# A Note on Extensions of Infinitary Logic*

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October 6, 2003

## Abstract

We show that a strong form of the so called Lindström's Theorem [4] fails to generalize to extensions of $L_{\kappa\omega}$ and $L_{\kappa\kappa}$: For weakly compact $\kappa$ there is no strongest extension of $L_{\kappa\omega}$ with the $(\kappa, \kappa)$-compactness property and the Löwenheim-Skolem theorem down to $\kappa$. With an additional set-theoretic assumption, there is no strongest extension of $L_{\kappa\kappa}$ with the $(\kappa, \kappa)$-compactness property and the Löwenheim-Skolem theorem down to $< \kappa$.

By a well-known theorem of Lindström [4], first order logic $L_{\omega\omega}$ is the strongest logic which satisifies the compactness theorem and the downward Löwenheim-Skolem theorem. For weakly compact $\kappa$, the infinitary logic $L_{\kappa\omega}$ satisfies both the $(\kappa, \kappa)$-compactness property and the Löwenheim-Skolem theorem down to $\kappa$. In [1] Jon Barwise pointed out that $L_{\kappa\omega}$ is not maximal with respect to these properties, and asked what is the strongest logic based on a weakly compact cardinal $\kappa$ which still satisfies the $(\kappa, \kappa)$-compactness property and some other natural conditions suggested by $\kappa$. We prove (Corollary 5) that for weakly compact $\kappa$ there is no strongest extension of $L_{\kappa\omega}$ with

the $(\kappa, \kappa)$-compactness property and the Löwenheim-Skolem theorem down to $\kappa$. This shows that there is no extension of $L_{\kappa\omega}$ which would satisify the most obvious generalization of Lindström's Theorem. A stronger result (Theorem 11) is proved under an additional assumption.

We use the notation and terminology of [2, Chapter II] as much as possible. We will work with concrete logics such as first order logic $L_{\omega\omega}$, infinitary logic $L_{\kappa\lambda}$ and their extensions $L_{\omega\omega}(\{Q_i : i \in I\})$ and $L_{\kappa\lambda}(\{Q_i : i \in I\})$ by generalized quantifiers. Therefore it is not at all critical which definition of a logic one uses as long as these logics are included and some basic closure properties are respected. We use $\mathcal{L} \leq \mathcal{L}'$ to denote the sublogic relation. Let $\mathcal{P}$ be a property of logics. A logic $\mathcal{L}^*$ is *strongest extension of $\mathcal{L}$ with $\mathcal{P}$*, if

1. $\mathcal{L} \leq \mathcal{L}^*$,

2. $\mathcal{L}^*$ has property $\mathcal{P}$,

and whenever $\mathcal{L}$ is a sublogic $\mathcal{L}'$ and $\mathcal{L}'$ has property $\mathcal{P}$, then $\mathcal{L}' \leq \mathcal{L}^*$.

Let $\mathcal{L}$ be a logic and $\kappa$ and $\lambda$ infinite cardinals. $\mathcal{L}$ is *$(\kappa, \lambda)$-compact* if for all $\Phi \subseteq \mathcal{L}$ of power $\kappa$, if each subset of $\Phi$ of cardinality $< \lambda$ has a model, then $\Phi$ has a model. $\mathcal{L}$ is *$\kappa$-compact* if it is $(\kappa, \omega)$-compact. $\mathcal{L}$ is *weakly $\kappa$-compact* if $\mathcal{L}$ is $(\kappa, \kappa)$-compact. $\mathcal{L}$ is *fully compact* if it is $\kappa$-compact for all $\kappa$. $\mathcal{L}$ has the *Löwenheim-Skolem property down to $\kappa$*, denoted by $\mathrm{LS}(\kappa)$ if every $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $\leq \kappa$. If every sentence $\phi \in \mathcal{L}$ which has a model, has a model of cardinality $< \kappa$, we say that $\mathcal{L}$ satisfies $\mathrm{LS}(< \kappa)$. The order-type of the well-ordering $R$ is denoted by $\mathrm{otp}(R)$.

**Theorem 1** *[4] The logic $L_{\omega\omega}$ is strongest extension of $L_{\omega\omega}$ with $\aleph_0$-compactness and $LS(\aleph_0)$.*

Let $C$ be a class of cardinals. Let

$$\mathfrak{A} \models \mathrm{Q}_C^{\mathrm{cf}} xy\phi(x, y, \vec{z}) \iff \mathrm{otp}(\{\langle a, b \rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}) \in C.$$

By [9], $L_{\omega\omega}(\mathrm{Q}_C^{\mathrm{cf}})$ is always fully compact. For $C$ an interval we use the notation $\mathrm{Q}_{[\kappa,\lambda)}^{\mathrm{cf}}$ and $\mathrm{Q}_{[\kappa,\lambda]}^{\mathrm{cf}}$.

**Proposition 2** *There is no strongest $\kappa$-compact extension of $L_{\omega\omega}$. In fact:*

1. *there are fully compact logics $\mathcal{L}_n$, $n < \omega$, such that $\mathcal{L}_n \leq \mathcal{L}_{n+1}$ for all $n < \omega$, but no $\aleph_0$-compact logic extends each $\mathcal{L}_n$.*

*2. There is an $\aleph_0$-compact logic $\mathcal{L}_1$ and a fully compact logic $\mathcal{L}_2$ such that no $\aleph_0$-compact logic extends both $\mathcal{L}_1$ and $L_2$.*

**Proof.** Let $\mathcal{L}_n = L_{\omega\omega}(\{Q^{\mathrm{cf}}_{[\aleph_\omega,\infty]}\} \cup \{Q^{\mathrm{cf}}_{\aleph_l} : l < n\})$. By [9], each $\mathcal{L}_n$ is fully compact. Clearly, no $\aleph_0$-compact logic can extend each $\mathcal{L}_n$.

For the second claim, let $\mathcal{L}_1$ be the logic $L_{\omega\omega}(Q_1)$, where $Q_1$ is the quantifier "there exists uncountable many" introduced by Mostowski [8]. This logic is $\aleph_0$-compact [3], see [2, Chapter IV] for more recent results. Let $\mathcal{L}_2$ be the logic $L_{\omega\omega}(Q_B)$, where $Q_B$ is the quantifier "there is a branch" introduced by Shelah [10]. More exactly,

$$Q_B xytuM(x)T(y)(t \leq u)$$

if and only if $\leq_T$ is a partial order of $T \subseteq M$ and there are $D, \leq_D, f$ and $B$ such that:

1. $\leq_D$ is a total order of $D \subseteq M$

2. $f : \langle T, \leq_T \rangle \to \langle D, \leq_D \rangle$ is strictly increasing

3. $\forall s \in D \exists p \in T(f(p) = s)$

4. $B \subseteq T$ is totally ordered by $\leq_T$

5. $\forall b \in B((p \in T \& p \leq_T b) \to p \in B)$

6. $\forall s \in D \exists b \in B(s \leq_D f(b))$.

The reader is referred to [10] for a proof of the full compactness of $\mathcal{L}_2$.

Suppose there were an $\aleph_0$-compact logic $\mathcal{L}$ containing both $\mathcal{L}_1$ and $\mathcal{L}_2$ as a sublogic. It is easy to see that the class of countable well-orders can be expressed as a relativized pseudoelementary class in $\mathcal{L}$. This contradicts $\aleph_0$-compactness of $\mathcal{L}$. $\square$

Lauri Hella pointed out that by elaborating the proof of claim (2) of the above proposition, we can make $\mathcal{L}_1$ fully compact. It was proved in [11] that, assuming GCH, there is no strongest extension of $\mathcal{L}_{\omega\omega}$ which is $\aleph_0$-compact. Our proof of (1) of the above proposition essentially occurs in a note, based on a suggestion of Paolo Lipparini, added after Theorem 8 of [11].

**Proposition 3** *Suppose $\kappa > \aleph_0$. There is no strongest extension of $L_{\kappa^+\omega}$ with $LS(\kappa)$*

**Proof.** Let $\mathcal{L}_1 = L_{\kappa^+\omega}(Q^{\mathrm{cf}}_{\aleph_0})$ and $\mathcal{L}_2 = L_{\kappa^+\omega}(Q^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. By using standard arguments with elementary chains of submodels, it is easy to see that both $\mathcal{L}_1$ and $\mathcal{L}_2$ have $\mathrm{LS}(\kappa)$, but the consistent sentence

$$R \text{ is a linear order with no last element } \wedge$$

$$\neg Q^{\mathrm{cf}}_{\aleph_0} xy R(x,y) \wedge \neg Q^{\mathrm{cf}}_{[\aleph_1,\kappa]} xy R(x,y)$$

has no models of size $\leq \kappa$. $\square$

It was proved in [11] that there is no strongest extension of $\mathcal{L}_{\omega\omega}$ with $\mathrm{LS}(\omega)$.

**Lemma 4** *Suppose $\kappa$ is weakly compact. Then $L_{\kappa\omega}(Q^{cf}_{\{\aleph_0\}})$ and $\mathcal{L}_{\kappa\omega}(Q^{cf}_{[\aleph_1,\kappa]})$ are weakly $\kappa$-compact. Moreover, if $\kappa > \omega$, these logics satisfy $LS(\kappa)$.*

**Proof.** The claim concerning $\mathrm{LS}(\kappa)$ is proved with a standard elementary chain argument. We prove the weak compactness of $\mathcal{L}_{\kappa\omega}(Q^{\mathrm{cf}}_{[\aleph_1,\kappa]})$. The case of $L_{\kappa\omega}(Q^{\mathrm{cf}}_{\{\aleph_0\}})$ is similar, but easier. For this end, suppose $T$ is a set of sentences of $\mathcal{L}_{\kappa\omega}(Q^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ and $|T| = \kappa$. We may assume $T \subseteq \kappa$. If $\alpha < \kappa$, then we assume that there is a model $\mathfrak{M}_\alpha \models T \cap \alpha$. In view of $\mathrm{LS}(\kappa)$, it is not a loss of generality to assume that $\mathfrak{M}_\alpha = \langle \kappa, R_\alpha \rangle$, where $R_\alpha \subseteq \kappa \times \kappa$. Let $R(\alpha, \beta, \gamma) \iff R_\alpha(\beta, \gamma)$. By weak compactness there is a transitive $M$ of cardinality $\kappa$ such that

$$\langle H(\kappa), \epsilon, T, R \rangle \prec_{L_{\kappa\kappa}} \langle M, \epsilon, T^*, R^* \rangle$$

and $\kappa \in M$. Let $\mathfrak{M} = \langle M, S \rangle$, where $S(x,y) \iff R^*(\kappa, x, y)$. We claim that $\mathfrak{M} \models T$. We need only worry about the cofinality-quantifier. Cofinalities $< \kappa$ can be expressed in $L_{\kappa\kappa}$, so they are preserved both ways. Therefore also cofinality $\kappa$ is preserved, and no other cofinalities can occur as the models have cardinality $\kappa$. $\square$

Since the logics $L_{\kappa\omega}(Q^{\mathrm{cf}}_{\aleph_0}))$ and $\mathcal{L}_{\kappa\omega}(Q^{\mathrm{cf}}_{[\aleph_1,\kappa]})$ cannot both be a sublogics of a logic with $\mathrm{LS}(\kappa)$, we get from the above lemma:

**Corollary 5** *Suppose $\kappa > \omega$ is weakly compact. Then there is no strongest weakly $\kappa$-compact extension of $L_{\kappa\omega}$ with $LS(\kappa)$.*

The logic $L_{\kappa\omega}$ actually satisfies the property $\mathrm{LS}(< \kappa)$ which is stronger than $\mathrm{LS}(\kappa)$. To prove a result like the above corollary for the property

$\mathrm{LS}(<\kappa)$ we have to work a little harder. At the same time we extend the proof to extensions of $L_{\kappa\kappa}$. Here the cofinality quantifiers $\mathrm{Q}_C^{\mathrm{cf}}$ will not help as $\mathrm{Q}_{\{\lambda\}}^{\mathrm{cf}}$ is definable in $L_{\kappa\kappa}$ for $\lambda < \kappa$. Therefore we use more refined order-type quantifiers.

**Definition 6** *Let $L_{\kappa\lambda}(Q)$ denote the formal extension of $L_{\kappa\lambda}$ by the generalized quantifier symbol $Qxy\phi(x, y, \vec{z})$. If $\mathcal{Y}$ is a class of ordinals, we get a logic $L_{\kappa\lambda}(Q, \mathcal{Y})$ from $L_{\kappa\lambda}(Q)$ by defining the semantics by*

$$\mathfrak{A} \models Qxy\phi(x, y, \vec{c}) \iff otp(\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}) \in \mathcal{Y}.$$

*If $\phi \in L_{\kappa\lambda}(Q, \mathcal{Y})$ and $\mathfrak{A} \models \phi$, we say that $\mathfrak{A} \models \phi$ holds* in the $\mathcal{Y}$-interpretation.

If $\mathfrak{A}$ is a model, then

$$o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda)$$

is the supremum of all $otp(\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\})$ where $\phi \in L_{\kappa\lambda}(\mathcal{Y})$, $\vec{c} \in A^{<\lambda}$ and $\{\langle a, b\rangle : \mathfrak{A} \models \phi(a, b, \vec{c})\}$ is well-ordered.

**Lemma 7** *Suppose $\kappa \geq \lambda$, $\phi \in L_{\kappa\lambda}(Q)$, $\mathfrak{A}$ is a model, $\vec{a} \in A^{<\lambda}$, and $\mathcal{Y}' \cap o(\mathfrak{A}, \mathcal{Y}, \kappa, \lambda) = \mathcal{Y}$. Then $\mathfrak{A} \models \phi(\vec{a})$ in the $\mathcal{Y}$-interpretation if and only if $\mathfrak{A} \models \phi(\vec{a})$ in the $\mathcal{Y}'$-interpretation.*

**Proof.** This is a straightforward induction of the length of the formula $\phi$.□

**Lemma 8**    *1. Suppose $\kappa > \omega$, $\phi \in L_{\kappa\kappa}(Q)$, and $\phi$ has a model $\mathfrak{A}$ in the $\mathcal{Y}$-interpretation. Then there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq 2^\kappa$ and $\mathcal{Y}' \subseteq (2^\kappa)^+$ such that $\mathcal{Y}' \cap \kappa = \mathcal{Y}$ and $\mathfrak{B} \models \phi$ in the $\mathcal{Y}'$-interpretation.*

*2. Suppose $\kappa = \kappa^{<\kappa}$, $T \subseteq L_{\kappa\kappa}(Q)$, $|T| \leq \kappa$ and $T$ has a model $\mathfrak{A}$ in the $\mathcal{Y}$-interpretation. Then for all $\xi < \kappa^+$ there is a submodel $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq \kappa$ and $\mathcal{Y}' \subseteq \kappa^+$ such that $\mathcal{Y} \cap \xi = \mathcal{Y}' \cap \xi$ and $\mathfrak{B} \models T$ in the $\mathcal{Y}'$-interpretation.*

**Proof.** We may assume $|A| \geq 2^\kappa$. Let us expand $\mathfrak{A}$ by

1. A well-ordering $\prec$ the order-type of which exceed all the order-types of well-orderings definable by subformulas of $\phi$ with parameters in $A$.

2. A new predicate $P$ which contains those elements $d$ of $A$ for which $otp(\{\langle a, b\rangle : a \prec b \prec d\}) \in \mathcal{Y}$

3. A prediacte $F$ which codes an isomorphism from each well-ordering, definable by a subformula of $\phi$ with parameters in $A$, onto an initial segment of $\prec$.

Let $\langle \mathfrak{A}, \prec, P, F \rangle$ be the expanded structure and $\langle \mathfrak{B}, \prec^*, P^*, F^* \rangle$ an $L_{\kappa\kappa}$-elementary substructure of it of cardinality $\leq 2^\kappa$. Let

$$\mathcal{Y}' = \{\operatorname{otp}(\{\langle a, b \rangle \in B^2 : a \prec^* b \prec^* d\}) : d \in P^*\}$$

It is easy to see that $\mathfrak{B} \models \phi$ in the $\mathcal{Y}'$-interpretation. The second claim is proved similarly. $\square$

Let $\pi$ be a canonical bijection from triples of ordinals to ordinals such that $\pi[\kappa^3] = \kappa$ for each infinite cardinal $\kappa$. We say that a pair $(\delta_1, Z_1)$ *codes* a pair $(\delta_2, Z_2)$ if $\delta_1$, $\delta_2$ are ordinals, $Z_1 \subseteq \delta_1$, $Z_2 \subseteq \delta_2$ and there is a bijection $f : \delta_2 \to \delta_1$ such that

1. $\delta_1$ is closed under $\pi$

2. $\pi(0, \alpha, \beta) \in Z_1 \iff f(\alpha) < f(\beta)$

3. $\pi(1, 0, \alpha) \in Z_1 \iff f(\alpha) \in Z_2$.

Suppose $\kappa$ is an uncountable cardinal. The *weakly compact ideal* on $\kappa$ is the ideal of subsets of $\kappa$ generated by the sets $\{\alpha : (H(\alpha), \epsilon, A \cap H(\alpha)) \models \neg\phi\}$, where $A \subseteq H(\kappa)$ and $\phi$ is a $\Pi_1^1$-sentence such that $(H(\alpha), \epsilon, A \cap H(\alpha)) \models \phi$.

**Definition 9** *A cardinal $\kappa$ satisfies $\diamondsuit(WC)$ if it is weakly compact and there is a sequence $\langle A_\alpha : \alpha < \kappa \rangle$ such that*

1. $A_\alpha \subseteq \alpha$ for $\alpha < \kappa$.

2. $(\forall A \subseteq \kappa)(\{\lambda < \kappa : A_\lambda = A \cap \lambda\} \in \mathcal{I}^+)$, where $\mathcal{I}$ is the weakly compact ideal on $\kappa$.

We make the following remarks without proof: If $\kappa$ is measurable $> \omega$, then $\kappa$ satisfies $\diamondsuit$(WC). If $\kappa$ is weakly compact $> \omega$, then there is a generic extension which preserves all cardinals and in which $\kappa$ satisfies $\diamondsuit$(WC). If V=L, then every weakly compact cardinal $> \omega$ satisfies $\diamondsuit$(WC).

**Theorem 10** *Suppose $\kappa > \omega$ satisfies $\diamondsuit(WC)$ and $2^\kappa = \kappa^+$. Then there is no strongest extension of $L_{\kappa\kappa}$ for which $\kappa$ is weakly compact and which has $LS(< \kappa)$.*

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**Proof.**    We shall construct two sets $\mathcal{Y}^1, \mathcal{Y}^2 \subseteq \kappa^+$ such that the logics $L_{\kappa\kappa}(Q, \mathcal{Y}^i)$ are weakly $\kappa$-compact for and these logics satisfy $\mathrm{LS}(< \kappa)$, but no logic containing both $L_{\kappa\kappa}(Q, \mathcal{Y}^1)$ and $L_{\kappa\kappa}(Q, \mathcal{Y}^2)$ satisfies $\mathrm{LS}(< \kappa)$. The sets $\mathcal{Y}^i$ are constructed by induction together with ordinals $\xi_\alpha^i < \kappa^+$ such that:

$$
\begin{array}{ll}
\mathcal{Y}^i = \bigcup_{\alpha < \kappa^+} \mathcal{Y}_\alpha^i & \\
\mathcal{Y}_0^i = \emptyset & \xi_0^i = 0 \\
\mathcal{Y}_\alpha^i = \mathcal{Y}_\beta^i \cap \xi_\alpha^i & \text{for } \alpha < \beta \\
\xi_\alpha^i \leq \xi_\beta^i & \text{for } \alpha < \beta \\
\mathcal{Y}_\nu^i = \bigcup_{\alpha < \nu} \mathcal{Y}_\alpha^i, & \xi_\nu^i = \bigcup_{\alpha < \nu} \xi_\alpha^i, \text{ for } \nu = \cup \nu \\
\mathcal{Y}_\alpha^1 \cap \mathcal{Y}_\alpha^2 = \emptyset & \text{for } \alpha < \kappa
\end{array}
$$

First we define $\mathcal{Y}_\alpha^i$ for $\alpha < \kappa$ in such a way that $L_{\kappa\kappa}(\mathcal{Y}^i)$ will in the end have the property $\mathrm{LS}(< \kappa)$.

Let $S_1, S_2$ be a partition of the set of cardinals $< \kappa$ into two stationary sets. Let $\{\phi_\nu^i : \nu \in S_i\}$ list all $L_{\kappa\kappa}(Q)$-sentences so that each sentence is listed as $\phi_\nu^i$ for stationary many $\nu \in S_i$.

Suppose $\alpha = \lambda + 1$ and $\xi_\lambda^i = \lambda$. Suppose $\lambda \in S_i$.

**Case 1.** Suppose that $(\lambda, A_\lambda)$ codes some pair $(\xi, Z)$. In this case we let

$$\mathcal{Y}_\alpha^i = \mathcal{Y}_\lambda^i \cup (Z \setminus \lambda), \xi_\alpha^i = \xi$$

$$\mathcal{Y}_\alpha^{3-i} = \mathcal{Y}_\lambda^{3-i}.$$

**Case 2.** Otherwise we let $\xi_\alpha^i = \lambda$, $\mathcal{Y}_\alpha^i = \mathcal{Y}_\lambda^i$, $\mathcal{Y}_\alpha^{3-1} = \mathcal{Y}_\lambda^{3-i}$.

Suppose then $\alpha = \lambda + 2$, $\xi_\lambda^i = \lambda \in S_i$ and we have defined $\xi_{\lambda+1}^i$ and $\mathcal{Y}_{\lambda+1}^i$.

**Case 3.**    The sentence $\phi_\lambda^i$ has a model in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \subseteq \kappa^+$ with $\mathcal{Y} \cap \xi_{\lambda+1}^i = \mathcal{Y}_{\lambda+1}^i$. By Lemma 8 part 2, $\phi_\lambda^i$ has a model $\mathfrak{A}$ of cardinality $< \kappa$ in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \subseteq \kappa$ of cardinality $< \kappa$ with $\mathcal{Y} \cap \xi_{\lambda+1}^i = \mathcal{Y}_{\lambda+1}^i$. Let $\mu$ be minimal such that $\phi_\lambda^i \in \mathcal{L}_{\mu\mu}(\mathcal{Y})$. Let $\xi_{\lambda+2}^i = o(\mathfrak{A}, \mathcal{Y}, \mu, \mu)$ and $\mathcal{Y}_{\lambda+2}^i = \mathcal{Y}$. Let $\mathcal{Y}_{\lambda+2}^{3-i} = \mathcal{Y}_{\lambda+1}^{3-i}$.

**Case 4.** Otherwise $\xi_{\lambda+1}^i = \xi_\lambda^i$, $\mathcal{Y}_\alpha^i = \mathcal{Y}_{\lambda+1}^i$, $\mathcal{Y}_\alpha^{3-1} = \mathcal{Y}_{\lambda+1}^{3-i}$.

Finally for all other $\alpha \leq \kappa$ we let $\xi_\alpha^i$ and $\mathcal{Y}_\alpha^i$ be defined canonically.

This ends the construction of $\mathcal{Y}_\alpha^i$ for $\alpha \leq \kappa$. Note that $\mathcal{Y}_\kappa^1 \cap \mathcal{Y}_\kappa^2 = \emptyset$. Moreover, if $\phi_\nu^i$ has a model in the $\mathcal{Y}$-interpretation for some $\mathcal{Y} \supseteq \mathcal{Y}_\kappa^i$, then, by construction, $\phi_\nu^i$ has a model of cardinality $< \kappa$ in the $\mathcal{Y}_\kappa^i$-interpretation.

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Let $\mathcal{Y}^i_{\kappa+1} = \mathcal{Y}^i_\kappa \cup \{\kappa\}$ and $\xi^i_{\kappa+1} = \kappa + 2$. Next we shall define $\mathcal{Y}^i_\alpha$ and $\xi^i_\alpha$ for $\kappa + 1 < \alpha < \kappa^+$. For this, let $\langle T_\alpha : \kappa < \alpha < \kappa^+ \rangle$ enumerate all $L_{\kappa\kappa}(Q)$-theories of cardinality $\leq \kappa$ in a language of cardinality $\leq \kappa$ which satisfy the condintion that every subset of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i_\kappa$-interpretation. Here we use the assumption $2^\kappa = \kappa^+$. We may assume $T_\alpha \subseteq \kappa$ for all $\alpha$.

Suppose $\mathcal{Y}^i_\beta$ and $\xi^i_\beta$ have been defined for $\beta < \alpha$. If $\alpha = \cup\alpha$, $\mathcal{Y}^i_\alpha$ and $\xi^i_\alpha$ are defined canonically. So assume $\alpha = \beta + 1$. Let $T : H(\kappa) \to H(\kappa)$ be the function $T(a) = T_\beta \cap a$. If $a \in H(\kappa)$, then $T(a)$ has a model $\mathfrak{B}_a$ in the $\mathcal{Y}^i_\kappa$-interpretation. By construction, we may assume $\mathfrak{B}_a \in H(\kappa)$. Let $B : H(\kappa) \to H(\kappa)$ be the function $B(a) = \mathfrak{B}_a$. Let $Z \subseteq \kappa$ code $(\xi^i_\beta, \mathcal{Y}^i_\beta)$. By $\diamondsuit(\mathrm{WC})$, $W = \{\lambda < \kappa : A_\lambda = Z \cap \lambda\} \in \mathcal{I}^+$, where $\mathcal{I}$ is the weakly compact ideal on $\kappa$. Let $A : \kappa \to H(\kappa)$ be the function $A(\alpha) = A_\alpha$. By the definition of $\mathcal{I}$, there are a transitive set $M$ and $Y^*, R^*$ such that

$$\langle H(\kappa), \epsilon, A, W, \mathcal{Y}^i_\kappa, B, T \rangle \prec_{\kappa\kappa} \langle M, \epsilon, A^*, W^*, Y^*, B^*, T^* \rangle$$

and $\kappa \in W^*$. Now $A^*(\kappa) = Z$ and, by construction, $Y^* \cap \xi^i_\beta = \mathcal{Y}^i_\beta$

It is clear now that $B(\kappa)$ is a model of $T_\alpha$ in the $Y^*$-interpretation. By Lemma 8 there is a model $\mathfrak{B}$ of cardinality $\leq \kappa$ of $T_\beta$ in the $Y^{**}$-interpretation for some $Y^{**}$ with $Y^{**} \cap \xi^i_\beta = \mathcal{Y}^i_\beta$. Let $\xi^i_\alpha = o(\mathfrak{B}, Y^{**}, \kappa, \kappa)$ and $\mathcal{Y}^i_\alpha = Y^{**} \cap \xi^i_\alpha$.

Finally, let $\mathcal{Y}^i = \bigcup_{\alpha < \kappa^+} \mathcal{Y}^i_\alpha$.

**Claim 1.** $L_{\kappa\kappa}(\mathcal{Y}^i)$ satisfies the $\mathrm{LS}(< \kappa)$-property.

Suppose $\phi$ is a sentence of $L_{\kappa\kappa}(\mathcal{Y}^i)$ with a model. Let $\lambda \in S_i$ such that $\xi^i_\lambda = \lambda$ and $\phi^i_\lambda = \phi$. By the construction of $\mathcal{Y}^i_{\lambda+2}$ there is a model of $\phi$ of cardinality $< \kappa$.

**Claim 2.** $L_{\kappa\kappa}(\mathcal{Y}^i)$ is weakly $\kappa$-compact.

Suppose $T \subseteq L_{\kappa\kappa}(\mathcal{Y}^i)$ is given and every subset of $T_\alpha$ of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i$-interpretation. Then $T = T_\alpha$ for some $\alpha$. By construction, every subset of $T_\alpha$ of cardinality $< \kappa$ has a model in the $\mathcal{Y}^i \cap \kappa$-interpretation. Thus the definition of $\mathcal{Y}^i_\alpha$ is made so that $T_\alpha$ has a model $\mathfrak{B}$ in the $\mathcal{Y}$-interpretation for some $\mathcal{Y}$ such that $\mathcal{Y} \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa) = \mathcal{Y}^i \cap o(\mathfrak{B}, \mathcal{Y}, \kappa, \kappa)$. Thus by Lemma 7, $\mathfrak{B} \models T_\alpha$ in the $\mathcal{Y}^i$-interpretation. The Claim is proved.

We can now finish the proof of the theorem. In a logic in which both the quantifier $Q_{\mathcal{Y}^1}$ and $Q_{\mathcal{Y}^2}$ are definable, we can say that the order-type of a well-ordering is in $\mathcal{Y}^1 \cap \mathcal{Y}^2$. Thus such a logic cannot satisfy $\mathrm{LS}(< \kappa)$. $\square$

It is interesting to note that a proof like above would not be possible for the following stronger Löwenheim-Skolem property: A *filter-family* is a family $\mathcal{F} = (\mathcal{F}(A))_{A \neq \emptyset}$, where $\mathcal{F}(A)$ is always a filter on the set $A$. Luosto [6] defines the concept of a $(\kappa^+, \omega)$-*neat* filter family. We will not repeat the definition here, its elements are closure under bijections, fineness, $\kappa^+$-completeness, normality and upward relativizability (all defined in [6]). Suppose $\mathcal{L}$ is a logic of the form $L_{\kappa\lambda}(\vec{Q})$ for some sequence $\vec{Q}$ of generalized quantifiers. We say that $\mathcal{L}$ has the $\mathcal{F}, \kappa$-*persistency property*, if for all models $\mathfrak{A}$ and $B \in \mathcal{F}(A)$, we have $\mathfrak{A} \upharpoonright B \prec \mathfrak{A}$. Luosto proves that if $\mathcal{L}_1$ and $\mathcal{L}_2$ both satisfy the $\mathcal{F}, \kappa$-persistency property, then there is $\mathcal{L}_3$ such that $\mathcal{L}_1 \leq \mathcal{L}_3$, $\mathcal{L}_2 \leq \mathcal{L}_3$ and $\mathcal{L}_3$ satisfies the $\mathcal{F}, \kappa$-persistency property. Lipparini [5] proves a similar result for families of limit ultrafilters related closely to compactness.

Tapani Hyttinen pointed out that the assumption $2^\kappa = \kappa^+$ is not needed in Theorem 10, if $\kappa$ is assumed to be measurable.

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