FALLEN CARDINALS

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ABSTRACT. We prove that for every singular cardinal μ of cofinality ω, the complete Boolean algebra comp $\mathcal{P}_\mu(\mu)$ contains a complete subalgebra which is isomorphic to the collapse algebra Comp $\mathrm{Col}(\omega_1, \mu^{\aleph_0})$. Consequently, adding a generic filter to the quotient algebra $\mathcal{P}_\mu(\mu) = \mathcal{P}(\mu)/[\mu]^{<\mu}$ collapses $\mu^{\aleph_0}$ to $\aleph_1$. Another corollary is that the Baire number of the space $U(\mu)$ of all uniform ultrafilters over μ is equal to ω_2. The corollaries affirm two conjectures of Balcar and Simon.

The proof uses pcf theory.

1. Introduction

1.1. **Forcing and distributivity of complete Boolean algebras.** Every separative poset P which may be used as a forcing notion, is embedded as a dense subset of a (unique) complete Boolean algebra, called the *completion of* P and denoted by comp P. The properties of the forcing extension V^P of the universe V of set theory, which is obtained by forcing with P, are tightly related to the Boolean-algebraic properties of comp P, in particular to the *distributivity* properties of comp P. The least cardinality of a new set in V^P, for example, is equal to the *distributivity number* of comp P, denoted $\mathfrak{h}(\text{comp}\, P)$, which should really be called the "non-distributivity number", since it is the least cardinality of a product of sums which violates distributivity (see [9] for more information). Finer properties of non-distributivity determine which cardinals of V are preserved and which are collapsed in the extension V^P. The non-distributivity property which is important for our context is the following:

Definition 1.1. *A complete Boolean algebra B is $(\kappa, \cdot, \lambda)$-nowhere distributive iff B contains partitions of unity P_α for $\alpha < \kappa$ (namely, $\sum P_\alpha = 1$ and $p \wedge q = 0$ for $p \neq q$ in P_α) so that for every $b \in B - \{0\}$ there exists $\alpha < \kappa$ so that $b \wedge p \neq 0$ for $\geq \lambda$ members $q \in P_\alpha$.*

Clearly, if $\lambda_1 < \lambda_2$ and B is $(\kappa, \cdot, \lambda_2)$ nowhere distributive, it is also $(\kappa, \cdot, \lambda_1)$-nowhere distributive. The systematic study of distributivity in Boolean algebras was pursued by the Czech school of set theory ever since the discovery of Forcing in 1963.

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1.2. **The quotient algebra $\mathcal{P}_\mu(\mu)$.** In 1972 Balcar and Vopěnka began the study non-distributivity in quotient algebras $\mathcal{P}_\kappa(\kappa)$ for infinite cardinals κ. For every infinite cardinal κ, the algebra $\mathcal{P}_\kappa(\kappa)$ is obtained as the quotient of the power set algebra $\mathcal{P}(\kappa)$ over the ideal $[\kappa]^{<\kappa}$ of all subsets of κ whose cardinality is strictly smaller than κ. It was first shown that for every singular cardinal μ of countable cofinality the distributivity number of $\mathcal{P}_\mu(\mu)$ is ω_1 and that for every cardinal κ of uncountable cofinality the distributivity number of $\mathcal{P}_\kappa(\kappa)$ is ω [6]. The distributivity number of $\mathcal{P}_\omega(\omega)$ was discussed separately in [5]. The exact nature of distributivity in various $\mathcal{P}_\kappa(\kappa)$ was addressed in a series of papers [1, 3, 2], usually under additional set theoretic assumptions. The optimal ZFC non-distributivity properties of $\mathcal{P}_\kappa(\kappa)$ were obtained in [2], from which we quote:

Theorem 1.2 (Balcar and Simon)**.**
(1) *For every singular μ of countable cofinality $\mathcal{P}_\mu(\mu)$ is $(\omega_1, \cdot, \mu^{\aleph_0})$-nowhere distributive*
(2) *For every singular κ of uncountable cofinality $\mathcal{P}_\kappa(\kappa)$ is $(\omega, \cdot, \kappa^+)$-nowhere distributive.*

1.3. **The collapse algebra Col $(\omega_1, \mu^{\aleph_0})$.** It was in [1] that it was first shown that under certain set theoretic assumptions $\mathcal{P}_\kappa(\kappa)$ (and some other factor algebras of $\mathcal{P}(\kappa)$) have completions which are isomorphic to suitable *collapse algebras.* Let us introduce collapse algebras. For cardinals $\kappa < \lambda$, κ regular, the poset $\mathrm{Col}\,(\kappa, l)$ is the natural κ-complete poset for introducing a function φ from κ onto λ, namely for "collapsing" λ to κ.

$$\begin{aligned} \mathrm{Col}\,(\kappa,\lambda) = \{f :\ & \text{for some } \alpha<\kappa,\ f \text{ is a functions,} \\ & \mathrm{dom} f = \alpha \text{ and } \mathrm{ran} f \subseteq \lambda\} \end{aligned} \tag{1}$$

The completion comp Col (κ, λ) is the collapse algebra for (κ, λ). The cardinality of Col (κ, λ) is clearly $\lambda^{<\kappa}$, and therefore, since Col (κ, λ) is dense in its completion, the density $\pi(\mathrm{comp\,Col}\,(\kappa, \lambda))$ is equal to $\lambda^{<k}$. For each $\alpha < \kappa$ let P_α be a maximal antichain in Col (κ, λ) composed of conditions which decide $\varphi\restriction\alpha$. The set $\{P_\alpha : \alpha < \kappa\}$ (which is, really, a *name* for φ) is also a witness to the fact that comp Col (κ, λ) is $(\kappa, \cdot, \lambda)$-*nowhere distributive*. The following characterization of comp Col $(\omega_1, \mu^{\aleph_0})$ for a singular μ of countable cofinality is a particular instance of a general characterization theorem for collapse algebras ([3], 1.15):

Theorem 1.3. *Let B be a complete $(\omega_1, \cdot, \mu)$ nowhere distributive Boolean algebra containing an $\aleph_1$-closed dense subset. If $\pi(B) = \mu^{\aleph_0}$, then B is isomorphic to comp Col $(\omega_1, \mu^{\aleph_0})$.*

Thus comp Col $(\omega_1, \mu^{\aleph_0})$ is characterized by the existence of an $\aleph_1$-closed dense subset, density $\mu^{\aleph_0}$ and $(\omega_1, \cdot, \mu)$-nowhere distributivity (which is a weaker condition than $(\omega_1, \cdot, \mu^{\aleph_0})$-nowhere distributivity which is actually satisfied).

1.4. **The problem.** Let μ be a singular cardinal of countable cofinality. Two of the properties which characterize comp Col $(\omega_1, \mu^{\aleph_0})$ hold also in the quotient algebra $\mathcal{P}_\mu(\mu)$: $\aleph_1$-completeness (easily) and $(\omega_1, \cdot, \lambda)$-nowhere distributivity (by Theorem 1.2). Could it be true that comp $\mathcal{P}_\mu(\mu)$ and comp Col $(\omega_1, \mu^{\aleph_0})$ are isomorphic? An old independence result of Baumgartner's rules that out. Baumgartner forced an almost disjoint family in $\mathcal{P}(\mu)$ of size $2^\mu > \mu^{\aleph_0}$, showing thus that it is consistent with ZFC that the cellularity, hence density, of $\mathcal{P}_\mu(\mu)$ strictly exceeds $\mu^{\aleph_0}$ ([7], 6.1).

In Baumgartner's model $\operatorname{Comp}\mathcal{P}_\mu(\mu)$ cannot be isomorphic to $\operatorname{Comp}\operatorname{Col}(\omega_1, \mu^{\aleph_0})$, whose density is exactly $\mu^{\aleph_0}$.

However, if one assumes that $\mu^{\aleph_0} = 2^\mu$, it follows trivially that the density of $\mathcal{P}_\mu(\mu)$ is $\mu^{\aleph_0}$, and hence, by the aforementioned characterization of $\operatorname{comp}\operatorname{Col}(\omega_1, \mu^{\aleph_0})$, it is isomorphic to $\operatorname{comp}\mathcal{P}_\mu(\mu)$. In particular, denoting by $\vdash$ provability in ZFC, we have [2]:

$$\mu^{\aleph_0} = 2^\mu \vdash \mu^{\aleph_0} \text{ collapses to } \aleph_1 \text{ in } V^{\mathcal{P}_\mu(\mu)} \tag{2}$$

What, then, is the precise relation between $\mathcal{P}_\mu(\mu)$ and Col $(\omega_1, \mu^{\aleph_0})$? Most importantly, does forcing with $\mathcal{P}_\mu(\mu)$ *always* collapse $\mu^{\aleph_0}$ to $\aleph_1$?

Balcar and Simon conjectured in [4] that the answer is "yes", namely, that the cardinal arithmetic assumption $\mu^{\aleph_0} = 2^\mu$ could be removed from (2). In the same paper they advance towards an affirmative solution of their conjecture by proving in ZFC that forcing with $\mathcal{P}_\mu(\mu)$ collapse the $2^{\aleph_0}$ to ω_1. Since for $\mu < 2^{\aleph_0}$ it holds trivially that $\mu^{\aleph_0} = 2^{\aleph_0}$, that proves their conjecture for all countably cofinal singular cardinals μ which are below the continuum:

$$\mu < 2^{\aleph_0} \vdash \mu^{\aleph_0} \text{ collapses to } \aleph_1 \text{ in } V^{\mathcal{P}_\mu(\mu)} \tag{3}$$

Finally, there was the problem of computing the *Baire number* of the space $U(\mu)$ of all uniform ultrafilters over μ. An ultrafilter u over μ is *uniform* if it does not contain a set of cardinality $< \mu$. With the usual topology, in which the basic open sets are $\hat{p} = \{u \in U(\mu) : p \in u\}$ for $p \in [\mu]^\mu$, the space $U(\mu)$ is a compact Hausdorff space and is therefore not coverable by ω_1 nowhere-dense sets. The *Baire number* of a space with no isolated points is the least number of nowhere-dense sets needed to cover the space. In [4] it was proved that the Baire number of $U(\mu)$ is ω_2 under any of the following assumptions: (*i*) $2^{\aleph_0} > \aleph_1$, (*ii*) $2^\mu = \mu^{\aleph_0}$ or (*iii*) $2^{\omega_1} = \omega_2$. It was conjectured that the Baire number of $U(\mu)$ could be shown to equal ω_2 in ZFC alone.

1.5. **The solution.** The main result in the present paper determines the precise relation between $\operatorname{Comp}\mathcal{P}_\mu(\mu)$ and $\operatorname{Comp}\operatorname{Coll}(\omega_1, \mu^{\aleph_0})$. The collapse algebra is isomorphic to a complete subalgebra of the quotient algebra (Theorem 2.1 below):

$$\vdash \operatorname{comp}\operatorname{Col}(\omega_1, \mu^{\aleph_0}) \lessdot \operatorname{comp}\mathcal{P}_\mu(\mu) \tag{4}$$

This implies that the universe $V^{\mathcal{P}_\mu(\mu)}$ contains $V^{\operatorname{Coll}(\omega_1, \mu^{\aleph_0})}$ as a subuniverse. Therefore,

$$\vdash \mu^{\aleph_0} \text{ collapses to } \omega_1 \text{ in } V^{\mathcal{P}_\mu(\mu)} \tag{5}$$

which proves the conjecture. An easy corollary of (4) is that the Baire number of $U(\mu)$ is equal to ω_2 (Theorem 2.15 below).

Balcar and Simon stated in [4] another ZFC conjecture concerning singular cardinals of uncountable cofinality. The authors will present a solution of that conjecture in a sequel paper.

1.6. **History.** B. Balcar presented this conjecture to the authors during a meeting in Hattingen, Germany, in June of 1999. Shelah then proved, using the Erdős-Rado theorem, that:

(6) $$\mu > 2^{\aleph_0} \vdash \mu^{\aleph_0} \text{ collapses to } 2^{\aleph_0} \text{ in } V^{\mathcal{P}_\mu(\mu)}$$

This affirmed the ZFC conjecture, since (3) and (6) together give (5) ($2^{\aleph_0} = \mu$ is of course impossible by König's Lemma). In August of 1999 Kojman found a direct ZFC proof of (4) by replacing Shelah's use of the Erdős-Rado theorem (which requires cardinal arithmetic assumptions) by a use of a pcf theorem. This proof is presented below.

1.7. **Description of the proof.** Let $P = \langle [\mu]^\mu, \leq\rangle$ where, for $p_1, p_2 \in P$, $p_1 \leq p_2 \iff |p_1 - p_2| < \mu$. For every $D \subseteq \mathcal{P}_\mu(\mu)$, D is dense in $\mathcal{P}_\mu(\mu)$ if and only if $\bigcup D$ is dense in P and D is a filter in $\mathcal{P}_\mu(\mu)$ if and only if $\bigcup D$ is a filter in P. Therefore, $G \subseteq \mathcal{P}_\mu(\mu)$ is a generic filter over $\mathcal{P}_\mu(\mu)$ if and only if $\bigcup G$ is a generic filter over P. Hence $V^{\mathcal{P}_\mu(\mu)} = V^P$. For convenience, we work with P rather than with $\mathcal{P}_\mu(\mu)$. Let λ denote $\mu^{\aleph_0}$.

The main point in finding a complete copy of $\operatorname{comp}\operatorname{Col}(\omega_1, \lambda)$ inside $\operatorname{comp}\mathcal{P}_\mu(\mu)$ is to overcome the large cellularity that $\operatorname{Comp} P$ may possess, e.g. in Baumgartner's model. This is achieved by forcing only with $Q \subseteq P$, which contains all *closed* conditions of P. It is not hard to verify that $\operatorname{Comp} Q$ is isomorphic to a complete subalgebra of $\operatorname{Comp} P$. Then it is shown that $\operatorname{Comp} Q \cong \operatorname{Comp}\operatorname{Col}(\omega_1, \lambda)$. To that end one needs to prove that $\pi(Q) = \lambda$. This fact is achieved by an old trick: club guessing. Once density is out of the way, it remains to establish $(\omega_1, \cdot, \mu)$-nowhere distributivity of $\operatorname{comp} Q$, to facilitate the use of Theorem 1.3 above. Here another pcf tool is used: the Trichotomy Theorem.

1.8. **Notation and preliminaries.** Our notation is mostly standard. One exception is that when the relations $f <_U g$, $f \leq_U g$ for ordinal functions f, g where U an ultrafilter over ω is extended to *partial* functions. We recall that if P and Q are posets and for some P-name $\underset{\sim}{G}$ it holds that $\Vdash_P$ "$\underset{\sim}{G}$ is a generic filter over Q" and for every $q \in Q$ there exists $p \in P$ such that $p \Vdash q \in \underset{\sim}{G}$, then $\operatorname{Comp} Q$ is isomorphic to a complete subalgebra of the $\operatorname{Comp} P$ via the embedding $b \mapsto \sum\{p \in P : p \Vdash_P \text{“}b \in \underset{\sim}{G}\text{''}\}$.

The following two theorems from pcf theory will be used:

Theorem 1.4 (Club Guessing). *If $\kappa^+ < \lambda$ and κ, λ are regular cardinals, then there exists a sequence $\overline{C} = \langle c_\delta : \delta < \lambda \wedge \operatorname{cf}\delta = \kappa\}$ so that:*

1. *For every $\delta < \lambda$ with $\operatorname{cf}\delta = \kappa$, c_δ is closed and unbounded in δ and $\operatorname{otp} c_\delta = \kappa$.*
2. *For every club E of λ there exists $\delta \in S^\lambda_\kappa$ so that $c_\delta \subseteq E$.*

Theorem 1.5. *(The Trichotomy) Suppose A is an infinite set, I an ideal over A and $\lambda > |A|^+$ a regular cardinal. If $\overline{f} = \langle f_\alpha : \alpha < \lambda\rangle$ is a $<_I$-increasing sequence of ordinal functions on A, then one of the following conditions holds:*

- *(Good) $\overline{f}$ has an exact upper bound f with $\operatorname{cf} f(a) > |A|$ for all $a \in A$;*
- *(Bad) there are sets $S(a)$ for $a \in A$ satisfying $|S(a)| \leq |A|$ and an ultrafilter D over A extending the dual of I so that for all $\alpha < \lambda$ there exists $h_\alpha \in \prod S(a)$ and $\beta < \lambda$ such that $f_\alpha <_D h_\alpha <_D f_\beta$.*
- *(Ugly) there is a function $g : A \to On$ such that letting $t_\alpha = \{a \in A : f_\alpha(a) > g(a)\}$, the sequence $\overline{t} = \langle t_\alpha : \alpha < \lambda\rangle$ does not stabilize modulo I.*

A proof of the club guessing Theorem can be found in [17], III,§1, [10] or the appendix to [12]. The Trichotomy Theorem is Lemma 3.1 in [17], and a shorter proof of it is available in the appendix to [11].

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2. The proof

Throughout this Section let μ be a fixed singular cardinal of countable cofinality, and let μ_n be a fixed strictly increasing sequence of regular cardinals with $\sum \mu_n = \mu$. Let $P = \{p \subseteq \mu : |p| = \mu\}$ and for $p_1, p_2 \in P$ let $p_1 \leq p_2$ iff $|p_1 - p_2| < \mu$. As pointed out in the introduction, forcing with P is equivalent to adding a generic filter to $\mathcal{P}_\mu(\mu)$.

Denote $\lambda = \mu^{\aleph_0} = |\prod \mu_n|$.

Theorem 2.1. *comp* $Col(\omega_1, \lambda) \lessdot$ *comp* $\mathcal{P}_\mu(\mu)$.

Proof. Let $Q = \{q \in P : q \text{ is closed}\}$. For a condition $p \in P$, let $\operatorname{acc} p$ be the set of all accumulation points of p. Clearly, $\operatorname{acc} p \in Q$

Lemma 2.2. *Let $G \subseteq P$ be a generic filter. Then $G_1 = \{q \in Q : (\exists p \in G)(\operatorname{acc} p \leq q\}$ is generic in Q.*

Proof. If $p_1, p_2 \in G$ then there is some $p_3 \in G$ so that $p_3 \subseteq p_1 \cap p_2$. So $\operatorname{acc} p_3 \leq \operatorname{acc} p_1, \operatorname{acc} p_2$ and $\operatorname{acc} p_3 \in \{\operatorname{acc} p : p \in G\}$. Thus G_1 is closed under finite intersections. Clearly, G_1 is upwards closed. Thus G_1 is a filter.

Suppose that $D \subseteq Q$ is dense and downwards closed. Let $p \in P$ be arbitrary, and consider $q = \operatorname{acc} p$. Let $q_1 \leq q$ be chosen in D.

For $\alpha \in q_1$ define $\beta_\alpha = \min(p - (\alpha + 1))$, and let $p_1 = \{\beta_\alpha : \alpha \in q_1\}$. $q_2 = \operatorname{acc} p_1 \subseteq \operatorname{acc} q_1 \leq q_1$ so there is some $p_1 \leq p$ with $\operatorname{acc} p_1 \in D$. □

By this Lemma it follows that

$$\operatorname{comp} Q \lessdot \operatorname{comp} P \tag{7}$$

We aim now to show that

$$\operatorname{comp} Q \cong \operatorname{comp} \operatorname{Col}(\omega_1, \lambda) \tag{8}$$

First, we shall see that $\pi(Q) = \lambda$.

Let $q \in Q$ be arbitrary. Let $a(p) = \{n : p \cap [\mu_n, \mu_{n+1}) \neq \emptyset\}$ and let $\{m_n : n < \omega\}$ be the increasing enumeration of $a(p)$.

Definition 2.3. *A condition $q \in Q$ is* normal *if it satisfies*

$$\operatorname{otp}[q \cap [\mu_{m_n}, \mu_{m_n+1})] = \mu_n + 1 \tag{9}$$

Lemma 2.4. *The set of normal conditions is dense in Q.*

Proof. Given a condition q, let m_n be the least so that $q \cap [\mu_{m_n}, \mu_{m_n+1})| \geq \mu_n$ and choose $c_n \subseteq q \cap [\mu_{m_n}, \mu_{m_n+1})$ of order type $\mu_n + 1$. Let $q' = \bigcup_n c_n$. Thus $q' \leq q$, and is of the form (9) above. □

Let M be a fixed elementary submodel of $\langle H(\Omega), \in \rangle$ for a sufficiently large regular cardinal Ω so that $\mu \subseteq M$, $[M]^{\aleph_0} \subseteq M$ and the cardinality of M is λ. Let $Q^M = Q \cap M$. Clearly, $|Q^M| = \lambda$.

Lemma 2.5. *Q^M is dense in Q.*

Proof. Let q be a condition in Q and assume, without loss of generality, that it is normal. Let $c_n = q \cap [\mu_{m_n}, \mu_{m_n+1})$.

Claim 2.6. *For every n, there exists a closed subset of c_{n+2} of order type $\mu_n + 1$ which belongs to M.*

Proof. Let $\gamma = \sup c_{n+2}$. In M, fix an increasing and continuous function $f : \mu_{n+2} \to \gamma$ with $\sup \operatorname{ran} f = \gamma$. Let $E = \{i < \mu_{n+2} : f(i) \in c_{n+2}\}$. Thus $E \subseteq \mu_{n+2}$ is a club in μ_{n+2}.

The club E itself may not belong to M (because c_{n+2} may not belong to M). But since $\mu_n^+ < \mu_{n+2}$ and both (regular) cardinals belong to M, M contains some club guessing sequence $\langle c_\delta : \delta < \mu_{n+2} \wedge \operatorname{cf}\delta = \mu_n \rangle$ by the club guessing Theorem 1.4 above. Thus there is some $\delta < \mu_{n+2}$ so that $c_\delta \subseteq E$. Clearly, $c_\delta \in M$. Since $f \in M$, also $\operatorname{ran}(f \restriction c_\delta) \cup \{\sup \operatorname{ran}(f \restriction c_\delta)\} \subseteq E$ belongs to M, and is a closed subset of c_{n+2} of order type $\mu_n + 1$. □

Using the claim, choose, for every n, a closed set b_n so that $b_n \subseteq c_{n+2}$, $\operatorname{otp} b_n = \mu_n + 1$ and $b_n \in M$. Since M is closed under countable sequences, $q' = \bigcup b_n \in Q^M$, and clearly $q' \leq q$ is a normal condition in Q. □

This has established that $\pi(Q) = \lambda$.

We need the following simple fact about Q and Q^M:

Fact 2.7. *Q is $\aleph_1$-complete and Q^M is $\aleph_1$-complete.*

Proof. Suppose that $q_0 \geq q_1 \geq \ldots$ is a decreasing sequence of conditions in Q. By induction on n, let m_n be chosen so that $\operatorname{otp}[q_n \cap [\mu_{m_n}, \mu_{m_n+1})] > \sum_{i<n} |q_n - q_i|^+$, and choose a closed subset c_{n+1} of $\bigcap_{i \leq n} q_i \cap [\mu_{m_n}, \mu_{m_n+1})$ with $\operatorname{otp} c_n = \mu_n + 1$. The condition $\bigcup c_n$ belongs to Q and $q \leq q_n$ for all n. If each q_n belongs to M then the sequence itself belongs to M because M is closed under taking ω-sequences, and hence some q which satisfies $q \leq q_n$ for all n belongs to M, by elementarity. (Alternatively, one can do the induction for proving completeness of Q inside M). □

Thus, $\operatorname{comp} Q$ contains an $\aleph_1$-complete dense set of size λ. To prove (8) from Theorem 1.3 it remains to show that $\operatorname{comp} Q$ is $(\omega_1, \cdot, \mu)$-nowhere distributive. For this purpose we inspect the generic cut which Q creates in $\prod \mu_n/U$, where U is the generic ultrafilter over ω which forcing with Q introduces.

Fact 2.8. *Suppose $G \subseteq Q$ is a generic filter. Then $\{a(q) : q \in G\}$ is an ultrafilter over ω.*

Proof. If $q_1 \leq q_2$ are normal conditions, then $a(q_1) - a(q_2)$ is finite. Thus $a : \{q \in Q : q \text{ is normal}\} \to \mathcal{P}(\omega)$ is an order preserving map onto $\langle \mathcal{P}(\omega), \subseteq^* \rangle$. Furthermore, if $t \leq a(q)$, then there is some $q' \leq q$ such that $a(q') = t$. Therefore the image of a generic $G \subseteq Q$ under a is an ultrafilter over ω. □

Given a normal condition $q \in Q$, define the following two functions on $a(q)$ by letting, for each $n \in a(q)$,

$$\chi_q^+(n) = \sup[q \cap [\mu_n, \mu_{n+1})]$$
$$\chi_q^-(n) = \min[q \cap [\mu_n, \mu_{n+1})]$$

The set of conditions q for which $\chi_q^+ \in \prod \mu_n$ is clearly dense in Q, so we always assume that $\chi_q^+ \in \prod \mu_n$.

Since $\chi_q^-(n) < \chi^- + q(n)$ for every $n \in a(q)$, and $q \Vdash$ "$a(q) \in U$", it follows that $q \Vdash$ "$\chi_q^- <_U \chi_q^+$".

Let $G \subseteq Q$ be a generic filter. Define

$$D_0^+ = \{\chi_q^+ : q \in G \text{ and } q \text{ is normal}\}$$
$$D_0^- = \{\chi_q^- : q \in G \text{ and } q \text{ is normal}\}$$

Now for each normal $q \in Q$, $q \Vdash \chi_q^+ \in D_0^+ \wedge \chi_q^- \in D_0^-$".

Let

$$D^+ = \{f \in \prod \mu_n : (\exists g \in D_0^+)(g \leq_U f)\}$$
$$D^- = \{f \in \prod \mu_n : (\exists g \in D_0^-)(f \leq_U g)\}$$

Lemma 2.9. *Suppose that $f \in \prod \mu_n$ and $q \in Q$ is a normal condition. Then there exists a normal condition $q' \leq q$ so that*

$$q' \Vdash \chi_{q'}^+ <_U f \vee f <_U \chi_{q'}^-$$

Proof. If for some infinite set $B \subseteq A(q)$, $m_n \in a(q) \Rightarrow f(m_n) < \chi_q^+(m_n)$ then $q' := \bigcup_{m_n \in B} c_n - (f(m_n) + 1)$ is a normal condition and for all $n \in a(q')$ it holds that $\chi_{q'}^-(n) > f(n)$. Since $q' \Vdash a(q') \in U$, the second alternative holds for q'.

If $\{n \in a(q) : f(n) < \chi_q^+(n)\}$ is finite, let n_0 be fixed so that for every $n > n_0$ it holds that $\sup c_n \leq f(n)$ and let, for $n > n_0$, $b_n \subseteq c_n$ be the initial segment of c_n whose order type is $\mu_{n-1}+1$. Now $\bigcup b_n$ is a normal condition and $q' \Vdash \chi_q^+ <_U f$. $\square$

Since Q is $\aleph_1$-complete, no new members are added to $\prod \mu_n$ after forcing with Q. Therefore, by Lemma 2.9, it holds that D^- is a lower half of a Dedekind cut in $\prod \mu_n / U$ whose upper half is D^+; that D^- has no last element and that D^+ has no first element. By the definition of D^-, it is clear that D_0^- is cofinal in $(D^-, <_U)$. Furthermore, if $\{f_i : i < \omega\}$ is a set of functions, $q \in Q$ and for all $i < \omega$ it holds that $q \Vdash f_i \in D^-$ then by iterated use of Lemma 2.9 and $\aleph_1$-completeness there exists $q' \leq q$ so that $q' \Vdash \bigwedge_i f_i <_U \chi_{q'}^-$. As a consequence, the cofinality of D^- is uncountable.

We shall need the following strengthening of Lemma 2.9, which says that the generic cut (D^-, D^+) is not trapped by any product of countable sets.

Lemma 2.10. *Suppose $A_n \subseteq [\mu_n, \mu_{n+1})$ is a countable set for each $n < \omega$, and $q \in Q$ is a normal condition. Then there is a condition $q' \leq q$ in Q so that for every $n \in a(q')$ it holds that $A_n \cap (\chi_{q'}^-, \chi_{q'}^+) = \emptyset$.*

Proof. Let $\varepsilon_n < \omega_1$ be the order type of A_n and let $\langle \alpha^n_i : i < \varepsilon_n \rangle$ be the increasing enumeration of A_n. Partition $[\mu_n, \mu_{n+1})$ to the intervals $[\mu_n, \alpha^n_0)$, $\{[\alpha^n_i, \alpha^n_{i+1}) : i+1 < \varepsilon_n\}$ and $[\sup A_n, \mu_{n+1})$. For every $n > 0$, choose an interval I_n in the partition of $[\mu_n, \mu_{n+1})$ so that $|I_n \cap q| = \mu_n$, and let $c_n \subseteq (I_n \cap q)$ be closed of order type $\mu_{n-1} + 1$. Let $q' = \bigcup I_n \cap q$. □

Lemma 2.11. *The cofinality of D^- is ω_1.*

Proof. We have seen that $\mathrm{cf}(D^-) > \aleph_0$. Suppose now, to the contrary, that $\kappa > \aleph_1$ is regular and that $q \Vdash$ "$\overline{f} = \langle f_i : i < \kappa \rangle$ is $<_U$ increasing and cofinal in D". The Trichotomy Theorem applies to $\overline{f}$, but:

The third condition ("Ugly") cannot hold , since U is an ultrafilter.

The first condition ("Good") cannot hold, because in D^+ there is no first element.

Let us see now that the second condition ("Bad") cannot hold either. Suppose that $q \Vdash$ "$\langle A_n : n < \omega \rangle$, witnesses 'Bad' for $\overline{f}$". Then,

$$q \Vdash (\forall \alpha < \kappa)(\exists h \in \prod A_n)(\exists \beta < \kappa)[f_\alpha <_U h <_U f_\beta] \tag{10}$$

By $\aleph_1$-completeness, we may assume that $\langle A_n : n < \omega \rangle$ and each A_n belong to the ground model. By Lemma 2.10 there is a condition $q' \leq q$ so that for all $n \in a(q)$,

$$A_n \cap \left(\chi^-_{q'}(n), \chi^+_{q'}(n) \right) = \emptyset \tag{11}$$

Since q forces that $\overline{f}$ is cofinal in D^- and $q' \Vdash \chi^-_{q'} \in D^- \wedge \chi^+_{q'} \in D^+$, there is some $\alpha < \kappa$ and $q'' \leq q'$ so that

$$q'' \Vdash \chi^-_{q'} <_U f_\alpha <_U \chi^+_{q'} \tag{12}$$

By strengthening q'' we may assume that for some $\beta < \kappa$ and $h \in \prod A_n$,

$$q'' \Vdash \chi^-_{q'} <_U f_\alpha <_U h <_U f_\beta <_U \chi^+_{q'} \tag{13}$$

So there is some n (in fact, infinitely many) so that

$$\chi^-_{q'}(n) < f_\alpha(n) < h(n) < f_\beta(n) < \chi^+_{q'}(n) \tag{14}$$

This is a contradiction to (11), since $h(n) \in A_n$.

Thus, the cofinality of D^- is at least ω_1 and no more than ω_1; so it is exactly ω_1. □

Since $\mathrm{cf}(D^-, <_U) = \omega_1$ and D^-_0 is cofinal in $(D^-, <_U)$, using $\aleph_1$-completeness of Q it is easy to find a sequence of conditions $\langle q(i) : i < \omega_1 \rangle \subseteq G$ such that $i < j < \omega_1 \Rightarrow q_(i) \geq q(j)$ and $\langle \chi^-_{q(i)} : i < \omega_1 \rangle$ is ($<_U$-increasing and) cofinal in $(D^-, <_U)$. Fix a Q-name $\underset{\sim}{q}$ for such a sequence. Observe that if q_1, q_2 are incompatible, then $\Vdash$ "$\neg(q_1 \in \mathrm{ran}\underset{\sim}{q} \wedge q_2 \in \mathrm{ran}\underset{\sim}{q})$", since any two conditions in $\mathrm{ran}\underset{\sim}{q}$ are compatible.

Lemma 2.12. *For every $q \in Q$ there is a set $\{q''_\alpha : \alpha < \lambda\}$ of pairwise incompatible conditions below q, so that for each $\alpha < \lambda$ there is $q'_\alpha \leq q$ and $i(\alpha)$ so that $q'_\alpha \Vdash \underset{\sim}{q}(i(\alpha)) = q''_\alpha$, and $\{q'_\alpha : \alpha < \lambda\}$ are pairwise incompatible.*

Proof. Let $q \in Q$ be a normal condition, and let $c_n = q \cap [\mu_{m_n}, \mu_{m_n+1})$. c_n is a closed set of order type μ_n. For each n let b_n be the initial segment of c_{n+1} of order type $(\prod_{i \le n} \mu_i, <_{\mathrm{lx}})$, the lexicographic ordering of all sequences $(x_0, x_1, \ldots, x_n)$ in the product $\mu_0 \times \mu_1 \cdots \times \mu_n$. Identify each member in b_n with the sequence in $\prod_{i \le n} \mu_i$ it corresponds to via the order isomorphism, and define a projection $\pi_{m,n} : b_n \to b_m$ for $m < n$ by mapping a sequence of length n to its initial segment of length m. The inverse limit of this system is the set of all functions $g \in \prod b_n$ with the property that for all $m < n$, $\pi_{m,n} g(n) = g(m)$. Denote this set of functions by $L \subseteq \prod b_n$.

Choose a set of λ different functions $\langle g_\alpha : \alpha < \lambda \rangle \subseteq L \cap \prod \operatorname{acc} b_n$ and for each α let $g'_\alpha(n) = g_\alpha(n) + 1$. Let $q_\alpha = \bigcup_{n>0} b_n \cap [g_\alpha(m_n), g'_\alpha(m_n))$. Thus each q_α is a condition below q. Furthermore, if $\alpha \neq \beta$ then from some point n_0 on, either $\chi^+_{q_\alpha}(n) < \chi^-_{q_\beta}(n)$ or $\chi^+_{q_\beta}(n) < \chi^-_{q_\alpha}(n)$. Thus $\{q_\alpha : \alpha < \lambda\}$ is a set of pairwise incompatible conditions below q.

For each $\alpha < \lambda$,

$$q_\alpha \Vdash \text{“}\chi^-_{q_\alpha} \in D^- \wedge \chi^+_{q_\alpha} \in D^+\text{”}$$
$$q_\alpha \Vdash \text{“}(\exists i < \omega_1)(\forall j < \omega_1)[i < j \Rightarrow \chi^-_{q_\alpha} <_U \chi^-_{\underset{\sim}{q}(j)}]\text{”}$$

Fix $q'_\alpha \le q_\alpha$ so that for some $i(\alpha) < \omega_1$ and $q''_\alpha \le q_\alpha$, $q'_\alpha \Vdash \underset{\sim}{q}(i(\alpha)) = q''_\alpha$. For $\alpha < \beta < \lambda$, since $q''_\alpha \le q'_\alpha \le q_\alpha$, $q''_\beta \le q'_\beta \le q_\beta$ and q_α, q_β are incompatible, q'_α is incompatible with q'_β and q''_α is incompatible with q''_β. □

Fix, for each $i < \omega_1$, a maximal antichain $P_i \subseteq Q$ of conditions that decides $\underset{\sim}{q}(\alpha)$.

Claim 2.13. *For every condition $q \in Q$ there exists some $i < \omega_1$ so that q is compatible with $\ge \mu$ members of P_i.*

Proof. Let $q \in Q$ be an arbitrary condition. By Lemma 2.12 there are λ pairwise incompatible conditions $\{q''_\alpha : \alpha < \lambda\}$ below q, each of which is forced to be $\underset{\sim}{q}(i(\alpha))$ for some $i(\alpha) < \omega_1$, by some extension $q'_\alpha \le q$, and $\{q'_\alpha : \alpha < \lambda\}$ are pairwise incompatible extensions of q. Since $\lambda > \mu$, there is necessarily some fixed $i < \omega_1$ so that $|\{\alpha < \lambda : \alpha(i) = i\}| > \mu$. Since different q'_α, q'_β in this set force different values for $\underset{\sim}{q}(i)$, they cannot be compatible with the same member of P_i. Thus q is compatible with $\ge \mu$ members of P_i. □

The last claim established $(\omega_1, \cdot, \mu)$-nowhere distributivity of $\operatorname{comp} Q$. By Theorem 1.3 $\operatorname{comp} \operatorname{Col}(\omega_1, \lambda) \cong \operatorname{comp} Q$, and since $\operatorname{comp} Q \lessdot \operatorname{comp} \mathcal{P}_\mu(\mu)$, the proof is complete.

□

Corollary 2.14. $V^{\mathcal{P}_\mu(\mu)} \models |\lambda| = \aleph_1$.

Proof. Since $\operatorname{comp} \operatorname{Col}(\omega_1, \lambda)$ is a complete subalgebra of $\operatorname{comp} \mathcal{P}_\mu(\mu)$, the universe $V^{\operatorname{Col}(\omega_1, \lambda)}$ is contained in $V^{\mathcal{P}_\mu(\mu)}$. Therefore, there is an onto function $\varphi : \omega_1 \to \lambda$ in $V^{\mathcal{P}_\mu(\mu)}$. Since $\mathcal{P}_\mu(\mu)$ is ω_1-complete, ω_1 is preserved in $V^{\mathcal{P}_\mu(\mu)}$. Thus, the cardinality of λ in $V^{\mathcal{P}_\mu(\mu)}$ is $\aleph_1$. □

Corollary 2.15. *For every singular cardinal μ with $\operatorname{cf}\mu = \aleph_0$ the Baire number of $U(\mu)$, the space of uniform ultrafillters over μ, is equal to ω_2.*

Proof. By Theorem 2.1 there exists a dense subset of Q which is isomorphic to the dense subset $D = \{f : \exists i < \omega_1[f : (i+1) \to \lambda]\}$ of $\mathrm{Col}\,(\omega_1, \lambda)$, namely, there are conditions $\{q_f : f \in D\} \subseteq Q$ so that $q_f \leq q_g \iff g \subseteq f$.

Let $W_{i,\alpha} = \{q_f : \mathrm{dom} f = i+1 \wedge f(i) = \alpha\}$. Define $V_{i,\alpha} = \{u \in U(\mu) : (\exists p \in u)(\exists q_f \in W_{i,\alpha})[\mathrm{acc} p \subseteq q_f]\}$. It should be clear that for each $i < \omega_1$ the collection $\{V_{i,\alpha} : \alpha < \lambda\}$ is a family of pairwise disjoint (nonempty) open sets in $U(\mu)$.

For each $\beta < \omega_2$ define $O_\beta = \bigcup\{V_{i,\beta} : i < \omega_1, \beta \leq \alpha < \omega_2\}$. Since O_β is a union of open sets in $U(\mu)$, it is open. We show that O_β is also dense in $U(\mu)$ for each $\beta < \omega_2$. Let $\beta < \omega_2$ be arbitrary and let $\hat{p}$ for $p \in P$ be an arbitrary basic open set in $U(\mu)$. By density of $\{q_f : f \in D\}$ there exists some $f \in \mathrm{Col}\,(\omega_1, \lambda)$ with domain $i + 1 < \omega$ so that $q_f \leq \mathrm{acc} p$. By 2.2, there exists some $p' \leq p$ so that $\mathrm{acc} p' \leq q_{f \cup \langle i+1, \beta \rangle}$. Thus $\hat{p}' \subseteq \hat{p} \cap O_\beta$.

Finally, we show that $\bigcap\{O_\beta : \beta < \omega_2\} = \emptyset$. Let $u \in U(\mu)$ be arbitrary and let $\alpha(u) = \sup\{\alpha < \omega_2 : (\exists i < \omega_1)(u \in V_{i,\alpha})\}$. Since for every $u \in U(\mu)$ and $i < \omega_1$ there is at most one $\alpha < \omega_2$ for which $u \in V_{i,\alpha}$, $\alpha(u) < \omega_2$. Now $u \notin O_{\alpha(u)+1}$.

Thus, by passing to the complements of O_β, $\beta < \omega_2$, it is seen that $U(\mu)$ is coverable by ω_2 nowhere-dense sets. Since it is known [4] that $U(\mu)$ cannot be covered by fewer than ω_2 nowhere-dense sets, its Baire number is equal to ω_2.

□

3. Concluding remarks

We first remark that the part of the proof between Lemma 2.7 and Lemma 2.12 can be applied verbatim to P instead of to Q to show that $\mathrm{comp}\,\mathcal{P}_\mu(\mu)$ is $(\omega_1, \cdot, \mu^{\aleph_0})$-nowhere distributive, and constitutes thus an alternative ZFC proof of $(\omega_1, \cdot, \mu^{\aleph_0})$-nowhere distributivity of $\mathcal{P}_\mu(\mu)$ from the Trichotomy theorem.

Next, we remark that Corollary 2.14 can be derived directly, without invoking Theorem 1.3, as follows: fix a 1-1 function $f : Q^M \to \lambda$ and apply Lemma 2.12 to Q^M, observing that the set $\{q''_\alpha : \alpha < \lambda\}$ belongs so M. Now fix a function $h : \lambda \to \lambda$ such that for every $A \in [\lambda]^\lambda \cap M$, $\mathrm{ran}(h \restriction A) = \lambda$. The function $h \circ \underset{\sim}{q}$ is a collapsing function by a simple density argument.

We devote now a few words to the role of pcf theory in this proof and in several other proofs. Pcf theory was developed to provide bounds on powers of strong limit singular cardinals, or, better, on the covering numbers of singular cardinals. The most well known discovery of the theory is that poset $\langle \mathcal{P}_{\aleph_0}(\aleph_\omega), \supseteq \rangle$ of countable subsets of $\aleph_\omega$ ordered by reverse inclusion has a dense subset of size $< \aleph_{\omega_4}$. In other words: the cardinality of this poset may be arbitrarily large, but its density is bounded.

From the point of view of pcf theory, powers of regular cardinals are the "soft" part of cardinal arithmetic, which envelopes the hard "skeleton" of powers of singulars that pcf theory addresses — the revised power set function pp. To read more about this philosophy the reader is referred to [17] (especially the analytical index. §14), [16] and [10].

The proof above is yet another example of the same theme: a complete subalgebra of density $\mu^{\aleph_0}$ is uncovered inside $\mathrm{Comp}\,\mathcal{P}_\mu(\mu)$, whose own density may be $2^\mu > \mu^{\aleph_0}$ in case the power function at regular cardinals assumes large values. The powers of regular uncountable cardinals may be "peeled off" from $\mathrm{comp}\,\mathcal{P}_\mu(\mu)$ by the club-guessing technique to get to the "skeleton" $\mathrm{comp}\,\mathrm{Col}\,(\omega_1, \mu^{\aleph_0})$.

Pcf methods are used also in other contexts to show that various structures on the power set of a singular cardinal contain "skeletons" of bounded cardinality. We quote the example [13] of a Dowker subspace of cardinality $\aleph_{\omega+1}$ inside M. E. Rudin's Dowker space [15], whose cardinality is $(\aleph_\omega)^{\aleph_0}$.

Pcf techniques were used for studying collapses of cardinals by Cummings [8] (see also [11]).

Lastly, we remark that while the role of closed unbounded subsets of *regular* cardinals in combinatorial set theory is so central that one could not imagine uncountable combinatorics without them, the proof above shows that also closed subsets of a *singular* cardinal may be sometimes useful.

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