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LARGE INTERVALS IN THE CLONE LATTICE

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Abstract. We give three examples of cofinal intervals in the lattice of (local) clones on an infinite set X, whose structure is on the one hand non-trivial but on the other hand reasonably well understood. Specifically, we will exhibit clones $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ such that:

(1) the interval $[\mathscr{C}_1, \mathscr{O}]$ in the lattice of local clones is (as a lattice) isomorphic to $\{0, 1, 2, \ldots\}$ under the divisibility relation,
(2) the interval $[\mathscr{C}_2, \mathscr{O}]$ in the lattice of local clones is isomorphic to the congruence lattice of an arbitrary semilattice,
(3) the interval $[\mathscr{C}_3, \mathscr{O}]$ in the lattice of all clones is isomorphic to the lattice of all filters on X.

1. Introduction

Definition 1.1. Let X be a nonempty set. The *full clone* on X, called $\mathscr{O}_X$ is the set of all finitary functions or ("operations") on X: $\mathscr{O}_X = \bigcup_{n=1}^\infty \mathscr{O}^{(n)}$, where $\mathscr{O}^{(n)}$ is the set of all functions from X^n into X. A *clone* (on X) is a set $\mathscr{C} \subseteq \mathscr{O}$ which contains all projections and is closed under composition. Alternatively, $\mathscr{C}$ is a clone if $\mathscr{C}$ is the set of term functions of some universal algebra over X. Identifying a clone $\mathscr{C}$ with the algebra $(X, \mathscr{C})$ (whose term functions are the elements of $\mathscr{C}$) allows us to talk about subalgebras and automorphisms of $\mathscr{C}$.

The set of clones over X forms a complete algebraic lattice with largest element $\mathscr{O}$. The coatoms of this lattice are called "precomplete clones" or "maximal clones". (See also [Szendrei-1986], [PK-1979]).

Definition 1.2. A clone $\mathscr{C}$ is called a *local clone*, iff each set $\mathscr{C} \cap \mathscr{O}^{(k)}$ is closed in the product topology (Tychonoff topology) on X^{X^k}, where X is taken to be discrete. In other words, $\mathscr{C}$ is local iff:

> Whenever $f \notin \mathscr{O}^{(k)}$, then there is a finite "witness" for it, i.e., there is a finite $A \subseteq X^k$ such that for all $g \in \mathscr{C}$: $g \restriction A \neq f \restriction A$.

The set of local clones over X forms again a complete lattice with largest element $\mathscr{O}$.

For any k-ary relation $\rho \subseteq X^k$ the set $\mathrm{Pol}(\rho)$ is the set of all functions preserving ρ. We will only need two special cases of this construction:

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Definition 1.3. For any $A \subseteq X$ we let

$$\mathrm{Pol}(A) := \bigcup_{n=1}^{\infty} \{f \in \mathscr{O}^{(n)} : f[A^n] \subseteq A\},$$

and for any unary (partial) function h we let

$$\mathrm{Pol}(h) := \bigcup_{n=1}^{\infty} \{f \in \mathscr{O}^{(n)} : \forall \bar{x} = (x_1, \dots, x_n) : f(h(x_1), \dots, h(x_n)) = h(f(\bar{x}))\}.$$

Definition 1.4.

- The ternary discriminator t (on the base set X) is defined to be the function $t : X^3 \to X$ satisfying $t(x, x, z) = z$ and $t(x, y, z) = x$ whenever $x \neq y$.
- An *internal isomorphism* of an algebra $(X, \mathscr{C})$ is a bijection $h : U \to V$ between two subalgebras of $(X, \mathscr{C})$ which is compatible with all operations of $\mathscr{C}$, i.e.: $\mathscr{C} \subseteq \mathrm{Pol}(h)$.
- A local clone $\mathscr{C}$ is called *locally quasiprimal* iff the elements of $\mathscr{C}$ are exactly the operations which are compatible with all internal isomorphism of $\mathscr{C}$, i.e., if

$$\mathscr{C} = \bigcap_{h \in \mathrm{Iso}(\mathscr{C})} \mathrm{Pol}(h).$$

(Note that the inclusion $\subseteq$ holds by definition of $\mathrm{Iso}(\mathscr{C})$.)

Theorem 1.5 (Pixley's theorem, see [Pixley-1971] and [Pixley-1982]). A local clone $\mathscr{C}$ is locally quasiprimal if $t \in \mathscr{C}$, where t is the ternary discriminator.

In the following sections we will use Pixley's theorem to describe intervals in the lattice of local clones. As a warmup, consider the following example:

Example 1.6. Fix an infinite set X, and let $s : X \to X$ be a 1-1 onto map without cycles. For $n > 0$, s^n is the n-th iterate of s, s^{-n} is the inverse of s^n. s^0 is the identity function.

Then $\mathscr{C}_1 := \mathrm{Pol}(s)$ is a local clone, and the local clones containing $\mathrm{Pol}(s)$ are exactly the clones $\mathrm{Pol}(s^n)$, $n \in \mathbb{Z}$; we have $\mathrm{Pol}(s^n) \subseteq \mathrm{Pol}(s^k)$ iff n divides k.

(These clones, and also the unbounded chain $\mathrm{Pol}(s^{2^n})$ were already considered in [RS-1984].)

Proof sketch. First we note that $\mathrm{Pol}(s)$ contains the discriminator, hence $\mathrm{Pol}(s)$, as well as any local clone containing it, must be locally quasiprimal.

Next, note that:

($*$) For all $a, b \in X$ there is a map $f \in \mathrm{Pol}(s)$ with $f(a) = b$.

[Why? Define $f(s^n(a)) := s^n(b)$ for all $n \in \mathbb{Z}$, and $f(x) = x$ for all x not of the form $s^n(a)$.]

Hence $(X, \mathrm{Pol}(s))$ has no proper subalgebras, so the internal isomorphisms of $(X, \mathrm{Pol}(s))$ are exactly the automorphisms of $(X, \mathrm{Pol}(s))$. Clearly s is an automorphism of this structure, and using ($*$) it is easy to see that every automorphism must be of the form s^n for some $n \in \mathbb{Z}$.

Now let $\mathscr{D}$ be a local clone above $\mathrm{Pol}(s)$. Let I be the set of internal isomorphisms (=automorphisms) of $(X, \mathscr{D})$. Then I is a subset and even a subgroup of $\{s^n : n \in \mathbb{Z}\}$, say $I = \{s^{nk_0} : n \in \mathbb{Z}\}$ for some $k_0 \in \mathbb{Z}$.

Hence $\mathscr{D}$ and $\mathrm{Pol}(s^{k_0})$ have the same set of internal isomorphisms; as both clones are locally quasiprimal, they must be equal. □

747 revision:2009-04-30 modified:2009-04-30

2. A large interval of local clones

Theorem 2.1. Let $(X, \vee)$ be a semilattice and let $Con(X, \vee)$ be the lattice of congruences on $(X, \vee)$. Let $\mathscr{C}_2$ be the clone of all operations that are bounded by the sup function of the appropriate arity:

$$\mathscr{C}_2 := \bigcup_{k=1}^{\infty} \{f \in \mathscr{O}^{(k)} : \forall x_1 \cdots x_k \; f(x_1, \ldots, x_k) \leq x_1 \vee \cdots \vee x_k\}.$$

(Here $x \leq y \;\Leftrightarrow\; x \vee y = y$ is the usual semilattice order.)

Then $[\mathscr{C}_2, \mathscr{O}_X] \simeq Con(X, \vee)$. That is, there is a lattice isomorphism between the set of local clones above $\mathscr{C}_2$ and the set of congruences of $(X, \vee)$.

Remark 2.2. If $\emptyset \subsetneq I \subsetneq X$ is an ideal, then the partition $\{I, X \setminus I\}$ corresponds to a congruence relation which is a coatom in $Con(X, \vee)$. In fact, all coatoms are obtained in this form. It is clear that $Con(X, \vee)$ is dually atomic.

It will be notationally more convenient to deal with congruence orders rather than congruence relations:

Definition 2.3. Let $(X, \vee)$ be a semilattice. We call $\preccurlyeq \subseteq X \times X$ a *congruence order* on $(X, \vee)$ if $\preccurlyeq$ is transitive, extends the semilattice order $\leq$ and satisfies

$$(**) \qquad \forall x, y, z \in X : \; x \preccurlyeq z \;\&\; y \preccurlyeq z \;\Rightarrow\; (x \vee y) \preccurlyeq z.$$

The following fact is easy to check:

Fact 2.4. The maps

$$\begin{array}{rcl} \preccurlyeq & \mapsto & \{\, (x, y) : x \preccurlyeq y \;\&\; y \preccurlyeq x \,\} \\ \theta & \mapsto & \{\, (x, y) : (x \vee y, y) \in \theta \,\} \end{array}$$

are monotone bijections between the congruence relations θ and congruence orders $\preccurlyeq$ on $(X, \vee)$, and they are inverses of each other.

Definition 2.5. For any clone $\mathscr{C}$ on the set X, and any subset $E \subseteq X$ we write $\langle E \rangle_{\mathscr{C}}$ for the subalgebra of $(X, \mathscr{C})$ that is generated by E. In other words: $\langle E \rangle_{\mathscr{C}} = \bigcup_{k=1}^{\infty} \{\, f(\bar{a}) : \bar{a} \in X^k,\; f \in \mathscr{C}^{(k)} \,\}$.

Definition 2.6. We define a correspondence between clones on X and preorders (quasiorders) on X through two maps $\mathscr{C} \mapsto R_{\mathscr{C}}$ and $\preccurlyeq \mapsto \mathscr{E}(\preccurlyeq)$.

- For any clone $\mathscr{C}$ on X, let $R_{\mathscr{C}}$ be the preorder on X defined by
$$x \; R_{\mathscr{C}} \; y \;\Leftrightarrow\; x \in \langle y \rangle_{\mathscr{C}}.$$
The associated equivalence relation $\sim_{\mathscr{C}}$ is then given by $\langle x \rangle_{\mathscr{C}} = \langle y \rangle_{\mathscr{C}}$, and the algebra $\langle x \rangle_{\mathscr{C}}$ generated by x is just the half-open interval
$$(x]_{R_{\mathscr{C}}} := \{y \in X : y \; R_{\mathscr{C}} \; x\}.$$
- For any preorder $\preccurlyeq$ on X let the clone $\mathscr{E}(\preccurlyeq)$ be defined by
$$\mathscr{E}(\preccurlyeq) = \bigcap_{a \in X} \operatorname{Pol}(\, (a]_{\preccurlyeq}).$$

Lemma 2.7. Let $\preccurlyeq$ be a preorder on X. Then the following are equivalent for all $a, b \in X$:

(i) $a \preccurlyeq b$.
(ii) $\chi_{a,b} \in \mathscr{E}(\preccurlyeq)$, where $\chi_{a,b}$ maps b to a, and is the identity otherwise.

747 revision:2009-04-30 modified:2009-04-30

(iii) There is a unary function $f \in \mathscr{E}(\preccurlyeq)$ with $f(b) = a$.
(iv) $a \in \langle b \rangle_{\mathscr{E}(\preccurlyeq)}$, i.e., $a \, R_{\mathscr{E}(\preccurlyeq)} \, b$.

Proof. The implications (i)⇒(ii)⇒(iii)⇒(i) and (iii)⇔(iv) are all easy. □

Lemma 2.8. Let $\mathscr{C}$ be a clone on X, $\mathscr{D} := \mathscr{E}(R_{\mathscr{C}})$. Then $\mathscr{C}$ and $\mathscr{D}$ have the same 1-generated subalgebras: $\forall b \in X : \langle b \rangle_{\mathscr{C}} = \langle b \rangle_{\mathscr{D}}$.

Proof. By the equivalence (i)⇔(iv) in Lemma 2.7, the relations $R_{\mathscr{C}}$ and $R_{\mathscr{D}}$ coincide. Now $\langle b \rangle_{\mathscr{C}} = (\, b \,]_{R_{\mathscr{C}}} = (\, b \,]_{R_{\mathscr{D}}} = \langle b \rangle_{\mathscr{D}}$. □

The relation $\langle x \rangle_{\mathscr{C}} \subseteq \langle y \rangle_{\mathscr{C}}$ carries information only about the unary functions of $\mathscr{C}$; in our context, however, this is sufficient, because our clones $\mathscr{C}$ are generated by $\mathscr{C}^{(1)} \cup \{\vee\}$. The "encoding" property defined below will help us to reduce questions about subalgebras to questions about 1-generated subalgebras.

Definition 2.9. Let $\mathscr{C}$ be a clone on X, $*$ a binary function in $\mathscr{C}$. We say that $(\mathscr{C}, *)$ *encodes pairs* iff $\langle x, y \rangle_{\mathscr{C}} = \langle x * y \rangle_{\mathscr{C}}$ for all $x, y \in X$.

Fact 2.10. Assume that both $(\mathscr{C}, *)$ and $(\mathscr{D}, *)$ encode pairs. Then the following are equivalent:

(1) $\mathscr{C}$ and $\mathscr{D}$ have the same subalgebras.
(2) $\mathscr{C}$ and $\mathscr{D}$ have the same finitely generated subalgebras.
(3) $\langle x \rangle_{\mathscr{C}} = \langle x \rangle_{\mathscr{D}}$ for all $x \in X$.

Lemma 2.11. Let $(X, \vee)$ be a semilattice and let $x \le y$ iff $x \vee y = y$. Let $\mathscr{C} := \mathscr{E}_{\le}$. Then

(1) $\mathscr{C}$ is a local clone containing the binary function $\vee$ as well as the ternary discriminator. In fact:
$\mathscr{C} = \bigcup_{k=1}^{\infty} \{\, f : \forall x_1, \ldots, x_n \, f(x_1, \ldots, x_n) \le x_1 \vee \cdots \vee x_n \,\}$.
(2) $\langle x, y \rangle_{\mathscr{C}} = \langle x \vee y \rangle_{\mathscr{C}}$, and similarly $\langle x, y \rangle_{\mathscr{D}} = \langle x \vee y \rangle_{\mathscr{D}}$ for all clones $\mathscr{D} \supseteq \mathscr{C}$.
(3) If $U, V \le (X, \mathscr{C})$, and $h : U \to V$ is an isomorphism with respect to the operations in $\mathscr{C}$, then one of the following holds:
   - $U = V$, and h is the identity on U.
   - U and V are singleton subalgebras.

Proof. (1) and (2) are obvious.

(3) For any a, b define $\psi_{a,b}(a, b) = a$, and $\psi_{a,b}(x, y) = y$ otherwise. Clearly $\psi_{a,b} \in \mathscr{C}$.

Assume that U contains at least 2 elements, and let $u \in U$ with $h(u) \neq u$. If there is some element $a < u$ in U, then the inequality

$$h(\psi_{a,u}(a, u)) = h(a) \neq h(u) = \psi_{a,u}(h(a), h(u))$$

shows that h is not an internal isomorphism. Otherwise let $u' \in U \setminus \{u\}$ and $b := u \vee u' \in U$, then $u < b$, and we get

$$h(\psi_{u,b}(u, b)) = h(u) \neq h(b) = \psi_{u,b}(h(u), h(b)).$$

□

747 revision:2009-04-30 modified:2009-04-30

Proof of Theorem 2.1. We just need to collect a few implications:

(1) The maps $\preccurlyeq \to \mathscr{E}(\preccurlyeq)$ and $\mathscr{D} \mapsto R_{\mathscr{D}}$ are monotone.
(2) For any local clone $\mathscr{D} \supseteq \mathscr{C}_2$, the relation $R_{\mathscr{D}}$ is a congruence order. [The main property to check is 2.3(∗∗): If $\langle x\rangle_{\mathscr{D}} \subseteq \langle z\rangle_{\mathscr{D}}$ and $\langle y\rangle_{\mathscr{D}} \subseteq \langle z\rangle_{\mathscr{D}}$, then $\langle x \vee y\rangle_{\mathscr{D}} = \langle x, y\rangle_{\mathscr{D}} \subseteq \langle z\rangle_{\mathscr{D}}$.]
(3) For any congruence order $\preccurlyeq$, the clone $\mathscr{E}(\preccurlyeq)$ is a local clone extending $\mathscr{C}_2$. [Obvious.]
(4) Any congruence order $\preccurlyeq$ coincides with $R_{\mathscr{E}(\preccurlyeq)}$. [By 2.7.]
(5) Finally we claim that each local clone $\mathscr{D} \subseteq \mathscr{C}$ coincides with $\mathscr{D}' := \mathscr{E}(R_{\mathscr{D}})$: From 2.8 we know that $\mathscr{D}$ and $\mathscr{D}'$ have the same 1-generated subalgebras, so from 2.10 we conclude that they have the same subalgebras. By 2.11, they have the same internal isomorphisms, so by Pixley's theorem they must be equal.

□

Example 2.12. Let $(X, <)$ be a linearly ordered set. Then the congruence relations on $(X, \max)$ are exactly the equivalence relations with convex classes. As a special case, consider the semilattice $(\mathbb{N}, \max)$. A congruence relation is just a partition of $\mathbb{N}$ into disjoint intervals.

The map $\theta \mapsto A_\theta := \{\max E : E \text{ is a finite congruence class}\}$ is an antitone 1-1 map from the congruence relations onto $\mathscr{P}(\mathbb{N})$, the power set of $\mathbb{N}$. The empty set corresponds to $\mathscr{O}$, or to the equivalence relation with a single class; the set $\mathbb{N}$ itself corresponds to $\mathscr{C}_2$, or to the equivalence with singleton classes.

3. A large interval of clones

On any infinite set X we will define a clone $\mathscr{C}_3$ such that the interval $[\mathscr{C}_3, \mathscr{O}]$ in the full clone lattice is very large (with $2^{2^{|X|}}$ precomplete elements), but still reasonably well understood.

Definition 3.1.

(1) For $A \subseteq X$, $n \geq 1$ let $\Delta_n(A) := \{(a, \dots, a) \in X^n : a \in A\}$.
(2) For any function $f \in \mathscr{O}^{(n)}$, let $f^{(1)} \in \mathscr{O}^{(1)}$ be defined by $f^{(1)}(x) = f(x, \dots, x)$.
(3) For any function $f \in \mathscr{O}^{(n)}$, we let
$$\mathrm{fix}(f) = \{x : f^{(1)}(x) = x\},\ \mathrm{nix}(f) = \{x : f^{(1)}(x) \neq x\}$$
(4) For any clone $\mathscr{C}$ we define $\mathrm{fix}(\mathscr{C}) := \{\mathrm{fix}(f) : f \in \mathscr{C}\}$, $\mathrm{nix}(\mathscr{C}) := \{\mathrm{nix}(f) : f \in \mathscr{C}\}$.
(5) For any family $T \subseteq \mathscr{P}(X)$ we define
$$\mathscr{E}_T := \{f \in \mathscr{O} : \mathrm{fix}(f) \in T\} = \bigcup_{A \in T} \bigcap_{a \in A} \mathrm{Pol}(\{a\})$$

Definition 3.2. Let $\mathscr{C}_3 := \mathscr{E}_{\{X\}}$ be the clone of "idempotent" functions, i.e., of all functions satisfying $f(x, \dots, x) = x$ for all x.

Theorem 3.3. The map $T \to \mathscr{E}_T$ is an order isomorphism between the set of all filters on X (including the improper filter $\mathscr{P}(X)$) and the set of all clones above $\mathscr{C}_3$.

In particular, the precomplete clones above $\mathscr{C}_3$ are exactly the clones of the form $\mathscr{E}_U$, where U is an ultrafilter on X.

747 revision:2009-04-30 modified:2009-04-30

Remark 3.4. The subalgebras of $\mathscr{C}_3$ are exactly all singleton sets, and $\mathscr{C}_3$ contains the ternary discriminator. Hence every local clone above $\mathscr{C}_3$ is determined by its subalgebras. For every $A \subseteq X$ the clone

$$\mathscr{E}_{\{A\}} := \bigcap_{a\in A} \mathrm{Pol}(\{a\}) = \{\, f : f(a,a,\dots,a) = a \text{ for all } a \in A\,\}$$

is a local clone whose subalgebras are exactly the singleton sets $\{a\}$ with $a \in A$. Hence the local clones above $\mathscr{C}_3$ are exactly the clones of the form $\mathscr{E}_{\{A\}}$, $A \subseteq X$. In the language of theorem 3.3: the local clones above $\mathscr{C}_3$ are exactly the clones corresponding to principal filters.

Lemma 3.5. Assume that $\mathscr{D} \supseteq \mathscr{C}_3$ is a clone, $f \in \mathscr{D}$, $\mathrm{fix}(f) \subseteq \mathrm{fix}(g)$. Then $g \in \mathscr{D}$. Hence, every clone $\mathscr{D} \supseteq \mathscr{C}_3$ is determined by $\mathrm{nix}(\mathscr{D})$: $f \in \mathscr{D} \Leftrightarrow \mathrm{nix}(f) \in \mathrm{nix}(\mathscr{D})$.

Proof. For $\bar{x} = (x_1, \dots, x_n)$ define

$$G(\bar{x}, y) = \begin{cases} x_1 & \text{if } x_1 = \cdots = x_n = y \\ g(\bar{x}) & \text{otherwise} \end{cases}$$

Clearly $G \in \mathscr{C}_3$. For $\bar{x} \in \Delta_n(\mathrm{fix}(f)) \subseteq \Delta_n(\mathrm{fix}(g))$, both $g(\bar{x})$ and $G(\bar{x}, f^{(1)}(x_1))$ have the value x_1. If $\bar{x} \in \Delta_n(X)\backslash\Delta_n(\mathrm{fix}(f))$, then $f^{(1)}(x_1) \neq x_1$, so $G(\bar{x}, f^{(1)}(x_1)) = g(\bar{x})$ by definition; the same holds for $\bar{x} \notin \Delta_n(X)$. Hence $g(\bar{x}) = G(\bar{x}, f^{(1)}(x_1))$ for all $\bar{x} \in X^n$. □

Lemma 3.6. Let $\mathscr{D}$ be a clone with $\mathscr{C}_3 \subseteq \mathscr{D}$, $\mathscr{D} \neq \mathscr{O}$. Then $\mathrm{nix}(\mathscr{D})$ is an ideal and $\mathrm{fix}(\mathscr{D})$ is a filter.

Proof. Lemma 3.5 shows that $\mathrm{nix}(\mathscr{D})$ is downward closed.

Now let $A_\ell = \mathrm{nix}(f_\ell)$, $f_\ell \in \mathscr{D}$ for $\ell = 1, 2$, and assume that $A_1 \cap A_2 = \emptyset$.

Let $B = X \setminus (A_1 \cup A_2)$. We may assume that either $|A_1| \geq 2$, or $B \neq \emptyset$ (or both).

In either case there is a unary function f_1' with $\mathrm{nix}(f_1') = A_1$, and f_1' maps A_1 into $A_1 \cup B$. By lemma 3.5, $f_1' \in \mathscr{D}$. So $f_2 \circ f_1' \in \mathscr{D}$. Since $\mathrm{nix}(f_2 \circ f_1') = A_1 \cup A_2$ we see that $\mathrm{nix}(\mathscr{D})$ is closed under $\cup$. □

Fact 3.7. For all filters T and all $A \in T$ there is a unary function f with $\mathrm{fix}(f) = A$.

Proof of Theorem 3.3. Again we just collect some implications.

(1) The maps $\mathscr{D} \mapsto \mathrm{fix}(\mathscr{D})$ and $T \mapsto \mathscr{E}_T$ are clearly monotone with respect to set inclusion.
(2) For every clone $\mathscr{C}$ the set $\mathrm{fix}(\mathscr{C})$ is a filter (by Lemma 3.6).
(3) For every filter T the set $\mathscr{E}_T$ is a clone. (Obvious.)
(4) For every filter T we have $T = \mathrm{fix}(\mathscr{E}_T)$. (By Fact 3.7.)
(5) For every clone $\mathscr{D} \supseteq \mathscr{C}_3$ we have $\mathscr{D} = \mathscr{E}_{\mathrm{fix}(\mathscr{D})}$:
Let $\mathscr{D}' := \mathscr{E}_{\mathrm{fix}(\mathscr{D})}$. From the previous item we conclude $\mathrm{fix}(\mathscr{D}') = \mathrm{fix}(\mathscr{D})$. Now by Lemma 3.5 we see $\mathscr{D} = \mathscr{D}'$.

□

Remark 3.8. If we regard the set X as a discrete topological space, then the Stone-Cech compactification of X is

$$\beta X = \{U : U \text{ is an ultrafilter on } X\}$$

There is a canonical 1-1 order-preserving correspondence between the filters on X (ordered by $\subseteq$) and the closed subsets of βX (ordered by $\supseteq$).

747 revision:2009-04-30 modified:2009-04-30

So the interval $[\mathscr{C}_3, \mathscr{O}]$ in the full clone lattice is isomorphic (as a complete lattice) to the family of closed subsets of βX, ordered by reverse inclusion: $\mathscr{O}$ corresponds to the empty set, each precomplete clone in $[\mathscr{C}_3, \mathscr{O}]$ corresponds to a singleton set.

Note that for any closed subset $F \subseteq \beta X$ and any $p \in \beta X \setminus F$, also $F \cup \{p\}$ is closed, and moreover:

F covers G (i.e., $F \supset G$, and the interval (G, F) is empty) iff $G = F \cup \{p\}$ for some $p \in \beta X \setminus F$

In particular, let $\mathscr{C}_{\rm bd} \supseteq \mathscr{C}_3$ be the clone corresponding to the ideal of small sets, i.e.,

$$\mathscr{C}_{\rm bd} := \{f \in \mathscr{O}^{(\cdot)} \exists B \subseteq X, |B| < |X|, \forall x \in X \setminus B : f(x, \dots, x) = x\}$$

Then every clone $\mathscr{C} \supsetneq \mathscr{C}_{\rm bd}$ has exactly $2^{2^{|X|}}$ lower neighbors in the clone lattice; the clone corresponds to a closed set F, and the lower neighbors correspond to closed sets $F \cup \{p\}$. This is a special case of a theorem of [Marchenkov-1981].

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747 revision:2009-04-30 modified:2009-04-30