THE NON-COMMUTATIVE SPECKER PHENOMENON IN THE UNCOUNTABLE CASE

SAHARON SHELAH AND KATSUYA EDA

1

An infinitary version of the notion of free products has been introduced and investigated by G. Higman [5]. Let $G_i (i \in I)$ be groups and $*_{i\in X} G_i$ the free product of $G_i (i \in X)$ for $X \Subset I$ and $p_{XY} : *_{i\in Y} G_i \to *_{i\in X} G_i$ the canonical homomorphism for $X \subseteq Y \Subset I$. ($X \Subset I$ denotes that X is a finite subset of I.)

Then, the unrestricted free product is the inverse limit $\varprojlim(*_{i\in X} G_i, p_{XY} : X \subseteq Y \Subset I)$.

We remark $*_{i\in\emptyset} G_i = \{e\}$.

For the simplicity, we abreviate $\varprojlim(*_{i\in X} G_i, p_{XY} : X \subseteq Y \Subset I)$ and $\varprojlim(*_{n\in\omega}\mathbb{Z}_n, p_{mn} : m \leq n < \omega)$ by $\varprojlim *G_i$ and $\varprojlim *\mathbb{Z}_n$ in the sequel.

For $S \subseteq I$, $p_S : \varprojlim *G_i \to \varprojlim *G_i$ be the canonical projection defined by: $p_S(x)(X) = x(X \cap S)$ for $X \Subset I$.

Theorem 1.1. *Let F be a free group. Then, for each homomorphism $h : \varprojlim *G_i \to F$ there exist countably complete ultrafilters $u_0, \cdots, u_m$ on I such that $h = h \cdot p_{U_0\cup\cdots\cup U_m}$ for every $U_0 \in u_0, \cdots, U_m \in u_m$.*

If the cardinality of the index set I is less than the least measurable cardinal, then there exists a finite subset X_0 of I and a homomorphism $\overline{h} : *_{i\in X_0} G_i \to F$ such that $h = \overline{h} \cdot p_{X_0}$, where $p_{X_0} : \varprojlim *G_i \to *_{i\in X_0} G_i$ is the canonical projection.

Previously, the second author showed the failure of the Specker phenomenon in the uncountable case in a different situation [6]. (See also [1].) We explain the difference between this result and Theorem 1.1 of the present paper. There is a canonical subgroup of the unrestricted free product, which is called the free complete product and denoted by $⋇_{i\in I} G_i$. When an index set I is countable, according to the Higman theorem (Lemma 1.2 and a variant for $⋇_{n<\omega}\mathbb{Z}_n$ [5, p.80]), a homomorphism from $\varprojlim *G_i$ or $⋇_{i\in I} G_i$ to a free group factors through a finite free product $*_{i\in F} G_i$. On the other hand, when the index set I is uncountable and each G_i is non-trivial, there exists a free retract of $⋇_{i\in I} G_i$ of large cardinality and there are homomorphisms not factoring through any finite free product $*_{i\in F}G_i$, which contrasts with the case when I is countable. This also contrasts with an abelian case, which is known as the Łoś theorem [4]. Theorem 1.1 says that differing from the case of the free complete product the non-commutative Specker phenomenon holds for the unrestricted free products similarly as in the abelian case.

Since the following lemma holds for the free σ-product $\times^{\sigma}_{i\in I}\mathbb{Z}_i$ instead of a free group F [3], Theorem 1.1 also holds for it. (We remark $\times^{\sigma}_{i\in I}\mathbb{Z}_i = \times_{i\in I}\mathbb{Z}_i$, when I is countable.)

Lemma 1.2. (*G. Higman* [5]) *For each homomorphism* $h : \varprojlim *\mathbb{Z}_n \to F$ *there exists* $m < \omega$ *and a homomorphism* $\overline{h} : *_{i<m}\mathbb{Z}_i \to F$ *such that* $h = \overline{h}\cdot p_m$*, where* $p_m : \varprojlim *\mathbb{Z}_n \to *_{n<m}\mathbb{Z}_n$ *is the canonical projection.*

Lemma 1.3. *Let* $I = \bigcup\{I_n : n < \omega\}$ *with* $I_n \subseteq I_{n+1}$ *and* $x_n \in G$ *be such that* $p_{I_n}(x_n) = e$*. Then, there exists a homomorphism* $\varphi : \varprojlim *\mathbb{Z}_n \to \varprojlim *G_i$ *such that* $\varphi(\delta_n) = x_n$ *for each* $n < \omega$*.*

For a homomorphism $h : \varprojlim *G_i \to F$, let $\mathrm{supp}(h) = \{X \subseteq I : p_X(g) = e$ implies $h(g) = 0$ for each $g\}$. In the sequel we assume that h is non-trivial. We remark the following facts:

(1) $p_X \cdot p_Y = p_{X\cap Y}$ for $X, Y \subseteq I$;
(2) $\mathrm{supp}(h) = \{X \subseteq I : h(g) = h(p_X(g))$ for each $g\}$;
(3) supp is a filter on I.

Lemma 1.4. *Let* $A_n \subseteq A_{n+1} \subseteq I$ *and* $A = \bigcup\{A_n : n < \omega\}$ *and* $B_{n+1} \subseteq B_n \subseteq I$ *and* $B = \bigcap\{B_n : n < \omega\}$*. If* $A_n \notin \mathrm{supp}(h)$ *for each* n*, then* $A \notin \mathrm{supp}(h)$ *and if* $B_n \in \mathrm{supp}(h)$ *for each* n*, then* $B \in \mathrm{supp}(h)$*.*

Proof. Suppose that $A \in \mathrm{supp}(h)$. Take g_n so that $h(g_n) \neq 0$ and $p_{A_n}(g_n) = e$ for each n and let $u_n = p_A(g_n)$. Since $I = \bigcup\{A_n \cup (I \setminus A) : n < \omega\}$ and $p_{A_n\cup(I\setminus A)}(u_n) = p_{A_n}(g_n) = e$, by Lemma 1.3 we have a homomorphism $\varphi : \varprojlim *\mathbb{Z}_n \to \varprojlim *G_i$ such that $\varphi(\delta_n) = u_n$ for each $n < \omega$. Then, $h \cdot \varphi(\delta_n) \neq 0$ for each n, which contradicts Lemma 1.2.

To show the second proposition by contradiction, suppose that $B \notin \mathrm{supp}(h)$. Then, we have $g \in G$ such that $p_B(g) = e$ but $h(g) \neq 0$. Let $v_n = p_{B_n}(g)$. Since $I = \bigcup\{B \cup (I \setminus B_n) : n < \omega\}$ and $p_{B\cup(I\setminus B_n)}(v_n) = p_B \cdot p_{B_n}(g_n) = e$, we apply Lemma 1.3 and have a homomorphism $\varphi : \varprojlim *\mathbb{Z}_n \to \varprojlim *G_i$ such that $\varphi(\delta_n) = v_n$ for each $n < \omega$. Then, we have a contradiction similarly as the above. □

Lemma 1.5. *Let* $A_0 \notin \mathrm{supp}(h)$*. Then, there exist* A *satisfying the following:*

(1) $A_0 \subseteq A \notin \mathrm{supp}(h)$*;*

(2) *for* $X \subseteq I$, $A \cup X \notin \mathrm{supp}(h)$ *imply* $(I \setminus X) \cup A \in \mathrm{supp}(h)$.

Proof. We construct $A_n \notin \mathrm{supp}(h)$ by induction as follows. Suppose that we have constructed $A_n \notin \mathrm{supp}(h)$. If A_n satisfies the required properties of A, we have finished the proof. Otherwise, there exist $A_n \subseteq A_{n+1} \subseteq I$ such that $A_{n+1} \notin \mathrm{supp}(h)$ and $(I \setminus A_{n+1}) \cup A_n \notin \mathrm{supp}(h)$. We claim that this process finishes in a finite step. Suppose that the process does not stop in a finite step. Then, we have A_n's and so let $A = \bigcup\{A_n : n < \omega\}$. Then, $A \notin \mathrm{supp}(h)$ by Lemma 1.4. Since $I \setminus A \subseteq I \setminus A_{n+1}$, $(I \setminus A) \cup A_n \notin \mathrm{supp}(h)$ for each $n < \omega$. Now, $I = \bigcup\{(I \setminus A) \cup A_n : n < \omega\}$ and and hence $I \notin \mathrm{supp}(h)$ by Lemma 1.4, which is a contradiction. □

Proof of Theorem 1.1.

Let $h : \varprojlim *G_i \to F$ be a non-trivial homomorphism. Apply Lemma 1.5 for $A_0 = \emptyset$ and we have A. We define u_0 as follows:

$X \in u_0$ if and only if $A \cup X \in \mathrm{supp}(h)$ for $X \subseteq I$. Then, u_0 is a countably complete ultrafilter on I by Lemma 1.4. We let $I_0 = I \setminus A$, then obviously $I \setminus I_0 \notin \mathrm{supp}(h)$.

When $I_0 \in \mathrm{supp}(h)$, then $h = h \cdot p_{U_0}$ for any $U_0 \in u_0$ and we have finished the proof. Otherewise, we construct $I_n \notin \mathrm{supp}(h)$ and countably complete ultrafilters u_n on I with $I_n \in u_n$ by induction as follows. Suppose that $\bigcup_{i=0}^n I_i \notin \mathrm{supp}(h)$, we aplly Lemma 1.5 for $A_0 = \bigcup_{i=0}^n I_i \notin \mathrm{supp}(h)$ and get a countably complete ultrafilter u_{n+1} on I wit $I_{n+1} \in u_{n+1}$ so that $I \setminus I_{n+1} \notin \mathrm{supp}(h)$.

To show that this procedure stops in a finite step, suppose the negation. Since $(I \setminus \bigcup_{k=0}^\infty I_k) \cup \bigcup_{k=0}^n I_k$ is disjoint from I_{n+1}, $(I \setminus \bigcup_{k=0}^\infty I_k) \cup \bigcup_{k=0}^n I_k \notin \mathrm{supp}(h)$ for each n. Then, we have $I \notin \mathrm{supp}(h)$ by Lemma 1.4, which is a contradiction.

Now, we have a finite partition $I_0, \cdots, I_n$ of I. By the construction, $X \in u_k$ if and only if $\bigcup_{i \neq k} I_i \cup X \in \mathrm{supp}(h)$ and hence for $U_k \in u_k$ $(0 \leq k \leq n)$, then $\bigcup_{i \neq k} I_i \cup U_k \in \mathrm{supp}(h)$. Since $\mathrm{supp}(h)$ is a filter, $\bigcup_{k=0}^n U_k \in \mathrm{supp}(h)$ and we have the first proposition.

If each u_j contains a singleton $\{i_j\}$, we have $\{i_0, \cdots, i_k\} \in \mathrm{supp}(h)$ and consequently the second one. □

Remark 1.6. (1) As we explained it, in the uncountable case the unrestricted free product and the free complete product behave differently concerning the Specker phenomenon. Therefore, there is a part in the above proof which cannot be converted to the case of the free complete product. It is an application of Lemma 1.3. The corresponding lemma to Lemma 1.3 is [2, Proposition 1.9], which holds under a more strict condition than Lemma 1.3, and we cannot apply it.

(2) In [3, theorem 1.2], we treated with general inverse limits for a countable index set. For general limits we cannot generalize to the uncountable case [3, Remark 1.2].

References

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Department of Mathematics, Hebrew University, Givat Ram, 91904 Jerusalem, Israel

School of Science and Engineering, Waseda University, Tokyo 169-0072, Japan
E-mail address: eda@logic.info.waseda.ac.jp