{\ell < m} a_{\alpha_\ell}$, so $\tau_1 \subseteq S \bmod J_2^+$.

Now assume $\tau_1 \cap S$ is infinite, hence $\lambda' > \mu$. So $\{d_z : z \in Y^*\}$ is a MAD family of subsets of $\lambda' \backslash \mu$, in fact is $\mathscr{A}''$. Hence $\{\{x^*_\gamma : \gamma \in d_z\} : z \in Y^*\}$ is a MAD family of subsets of $S = \{x^*_\gamma : \gamma \in [\mu, \lambda')\}$. So necessarily for some $z \in Y^*$ the set $\tau_1 \cap S \cap \{x^*_\gamma : \gamma \in d_z\}$ is infinite. As $\tau_1 \cap S \cap \{x^*_\gamma : \gamma \in d_z\} \subseteq c_{y_\ell}$ for $\ell < n_1$, and $\mathrm{id}_1(\mathbb{B}^*/J_1)$ is a maximal ideal and the choice of Y, Y^* necessarily $y_\ell = z$, hence $y_0 = z, n_1 = 1$. Similarly $\ell \in [n_1, n_2) \Rightarrow y_\ell \ne z$ hence $\ell \in [n_1, n) \Rightarrow y_\ell \cap y_0 = y_\ell \cap z \in \mathrm{id}_{\mathrm{rk}(z, \mathbb{B}^*)}(\mathbb{B}^*) \Rightarrow |\{x^*_\gamma : \underset{\sim}{\gamma} \in d_z\} \cap c_{y_\ell}| < \aleph_0$. Hence clearly $\ell \in [n_1, n) \Rightarrow y_\ell \notin D_z$ but $y_0 \in D_z$ and $\alpha < \mu \Rightarrow \{\alpha\} \notin D_z$ (as $z \in Y!$) hence $\mathbb{B}^* \notin \mathrm{id}_1(\mathbb{B}^*)$, contradiction to our present assumption, so necessarily $\tau_1 \cap S$ is finite. So $\tau_1 \cap S \in J_1^+$. Together with the previous paragraph, $\tau_1 \in J_2^+$, but $\tau_1 \in \mathbf{B}$ hence $\tau_1 \in J_2$ as required that is really $\tau_2 \in \mathrm{id}_1(\mathbb{B}) \Rightarrow \tau_1 \in J_2$.

So we have proved $(*)_2$.]

[4]really we get "iff" but no need

As $\mathbb{B}^*$ is superatomic and the choice of $Y \cup Y^{\max}$ clearly by $(*)$ the statement $\otimes_7$ follows, in fact h induces an isomorphism $\hat{h}$ from $\mathbf{B}/J_2$ onto $\mathbb{B}^*$. But $\mathbb{B}^*$ is superatomic and $J_2 \subseteq \mathrm{id}_\infty(\mathbb{B})$ by $\otimes_6(i)$ hence

$\otimes_8$ $\mathbf{B}$ is superatomic.

Now as $\{\{\alpha\} : \alpha < \mu\}$ are the atoms of $\mathbb{B}^*$ clearly and recall $\{X_\alpha \cup a_\alpha / J_1 : \alpha < \mu\}$ are the atoms of $\mathbf{B}/J_1$ by $\otimes_6(iii)$ and as $J_1 \subseteq [X^*]^{<\theta}$ while $|X_\alpha \cup a_\alpha| \geq \theta$, clearly

$\otimes_9$ $J_1 = \mathbf{B} \cap [X^*]^{<\theta}$.

For the rest of the proof let $f \in \mathrm{AUT}(\mathbf{B})$ and toward contradiction we assume $\sup(f) = \{x \in \mathrm{atom}(\mathbb{B}) : f(x) \neq x\}$ has cardinality $\geq \theta$.
Recall that $J_1 = \{a \in \mathbf{B} : |a| < \theta\}$ and $\{\{x\} : x \in X^*\}$ are the atoms of $\mathbf{B}$ so necessarily f maps J_1 onto itself. Note that $\{(X_\alpha \cup a_\alpha)/J_1 : \alpha < \mu\}$ list the atoms of $\mathbf{B}/J_1$ by $\otimes_6 + \otimes_7$. Assume $f(X_\alpha \cup a_\alpha)/J_1 = (X_\beta \cup a_\beta)/J_1$ and $\alpha \neq \beta$; let $c_1 = (X_\alpha \cup a_\alpha) - f^{-1}(X_\beta \cup a_\beta)$ and $c_2 = (X_\beta \cup a_\beta) - f(X_\alpha \cup a_\alpha)$, so both being the difference of two members of $\mathbf{B}$ are in $\mathbf{B}$ and $c_1 \leq X_\alpha \cup a_\alpha, c_2 \leq X_\beta \cup a_\beta$ and by the present assumption of course $c_1, c_2 \in J_1$ hence $|c_1| < \theta$ and $|c_2| < \theta$. Now $c_1 \leq X_\alpha \cup a_\alpha, |c_1| < \theta$ implies $c_1 \in I^+_\alpha$ so $c_1 \cap X_\alpha \in I_\alpha$ and $c_1 \backslash X_\alpha$ is finite; similarly $c_2 \cap X_\beta \in I_\beta, c_2 \backslash X_\beta$ is finite. Clearly $f \restriction (\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c_1))$ is an isomorphism from $\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c_1)$ onto $\mathbf{B} \restriction (X_\beta \cup a_\beta - c_2)$, contradicting $\otimes_5$ by the "moreover" part of the clause (d) of the assumption of Lemma 2.1. Hence the automorphism which f induced on $\mathbb{B}^*/J_1$ maps each atom to itself hence is the identity. Also for $\alpha < \mu$ we have $(X_\alpha \cup a_\alpha) \Delta f(X_\alpha \cup a_\alpha) \in J_1$, that is, has cardinality $< \theta$. So

$\boxtimes_3$ for each $\alpha < \mu$, letting $c^1_\alpha =: ((X_\alpha \cup a_\alpha) - f^{-1}(X_\alpha \cup a_\alpha)) \in J_1$ and $c^2_\alpha =: (X_\alpha \cup a_\alpha) - f(X_\alpha \cup a_\alpha) \in J$ we have $f \restriction (\mathbb{B}_\alpha \restriction (1 - c^1_\alpha))$ is an isomorphism from $\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c^1_\alpha)$ onto $\mathbf{B} \restriction (X_\alpha \cup a_\alpha - c^2_\alpha)$
hence

$\boxtimes_4$ $Z_\alpha =: \{x : x$ an atom of $\mathbb{B}_\alpha$ and $x \leq_{\mathbb{B}_\alpha} c^1_\alpha \vee f(x) \neq x\}$ has cardinality $< \theta$

by clause (d) of the assumptions on $\mathbb{B}_\alpha$. Let $v =: \{\alpha < \mu :$ for some $x \in X_\alpha$ we have $f(\{x\}) \neq \{x\}\}$. Assume (toward contradiction) for the time being

$\boxtimes_5$ v has cardinality $\geq \mathrm{cf}(\theta)$.

For $\alpha \in v$ choose $x_\alpha \in X_\alpha$ such that $f(\{x_\alpha\}) \neq \{x_\alpha\}$ and possibly shrinking v without loss of generality $\alpha, \beta \in v \Rightarrow \{x_\alpha\} \neq f(\{x_\beta\})$. Let $g' : v \to \mu + 1$ be such that $f(\{x_\alpha\}) \subseteq X_{g'(\alpha)}$ where we stipulate $X_\mu = S$. Applying the above to f^{-1} we could have chosen $(x_i, \alpha_i, \gamma_i)$ by induction on $i < \mathrm{cf}(\theta)$ such that $\alpha_i \in v, f(\{x_i\}) \neq \{x_i\}, x_i \in X_{\alpha_i}, f(\{x_i\}) \subseteq X_{\gamma_i}$ and $\alpha_i, \gamma_i \notin \{\alpha_j, \gamma_j : j < i\} \backslash \{\mu\}$, and let $v = \{\alpha_i : i < \mathrm{cf}(\theta)\}$ without loss of generality either g' is one-to-one into μ or g'

is constantly μ. Now by clause (b) of the assumption without loss of generality for some $A \in \mathscr{A}$ we have $A \subseteq \{x_\alpha : \alpha \in v\}$. So $\alpha[A] < \mu$ is well defined and $\{x \in X_{\alpha[A]} \cup a_{\alpha[A]} : f(\{x\}) \le_{\mathbf{B}} X_{\alpha[A]} \cup a_{\alpha[A]}\}$ does not belong to $I^+_{\alpha[A]}$; so by $\boxtimes_3$ (applied to $\alpha = \alpha[A]$ and the properties of $c^1_{\alpha[A]}, c^2_{\alpha[A]}$) an easy contradiction.
We can conclude that $\neg\boxtimes_5$ hence v has cardinality $<$ cf(θ) hence $|\{x \in X : f(x) \neq x\}| < \theta$. If $\mu = \lambda'$ we are done so assume $\mu < \lambda'$.
Now $S = \{x^*_\gamma : \gamma \in [\mu, \lambda')\} = X^* \backslash X \subseteq X^*$ satisfies:

$\otimes_9(\alpha)$ $(\forall b \in \mathbf{B})(b \cap S \text{ infinite} \wedge \bigwedge_{\alpha \in v} b \cap X_\alpha = \emptyset \Rightarrow 1 \le \text{ rk}(b/J_1, \mathbf{B}/J_1))$ and

(β) if S' satisfies the property of S in clause (α), then $|S' \backslash S| < \theta$
[Why? Clause (α) is proved by inspecting the definition of $\mathbf{B}$. As for clause (β), if $|S' \backslash S| \ge \theta$ as $S' \backslash S \subseteq X$ clearly then there is $A \in \mathscr{A}$ such that $\{g(i) : i \in A\} \subseteq^* S' \backslash S$. First if $\alpha =: \alpha[A]$ is well defined then $X_\alpha \cup a_\alpha \in \mathbf{B}$, rk$((X_\alpha \cup a_\alpha))/J_1, \mathbf{B}/J_1) = 0 < 1$ but $(X_\alpha \cup a_\alpha) \cap S' \supseteq a_\alpha$ is infinite; contradiction. Second if $\alpha[A]$ is not well defined then for some $\alpha < \mu$ we have $\{g(i) : i \in A\} \cap X_\alpha$ is infinite and we get a similar contradiction.]

Hence for $n = 1, -1$ the set $S^n_f =: \{x^*_\gamma : \gamma \in [\mu, \lambda')$ and $f^n(\{x^*_\gamma\}) \subseteq X\}$ has cardinality $< \theta$. Let $S^*_f = S^{-1}_f \cup S^1_f$.

Also for every $y \in Y^*$ letting $\gamma =$ rk$(y, \mathbb{B}^*)$ we have $c_y \Delta f(c_y) \in J_1$, (just recall that the automorphism that f induced on $\mathbf{B}/J_1$ is the identity, and recall that $[d \subseteq S \ \ \& \ \ d \in J_1 \Rightarrow d$ is finite by $\otimes_6]$, hence the symmetric difference of $\{\{x^*_\gamma\} : \gamma \in d_y\} \backslash S^*_f$ and $\{f\{x^*_\gamma\} : \gamma \in d_y\} \backslash S^*_f$ is finite.
As $\mathscr{A}'' = \{d_y : y \in Y^*\}$ is a MAD family of subsets of $\lambda' \backslash \mu$ as in clause $(k)(\alpha)$ of the assumption, the set $\{\gamma \in [\mu, \lambda') : f(\{x^*_\gamma\}) \neq \{x^*_\gamma\}\}$ is of cardinality $< \theta$; so we are done.

Not exactly: we have assumed $\boxtimes_1$.

To eliminate this extra assumption we make some minor changes. First without loss of generality $\mathbb{B}^*$ is a Boolean Algebra of subsets of $\{\alpha : \alpha < \mu \text{ even}\}$ with the singletons being its atoms. Second, for $A \in \mathscr{A}$, if possible we choose $u = u_A$ as follows, as we can replace u_A by any infinite subset, without loss of generality [clause (c) is possible as in the justification of $\boxtimes_0$ above]:

(a) if case (α) occurs in (b) below then
$\langle g^*(\gamma_{A,k}) : k \in u\rangle$ is with no repetitions

(b) either (α) or (β) where

 (α) $g^*(\gamma_{A,k})$ is odd for every $k \in u$

 (β) $g^*(\gamma_{A,k})$ is even for every $k \in u$

(c) if case (β) occurs in clause (b), then there is a unique $y = y_A \in Y$ such that $\langle \{g^*(\gamma_{A,k})\} : k \in u\rangle$ converge to D_{y_A}.

Note

$(*)$ if u_A is not well defined then for some finite $w \subseteq \mu$ we have $\{g(\gamma_{A,k}) : k < \omega\} \subseteq \bigcup\limits_{\alpha \in w} X_\alpha$.

Now we choose $\langle \alpha[A] : A \in \mathscr{A}, u_A$ well defined $\rangle$ such that:

$(**)$ $\langle \alpha[A] : A \in \mathscr{A}, u_A$ well defined$\rangle$ is with no repetitions, each $\alpha[A]$ is an odd ordinal $< \mu$ and if possible it lists all of them.

Clearly without loss of generality $\mathbb{B}^*/\mathrm{id}_1(\mathbb{B}^*)$ is nontrivial hence $Y \neq \emptyset$ so choose $y^* \in Y$. Now we define a function g from $\mathbb{B}^*$ into $\mathscr{P}(\mu)$ as follows:

$$\begin{aligned} g(x) = \{\alpha < \mu : & \alpha \text{ is even and } \in x \ \underline{\text{or}}\ \alpha = \alpha[A] \text{ hence odd,} \\ & A \in \mathscr{A}, u_A \text{ and } y_A \text{ are well defined and } x \cap y_A \notin \mathrm{id}_{\mathrm{rk}(y_A,\mathbb{B}^*)}(\mathbb{B}^*)\ \underline{\text{or}} \\ & \alpha \text{ is odd but } \notin \{\alpha[A] : A \in \mathscr{A}, u_A \text{ and } y_A \text{ are well defined}\} \text{ and} \\ & x \cap y^* \notin \mathrm{id}_{\mathrm{rk}(y^*,\mathbb{B})}(\mathbb{B}^*)\}. \end{aligned}$$

Easily g is a homomorphism from $\mathbb{B}^*$ into $\mathscr{P}(\mu)$ as $\mathbb{B}^*$ is superatomic. Let $\mathbb{B}^{**}$ be the Boolean Algebra of subsets of μ generated by $\mathrm{Rang}(g) \cup \{(\alpha) : \alpha < \mu\}$. Now we just replace $\mathbb{B}^*$ by $\mathbb{B}^{**} \subseteq \mathscr{P}(\mu)$. $\square_{2.1}$

2.2 Discussion:
Why do we use MAD families $\mathscr{A} \subseteq [\mu]^{\aleph_0}$ and not $\subseteq [\mu]^{\aleph_1}$? If we use the latter, we have to take more care about superatomicity as the intersections of such members may otherwise contradict superatomicity.

§3 Sufficient conditions for the construction's assumptions

Here we shall show that the assumptions of 2.1 are reasonable. Now in 3.2 we shall reduce the clause (k) of 2.1 to $\Pr(\lambda', \theta)$ where Pr formalizes clause (b) there. In 3.3, 3.5 we give sufficient conditions for $\Pr(\mu, \sigma)$. In fact, it is clear that (high enough) it is not easy to fail it. In 3.10 we give a sufficient condition for a strong version of clauses (e) - (f) of 2.1 (and earlier deal with the conditions appearing in it). So at least for some cardinals θ the statement "not having the assumptions of 2.1" with $\theta = \sigma^+$ (for simplicity) $\kappa = \beth_2(\sigma), \mu = \beth_3(\sigma)$ and λ such that (h) + (i) + (j) of 2.1 holds has large consistency strength.

3.1 Definition. 1) $\Pr(\chi, \mu, \theta)$ means that $\mu \geq \theta$ and for some $\mathscr{A}$ we have:

- (a) $\mathscr{A} \subseteq [\mu]^{\aleph_0}$
- (b) $\mathscr{A}$ is almost disjoint, i.e. $A \neq B \in \mathscr{A} \Rightarrow |A \cap B| < \aleph_0$
- (c) $|\mathscr{A}| = \chi$
- (d) $(\forall B \in [\mu]^\theta)(\exists A \in \mathscr{A})[A \subseteq^* B]$.

2) If we omit χ we mean "some χ".
3) We call $\mathscr{A} \subseteq [\lambda]^{\aleph_0}$ saturated <u>if</u> for every $A \in [\lambda]^{\aleph_0}$ not almost contained[5] in a finite union of members of $\mathscr{A}$, almost contains a member of $\mathscr{A}$.

<u>3.2 Fact</u>: 1) Clause (b) of the assumption of 2.1 is equivalent to $\Pr(\mu, \mu, \mathrm{cf}(\theta))$.
2) Clause (k)$(\alpha) + (\gamma)$ of the assumption of 2.1 follows from $\Pr(\chi', \lambda', \theta)$ & $\chi = \chi' + 2^{\aleph_0}$.
3) If $\mathscr{A} \subseteq [\mu]^{\aleph_0}$ is almost disjoint and is saturated <u>then</u> $\Pr(|\mathscr{A}|, \mu, \aleph_1)$.
4) If $\mu = \mu^{\aleph_0} \geq \theta$ <u>then</u> $\Pr(\mu, \theta) \equiv \Pr(\mu, \mu, \theta)$ and $\chi \neq \mu \Rightarrow \neg \Pr(\chi, \mu, \theta)$.
5) If $\theta < \mu_1 \leq \mu_2$ and $\Pr(\mu_2, \theta)$ <u>then</u> $\Pr(\mu_1, \theta)$.

Proof. 1) Read the two statements.
2) Let $\mathscr{A} \subseteq [\lambda']^{\aleph_0}$ exemplify $\Pr(\chi', \lambda', \theta)$. For each $A \in \mathscr{A}$ we can find $\langle B_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$ such that:

- $(*)(i)$ $B_{A,\zeta} \in [A]^{\aleph_0}$
- (ii) $\zeta \neq \varepsilon \Rightarrow B_{A,\zeta} \cap B_{A,\varepsilon}$ is finite
- (iii) if π is a partial one-to-one function from A to A such that $x \in \mathrm{Dom}(\pi) \rightarrow x \neq \pi(x)$ <u>then</u> for some $\zeta < 2^{\aleph_0}$ we have $\alpha \in B_{A,\zeta} \Rightarrow \alpha \notin \mathrm{Dom}(\pi) \vee \pi(\alpha) \notin B_{A,\zeta}$.

[Why? First find $\langle B'_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$ satisfying (i), (ii), let $\langle \pi_\zeta : \zeta < 2^{\aleph_0} \rangle$ list the π's from (iii) and choose $B_{A,\zeta} \in [B'_{A,\zeta}]^{\aleph_0}$ to satisfy clause (iii) for π_ζ.
Lastly, let $\mathscr{A}'$ be any MAD family of subsets of A extending $\{B_{A,\zeta} : A \in \mathscr{A}$ and $\zeta < 2^{\aleph_0}\}$.]
Having found $\langle B_{A,\zeta} : \zeta < 2^{\aleph_0} \rangle$ we let $\mathscr{A}' = \{B_{A,\zeta} : A \in \mathscr{A}$ and $\zeta < 2^{\aleph_0}\}$, it has cardinality $|\mathscr{A}| + 2^{\aleph_0} = \chi' + 2^{\aleph_0}$ and is as required in clauses $(k)(\alpha) + (\gamma)$ of 2.1.
3), 4), 5) Easy. $\square_{3.2}$

[5] "A is almost contained in B", i.e. $A \subseteq^* B$ means that $A \backslash B$ is finite.

3.3 Claim. *1) Assume*

- (a) $\kappa_n < \kappa_{n+1} < \kappa < \mu_n < \mu_{n+1} < \mu$ *for* $n < \omega$
- (b) $\kappa = \sum \kappa_n, \mu = \sum \mu_n$ and $\max \mathrm{pcf}\{\kappa_n : n < \omega\} > \mu$
- (c) κ *strong limit and* $2^\kappa \geq \mu^+$
- (d) $\langle \mu_n : n < \omega \rangle$ *satisfies the requirements from [Sh 513, §1] or at least the conclusion, i.e.*
  - $\odot$ *for every* $\lambda \geq \mu$ *for some* n *we have: if* $\mathfrak{a} \subseteq$ *Reg* $\cap \lambda \backslash \mu$ *and* $|\mathfrak{a}| < \mu$ *then sup* $pcf_{\mu_n\text{-complete}}(\mathfrak{a}) \leq \lambda$.

Then for every $\lambda \geq \kappa$:

$\otimes_{\lambda,\kappa}$ *we can find* $\{\bar{A}_\alpha : \alpha < \alpha^*\}$ *such that*

- (α) *each* $\bar{A}_\alpha$ *has the form* $\langle A_{\alpha,n} : n < \omega \rangle$, *it belongs to* $\prod\limits_{n<\omega} [\lambda]^{\kappa_n}$ *and for each* α *we have* $\langle A_{\alpha,n} : n < \omega \rangle$ *pairwise disjoint*
- (β) *if* $\alpha \neq \beta$, *then* $\bar{A}_\alpha, \bar{A}_\beta$ *are almost disjoint which means that* $f \in \prod\limits_{n<\omega} A_{\alpha,n}$ & $f' \in \prod\limits_{n<\omega} A_{\beta,n} \Rightarrow |\mathrm{Rang}(f) \cap \mathrm{Rang}(f')| < \aleph_0$
- (γ) *if* $\bar{A} \in \prod\limits_{n<\omega} [\lambda]^{\kappa_n}$ *then for some* $\alpha < \alpha^*$ *and one to one function* $h_1, h_2 \in {}^\omega\omega$ *we have* $\kappa = \lim \langle |A_{\alpha,h_1(n)} \cap A_{\alpha,h_2(n)}| : n < \omega \rangle$.

2) If $\kappa = \aleph_0, \kappa_n = 1, \mu_n < \mu_{n+1} < \mu = \Sigma\{\mu_\ell : \ell < \omega\} < 2^{\aleph_0}$ *and* $\odot$ *of (1), then the conclusion of (1) holds.*
3) We can conclude in (1) that: there is $\mathscr{A} \subseteq [\lambda]^{\aleph_0}$, *an almost disjoint family such that* $(\forall B \in [\lambda]^\kappa)(\exists A \in \mathscr{A})(A \subseteq B)$.

Proof. By [Sh 460], [Sh 668, §3] (even more).

3.4 Remark. 1) Are the hypotheses of 3.3(1) reasonable?
1a) Assume that κ is strong limit of cofinality $\aleph_0 < \kappa$ and $2^\kappa > \kappa^{+\omega}$. We let $\mu_n = \kappa^{+1+n}$. There is a sequence $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ as in clause (a), (b), (c) of 3.3(1); such $\bar{\kappa}$ exists (by [Sh:g, Ch.IX,§5], and it is hard not to satisfy clause (d) (see [Sh 513]).

1b) Clause (c), i.e. κ strong limit, is needed just to start the induction.
2) Similarly for 3.3(2).

We quote Goldstern Judah and Shelah [GJSh 399] which implies 3.5(1) and (2).

3.5 Claim. *Assume CH + SCH + $(\forall\mu)(\mathrm{cf}(\mu) = \aleph_0 \;\;\&\;\; 2^{\aleph_0} < \mu \rightarrow \square_{\mu^+})$. Then there is a saturated MAD family $\mathscr{A}_\lambda \subseteq [\lambda]^{\aleph_0}$ for every uncountable λ (of cardinality $\lambda^{\aleph_0}$).*

Proof. This is the main result of Goldstern, Judah and Shelah [GJSh 399].

3.6 Definition. Let $\mu \geq \theta$.
1) Let $\mathscr{S}_\theta$ be the class of $\bar a = \langle a_n : n < \omega\rangle$ such that $|a_n| \leq \theta, a_n \subseteq a_{n+1}, [\mathrm{cf}(\theta) = \aleph_0 \Rightarrow |a_n| < \theta]$ and $\theta = \limsup_n |a_{n+1} \backslash a_n|$. Let $\mathscr{S}_{\theta,\mu} =: \{\bar a : \bar a = \langle a_n : n < \omega\rangle \in \mathscr{S}_\theta$ and $a_n \in [\mu]^{\leq\theta}\}$.
3) For $\bar a \in \mathscr{S}_\theta$ let $\mathrm{set}(\bar a) = \{w : |w| = \aleph_0$ and $w \subseteq \bigcup\limits_{n<\omega} a_n$ and $n < \omega \Rightarrow |w \cap a_n \backslash \bigcup\limits_{\ell<n} a_\ell| < \aleph_0\}$.
4) For $\bar a, \bar b \in \mathscr{S}_\theta$ let $\bar a \leq^* \bar b$ mean $\mathrm{set}(\bar a) \supseteq \mathrm{set}(\bar b)$.
5) We say $\bar a, \bar b \in \mathscr{S}_\theta$ are compatible <u>if</u> $(\exists \bar c \in \mathscr{S}_\theta)[\bar a \leq^* \bar c \;\;\&\;\; \bar b \leq^* \bar c \;\;\&\;\; \bigcup\limits_n c_n \subseteq \bigcup\limits_n a_n \cap \bigcup\limits_n b_n]$ (if $\mathrm{cf}(\theta) = \aleph_0 < \theta$, this is equivalent to "$\bigcup\limits_{n<\omega} a_n \cap \bigcup\limits_n b_n$" has cardinality θ).

3.7 Definition. 1) For $\theta \leq \mu$ let $\boxtimes_{\theta,\mu}$ be

$\boxtimes_{\theta,\mu}$ there is $\mathscr{S}^* \subseteq \mathscr{S}_{\theta,\mu}$ such that:
- (a) for every $\bar a \in \mathscr{S}_{\theta,\mu}$ there is $\bar b \in \mathscr{S}^*$ compatible with $\bar a$
- (b) if $\bar a \neq \bar b \in \mathscr{S}^*$ then $\mathrm{set}(\bar a) \cap \mathrm{set}(\bar b) = \emptyset$.

2) For $\theta \leq \mu$, let $\boxtimes'_{\theta,\mu}$ mean:

$\boxtimes'_{\theta,\mu}$ if $\mathscr{S} \subseteq \mathscr{S}_{\theta,\mu}$ has cardinality $\leq \mu$ then we can find $\mathscr{S}^* \subseteq \mathscr{S}_{\theta,\mu}$ such that:
- (a) for every $\bar a \in \mathscr{S}$ there is $\bar b \in \mathscr{S}^*$ such that $\bar b \leq^* \bar a$
- (b) for every $\bar b \in \mathscr{S}^*$ there is $\bar a \in \mathscr{S}$ such that $\bar b \leq^* \bar a$
- (c) $\langle \mathrm{set}(\bar b) : \bar b \in \mathscr{S}^*\rangle$ are pairwise disjoint.

3) We may replace μ by a set A (but obviously $\boxtimes_{\theta,A}$ is equivalent to $\boxtimes_{\theta,|A|}$ and $\boxtimes'_{\theta,A}$ to $\boxtimes_{\theta,|A|}$).

3.8 Fact. 1) Assume $\theta > \text{cf}(\theta) = \aleph_0$ is strong limit, $\theta = \Sigma\{\theta_n : n < \omega\}, \theta_n < \theta_{n+1}$ and $\bar{b} \in \mathscr{S}_{\theta,\mu}$. <u>Then</u> we can find $\mathscr{A} \subseteq \mathscr{S}_\theta$ such that:

- (a) if $\bar{a} \in \mathscr{A}$ then $(\forall n)(\exists m)(a_n \subseteq b_m)$ (so $\bar{a} \leq \bar{b}$)
- (b) if $\bar{a} \in \mathscr{A}$ then $|a_n| = \theta_n$ moreover $\text{otp}(a_n) = \theta_n$ and a_{n+1} is an end extension of a_n
- (c) if $\bar{a} \in \mathscr{A}$ then $\langle a_n : n < \omega \rangle$ is increasing
- (d) $\bar{a}^1 \neq \bar{a}^2$ then $\text{set}(\bar{a}^1) \cap \text{set}(\bar{a}^2) = \emptyset$
- (e) if $\bar{c} \in \mathscr{S}_\theta$ is compatible with $\bar{b}$ then it is compatible with $\bar{a}$ for some $\bar{a} \in \mathscr{A}$.

2) If $(\forall \alpha < \theta_n)(|\alpha|^\sigma < \theta_n = \text{cf}(\theta_n))$ and $<_\alpha$ is a well ordering of $\cup\{b_n : n < \omega\}$ for $\alpha < \sigma$, <u>then</u> we can strengthen (b) to

- $(b)^+$ for $\alpha < \sigma$ and $\bar{a} \in \mathscr{A}$ and $n < \omega$, $\text{otp}(b_n, <_\alpha \restriction b_n) = \theta_n$ and if $\sigma < \aleph_0$ then b_{n+1} is a $<_\alpha$-end extension of b_n.

3) $\bar{a}, \bar{b} \in \mathscr{S}_{\theta,A}$ are incompatible <u>iff</u> $\bigcup\limits_{n<\omega} a_n \cap \bigcup\limits_{n<\omega} b_n$ has cardinality $< \theta$ ($\text{cf}(\theta) = \aleph_0 < \theta$ suffice).
4) (a) $\boxtimes_{\theta,\mu}$ implies $\boxtimes'_{\theta,\mu}$.
(b) $\boxtimes'_{\theta,\mu}$ is equivalent of $\boxtimes_{\theta,\mu}$ if $\mu = \mu^\theta$.

Proof. As in 3.9 below.

3.9 Claim. *Assume θ is strong limit, $\theta > \text{cf}(\theta) = \aleph_0$.*
1) $\mu \in (\theta, 2^\theta]$ <u>then</u> $\boxtimes'_{\theta,\mu}$ from 3.7 holds.
2) Also if $\theta < \mu < (2^\theta)^{+2^\theta}$ then $\boxtimes'_{\theta,\mu}$.
3) If $2^\theta < \mu$ and $(\forall \lambda)(2^\theta < \lambda < \mu,\ cf(\lambda) = \aleph_0 \rightarrow \lambda^{\aleph_0} = \lambda^+ + \Box_\lambda)$ <u>then</u> $\boxtimes'_{\theta,\mu}$.

Proof. 1) Straight, as $|\mathscr{S}_{\theta,\mu}| = \mu^\theta = 2^\theta$ we can find $\langle \bar{a}^\alpha : \alpha < 2^\theta \rangle$ listing $\mathscr{S}_{\theta,\mu}$. Now we choose $\gamma(\alpha), \bar{b}^\alpha$ by induction on $\alpha < 2^\theta$ such that

(a) $\bar b^\alpha \in \mathscr{S}_{\theta,\mu}$
(b) $\beta < \alpha \Rightarrow$ set$(\bar b^\beta) \cap$ set$(\bar b^\alpha) = \emptyset$
(c) $\bar c^\alpha \le a^{\gamma(\alpha)}$
(d) $\gamma(\alpha) =$ Min$\{\gamma : \bar a^\gamma$ incompatible with $\bar b^\beta$ for every $\beta < \alpha\}$.

Arriving to α choose $\gamma(\alpha)$ by clause (d), we note that $\beta < \gamma(\alpha) \Rightarrow c^\alpha_\beta = \bigcup\limits_n a^{\gamma(\alpha)}_n \cap \bigcup\limits_n b^\beta_n$ has cardinality $< \theta$, hence we can find $\bar b_{\alpha,\varepsilon} \le \bar a^{\gamma(\alpha)}$ for $\varepsilon < 2^\theta$ with $\langle$set$(\bar b_{\alpha,\varepsilon}) : \varepsilon < 2^\theta\rangle$ pairwise disjoint. So for all but $\le \theta + |\alpha|$ of the $\varepsilon < 2^\theta, \bar b_\alpha = \bar b_{\alpha,\varepsilon}$ is as needed.
2) After reading [Sh 460] this is easy and anyhow in subsequent work we give fuller answers.
3) As in [GJSh 399]. $\square_{3.9}$

3.10 Claim. *1) Assume*

$\boxtimes_{\theta,\kappa,\mu}$ *θ is strong limit, $\aleph_0 = \mathrm{cf}(\theta) < \theta$ and $\theta \le \kappa \le 2^{2^\theta}, \mu = 2^\kappa$ and $\boxtimes_{\theta,\kappa}$ (from 3.7) holds so $\mu = \mu^{\aleph_0}$.*

Then some $\bar{\mathbb{B}} = \langle \mathbb{B}_\alpha : \alpha < \mu\rangle$ satisfies clauses (c) - (g) of 2.1; in fact $\mathbb{B}_\alpha$ is a subalgebra of $\mathscr{P}(\kappa)$ with 2 levels and $id_{<\infty}(\mathbb{B}_\alpha)$ is included in $[\kappa]^{<\aleph_1}$ hence $\mathbb{B}_\alpha \subseteq \{a \subseteq \kappa$: a countable or co-countable$\}$.
2) As above except that instead "θ strong limit, $\mathrm{cf}(\theta) = \aleph_0 < \theta$" we demand $2^\theta = \theta^{\aleph_0} > 2^{\aleph_0}$ & $\theta > \mathrm{cf}(\theta) = \aleph_0$ or $\theta = \aleph_0 +$ "there is no infinite MAD family $\mathscr{A} \subseteq [\omega]^{\aleph_0}$ of cardinality $<$ the continuum".

Proof. 1) Let $\theta = \sum\limits_{n<\omega} \theta_n, \theta_n < \theta_{n+1} < \theta$.

Fact: Letting $\bar a^* = \langle \theta_n : n < \omega\rangle$, i.e. $a^*_n = \theta_n$ we can find $\overline{t^a} = \langle t_{\ell,\alpha} : \ell < 3, \alpha < 2^\theta\rangle$ such that:

(i) $t_{\ell,\alpha} \in$ set$(\bar a^*)$ has order type ω
(ii) for some one to one onto $\pi : 2^\theta \times 2^\theta \to 2^\theta$ we write $t_{2,\alpha,\beta}$ for $t_{2,\pi(\alpha,\beta)}$
(iii) if $(\ell_1,\alpha_1) \ne (\ell_2,\alpha_2)$ then $t_{\ell_1,\alpha_1} \cap t_{\ell_2,\alpha_2}$ is finite
(iv) if $\bar a \in \mathscr{S}_{\theta,\kappa}$ and $\bigcup\limits_{n<\omega} a_n \subseteq \theta$ then for some $\alpha < 2^\theta$ we have $[\beta < 2^\theta \Rightarrow t_{2,\alpha,\beta} \in$ set$(\bar a)]$

(v) if $\bar{a}, \bar{b} \in \mathscr{S}_{\theta,\kappa}$ and $\bigcup\limits_{n<\omega} a_n \cup \bigcup\limits_{n<\omega} b_n \subseteq \theta$ and $\mathrm{set}(\bar{a}) \cap \mathrm{set}(\bar{b}) = \emptyset$ and $h : \bigcup\limits_{n<\omega} a_n \to \bigcup\limits_{n<\omega} b_n$ is one to one and maps a_n onto b_n <u>then</u> for some $\alpha, t_{0,\alpha} \in \mathrm{set}(\bar{a}), t_{1,\alpha} \in \mathrm{set}(\bar{b})$ and h maps $t_{0,\alpha}$ into a co-infinite subset of $t_{1,\alpha}$.

Proof of the fact. Straight.

<u>Construction</u>: Let $\mathscr{S}^* = \{\bar{a}^\gamma : \gamma < \gamma^*\}$ exemplify $\boxtimes_{\theta,\kappa}$ so $|\gamma^*| \le \kappa^\theta$, without loss of generality $\bar{a} \in \mathscr{S}^*$ & $n < \omega \Rightarrow (\mathrm{otp}(a_n)$ & a_{n+1} is an end extension of $a_n)$; (by 3.8, i.e. by replacing $\bar{a}^\gamma$ by a suitable family $\subseteq \{\bar{b} : \bar{b} \le \bar{a}^\gamma\}$). Let $\{X_\gamma : \gamma < \kappa\}$ be a sequence of subsets of 2^θ such that $\gamma_1 \ne \gamma_2 \Rightarrow |X_{\gamma_1} \backslash X_{\gamma_2}| = 2^\theta$; let $\langle Y_j : j < \mu \rangle$ be a sequence of subsets of κ such that $j_1 \ne j_2 \Rightarrow |Y_{j_1} \backslash Y_{j_2}| = \kappa$, let g_γ be a one to one mapping from θ into $\bigcup\limits_{n<\omega} a^\gamma_n$ mapping θ_n onto a^γ_n, and lastly let $t^\gamma_{\ell,\alpha} = g''_\gamma(t_{\ell,\alpha}) = \{g_\gamma(\zeta) : \zeta \in t_{\ell,\alpha}\}$ for $\ell < 3, \alpha < \gamma^+$ hence $t^\gamma_{2,\alpha,\beta} = g''_\gamma(t^\gamma_{2,\alpha,\beta})$. Let $t^\gamma_{3,\alpha,\beta} = \{g_\gamma(\varepsilon) : \varepsilon \in t_{2,\alpha,\beta}$ and $|t_{2,\alpha,\beta} \cap \varepsilon|$ is even$\}$.

For $j < \mu$, let $\mathscr{A}_j$ be the following family of subsets of κ

$$t^\gamma_{0,\alpha}, t^\gamma_{1,\alpha} \text{ when } \gamma < \gamma^*, \alpha < 2^\theta$$

$$t^\gamma_{2,\alpha,1+\beta} \text{ when } \gamma < \gamma^*, \beta \notin X_\gamma, \alpha < 2^\theta$$

$$t^\gamma_{3,\alpha,1+\beta} \text{ when } \gamma < \gamma^*; \beta \in X_\gamma, \alpha < 2^\theta$$

$$t^\gamma_{2,\alpha,0} \text{ when } \gamma < \gamma^*, \alpha < 2^\theta, \gamma \notin Y_j \text{ and}$$

$$t^\gamma_{3,\alpha,0} \text{ when } \gamma \in Y_j.$$

Clearly

$\bigodot_1$ $t' \ne t'' \in \mathscr{A}_j \Rightarrow |t' \cap t''| < \aleph_0 = |t'|$.

Let $\mathscr{A}^+_j$ be a maximal almost disjoint family of countable subsets of κ extending $\mathscr{A}_j$. Let I_j be the Boolean ring of subsets of κ generated by $\mathscr{A}^+_j \cup \{\{\varepsilon\} : \varepsilon < \kappa\}$ and $\mathbb{B}_j$ be the Boolean algebra of subsets of κ generated by I_j. Now

$\odot_2$ if $i_0, i_1 < \mu$ and $b_0, b_1 \in [\kappa]^\theta$ and h is a one to one mapping from b_0 onto b_1 such that $\alpha \in \mathrm{Dom}(h) \Rightarrow h(\alpha) \neq \alpha$, <u>then</u> for some $t^0 \in \mathscr{A}^+_{i_0}, t^1 \in \mathscr{A}^+_{i_1}$ we have: $t^0 \subseteq^* b_0, t^1 \subseteq^* b_1$ and h maps t^0 into a co-infinite subset of t^1
[why? for some $\gamma_0 < \kappa$ the set $b_0 \cap \bigcup\limits_{n<\omega} a^{\gamma_0}_n$ have cardinality θ, so without loss of generality $b_0 \subseteq \bigcup\limits_{n<\omega} a^{\gamma_0}_n$ and similarly for some $\gamma_1 < \kappa$ without loss of generality $b_1 \subseteq \bigcup\limits_{n<\omega} a^{\gamma_1}_n$. For $\ell = 0,1$ let $b^-_\ell \in [\theta]^\theta$ be such that g_{γ_ℓ} maps b^-_ℓ onto b_ℓ. Now without loss of generality $b^-_0 \cap b^-_1 = \emptyset$ or $b^-_0 = b^-_1$ (recall we have to preserve "h is from b_0 onto b_1", too!). If $b^-_0 \cap b^-_1 = \emptyset$ then by clause (v) of the fact some $t^{\gamma_0}_{0,\alpha_0} \in \mathscr{A}_{i_0} \subseteq \mathscr{A}^+_{i^0}$ and $t^{\gamma_1}_{0,\alpha_1} \in \mathscr{A}_{i_1} \subseteq \mathscr{A}^+_{i_1}$ will be as required in the conclusion of $\odot_2$. So assume $b^-_0 = b^-_1$, let $b^*_0 = \{\alpha \in b^-_0 : h \circ g_{\gamma_0}(\alpha) \neq g_{\gamma_1}(\alpha)\}$. If b^*_0 has cardinality θ, we get the desired conclusion (in $\odot_2$) as above, so assume $|b^*_0| < \theta$ hence without loss of generality $b^*_0 = \emptyset$. Also if $\gamma_0 \neq \gamma_1$ then $|X_{\gamma_0} \backslash X_{\gamma_1}| = 2^\theta$ hence we can find a non zero ordinal $\beta \in X_{\gamma_0} \backslash X_{\gamma_1}$ and by clause (ii) of the fact we can find an ordinal $\alpha < 2^\theta$ such that $(\forall \beta < 2^\theta)[t^\gamma_{2,\alpha,\beta} \subseteq b^-_0]$ hence we can use $t^\gamma_{3,\alpha,\beta}, t^\gamma_{2,\alpha,\beta}$. So we have to assume $\gamma_0 = \gamma_1$ but then $g_{\gamma_0} = g_{\gamma_1}$ so $h \restriction (b_0 \backslash b^*_0)$ is the identity, a contradiction.]

$\odot_3$ if $i_0 \neq i_1 < \mu$ and[6] $Z \in [\kappa]^{<\kappa}$ and h is a one to one function from $\kappa \backslash Z$ onto $\kappa \backslash Z$ <u>then</u> for some $t^0 \in \mathscr{A}^+_{i_0}$ satisfying $t^0 \subseteq^* \mathrm{Dom}(h)$ and $t^1 \in \mathscr{A}^+_{i_1}$ we have: $h''(t^0) \subseteq^* t^1$ and $t^1 \backslash h''(t^0)$ is infinite.
[Why? Let $Z_1 = \{\alpha \in \mathrm{Dom}(h) : h(\alpha) \neq \alpha\}$, so by $\odot_2$ we know $|Z_1| < \theta$. We know that $Y_{i_0} \backslash Y_{i_1}$ has cardinality μ, hence for some $\gamma \in Y_{i_0} \backslash Y_{i_1}$ we have $\mathrm{set}(\bar{a}_\gamma) \cap [Z \cup Z_1]^{\aleph_0} = \emptyset$. So $t^\gamma_{3,\alpha,0} \in \mathscr{A}_{i_0} \subseteq \mathscr{A}^+_j$ and $t^\gamma_{2,\alpha,0} \in \mathscr{A}_{i_1} \subseteq \mathscr{A}^+_{i_1}$, so $t^\gamma_{3,\alpha,0}$ is a co-infinite subset of $t^\gamma_{2,\alpha,0}, t^\gamma_{2,\alpha,0} \subseteq^* \kappa \backslash Z \backslash Z_0$ and h maps $t^\gamma_{3,\alpha,0} \backslash Z \backslash Z_0$ to itself a co-infinite subset of $t^\gamma_{2,\alpha,0}$.]

Clearly $\langle \mathbb{B}_j : j < \mu \rangle$ is as required so we are done.
2) Similar proof. $\square_{3.10}$

3.11 Conclusion. 1) Under the assumption $\boxtimes_{\theta,\kappa,\mu}$ of 3.10, let $\lambda^* = \mathrm{Ded}^+(\mu) = \mathrm{Min}\{\lambda$: there is no tree with $\leq \mu$ nodes and $\geq \lambda$ branches (equivalently, a linear order of cardinality λ and density $\leq \mu\}$. <u>Then</u> for any λ satisfying $\mu \leq \lambda < \lambda^*$, there is a superatomic Boolean Algebra of cardinality λ and μ atoms with no

[6] by a little more care in indexing, $Z \in [\mu]^{<\mu}$ is O.K. and we can choose γ such that $\bigcup\limits_n a_{\gamma,n} \subseteq \kappa \backslash Z \backslash Z_0$

automorphism moving $\geq \theta$ atoms.
2) Assume: θ is uncountable strong limit of cofinality $\aleph_0$, $\mathrm{pp}_{J^{\mathrm{bd}}_\omega}(\theta) = 2^\theta$ (see [Sh:g, Ch.IX,§5] why this is reasonable) and $\kappa = (2^\theta)^{+\alpha} \leq 2^{2^\theta}, \alpha < (2^\theta)^+, \mu = 2^\kappa$ and $\mu < \lambda < \mathrm{Ded}^+(\mu)$, e.g. $\lambda = 2^\chi$ for $\chi = \mathrm{Min}\{\chi : 2^\chi > \mu\}$. Then there is a superatomic Boolean Algebra of cardinality λ and μ atoms, with no automorphism moving $\geq \theta$ atoms.
3) In part (2) we can replace $\kappa = (2^\theta)^{+\alpha}$ by $\kappa = 2^{2^\theta}$, if some very weak pcf hypothesis (whose negation is not known to be consistent and also of §4), e.g.

- $(*)$ if $\mathfrak{a}$ is a countable set of regular cardinal then $\mathrm{pcf}(\mathfrak{a})$ is countable (or just $\leq \aleph_{n(*)}$).

Proof. 1) We, of course, use Lemma 2.1 with θ^+ here standing for θ there, so we have to show that the assumptions there holds.
Clause (a) of 2.1 holds trivially.
Clause (b) of 2.1 follows from $\boxtimes_{\theta,\kappa}$ (every $(\forall A \in [\mu]^\theta)(\exists B \in \mathscr{A})(B \subseteq A)$) rather than just $(\forall A \in [\mu]^{\theta^+})$. There is a sequence $\langle \mathbb{B}_\alpha : \alpha < \mu \rangle$ satisfying clauses (c) - (g) of 2.1 by 3.10. There is a Boolean Algebra $\mathbb{B}^*$ satisfying clauses $(h), (i), (j)$ of 2.1 because $\lambda < \lambda^*$, so there is a tree $\mathscr{T}$ with μ nodes and $\geq \lambda$ branches, let $\mathscr{Y}$ be a set of λ branches of $\mathscr{T}$ and let $\mathbb{B}$ be the Boolean Algebra of subsets of $\mathscr{T}$ generated by $\{a : a \subseteq T, a$ is linearly ordered by $<_T$ and $x \in a \ \& \ y <_T x \Rightarrow y \in a$ and a is bounded on $a \in \mathscr{Y}\}$.

Lastly, clause (k) of 2.1 hold vacuously as we choose $\lambda' = \mu$. $\square_{3.11}$

3.12 Claim. *Assume*

- (a) $\mathrm{Pr}(\beth_3, \aleph_1)$
- (b) $\lambda^* = \mathrm{Min}\{\lambda'$*: there is a tree with* $\beth_3$ *models of* $\geq \lambda'$ *branches*$\}$
- (c) $\beth_3 \leq \lambda < \lambda^*$.

Then there is a superatomic Boolean Algebra with λ *elements* $\beth_3$ *atoms and no automorphisms moving uncountably many atoms.*

Proof. The main new point is that we can prove a parallel of 3.10 noting that as $\mathrm{Pr}(\beth_3, \aleph_1)$ holds also $\mathrm{Pr}(\beth_2, \aleph_1)$ holds. $\square_{3.12}$

3.13 Remark. 1) So clearly in many models of ZFC we get that the bound in 1.1 cannot be improved.

2) The question is whether inductively we can get for many θ's the parallel of 3.10.
3) We can under weak assumptions add $\lambda', \mu \le \lambda' \le (\lambda')^{\aleph_0} \le \lambda$ and demand that the Boolean algebra has μ' atoms. For this we need to check condition $(k)(\alpha)$. We probably can omit the demand "$(\lambda')^{\aleph_0} \le \lambda$" in the generalization of 3.11 indicated above, for this we just need to weaken "$\mathscr{A}$ is MAD" in 2.1.

3.14 Claim. *1) Let $\lambda > \aleph_0$. A sufficient condition for the existence of a saturated* MAD *family $\mathscr{A} \subseteq [\lambda]^{\aleph_0}$ is*

$\boxplus_{\lambda,\theta}$ *letting i.e. $\theta = \mathrm{Min}\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^{\aleph_0}$ is an infinite* MAD *family,* *then* *for every μ satisfying $2^{\aleph_0} < \mu \le \lambda^{\aleph_0}$ we have $\neg(a)_{\mu,\theta}$ and $\aleph_0 < \sigma = \mathrm{cf}(\sigma) \le \theta \Rightarrow \neg(b)_{\mu,\theta}$ where*

- $(a)_{\mu,\theta}$ *there is a set $\mathfrak{b} \subseteq \mathrm{Reg} \cap \mu \backslash 2^{\aleph_0}$ of cardinality $\le \theta$ such that $\Pi\mathfrak{b}/[\mathfrak{b}]^{<\aleph_0}$ is μ-directed, moreover, for no sequence $\bar{b} = \langle \mathfrak{b}_i : i < \theta \rangle$, each $\mathfrak{b}_i \subseteq \mathrm{Reg} \cap \mu \backslash 2^{\aleph_0}$ finite and $\mathfrak{c} \subseteq \cup\{\mathfrak{b}_i : i < \theta\}$ & $\max \mathrm{pcf}(\mathfrak{c}) < \mu \Rightarrow \aleph_1 > |\{i < \theta : \mathfrak{b}_i \subseteq \mathfrak{c}\}|$*
- $(b)_{\mu,\theta}$ *μ is regular, $S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \mathrm{cf}(\theta)\}$ is stationary, $\bar{A} = \langle A_\delta : \delta \in S \rangle, A_\delta \subseteq \delta$, $\mathrm{otp}(A_\delta) = \theta, \delta_2 \ne \delta_2 \Rightarrow A_{\delta_1} \cap A_{\delta_2}$ finite.*

2) Similarly concerning $\boxtimes'_{\theta,\mu}$

Proof. As in [Sh 668].

§4 On independence

In the bound $\beth_4(\sigma)$, the last exponentiation was really $\mathrm{sa}(\mu)$ where

4.1 Definition. 1) $\mathrm{sa}^+(\mu) = \sup\{|\mathbb{B}|^+ : \mathbb{B}$ is a superatomic Boolean Algebra with μ atoms$\}$.

2) $\mathrm{sa}(\mu) = \sup\{|\mathbb{B}| : \mathbb{B}$ is a superatomic Boolean Algebra with μ atoms$\}$.
3) $\mathrm{sa}^+(\mu,\theta) = \sup\{|\mathbb{B}|^+ : \mathbb{B}$ is a superatomic Boolean subalgebra of $\mathscr{P}(\mu)$ extending $\{a \subseteq \mu : a$ finite or cofinite$\}$ such that $a \in \mathbb{B} \Rightarrow |a| < \theta \vee |\mu \backslash a| < \theta\}$.
4) $\mathrm{sa}(\mu,\theta) = \sup\{|\mathbb{B}| : \mathbb{B}$ is as in (3)$\}$.
5) $\mathrm{sa}^*(\theta) = \mathrm{Min}\{\lambda : \mathrm{cf}(\lambda) \ge \theta$ and if $\mu < \lambda$ then $\mathrm{sa}^+(\mu,\theta) \le \lambda\}$.

That is, by the proof of Theorem 1.1

4.2 Claim. *If $\mathbb{B}$ is a superatomic Boolean Algebra with no automorphism moving $\geq \theta$ atoms, $\theta = \mathrm{cf}(\theta) > \aleph_0$ <u>then</u> $|\mathbb{B}| < sa^+(\beth_3(< \theta))$, moreover $|\mathbb{B}| < sa^+(\beth_2(sa^*(\theta)))$.*

<u>4.3 Discussion</u>: 1) Now consistently $\mathrm{sa}(\aleph_1) < 2^{\aleph_1}$. Why? Because [Sh 620, 8.1] show the consistency of a considerably stronger statement. It proves that e.g. if we start with $\mathbf{V} \models$ GCH and $\mathbb{P}$ is adding $\aleph_{\omega_1}$ Cohen reals <u>then</u> in $\mathbf{V}^{\mathbb{P}}$, ($2^{\aleph_0} = \aleph_{\omega_1} < 2^{\aleph_1} = \aleph_{\omega_1+1}$ and) among any $\aleph_{\omega_1+1}$ members of $\mathscr{P}(\omega_1)$ there are $\aleph_{\omega_1+1}$ which form an independent family, i.e. any finite nontrivial Boolean combination of them is nonempty, in other words "$\mathscr{P}(\omega_1)$ has $\aleph_{\omega_1+1}$-free precaliber in Monk's question definition". (Not surprising this is the same model for "no tree with $\aleph_1$ nodes has $2^{\aleph_1}$ branches" in [B1]).
2) So the bound $\beth_4(\theta)$ is not always the right ones though this needs use of more complicated functions.
3) We have not looked at the question: does the use of $\mathrm{sa}^*(\theta)$ in claim 4.2 really help?

4.4 Claim. *Assume*

- (a) $\Upsilon = \Upsilon^{<\Upsilon} < \mu = \mathrm{cf}(\mu) < \chi$
- (b) $\mathrm{cf}(\chi) = \mu$ *and* $(\forall \alpha < \chi)(|\alpha|^\mu < \chi)$ *and* $(\forall \alpha < \mu)(|\alpha|^{<\Upsilon} < \mu)$
- (c) $\mathbb{Q}$ *is a forcing notion of cardinality* $< \chi$ *such that in* $\mathbf{V}^{\mathbb{Q}}$: μ *is a regular cardinal and* $(\forall a \in [\chi]^{<\mu})(\exists b)[a \subseteq b \in ([\chi]^{<\mu})^{\mathbf{V}}]$
- (d) $\mathbb{P} = \{f : f$ *a partial function from* χ *to* $\{0,1\}$ *of cardinality* $< \Upsilon\}$ *order by inclusion (that is, adding a* χ Υ*-Cohen).*

<u>Then</u> in $\mathbf{V}^{\mathbb{Q}\times\mathbb{P}}$ we have: ($2^\Upsilon = 2^{<\mu} = \chi, 2^\mu = \chi^\mu = (\chi^\mu)^{\mathbf{V}}$ and) $sa(\mu) = \chi < 2^\mu$, moreover the Boolean Algebra $\mathscr{P}(\mu)$ has χ^+-free precaliber.

Proof. Work in $\mathbf{V}^{\mathbb{Q}}$, like [Sh 620, 8.1], not using "$\mathbb{P}$ is σ-complete" which may fail in $\mathbf{V}^{\mathbb{Q}}$. $\square_{4.4}$

On the other hand

4.5 Claim. *Assume $\bar{\lambda} = \langle \lambda_n : n < \omega \rangle$ satisfies $\lambda_{n+1} = \mathrm{Min}\{\lambda : 2^\lambda > 2^{\lambda_n}\}$. <u>Then</u> for infinitely many n's for some $\mu_n \in [\lambda_n, \lambda_{n+1})$ we have $\mathrm{sa}(\mu_n) = 2^{\mu_n} = 2^{\lambda_n}$ (in fact $\mathrm{sa}^+(\mu_n) = (2^{\mu_n})^+ = (2^{\lambda_n})^+$ except possibly when $\mathrm{cf}(2^{\lambda_n}) \leq 2^{\lambda_{n-1}}$).*

Proof. By [Sh 430, 3.4] we have for infinitely many n's $\mu_n \in [\lambda_n, \lambda_{n+1})$ and for every regular $\chi \leq 2^{\lambda_n} = 2^{\mu_n}$, a tree with $\leq \mu_{n+1}$ nodes, λ_n levels and $\geq \chi$ λ_n-branches. $\square_{4.5}$

<u>4.6 Conclusion</u>: 1) Assume θ is strong limit, $\theta > \mathrm{cf}(\theta) = \aleph_0$ and $\Pr(2^{2^\theta}, \theta)$ and $\lambda < \mathrm{sa}^+(\beth_3(\theta))$. <u>Then</u>

$(*)_{\theta,\lambda}$ there is a superatomic Boolean Algebra without any automorphism moving $\geq \theta$ atoms such that $\mathbb{B}$ has cardinality λ (and has $\beth_3(\theta)$ atoms[7]).

2) Assume $\Pr(\beth_2, \aleph_1)$ and $\lambda < sa^+(\beth_3)$. <u>Then</u> $(*)_{\theta,\lambda}$ holds.

Proof. 1) Use 3.10 and 2.1.
2) Similar only replace 3.10 by a parallel claim. $\square_{4.6}$

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[7] we can allow less atoms and less elements

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