

PERFECTLY MEAGER SETS AND UNIVERSALLY NULL SETS

TOMEK BARTOSZYNSKI AND SAHARON SHELAH

ABSTRACT. We will show that there is no ZFC example of a set distinguishing between universally null and perfectly meager sets.

1. INTRODUCTION

Consider the following three families of sets of reals:

Definition 1. Let $X \subseteq \mathbb{R}$.

- (1) X is perfectly meager if for every perfect set $P \subseteq \mathbb{R}$, $P \cap X$ is meager in P .
- (2) X is universally meager if every Borel isomorphic image of X is meager
- (3) X is universal null if every Borel isomorphic image of X has Lebesgue measure zero.

Let **PM**, **UM** and **UN** denote these families respectively.

The family **UM** was studied recently by Zakrzewski [13], and identified as an analog of **UN**.

One gets an equivalent definition of **UN** by replacing “Borel isomorphic” by “homeomorphic”, but this is not the case with **UM**.

Let \mathcal{M} and \mathcal{N} denote the σ -ideals of meager and of measure zero subsets of the reals, respectively.

For a σ -ideal $\mathcal{J} \subseteq P(\mathbb{R})$ let

$$\text{non}(\mathcal{J}) = \min\{|X| : X \subseteq \mathbb{R} \text{ \& } X \notin \mathcal{J}\}.$$

There are many ZFC examples of uncountable sets that are in $\mathbf{UM} \cap \mathbf{UN}$. These include $\omega_1 \omega_1^*$ -gaps, a selector from the constituents of a non-Borel $\mathbf{\Pi}_1^1$ set, etc. (see [9]) All these sets have size \aleph_1 , since Miller [8] showed that, consistently, no set of size 2^{\aleph_0} is in $\mathbf{UM} \cup \mathbf{UN}$.

Grzegorek found other constructions in ZFC that produce sets of (consistently) different sizes.

Theorem 2 (Grzegorek, [6]).

- (1) There exists a set $X \in \mathbf{UN}$ such that $|X| = \text{non}(\mathcal{N})$,
- (2) There exists a set $X \in \mathbf{UM}$ such that $|X| = \text{non}(\mathcal{M})$.

The problem whether the equality $\mathbf{UM} = \mathbf{UN}$ is consistent is open. However, both inclusions are consistent with ZFC; $\mathbf{UM} \subsetneq \mathbf{UN}$ holds in a model obtained by

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adding \aleph_2 Cohen reals, and $\mathbf{UN} \subsetneq \mathbf{UM}$ holds in a model obtained by adding \aleph_2 random reals (side-by-side) (see [4], [9], [8]).

In this paper we investigate the connection between families \mathbf{UN} and \mathbf{PM} , and show that both inclusions $\mathbf{PM} \subseteq \mathbf{UN}$ and $\mathbf{UN} \subseteq \mathbf{PM}$ are consistent with ZFC as well. Observe that trivially $\mathbf{UM} \subseteq \mathbf{PM}$, thus we only need to check that $\mathbf{PM} \subseteq \mathbf{UN}$ is consistent. Recall that $\mathbf{PM} \neq \mathbf{UM}$ is consistent ([12]) as well as $\mathbf{PM} = \mathbf{UM}$ ([2]). We will show that:

Theorem 3. *It is consistent with ZFC that $\mathbf{PM} \subseteq [\mathbb{R}]^{\leq \aleph_1} \subseteq \mathbf{UN}$.*

2. FORCING

Suppose that $X \subseteq 2^\omega$ is a perfectly meager set in \mathbf{V} . Let \tilde{P} be a fixed closed subset of $2^\omega \times 2^\omega$ which is universal for perfect sets in 2^ω . In other words, for every perfect set $P \subseteq 2^\omega$ there exists an x such that $P = (\tilde{P})_x = \{y : (x, y) \in \tilde{P}\}$. Note that this property is absolute. Since X is perfectly meager, we can find sets $\tilde{Q}^n \subseteq 2^\omega \times 2^\omega$ such that for every $x \in 2^\omega$ and $n \in \omega$,

- (1) $(\tilde{Q}^n)_x$ is a closed nowhere dense subset of $(\tilde{P})_x$,
- (2) $X \cap (\tilde{P})_x \subseteq \left(\bigcup_{n \in \omega} \tilde{Q}^n\right)_x$.

Clearly, the set $\bigcup_{n \in \omega} \tilde{Q}^n$ witnesses that $X \in \mathbf{PM}$ since

$$X \subseteq 2^\omega \setminus \bigcup_{x \in 2^\omega} \left(\tilde{P} \setminus \bigcup_{n \in \omega} \tilde{Q}^n \right)_x.$$

Note that the last inclusion makes sense even if X is not a subset of \mathbf{V} . Suppose that $\mathbf{V}' \subseteq \mathbf{V}$ and $X \subseteq \mathbf{V}$ is a set of reals. We will say $\mathbf{V}' \models X \in \mathbf{PM}$ if there exists a family $\{\tilde{Q}^n : n \in \omega\} \in \mathbf{V}'$ such that $X \cap (\tilde{P})_x \subseteq \left(\bigcup_{n \in \omega} \tilde{Q}^n\right)_x$ for every real $x \in \mathbf{V}'$.

The property of being perfectly meager is not absolute so whether X is perfectly meager in \mathbf{V}' has no bearing onto whether X is perfectly meager in \mathbf{V} . For example, if $x \in \mathbf{V}$ is a Cohen real over \mathbf{V}' then the set $\{x\}$ is perfectly meager in \mathbf{V} but not in \mathbf{V}' .

Lemma 4. *Let $\langle \mathcal{P}_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ be a countable support iteration of proper forcing notions over $\mathbf{V} \models \text{CH}$. Suppose that $X \subseteq \mathbf{V}^{\mathcal{P}_{\omega_2}} \cap \mathbb{R}$ is a perfectly meager set. Then there exists an ω_1 -club $C \subseteq \omega_2$ such that for every $\alpha \in C$,*

$$\mathbf{V}^{\mathcal{P}_\alpha} \models X \in \mathbf{PM}.$$

Proof. Let $\{\tilde{Q}^n : n \in \omega\} \in \mathbf{V}^{\mathcal{P}_{\omega_2}}$ be a family witnessing that X is perfectly meager. Let C consist of those ordinals of cofinality ω_1 that for every n , $\tilde{Q}^n \cap ((2^\omega \cap \mathbf{V}^{\mathcal{P}_\alpha}) \times 2^\omega) \in \mathbf{V}^{\mathcal{P}_\alpha}$. The usual argument involving Skolem-Löwenheim theorem shows that C has the required property. \square

Our objective is to find a set of general conditions on a forcing notion \mathbb{P} such that the countable support iteration of \mathbb{P} of length ω_2 produces a model where $\mathbf{PM} \subseteq [\mathbb{R}]^{\leq \aleph_1} \subseteq \mathbf{UN}$. These conditions are sufficient for the class of forcing notions defined using norms [10].

These conditions are the following:

- (1) $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap 2^\omega \in \mathcal{N}$,
- (2) $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap 2^\omega \notin \mathcal{M}$,

- (3) \mathbb{P} is ω^ω -bounding, that is $\omega^\omega \cap \mathbf{V}$ is a dominating family in $\omega^\omega \cap \mathbf{V}^{\mathbb{P}}$,
- (4) \mathbb{P} adds a real $x_{\mathbb{P}} \in 2^\omega$ such that $\mathbf{V} \models \{x_{\mathbb{P}}\} \notin \mathbf{PM}$.
- (5) \mathbb{P} generic real is minimal, that is, if g is \mathbb{P} -generic over \mathbf{V} and $x \in \mathbf{V}[g] \cap 2^\omega$ then $x \in \mathbf{V}$ or $g \in \mathbf{V}[x]$.

Condition (1) is necessary to make all sets of size \aleph_1 universally null, and condition (2) is necessary to avoid making all \aleph_1 sets perfectly meager. Recall that (2) and (3) together are essentially equivalent to $\mathbf{V}^{\mathbb{P}} \models \mathbf{V} \cap \mathcal{M}$ is cofinal in \mathcal{M} .

For the forcing notions \mathbb{P} that we have in mind the following property holds: for every real $x \in \mathbf{V}^{\mathbb{P}}$ there exists a continuous function $f \in \mathbf{V}$ such that $x = f(x_G)$, where x_G is a generic real.

Condition (5), guarantees that in the above context f can be chosen to be a homeomorphism. In particular, if X is a set of reals of size \aleph_2 then X will contain a homeomorphic image of a sequence of generic reals.

The following forcing notion appeared in [5], it is similar (but not identical) to the infinitely equal real forcing from [7].

For a tree p and $t \in p$, let $\text{succ}_p(t)$ be the set of all immediate successors of t in p , $p_t = \{v \in p : t \subseteq v \text{ or } v \subseteq t\}$ the subtree of p determined by t , $p|n$ the n -th level of p , and let $[p]$ be the set of branches of p . By identifying $s \in \omega^{<\omega}$ with the full-branching tree having root s , we can also denote $[s] = \{f \in \omega^\omega : s \subseteq f\}$.

Fix a strictly increasing function $f \in \omega^\omega$ and let $\mathbf{X} = \prod_{n \in \omega} f(n)$. Note that \mathbf{X} is a Polish space homeomorphic to 2^ω . For technical reasons we require that $f(n) = 2^{\tilde{f}(n)}$ for $n \in \omega$.

Let $\mathbb{E}\mathbb{E}$ be the following forcing notion: $p \in \mathbb{E}\mathbb{E}$ if

- (1) p is a nonempty subtree of $\omega^{<\omega}$,
- (2) $s(n) < f(n)$ for all $s \in p$ and $n \in \text{dom}(s)$,
- (3) for all $s \in p$ there exists an extension t of s such that $t \cap n \in p$ for all $n < f(|t|)$.

For $p, q \in \mathbb{E}\mathbb{E}$, $p \geq q$ if $p \subseteq q$. Without loss of generality we can assume that $|\text{succ}_p(s)| = 1$ or $\text{succ}_p(s) = f(|p|)$ for all $p \in \mathbb{E}\mathbb{E}$ and $s \in p$. Conditions of this type form a dense subset of $\mathbb{E}\mathbb{E}$.

Let

$$\text{split}(p) = \{s \in p : |\text{succ}_p(s)| > 1\} = \bigcup_{n \in \omega} \text{split}_n(p),$$

where $\text{split}_n(p) = \left\{ s \in \text{split}(p) : \left| \{t \subsetneq s : t \in \text{split}(p)\} \right| = n \right\}$.

For $p, q \in \mathbb{E}\mathbb{E}$, $n \in \omega$, we let

$$p \geq_n q \iff p \geq q \ \& \ \text{split}_n(q) = \text{split}_n(p).$$

Lemma 5 ([5]). (1) $\mathbb{E}\mathbb{E}$ satisfies Axiom A, so it is proper,

- (2) $\mathbf{V}^{\mathbb{E}\mathbb{E}} \models \mathbf{V} \cap 2^\omega \in \mathcal{N}$,
- (3) $\mathbf{V}^{\mathbb{E}\mathbb{E}} \models \mathbf{V} \cap 2^\omega \notin \mathcal{M}$,
- (4) for every maximal antichain $\mathcal{A} \subseteq \mathbb{E}\mathbb{E}$, $p \in \mathbb{E}\mathbb{E}$, and $n \in \omega$ there exists $q \geq_n p$ such that $\{r \in \mathcal{A} : r \text{ is compatible with } q\}$ is finite.
- (5) for every family of maximal antichains $\{\mathcal{A}_n : n \in \omega\}$ and $p \in \mathbb{E}\mathbb{E}$ there exists $q \geq p$ such that for every n , $\{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$ is finite.
- (6) $\mathbb{E}\mathbb{E}$ is ω^ω bounding,
- (7) $\mathbf{V}^{\mathbb{E}\mathbb{E}} \models \mathbf{V} \cap \mathcal{M}$ is cofinal in \mathcal{M} . \square

Note that for $p \in \mathbb{E}\mathbb{E}$ the set $[p]$ is a compact subset of $\mathbf{X} = \prod_n f(n)$. Moreover, there is a canonical isomorphism between $[p]$ and 2^ω defined as follows:

For every n let $\{s_0^n, \dots, s_{f(n)}^n\}$ be a fixed enumeration of 0-1 sequences of length $\tilde{f}(n)$ (recall that $f(n) = 2^{\tilde{f}(n)}$). Define $F : [p] \longrightarrow 2^\omega$ as

$$F(x) = s_{x(n_0+1)}^{n_0} \widehat{\ } s_{x(n_1+1)}^{n_1} \widehat{\ } \dots,$$

where n_0, n_1, \dots is the increasing enumeration of the set $\{n : x \upharpoonright n \in \text{split}(p)\}$.

Lemma 6. *Let $p \in \mathbb{E}\mathbb{E}$ and suppose that $H \subseteq [p]$ is a meager set in $[p]$. For every $n \in \omega$ there exists $q \geq_n p$ such that $[q] \cap H = \emptyset$. In particular, $\Vdash_{\mathbb{E}\mathbb{E}} \text{“}\mathbf{V} \models \{\dot{g}\} \notin \mathbf{PM}\text{”}$.*

Proof. Let $H \subseteq [p]$ be a meager set, and let $n \in \omega$. Fix a descending sequence of open sets $\langle U_k : k \in \omega \rangle$ such that each U_k is dense in $[p]$ and $H \cap \bigcap_k U_k = \emptyset$. By induction build a sequence $\langle p_k : k \in \omega \rangle$ such that $p_0 = p$, and for every k ,

- (1) $p_{k+1} \geq_{n+k+1} p_k \in \mathbb{E}\mathbb{E}$,
- (2) $[p_{k+1}] \subseteq U_k$.

Suppose that p_k is given. For every $v \in \text{split}_{n+k+1}(p_k)$ find $q_v \geq (p_k)_v$ such that $[q_v] \subseteq U_k$. Let $p_{k+1} = \bigcup \{q_v : v \in \text{split}_{n+k+1}(p_k)\}$. Condition $q = \lim_k p_k$ has the required property.

Suppose that $\{\widetilde{Q}^n : n \in \omega\} \in \mathbf{V}$ is a possible witness that $\{\dot{g}\}$ is perfectly meager, and let $p \in \mathbb{E}\mathbb{E}$. Find $x \in \mathbf{V}$ such that $[p] = (P)_x$ and let $q \geq p$ be such that $[q] \cap \left(\bigcup_n \widetilde{Q}^n\right)_x = \emptyset$. Clearly,

$$q \Vdash_{\mathbb{E}\mathbb{E}} \{\dot{g}\} \in \bigcup_{x \in \mathbf{V}} \left(P \setminus \bigcup_n \widetilde{Q}^n \right)_x.$$

In particular, $q \Vdash_{\mathbb{E}\mathbb{E}} \text{“}\mathbf{V} \models \{\dot{g}\} \notin \mathbf{PM}\text{”}$. □

Lemma 7. *Suppose that $p \in \mathbb{E}\mathbb{E}$ and $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \in 2^\omega$. For every $n \in \omega$ there exists $q \geq_n p$ and a continuous function $F : [q] \longrightarrow 2^\omega$ such that $q \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} = F(\dot{g})$, where \dot{g} is the canonical name for the generic real.*

Moreover, we can require that for every $v \in \text{split}_n(q)$ and any $x_1, x_2 \in [q_v]$, $F(x_1) \upharpoonright n = F(x_2) \upharpoonright n$.

Proof. The first part is a special case of a more general fact. For $n \in \omega$ let $\mathcal{A}_n \subseteq \mathbb{E}\mathbb{E}$ be a maximal antichain below p such that $\forall r \in \mathcal{A}_n \exists s \in 2^n r \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \upharpoonright n = s$. Use 5(5) to find $q \geq p$ such that for every $n \in \omega$, $\{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$ is finite. Let $\mathcal{A}'_n = \{r \in \mathcal{A}_n : r \text{ is compatible with } q\}$. Without loss of generality we can assume that $[q] \subseteq \bigcup_{r \in \mathcal{A}'_n} [r]$. It follows that $[r] \cap [q]$ is clopen in $[q]$ for every $r \in \mathcal{A}'_n$. Define $F : [q] \longrightarrow 2^\omega$ as $F(x) = y$ if for every $n \in \omega$ there exists $r \in \mathcal{A}'_n$ such that $x \in [r]$ and $r \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \upharpoonright n = y \upharpoonright n$. It is easy to see that F is a continuous function that has the required properties.

To show the second part we need to build q in such a way that for every $v \in \text{split}_n(q)$, there is $r \in \mathcal{A}'_n$ such that $q_v \geq r$. □

Lemma 8. *Suppose that $p \in \mathbb{E}\mathbb{E}$, $n \in \omega$ and $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \in 2^\omega$. Let $F : [q] \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} = F(\dot{g})$.*

There exists $q \geq p$ such that $F \upharpoonright [q]$ is constant, or there exists $q \geq_n p$ such that $F \upharpoonright [q]$ is one-to-one. In particular, the generic real is minimal.

Proof. Consider the following two cases.

CASE 1 $p \not\Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \notin \mathbf{V}$. Let $x \in \mathbf{V}$ and $q \geq p$ be such that $q \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} = x$. Clearly $F \upharpoonright [q]$ is constant with value x .

CASE 2 $p \Vdash_{\mathbb{E}\mathbb{E}} \dot{x} \notin \mathbf{V}$.

Build by induction a sequence of conditions $\langle p_k : k \in \omega \rangle$ such that $p_0 = p$ and for every k ,

- (1) $p_{k+1} \geq_{n+k+1} p_k$,
- (2) sets $\left\{ F^m \left([(p_{k+1})_s] \right) : s \in \text{split}_{n+k+1}(p_{k+1}) \right\}$ are pairwise disjoint and have diameter $< 2^{-k}$

Suppose that p_k is given. Note that $F^m \left([(p_k)_s] \right)$ is uncountable for every $s \in p_k$. For $v \in \text{split}_{n+k+1}(p_k)$ choose pairwise different reals $x_v \in F^m \left([(p_k)_v] \right)$. It is not important now but will be relevant in the sequel, that we can choose these reals “effectively” from a fixed countable subset of $[p_k]$. Let $\ell > k$ be such that sequences $x_v \upharpoonright \ell$ are also pairwise different. For every $v \in \text{split}_{n+k+1}(p_k)$ let $s_v \in \text{split}(p_k)$ be such that for every $z \in [(p_k)_{s_v}]$, $F(z) \upharpoonright \ell = x_v \upharpoonright \ell$. If F is as in the second part of lemma 7 then we can find s_v in $\text{split}_\ell(p_k)$. Define $p_{k+1} = \bigcup \{ (p_k)_{s_v} : v \in \text{split}_{n+k+1}(p_k) \}$. Observe that $q = \lim_k p_k$ has the required property. \square

Note that the above lemma shows that the reals added by $\mathbb{E}\mathbb{E}$ are minimal. Infinitely equal forcing from [7] or [4] does not have this property.

3. ITERATION OF $\mathbb{E}\mathbb{E}$.

Let $\alpha \leq \omega_2$ be an ordinal and suppose that $\mathbb{E}\mathbb{E}_\alpha$ is a countable support iteration of $\mathbb{E}\mathbb{E}$ of length α . In other words, $p \in \mathbb{E}\mathbb{E}_\alpha$ is

- (1) p is a function and $\text{dom}(p) = \alpha$,
- (2) $\text{supp}(p) = \{ \beta : p(\beta) \neq \emptyset \}$ is countable,
- (3) $\forall \beta < \alpha \ p \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_\beta} p(\beta) \in \mathbb{E}\mathbb{E}$.

For $F \in [\alpha]^{<\omega}$, $n \in \omega$, and $p, q \in \mathbb{E}\mathbb{E}_\alpha$ define

$$q \geq_{F,n} p \iff q \geq p \ \& \ \forall \beta \in F \ q \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_\beta} q(\beta) \geq_n p(\beta).$$

The following fact is well-known.

Theorem 9 ([5], [7], [3]). *Suppose that $p \in \mathbb{E}\mathbb{E}_\alpha$, $F \in [\alpha]^{<\omega}$, and $n \in \omega$.*

- (1) *for every maximal antichain $\mathcal{A} \subseteq \mathbb{E}\mathbb{E}_\alpha$, there exists $q \geq_{F,n} p$ such that $\{ r \in \mathcal{A} : r \text{ is compatible with } q \}$ is finite.*
- (2) *for every family of maximal antichains $\{ \mathcal{A}_n : n \in \omega \}$ there exists $q \geq p$ such that for every n , $\{ r \in \mathcal{A}_n : r \text{ is compatible with } q \}$ is finite.*
- (3) $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models [\mathbb{R}]^{<2^{\aleph_0}} \subseteq \mathcal{N}$.
- (4) $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models \mathcal{M} \cap \mathbf{V}$ is cofinal in \mathcal{M} . \square

For $p \in \mathbb{E}\mathbb{E}_\alpha$ let $\text{cl}(p)$ be the smallest set $w \subseteq \alpha$ such that p can be evaluated using generic reals $\langle \dot{g}_\beta : \beta \in w \rangle$. In other words, $\text{cl}(p)$ consists of those $\beta < \alpha$ such that the transitive closure of p contains $\mathbb{E}\mathbb{E}_\beta$ name for an element of $\mathbb{E}\mathbb{E}$. It is well-known [11] that $\{ p \in \mathbb{E}\mathbb{E}_\alpha : \text{cl}(p) \in [\alpha]^{<\omega} \}$ is dense in $\mathbb{E}\mathbb{E}_\alpha$.

Suppose that $p \in \mathbb{E}\mathbb{E}_\alpha$, $w = \text{cl}(p)$ is countable and $\alpha_p = \text{ot}(\text{cl}(p))$. Let $\mathbb{E}\mathbb{E}_w$ be the countable support iteration of $\mathbb{E}\mathbb{E}$ with the domain w . In other words, consider

The countable support iteration $\langle \mathcal{P}_\beta, \dot{Q}_\beta : \beta < \text{sup}(w) \rangle$ such that

$$\forall \beta < \text{sup}(w) \Vdash_{\mathcal{P}_\beta} \dot{Q}_\beta \simeq \begin{cases} \mathbb{E}\mathbb{E} & \text{if } \beta \in w \\ \emptyset & \text{if } \beta \notin w \end{cases}.$$

It is clear that $\mathbb{E}\mathbb{E}_w \simeq \mathbb{E}\mathbb{E}_{\alpha_p}$. Moreover, we can view condition p as a member of $\mathbb{E}\mathbb{E}_w$.

For the rest of the section we will consider only the iteration of $\mathbb{E}\mathbb{E}$ of countable length α and show that $\mathbb{E}\mathbb{E}_\alpha$ has the same properties that $\mathbb{E}\mathbb{E}$ has.

Let α be a countable ordinal and $p \in \mathbb{E}\mathbb{E}_\alpha$. Define $\bar{p} \subseteq \mathbf{X}^\alpha$ as follows:

$\langle x_\beta : \beta < \alpha \rangle \in \bar{p}$ if for every $\beta < \alpha$,

$$x_\beta \in \left[p(\beta) [\langle x_\gamma : \gamma < \beta \rangle] \right].$$

Note that $p(\beta) [\langle x_\gamma : \gamma < \beta \rangle]$ is the interpretation of $p(\beta)$ using reals $\langle x_\gamma : \gamma < \beta \rangle$ so may be undefined if these reals are not sufficiently generic.

For a set $G \subseteq \mathbf{X}^\alpha$, $u \subseteq \alpha$, and $x \in \mathbf{X}^u$ let

$$(G)_x = \{y \in \mathbf{X}^{\alpha \setminus u} : \exists z \in G \ z \upharpoonright u = x \ \& \ z \upharpoonright (\alpha \setminus u) = y\},$$

and for $\beta \in \alpha$ let $(G)_\beta = \{x(\beta) : x \in G\}$.

We say that $p \in \mathbb{E}\mathbb{E}_\alpha$ is good if

- (1) \bar{p} is compact,
- (2) for every $\beta < \alpha$ and $x \in \overline{p \upharpoonright \beta}$, $\overline{p[x]} = (p)_x$ and $\overline{p(\beta)[x]} = ((p)_x)_\beta$.
- (3) \bar{p} is homeomorphic to \mathbf{X}^α via a homeomorphism h such that for every $\beta < \alpha$ and $x \in \overline{p \upharpoonright \beta}$, $h \upharpoonright ((p)_x)_\beta$ is a homeomorphism between $((p)_x)_\beta$ and \mathbf{X} .

Lemma 10. $\{p \in \mathbb{E}\mathbb{E}_\alpha : \bar{p} \text{ is good}\}$ is dense in $\mathbb{E}\mathbb{E}_\alpha$.

Proof. CASE 1. $\alpha = \beta + 1$.

Fix $p \in \mathbb{E}\mathbb{E}_\alpha$ and for $n \in \omega$ let \mathcal{A}_n be a maximal antichain below $p \upharpoonright \beta$ such that

- (1) $\forall r \in \mathcal{A}_n \ \bar{r}$ is compact.
- (2) $\forall r \in \mathcal{A}_n \ \exists t \subseteq \prod_{j < n} f(j) \ r \Vdash_{\mathbb{E}\mathbb{E}_\beta} p(\beta) \upharpoonright n = t$.

Fix a sequence $\langle F_n : n \in \omega \rangle$ such that for $n \in \omega$,

- (1) $F_n \in [\beta]^{<\omega}$,
- (2) $F_n \subseteq F_{n+1}$,
- (3) $\bigcup_n F_n = \beta$.

By induction build a sequence $\langle q_n : n \in \omega \rangle$ such that for $n \in \omega$,

- (1) $\overline{q_n}$ is compact,
- (2) $q_{n+1} \geq_{F_n, n} q_n$,
- (3) $\exists \mathcal{A}'_n \in [\mathcal{A}_n]^{<\omega} \ \overline{q_n} \subseteq \bigcup_{r \in \mathcal{A}'_n} \bar{r}$.

Let $q_\omega = \lim_n q_n$. As in the proof of 7 we show that there exists a continuous function $F : \overline{q_\omega} \rightarrow \mathbb{E}\mathbb{E}$ (encode elements of $\mathbb{E}\mathbb{E}$ as reals) such that

$$q_\omega \Vdash_{\mathbb{E}\mathbb{E}_\beta} p(\beta) = F(\langle \dot{g}_\gamma : \gamma < \beta \rangle).$$

Consider $q = q_\omega \widehat{\ } p(\beta) \geq p$. Clearly, $\bar{q} = \{\langle x, y \rangle : x \in \overline{q_\omega}, y \in [F(x)]\}$ is compact in \mathbf{X}^α . Remaining requirements are met as well.

CASE 2. α is limit.

Given $p \in \mathbb{E}\mathbb{E}_\alpha$ fix sequences $\langle F_n : n \in \omega \rangle$ and $\langle \alpha_n : n \in \omega \rangle$ such that

- (1) $F_n \in [\alpha_n]^{<\omega}$,
- (2) $F_n \subseteq F_{n+1}$,

- (3) $\bigcup_n F_n = \alpha$,
- (4) $\sup_n \alpha_n = \alpha$.

By induction build a sequence $\langle q_n : n \in \omega \rangle$ such that for $n \in \omega$,

- (1) $q_n \in \mathbb{E}\mathbb{E}_\alpha$,
- (2) $\text{supp}(q_n) \subseteq \alpha_n$,
- (3) $q_{n+1} \geq_{F_n, n} q_n$,
- (4) $q_n \upharpoonright \alpha_n \geq p \upharpoonright \alpha_n$,
- (5) $q_n \upharpoonright \alpha_n$ is compact in \mathbf{X}^{α_n} .

Let $q = \lim_n q_n$. Note that $\bar{q} = \bigcap_n \overline{q_n \upharpoonright \alpha_n} \times \mathbf{X}^{\alpha \setminus \alpha_n}$ is as required. \square

From now on we will always work with conditions p such that \bar{p} is good. We noticed earlier that for every condition $p \in \mathbb{E}\mathbb{E}$, $[p]$ is canonically isomorphic to 2^ω , in exactly the same way we can verify that if $p \in \mathbb{E}\mathbb{E}_\alpha$ and \bar{p} is good then \bar{p} is isomorphic to $(2^\omega)^\alpha$.

As in the lemma 7 we show that:

Lemma 11. *Suppose that $p \in \mathbb{E}\mathbb{E}_\alpha$ and $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} \in 2^\omega$. Then there exists $q \geq p$ and a continuous function $F : \bar{p} \rightarrow 2^\omega$ such that $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \alpha \rangle$ is the sequence of generic reals.*

Lemma 12. *Let $p \in \mathbb{E}\mathbb{E}_\alpha$ and suppose that $H \subseteq \bar{p}$ is a meager set in \bar{p} . For every $F \in [\alpha]^{<\omega}$ and $n \in \omega$ there exists $q \geq_{F, n} p$ such that $\bar{q} \cap H = \emptyset$.*

Proof. As before, without loss of generality we can assume that α is countable.

Induction on α .

CASE 1. $\alpha = \beta + 1$.

Suppose that $p \in \mathbb{E}\mathbb{E}_\alpha$ and $H \subseteq \bar{p} \subseteq \mathbf{X}^\beta \times \mathbf{X}$ is meager, and let $F \in [\alpha]^{<\omega}$ and $n \in \omega$ be given.

Let

$$H' = \{x \in \overline{p \upharpoonright \beta} : (H)_x \text{ is not meager in } [p(\beta)[x]] = ((\bar{p})_x)_\beta\}.$$

Using the fact that \bar{p} is homeomorphic to $(2^\omega)^\alpha$ via homeomorphism respecting vertical sections, and by Kuratowski-Ulam theorem, we conclude that H' is a meager set in $\overline{p \upharpoonright \beta}$.

Recall the following classical lemma:

Lemma 13 ([1]). *Suppose that $H \subseteq 2^\omega \times 2^\omega$ is a Borel set.*

- (1) *Assume $(H)_x$ is meager for all x . Then there exists a sequence of Borel sets $\{G_n : n \in \omega\} \subseteq 2^\omega \times 2^\omega$ such that*
 - (a) *$(G_n)_x$ is a closed nowhere dense set for all $x \in 2^\omega$,*
 - (b) *$H \subseteq \bigcup_{n \in \omega} G_n$.*

By the inductive hypothesis we can find $q^* \geq_{F \cap \beta, n} p \upharpoonright \beta$ such that $\overline{q^*} \cap H' = \emptyset$. By lemma 6 for every $x \in \overline{q^*}$ there exists $q_x \geq_n p(\beta)[x]$ such that $[q_x] \cap (H)_x = \emptyset$. Moreover, by 13, the mapping $x \mapsto q_x$ is can be chosen to be Borel, and subsequently, by shrinking q^* , continuous. Let $q \in \mathbb{E}\mathbb{E}_\alpha$ be defined such that $q \upharpoonright \beta = q^*$ and $q^* \Vdash_{\mathbb{E}\mathbb{E}_\beta} q(\beta) = q_{\dot{g}_\beta}$. It is clear that q has the required properties.

CASE 2. α is limit.

Fix sequences $\langle F_n : n \in \omega \rangle$ and $\langle \alpha_n : n \in \omega \rangle$ such that

- (1) $F_n \in [\alpha_n]^{<\omega}$,
- (2) $F_n \subseteq F_{n+1}$,

- (3) $\bigcup_n F_n = \alpha$,
- (4) $\sup_n \alpha_n = \alpha$.

By induction build a sequence $\langle q_n : n \in \omega \rangle$ such that for $n \in \omega$,

- (1) $q_n \in \mathbb{E}\mathbb{E}_\alpha$,
- (2) $\text{supp}(q_n) \subseteq \alpha_n$,
- (3) $q_{n+1} \geq_{F_n, n} q_n$,
- (4) $q_n \upharpoonright \alpha_n \geq p \upharpoonright \alpha_n$,
- (5) $\overline{q_n \upharpoonright \alpha_n} \cap H_n = \emptyset$, where $H_n = \left\{ x \in \overline{q_n \upharpoonright \alpha_n} : (H)_x \text{ is not meager in } \overline{p \upharpoonright x} \right\}$.

As before (5) is possible by Kuratowski-Ulam theorem. Let $q = \lim_n q_n$. It is clear that $\overline{q} \cap H = \emptyset$. \square

The following lemma is an analog of lemma 8.

Lemma 14. *Suppose that $p \in \mathbb{E}\mathbb{E}_\alpha$, $n \in \omega$ and $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} \in 2^\omega$. Let $F : \overline{p} \rightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \alpha \rangle$ is the sequence of generic reals. There exists $q \geq p$ such that exactly one of the following conditions hold:*

- (1) $F \upharpoonright \overline{q}$ is constant,
- (2) there exists $\beta < \alpha$ such that $F \upharpoonright \overline{q \upharpoonright \beta}$ is one-to-one and for every $x \in \overline{q \upharpoonright \beta}$, $F \upharpoonright (q \upharpoonright \beta)_x$ is constant,
- (3) $F \upharpoonright \overline{q}$ is one-to-one.

Proof. We have three cases:

CASE 1. There exists $q \geq p$ such that $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} \in \mathbf{V}$. Without loss of generality we can assume that for some $x \in \mathbf{V} \cap 2^\omega$ $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = x$. It follows that $F \upharpoonright \overline{q}$ is constant.

CASE 2. There exists $q \geq p$ such that $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \exists \beta < \alpha \dot{x} \in \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}$. By shrinking q we can assume that there exists a continuous function $G : \overline{q \upharpoonright \beta} \rightarrow 2^\omega$ such that $q \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = G(\dot{\mathbf{g}} \upharpoonright \beta)$. In particular, for $x \in [q]$, $F(x) = G(x \upharpoonright \beta)$. If β was minimal then, using the argument below, we can also assume that G is one-to-one.

Suppose that $q \in \mathbb{E}\mathbb{E}_\alpha$, $F \in [\alpha]^{<\omega}$, and $n \in \omega$. Without loss of generality we can assume that for every $\beta \in F$, $q \upharpoonright \beta$ determines the value of $\text{split}_n(q(\beta))$ (up to finitely many values). Suppose that $\sigma : F \rightarrow \omega^{<\omega}$ is a function such that $\sigma(\beta) \in \text{split}_n(q(\beta))$ for $\beta \in F$. Let $(q)_\sigma$ be the condition defined as

$$\forall \beta < \alpha (q)_\sigma \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_\beta} (q)_\sigma(\beta) = \begin{cases} q(\beta) & \text{if } \beta \notin F \\ (q(\beta))_{\sigma(\beta)} & \text{if } \beta \in F \end{cases} .$$

Let $\Sigma_{F,n}$ be the finite set of all mappings σ satisfying the requirements.

Lemma 15. *Suppose that $F \in [\alpha]^{<\omega}$, $n \in \omega$ and*

$$p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \dot{x} = F(\dot{\mathbf{g}}) \ \& \ \forall \beta < \alpha \dot{x} \notin \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta} .$$

There exists $q \geq_{F,n} p$ such that the sets $\left\{ F''(\overline{(q)_\sigma}) : \sigma \in \Sigma_{F,n} \right\}$ are pairwise disjoint.

Proof. Induction on $|F|$ and α . If $F = \{\beta\}$ this is essentially lemma 8.

Let $\{v_j : j < k^*\}$ be an enumeration of $\text{split}_n(p(\beta))$. For $v \in \text{split}_n(p)$ choose pairwise different reals $x_v \in F''(\overline{(p)_v})$. Note that this choice can be made canonically from, for example, the countable dense set of leftmost branches of subtrees

of p . Let $\ell > k$ be such that sequences $x_v \upharpoonright \ell$ are also pairwise different. Define conditions $\langle r_j : j \leq k^* \rangle, \langle q_j : j \leq k^* \rangle$ such that for every $j \leq k^*$,

- (1) $r_j \in \mathbb{E}\mathbb{E}_\beta$,
- (2) $r_{j+1} \geq r_j$,
- (3) $r_j \Vdash_{\mathbb{E}\mathbb{E}_\beta} q_j \geq (p)_{v_j} \upharpoonright [\beta, \alpha]$,
- (4) $\forall z \in \overline{r_j \frown q_j}, F(z) \upharpoonright \ell = F(x_{v_j}) \upharpoonright \ell$.

Let $q \upharpoonright \beta = q_{k^*}$ and $q \upharpoonright [\beta, \alpha] = \bigcup_{j < k^*} q_j$.

Suppose that $|F| = k + 1$ and let $\beta = \max(F)$.

By the part already proved, for each $\mathbf{x} = \langle x_\gamma : \gamma < \beta \rangle \in \overline{p \upharpoonright \beta}$ find a condition $q_{\mathbf{x}} \geq_n p \upharpoonright [\beta, \alpha][\mathbf{x}]$ such that the sets $\left\{ F''(\overline{(q_{\mathbf{x}})_s}) : s \in \text{split}_n(q_{\mathbf{x}}) \right\}$ are pairwise disjoint. Note that we can do it in such a way that the mapping $\mathbf{x} \mapsto q_{\mathbf{x}}$ is continuous (As before we first choose $q_{\mathbf{x}}$ in a Borel way, and then shrink $p \upharpoonright \beta$ to make this mapping continuous). That defines a $\mathbb{E}\mathbb{E}_\beta$ -name for an element of $\mathbb{E}\mathbb{E}_{\beta, \alpha}$, which we call q^* .

Next, let $F' = F \setminus \{\beta\}$ and apply the inductive hypothesis to find $q' \geq_{F', n} p \upharpoonright \beta$ such that $\left\{ F''(\overline{(q')_\sigma}) : \sigma \in \Sigma_{F', n} \right\}$ are pairwise disjoint. Let $q \in \mathbb{E}\mathbb{E}_\alpha$ be defined as $q \upharpoonright \beta = q'$ and $q \upharpoonright \beta \Vdash_{\mathbb{E}\mathbb{E}_\beta} q \upharpoonright [\beta, \alpha] = q^*$.

It is clear that q is as required. \square

CASE 3. $p \Vdash_{\mathbb{E}\mathbb{E}_\alpha} \forall \beta < \alpha \dot{x} \notin \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}$.

Let $\langle F_n : k \in \omega \rangle$ be an increasing sequence of finite sets such that $\bigcup_n F_n = \alpha$.

By induction build a sequence of conditions $\langle p_n : n \in \omega \rangle$ such that $p_0 = p$ and for every n ,

- (1) $p_{n+1} \geq_{F_n, n} p_n$,
- (2) sets $\left\{ F''(\overline{(p_n)_\sigma}) : \sigma \in \Sigma_{F_n, n} \right\}$ are pairwise disjoint.

Let $q = \lim_n p_n$.

Suppose that $\mathbf{x} = \langle x_\beta : \beta < \alpha \rangle$ and $\mathbf{x}' = \langle x'_\beta : \beta < \alpha \rangle$ are two distinct points in \overline{q} . Let β be the first ordinal such that $x_\beta \neq x'_\beta$. Let n be so large that $\beta \in F_n$ and there are two distinct $\sigma, \sigma' \in \Sigma_{F_n, n}$ such that $\mathbf{x} \in \overline{(p_n)_\sigma}$ and $\mathbf{x}' \in \overline{(p_n)_{\sigma'}}$. Since $F''(\overline{(p_n)_\sigma}) \cap F''(\overline{(p_n)_{\sigma'}}) = \emptyset$, it follows that $F(\mathbf{x}) \neq F(\mathbf{x}')$. \square

4. A MODEL WHERE $\mathbf{PM} \subseteq \mathbf{UN}$.

Let $\mathbb{E}\mathbb{E}_{\omega_2}$ be the countable support iteration of $\mathbb{E}\mathbb{E}$ of length \aleph_2 . We will show that in $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}}$, $\mathbf{PM} \subseteq \mathbf{UN}$.

By theorem 9(2), $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models [\mathbb{R}]^{< 2^{\aleph_0}} \subseteq \mathbf{UN}$, thus we have to show that

$$\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}} \models \mathbf{PM} \subseteq [\mathbb{R}]^{< 2^{\aleph_0}}.$$

Suppose that $X \in \mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}}$ is a set of reals of size \aleph_2 . Let $\{\dot{x}_\alpha : \alpha < \omega_2\}$ be the set of names for elements of X such that $\Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \forall \alpha \neq \beta \dot{x}_\alpha \neq \dot{x}_\beta$. Apply lemma 11 and find for each $\alpha < \omega_2$ a set $w_\alpha \in [\omega_2]^{\leq \omega}$, a condition $p_\alpha \in \mathbb{E}\mathbb{E}_{w_\alpha}$, and a continuous function $F_\alpha : \overline{p_\alpha} \rightarrow 2^\omega$ such that $p_\alpha \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \dot{x}_\alpha = F_\alpha(\langle \dot{g}_\beta : \beta \in w_\alpha \rangle)$. We can assume that w_α is minimal. In other words, $p_\alpha \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \forall \beta < \sup(w_\alpha) \dot{x}_\alpha \notin \mathbf{V}^{\mathbb{E}\mathbb{E}_\beta}$. In particular, without loss of generality we can assume F_α is one-to-one, so it is a homeomorphism.

By thinning out we can assume that $\text{ot}(w_\alpha) = \gamma$, $F_\alpha = F$ and $\overline{p_\alpha} = \overline{p}$. Moreover, since $\mathbf{V} \models \text{CH}$, we can assume that $w_\alpha \cap w_\beta = w^*$ for $\alpha \neq \beta$. Finally, without loss of generality we can assume that $w^* = \emptyset$.

Let $P = F''(\overline{p})$. Since F is a homeomorphism, P is perfect. We will show that $X \cap P$ is not meager in $\mathbf{V}^{\mathbb{E}\mathbb{E}_{\omega_2}}$ (relative to P).

Assume otherwise and let $H \subseteq P$ be a meager set such that for some $p^* \in \mathbb{E}\mathbb{E}_{\omega_2}$, $p^* \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} X \cap P \subseteq H$. By 9(4) we can assume that $H \in \mathbf{V}$. Set $G = (F)^{-1}(H)$ and notice that G is a meager subset of \overline{p} .

Find $\alpha < \omega_2$ such that $w_\alpha \cap \text{cl}(p^*) = \emptyset$. By lemma 12 there exists $q \geq p$, $q \in \mathbb{E}\mathbb{E}_{w_\alpha} \simeq \mathbb{E}\mathbb{E}_\gamma$ such that $\overline{q} \cap G = \emptyset$.

Since p^* and q are compatible let $r \geq p^*, q$. It follows that

$$r \Vdash_{\mathbb{E}\mathbb{E}_{\omega_2}} \dot{x}_\alpha = F_\alpha(\langle \dot{q}_\beta : \beta \in w_\alpha \rangle) \notin H,$$

which finishes the proof.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, BOISE STATE UNIVERSITY, BOISE, IDAHO 83725 U.S.A.

E-mail address: tomek@math.boisestate.edu, <http://math.boisestate.edu/~tomek>

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL AND DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NJ

E-mail address: shelah@math.huji.ac.il, <http://math.rutgers.edu/~shelah/>