WEAK DIAMOND

SAHARON SHELAH

Abstract. Under some cardinal arithmetic assumptions, we prove that every stationary subset of $\lambda$ of a right cofinality has weak diamond. This is a strong negation of uniformization. We then deal with a weaker version of the weak diamond that involves restricting the domain of the colourings. We then deal with semi-saturated (normal) filters.

Key words and phrases. Set theory, Normal ideals, Weak diamond, precipitous filters, semi saturated filters.

Research supported by the United States-Israel Binational Science Foundation Publication 755. Corrections after proof reading to the journal.
Annotated Content

§1. Weak Diamond: sufficient condition

We prove that if $\lambda = 2^\mu = \lambda^{<\lambda}$ is weakly inaccessible,

$$\Theta = \{ \theta : \theta = \text{cf}(\theta) < \lambda \quad \text{and} \quad \alpha < \lambda \Rightarrow |\alpha^{(\text{tr}, \theta)}| < \lambda \} \quad \text{and} \quad S \subseteq \{ \delta < \lambda : \text{cf}(\delta) \in \Theta \}$$

is stationary then it has weak diamond. We can omit or weaken the demand $\lambda = \lambda^{<\lambda}$ if we restrict the colouring $F$ (in the definition of the weak diamond) such that for $\eta \in \delta \delta$, $F(\eta)$ depends only on $\eta \upharpoonright C_\delta$ where $C_\delta \subseteq \delta$, $\lambda = \lambda|C_\delta|$. 

§2. On versions of precipitousness

We show that for successor $\lambda > \beth_\omega$, the club filter on $\lambda$ is not semi-saturated (even every normal filter concentrating on any $S \in I[\lambda]$ of cofinality from a large family). Woodin had proved $D_{\omega_1} + S_\delta^\delta$ consistently is semi-saturated].
1. Weak Diamond: sufficient condition

On the weak diamond see [DvSh 65], [Sh:f, Appendix §1], [Sh 208], [Sh 638]; there will be subsequent work on the middle diamond.

Definition 1.1. For regular uncountable $\lambda$,

1. We say $S \subseteq \lambda$ is small if it is $F$-small for some function $F$ from $\lambda^+ \lambda$ to $\{0,1\}$, which means $(*)_{F,S}$ for every $\delta \in S^2$ there is $\eta \in \lambda \lambda$ such that $\{\lambda \in S : F(\eta \downarrow \delta) = c_\delta\}$ is not stationary.

2. Let $D^\text{wd}_\lambda = \{A \subseteq \lambda : \lambda \setminus A$ is small $\}$, it is a normal ideal (the weak diamond ideal).

Claim 1.2. Assume

(a) $\lambda = \lambda^{<\lambda} = 2^\mu$

(b) $\Theta = \{\theta : \theta = cf(\theta)$ and for every $\alpha < \lambda$, we have $|\alpha|^{<\theta} < \lambda$ or just $|\alpha|^{<|\theta|} < |\lambda|$ (see below; so if $\lambda > \sum_\omega$ every large enough regular $\theta < \sum_\omega$ is in $\Theta$)

(c) $S \subseteq \{\delta < \lambda : cf(\delta) \in \Theta$, and $\mu^\omega$ divides $\delta$ $\}$ is stationary.

Then $S$ is not in the ideal $D^\text{wd}_\lambda$ of small subsets of $\lambda$.

Definition 1.3. (1) Let $\chi^{(\theta)} = \text{Min}\{|P| : P \subseteq [\chi]^\theta$ and every $A \in [\chi]^\theta$ is included in the union of $< \theta$ members of $P\}$.

(2) $\chi^{(\theta)} = \text{sup}\{|\text{lim}_{\theta}(t)| : t$ is a tree with $\leq \chi$ nodes and $\theta$ levels $\}$

Remark 1.4. (1) On $\chi^{(\theta)}$, see [Sh 589], on $\chi^{(\theta)}$ see there and in [Sh 460] but no real knowledge is assumed here.

(2) The interesting case of 1.2 is $\lambda$ (weakly) inaccessible; for $\lambda$ successor we know more; but in later results even if $2^\mu$ is successor we say on it new things.

(3) Actually only $F \upharpoonright (\bigcup_{\delta \in S} \delta \delta)$ mark. ??

Proof. Let $F$ be a function from $\bigcup_{\delta \in S} \delta \lambda$ to $\{0,1\}$, i.e., $F$ is a colouring, and we shall find $f \in S^\lambda$ as required for it.

Let $\{\nu_i : i < \lambda\}$ list $\bigcup_{\alpha < \lambda} \alpha \lambda$ such that

$$\alpha < \text{lg}(\nu_i) \Rightarrow \nu_i \upharpoonright \alpha \in \{\nu_j : j < i\}.$$ For $\delta \in S$ let $P_\delta = \{\eta \in \delta \delta : (\forall \alpha < \delta)(\eta \downarrow \alpha \in \{\nu_i : i < \delta\})\}.$

Clearly $\delta \in S \Rightarrow |P_\delta| \leq |\delta|^{<|\theta|} < \lambda$ by assumption (c). For each $\eta \in P_\delta$ we define $h_\eta \in \mu^2$ by: $h_\eta(\varepsilon) = F(g_\eta,\varepsilon)$ where for $\varepsilon < \mu$, we let $g_\eta,\varepsilon \in \delta \delta$ be defined by $g_\eta,\varepsilon(\alpha) = \eta(\mu \alpha + \varepsilon)$ for $\alpha < \delta$, recalling that $\mu^\omega$ divides $\delta$ as $\delta \in s$. So $\{h_\eta : \eta \in P_\delta\}$ is a subset of $\mu^2$ of cardinality $|P_\delta| < \lambda = 2^\mu$ hence we can choose $g_\delta \in \mu^2 \setminus \{g_\eta : \eta \in P_\delta\}$. For $\varepsilon < \mu$ let $f_\varepsilon \in S^\lambda$ be $f_\varepsilon(\delta) = 1 - g_\delta(\varepsilon)$. If for some $\varepsilon < \mu$ the function $f_\varepsilon$ serve as a weak diamond
sequence for $F$, we are done so assume that (for each $\varepsilon < \mu$) there are $\eta_\varepsilon$ and $E_\varepsilon$ such that:

(a) $E_\varepsilon$ is a club of $\lambda$.

(b) $\eta_\varepsilon \in \check{\lambda}^\varepsilon$.

(c) if $\delta \in E_\varepsilon \cap S$ then $F(\eta_\varepsilon \restriction \delta) = 1 - f_\varepsilon(\delta)$ and $\eta_\varepsilon \restriction \delta \in \delta^\varepsilon$.

Now define $\eta \in \check{\delta}^2$ by $\eta(\mu \alpha + \varepsilon) = \eta_\varepsilon(\alpha)$ for $\alpha < \lambda, \varepsilon < \mu$.

Let $E = \{ \delta < \lambda : \delta \text{ is divisible by } \mu^\varepsilon \text{ and } \varepsilon < \mu \Rightarrow \delta \in E_\varepsilon \text{ and } (\forall \alpha < \delta)[\eta \restriction \alpha \in \{ \eta_i : i < \delta \}] \}$. Clearly $E$ is a club of $\lambda$ hence we can find $\delta \in E \cap S$. So by the definition of $\mathcal{P}_\delta$ we have $\eta \restriction \delta \in \mathcal{P}_\delta$ and for $\varepsilon < \mu$ we have $g_{\eta \restriction \delta, \varepsilon} \in \delta^\varepsilon$ is equal to $\eta_\varepsilon \restriction \delta$ (Why? note that $\mu \delta = \mu$ as $\delta \in E$ and see the definition of $g_{\eta \restriction \delta, \varepsilon}$ and of $\eta$, so: $\alpha < \delta \Rightarrow g_{\eta \restriction \delta, \varepsilon}(\alpha) = (\eta_\varepsilon(\mu \alpha + \varepsilon) = \eta_\varepsilon(\alpha)$). Hence $h_{\eta \restriction \delta} \in \check{\mu}_2$ is well defined and by the choice of $\eta$ we have $\varepsilon < \mu \Rightarrow g_{\eta \restriction \delta, \varepsilon} = \eta_\varepsilon \restriction \delta$ so by its definition, $h_{\eta \restriction \delta}$ for each $\varepsilon < \mu$ satisfies $h_{\eta \restriction \delta}(\varepsilon) = F(g_{\eta \restriction \delta, \varepsilon}) = F(\eta_\varepsilon \restriction \delta)$.

Now by clause (c) and the choice of $f_\varepsilon$ we have $F(\eta_\varepsilon \restriction \delta) = 1 - f_\varepsilon(\delta) = g_\varepsilon^\delta(\varepsilon)$ so $h_{\eta \restriction \delta} = g_\varepsilon^\delta$, but $h_{\eta \restriction \delta} \in \mathcal{P}_\delta$ whereas we have chosen $g_\varepsilon^\delta$ such that $g_\varepsilon^\delta \notin \mathcal{P}_\delta$, a contradiction.

We may consider a generalization.

**Definition 1.5.**

(1) We say $C$ is a $\lambda-$Wd-parameter if:

(a) $\lambda$ is a regular uncountable,

(b) $S$ a stationary subsets of $\lambda$,

(c) $C = \langle C_\delta : \delta \in S \rangle, C_\delta \subseteq \delta$

(1A) We say $C$ is a $(\lambda, \kappa, \chi)$-Wd-parameter if in addition $(\forall \delta \in S)[\text{cf}(\delta) = \kappa \land |C_\delta| < \chi]$. We may also say that $C$ is $(S, \kappa, \chi)$-parameter.

(2) We say that $F$ is a $C$-colouring if $C$ is a $\lambda$-Wd-parameter and $F$ is a function from $\lambda^\lambda \triangleleft \lambda$ to $2$ such that:

If $\delta \in S$, $\eta_0, \eta_1 \in \delta^\varepsilon$ and $\eta_0 \restriction \delta = \eta_1 \restriction \delta$ then $F(\eta_0) = F(\eta_1)$.

(2A) If $C = \langle \check{\delta} : \delta \in S \rangle$ we may omit it writing $S$ - colouring

(2B) In part (2) we can replace $F$ by $\langle F_\delta : \delta \in S \rangle$ where $F_\delta : \langle C_\delta \rangle \delta \rightarrow 2$ such that $\eta \in \delta^\varepsilon \land \delta \in S \Rightarrow F(\eta) = F_\delta(\eta \restriction \delta \subseteq C_\delta)$. So abusing notation we may write $F(\eta \restriction \delta)$

(3) Assume $F$ is a $C$-clauing, $C$ a $\lambda$-Wd-parameter.

We say $\check{c} \in \check{S}^2$ (or $\check{c} \in \lambda^2$) is an $F$-wd-sequence if:

(*) for every $\eta \in \lambda^\lambda$, the set $\{ \delta \in S : F(\eta \restriction \delta) = c_\delta \}$ is a stationary subset of $\lambda$.

We also may say $\check{c}$ is an $(F, S)$-Wd-sequence.

(3A) We say $\check{c} \in \check{S}^2$ is a $D - F$-Wd-sequence if $D$ is a filter on $\lambda$ to which $S$ belongs and

(*) for every $\eta \in \lambda^\lambda$ we have

$\{ \delta \in S : F(\eta \restriction \delta) = c_\delta \} \neq \emptyset \mod D$

(4) We say $C$ is a good $\lambda$-Wd-parameter, if for every $\alpha < \lambda$ we have $\lambda > |\{ C_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in C_\delta \}|$.
Similarly to 1.2 we have

**Claim 1.6.** Assume

(a) \( \mathcal{C} \) is a good \((\lambda, \kappa, \chi)\)-Wd-parameter.
(b) \( |\alpha|^{(\kappa, \kappa)} < \lambda \) for every \( \alpha < \lambda \).
(c) \( \lambda = 2^\mu \) and \( \lambda = \lambda^{\lt \chi} \).
(d) \( F \) is a \( C \)-colouring.

Then there is a \( F \)-Wd-sequence.

**Proof.** Let \( \text{cd} \) be a 1-to-1 function from \( \mu \lambda \) onto \( \lambda \), for simplicity, and without loss of generality

\[ \alpha = \text{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu \rangle) \Rightarrow \alpha \geq \sup \{ \alpha_\varepsilon : \varepsilon < \mu \} \]

and let the function \( \text{cd}_i : \lambda \to \lambda \) for \( i < \mu \) be such that \( \text{cd}_i((\text{cd}(\alpha_\varepsilon : \varepsilon < \mu))) = \alpha_i \).

Let \( T = \{ \eta \colon \text{for some } C \subseteq \lambda \text{ of cardinality } < \chi, \text{ we have } \eta \in C \lambda \} \), so by assumption (c) clearly \( |T| = \lambda \), so let us list \( T \) as \( \{ \eta_\alpha : \alpha < \lambda \} \) with no repetitions, and let \( T_{\alpha<\lambda} = \{ \eta_\beta : \beta < \alpha \} \). For \( \delta \in S \) let \( \mathcal{P}_\delta = \{ \eta : \eta \text{ a function from } C_\delta \text{ to } \delta \text{ such that for every } \alpha \in C_\delta \text{ we have } \eta | (C_\delta \cap \alpha) \in T_{\delta<\alpha} \} \).

By \( C \) being good and clause (b) of the assumption necessarily \( \mathcal{P}_\delta \) has cardinality \( < \lambda \). For each \( \eta \in \mathcal{P}_\delta \) and \( \varepsilon < \mu \) we define \( \nu_{\eta, \varepsilon} \in C_\delta \) by \( \nu_{\eta, \varepsilon}(\alpha) = \text{cd}_\varepsilon(\eta(\alpha)) \) for \( \alpha \in C_\delta \). Now for \( \eta \in \mathcal{P}_\delta \), clearly \( \rho_\eta = : \langle F(\nu_{\eta, \varepsilon}) : \varepsilon < \mu \rangle \) belongs to \( \mu^2 \). Clearly \( \{ \rho_\eta : \eta \in \mathcal{P}_\delta \} \) is a subset of \( \mu^2 \) of cardinality \( \leq |\mathcal{P}_\delta| \) which as said above is \( < \lambda \). But \( |\mu^2| = 2^\mu = \lambda \) by clause (c) of the assumption, so we can find \( \rho_\delta^* \in \mu^2 \setminus \{ \rho_\eta : \eta \in \mathcal{P}_\delta \} \).

For each \( \varepsilon < \mu \) we can consider the sequence \( \vec{c}^\varepsilon = \langle 1 - \rho_\delta^*(\varepsilon) : \delta \in S \rangle \) as a candidate for being an \( F \)-Wd-sequence. If one of them is, we are done.

So assume toward contradiction that for each \( \varepsilon < \mu \) there is \( \eta_\varepsilon \in \lambda^\lambda \) which exemplify its failure, so there is a club \( E_\varepsilon \) of \( \lambda \) such that

\[ \exists_1 \delta \in S \cap E_\varepsilon \Rightarrow F(\eta_\varepsilon | C_\delta) \neq c_\varepsilon^\delta \]

and without loss of generality

\[ \exists_2 \alpha < \delta \in E_\varepsilon \Rightarrow \eta_\varepsilon(\alpha) < \delta. \]

But \( c_\varepsilon^\delta = 1 - \rho_\delta^*(\varepsilon) \) and so \( z \in \{0, 1\} \setminus z \neq c_\varepsilon^\delta \Rightarrow z = \rho_\delta^*(\varepsilon) \) hence we have got

\[ \exists_3 \delta \in S \cap E \Rightarrow F(\eta_\varepsilon | C_\delta) = \rho_\delta^*(\varepsilon) \]

Define \( \eta^* \in \lambda^\lambda \) by \( \eta^*(\alpha) = \text{cd}(\langle \eta_\varepsilon(\alpha) : \varepsilon < \mu \rangle) \), now as \( \lambda \) is regular uncountable clearly \( E^\prime = \{ \delta \colon \text{for every } \alpha < \delta \text{ we have } \eta^*(\alpha) < \delta \} \) and if \( \delta' \in S, C' = C_\delta \cap \alpha \in C_\delta \) then \( \eta^* | C' \in T_{\delta<\alpha} \) is a club of \( \lambda \) (see the choice of \( T, T_{\delta<\alpha} \), recall that by assumption (a) the sequence \( \vec{C} \) is good, see Definition 1.5(4)).

Clearly \( E^* = \cap \{ E_\varepsilon : \varepsilon < \mu \} \cap E \) is a club of \( \lambda \). Now for each \( \delta \in E^* \cap S \), clearly \( \eta^* \in C_\delta \in \mathcal{P}_\delta \); just check the definitions of \( \mathcal{P}_\delta \) and \( E, E^* \). Now recall \( \nu_{\eta^* | C_\delta, \varepsilon} \) is the function from \( C_\delta \) to \( \delta \) defined by

\[ \nu_{\eta^* | C_\delta, \varepsilon}(\alpha) = \text{cd}_\varepsilon(\eta^*(\alpha)). \]
But by our choice of \( \eta^* \) clearly \( \text{cd}_\varepsilon(\alpha) = \eta_\varepsilon(\alpha) \), so 
\[
\alpha \in C_\delta \Rightarrow \nu_{\eta^*|C_\delta,\varepsilon}(\alpha) = \eta_\varepsilon(\alpha) \quad \text{so} \quad \nu_{\eta^*|C_\delta,\varepsilon} = \eta_\varepsilon \upharpoonright C_\delta,
\]
Hence \( \mathbf{F}(\nu_{\eta^*|C_\delta,\varepsilon}) = \mathbf{F}(\eta_\varepsilon \upharpoonright C_\delta) \), however as \( \delta \in E^* \subseteq E_\varepsilon \) clearly \( \mathbf{F}(\eta_\varepsilon \upharpoonright C_\delta) = \rho^*_\delta(\varepsilon) \), together \( \mathbf{F}(\nu_{\eta^*|C_\delta,\varepsilon}) = \rho^*_\delta(\varepsilon) \).

As \( \eta^* \upharpoonright C_\delta \in \mathcal{P}_\delta \) clearly \( \rho^*_\eta \upharpoonright C_\delta \in 2^\lambda \), moreover for each \( \varepsilon < \mu \) we know that \( \rho^*_\eta \upharpoonright (C_\delta(\varepsilon)) \), see its definition above, is equal to \( \mathbf{F}(\nu_{\eta^*|C_\delta,\varepsilon}) \) which by the previous sentence is equal to \( \rho^*_\delta(\varepsilon) \). As this holds for every \( \varepsilon < \mu \) and \( \rho^*_\eta \upharpoonright C_\delta \), \( \rho^*_\delta \) are members of \( 2^\lambda \), clearly they are equal. But \( \eta^* \upharpoonright C_\delta \in \mathcal{P}_\delta \) so \( \rho^*_\eta \upharpoonright C_\delta \in \{ \eta : \eta \in \mathcal{P}_\delta \} \) whereas \( \rho^*_\delta \) has been chosen outside this set, contradiction. \( \square \)

Well, are there good \( (\lambda, \kappa, \kappa) \)-parameters? (on \( I[\lambda] \) see [Sh 420, §31]).

**Claim 1.7.**

1. If \( S \) is a stationary subset of the regular cardinal \( \lambda \) and \( S \in I[\lambda] \) and \( (\forall \delta \in S) \text{cf}(\delta) = \kappa \) then for some club \( E \) of \( \lambda \), there is a good \( (S \cap E, \kappa, \kappa) \)-parameter.
2. If \( \kappa = \text{cf}(\kappa), \kappa^+ < \lambda = \text{cf}(\lambda) \) then there is a stationary \( S \in I[\lambda] \) with \( (\forall \delta \in S)[\text{cf}(\delta) = \kappa] \).

**Proof.**

1. By the definition of \( I[\lambda] \)
2. By [Sh 420, §31].  \( \square \)

We can note

**Claim 1.8.**

1. Assume the assumption of 1.6 or 1.2 with \( C_\delta = \delta \) and \( D \) is a \( \mu^+ \)-complete filter on \( \lambda, S \in D \), and \( D \) include the club filter. Then we can get that there is a \( D - \mathbf{F} \)-Wd-sequence.
2. In 1.6, we can weaken the demand \( \lambda = 2^\mu \) to \( \lambda = \text{cf}(2^\mu) \) that is, assume
   a. \( C \) is a good \( (\lambda, \kappa, \chi) \)-Wd-parameter.
   b. \( |\alpha|^{(\tau, \kappa)} < 2^\mu \) for every \( \alpha < \lambda \).
   c. \( \lambda = \text{cf}(2^\mu) \) and \( 2^\mu = (2^\mu)^< \chi \)
   d. \( \mathbf{F} \) is a \( \check{C} \)-colouring
   e. \( D \) is a \( \mu^+ \)-complete filter on \( \lambda \) extending the club filter to which \( \text{Dom}(\check{C}) \) belongs.
   
   Then \(^{1}\) there is a \( D - \mathbf{F} \)-Wd-sequence.
3. In 1.6+1.8(2) we can omit \( \lambda \) regular”.

**Proof.**

1. The same proof.
2. Let \( H^* : \lambda \to 2^\mu \) be increasing continuous with unbounded range and let \( S \in I[\lambda] \) be stationary, such that \( (\forall \delta \in S) \text{cf}(\delta) = \kappa \), and \( \check{C} = \langle C_\delta : \delta \in S \rangle \) is a good \( (\text{cf}(\lambda), \kappa, \kappa) \)-Wd- parameter, let
   \[
   S' = \{ h^*(\alpha) : \alpha \in S \}, \quad C_{h^*(\delta)} = \{ h^*(\alpha) : \alpha \in C_\delta \}, \quad \check{C}' = \langle C_\beta : \beta \in S' \rangle
   \]

\(^{1}\)in fact if \( \lambda = \text{cf}(2^\mu) < 2^\mu \) then the demand \( "\check{C} \) is good” is not necessary; see more in [Sh 775]
and repeat the proof using \( \lambda' = 2^\mu, C' = \langle C'_\delta : \delta \in S' \rangle \) instead of \( \lambda, C \).
Except that in the choice of the club \( E \) we should use \( E' = \{ \delta < \lambda : \forall\alpha \in \delta \cap \text{Rang}(h^*) \text{ we have } \eta^*(\alpha) < \delta \text{ and } \delta \text{ is a limit ordinal and } \delta' \in S' \wedge C' = C'_\delta \cap \alpha \Rightarrow \eta^* | C' \in T_{<\delta} \} \).

(3) Similarly.

\[ \square \]

This lead to considering the natural related ideal.

**Definition 1.9.** Let \( \bar{C} \) be a \((\lambda, \kappa, \chi)\)-parameter.

1. For a family \( \mathcal{F} \) of \( \bar{C} \)-colouring and \( \mathcal{P} \subseteq \lambda^2 \), let \( id_{\bar{C}, \mathcal{F}, \mathcal{P}} \) be
   \[
   \{ \delta \in \lambda : \text{for some } \mathcal{F} \in \mathcal{F} \text{ for every } \bar{c} \in \mathcal{P} \text{ for some } \eta \in \lambda \lambda \text{ the set}
   \{ \delta \in \lambda \cap S : \mathcal{F}(\eta | C_\delta) = c_\delta \} \text{ is not stationary} \}.
   \]

2. If \( \mathcal{P} \) is the family of all \( \bar{C} \)-colouring we may omit it. If we write Def instead of \( \mathcal{F} \) this mean as in [Sh 576, §1].

We can strengthen 1.6 as follows.

**Definition 1.10.** We say the \( \lambda \)-colouring \( F \) is \((S, \chi)\)-good if:

1. \( S \subseteq \{ \delta < \lambda : cf(\delta) < \chi \} \) is stationary
2. we can find \( E \) and \( \langle C_\delta : \delta \in S \cap E \rangle \) such that
   \begin{itemize}
   \item \( E \) a club of \( \lambda \)
   \item \( C_\delta \) is an unbounded subset of \( \delta, |C_\delta| < \chi \).
   \item if \( \rho, \rho' \in C_\delta, \delta \in S \cap E, \) and \( \rho' \cap C_\delta = f \cap C_\delta \)
   \renewcommand{\arraystretch}{1.5}
   \begin{align*}
   \text{then } F(\rho') &= F(\rho) \\
   \text{for every } \alpha < \lambda \text{ we have } \lambda > |\{ C_\delta \cap \alpha : \delta \in S \cap E \text{ and } \alpha \in C_\alpha \}| \\
   \text{we have } \delta \in S \Rightarrow |\delta^{cf(\delta)}| : \\
   \end{align*}
   \end{itemize}

**Claim 1.11.** Assume

1. \( \lambda = cf(2^\mu) \)
2. \( F \) is an \((S, \kappa)\)-good \( \lambda \)-colouring.

Then there is a \((F, S)\)-Wd-sequence, see Definition 1.5(3).

**Remark 1.12.** So if \( \lambda = cf(2^\mu) \) and we let \( \Theta_\lambda =: \{ \theta = cf(\theta) \text{ and } (\forall \alpha < \lambda)(\alpha^{tr(\theta)} < \lambda) \} \) then

1. \( \Theta_\lambda \) “large” (e.g. contains every large enough \( \theta \in \text{Reg} \cap \omega_2 \) if \( \omega_2 < \lambda \)) and
2. if \( \theta = cf(\theta) \land \theta^+ < \lambda \) then there is a stationary \( S \in I[\lambda] \text{ such that } \delta \in S \Rightarrow cf(\delta) = \theta. \)
3. if \( \theta \in \Theta, S \) are as above then there is a good \( \langle C_\delta : \delta \in S \rangle \)
4. for \( \theta, S, C \) as above, if \( F = \langle F_\delta : \delta \in S \rangle \) and \( F_\delta(\eta) \) depend just on \( \eta \cap C_\delta \) and \( D \) is a normal ultrafilter on \( \lambda \) (or less), and lastly \( S \in D \) then there is an \( D - F \)-Wd-sequence; see Definition 1.5(3A).
2. On versions of precipitousness

Definition 2.1. (1) We say the \( D \) is \((\mathbb{P}, D)\)-precipitous if
(a) \( D \) is a normal filter on \( \lambda \), a regular uncountable cardinal.
(b) \( \mathbb{P} \) is forcing notion with \( \emptyset \mathbb{P} \) minimal.
(c) \( D \) a \( \mathbb{P} \)-name of an ultrafilter of the Boolean Algebra \( \mathcal{P}(\lambda) \)
(d) letting for \( p \in \mathbb{P} \)
\[ D_{p,D} = \{ A \subseteq \lambda : p \Vdash A \in D \} \]
we have:
(\( \alpha \)) \( D_{\emptyset, D} = D \) and
(\( \beta \)) \( D_{p,D} \) is normal filter on \( \lambda \)
(\( \varepsilon \)) \( p \Vdash \text{"} V^\lambda / D \text{" is well founded"}. \)
(2) For \( \lambda \) regular uncountable and \( D \) a normal filter on \( \lambda \) let
\[ \text{NOR} \lambda D = \{ D' : D' \text{ a normal filter on } \lambda \text{ extending } D \} \]
Woodin [W99] defined and was interested in semi-saturation for \( \lambda = \aleph_2 \), where!

(1A) If \( D \) is clear from the context (as in part (2)) we may omit \( D \).

Definition 2.2. For \( \lambda \) regular uncountable cardinal, a normal filter \( D \) on \( \lambda \) is called semi-saturated when for every forcing notion \( \mathbb{P} \) and \( \mathbb{P} \)-name \( D \) of a normal (for regressive \( f \in V \)) ultrafilter on \( \mathcal{P}(\lambda)^V \), we have: \( D \) is \((\mathbb{P}, D)\)-precipitous.

Woodin proved \( \text{Con}(\omega_2 \upharpoonright S^0_{\aleph_2} \text{ is semi saturated}) \), he proved that the existence of such filter has large consistency strength by proving 2.3 below. This is related to [Sh:g, V].

Claim 2.3. If \( \lambda = \mu^+ \), \( D \) a semi-saturated filter or \( \lambda \), then every \( f \in \lambda^\lambda \) is \( <_D \) than the \( \alpha \)-th function for some \( \alpha < \lambda^+ \) (on the \( \alpha \)-th function see e.g [Sh:g, XVII, §3])

In fact

Claim 2.4. If \( \lambda = \mu^+ \) and \( D \) is \( \text{NOR}_\lambda \)-precipitous then every \( f \in \lambda^\lambda \) is \( <_D \) smaller than the \( \alpha \)-th function for some \( \alpha < \lambda^+ \)

Proof. The point is that
(a) if \( D \) is a normal filter on \( \lambda \), \( \langle f_\alpha : \alpha < \lambda^+ \rangle \) is \( <_D \) -increasing in \( \lambda \) and \( f, \lambda, \alpha < \lambda^+ \Rightarrow \neg(f \leq_D f_\alpha) \) then there is a normal filter \( D_1 \) on \( \lambda \) extending \( D \) such that \( \alpha < \lambda^+ \Rightarrow f_\alpha < _D f \)
(b) if \( \langle f_\alpha : \alpha \leq \lambda^+ \rangle \) is \( <_D \) - increasing \( f_\alpha \in \lambda^\lambda \), and \( \lambda = \mu^+ \) and \( X \subseteq \delta < \lambda : \text{cf}(f_{\lambda^+}(\delta)) = \theta \not\in \emptyset \bmod D \) then there are functions \( g_i \subseteq \lambda^\lambda \) for \( i < \theta \) such that \( g_i < f_{\lambda^+} \bmod (D + X) \), and \( \forall \alpha < \lambda^+ (\exists i < \theta)(\neg g_i <_D f_\alpha) \).

[In details let \( \Gamma = \{ (D_1, f, \alpha) : D_1 \in \text{NOR}_\lambda, f \in \lambda^\lambda, D_1 \Vdash \text{NOR}_\lambda \text{ "} f / D \text{ is the } \alpha \text{-th ordinal in } V^\lambda / D \text{ and } \neg f \leq f_\alpha \text{mod } D_1 \text{ for } \alpha < \lambda^+ \}, \text{ for some} \)
Claim 2.5.  

(1) If \( \lambda = \mu^+ \geq \beth_\omega \) then the club filter on \( \lambda \) is not semi-saturated.

(2) If \( \lambda = \mu^+ \geq \beth_\omega \) then for every large enough regular \( \kappa < \beth_\omega \), there is no semi-saturated normal filter \( D^* \) on \( \lambda \) to which \( S^\lambda_\kappa = \{ \delta < \lambda : cf(\delta) = \kappa \} \) belongs.

(3) If \( 2^{\kappa_0} < \lambda = \mu^+ > \kappa = cf(\kappa) > \aleph_0 \) and for every \( f \in {}^\kappa \lambda \) we have \( rk_{\kappa_0}(f) < \lambda \) then there is no semi-saturated normal filter \( D^* \) on \( \lambda \) to which \( \{ \delta < \lambda : cf(\delta) = \kappa \} \) belongs.

(4) In 1), 2), 3), if \( "D \) is Nor\( \mathcal{D} \)-semi-saturated" then the conclusion holds for \( D \).

Remark: We can replace \( \beth_\omega \) by any strong limit uncountable cardinal.

Proof.  

(1) Follows by (2)

(2) By [Sh 460] for some \( \kappa_0 < \beth_\omega \), for every regular \( \kappa \in (\kappa_0, \beth_\omega) \) we have: \( \mu^{(\kappa)} = \mu \), see 1.3. Let \( D = \{ A \subseteq \kappa : \text{sup}(\kappa \setminus A) < \kappa \} \).

By part (3) it is enough to prove

\( \square \) if \( f \in {}^\kappa \lambda \) then \( rk_D(f) < \lambda \)

proof of \( \square \) If not then for every \( \alpha < \lambda \) there is

\[ f_\alpha \in {}^\kappa \lambda \quad \text{such that} \quad f_\alpha <_D f \quad \text{and} \quad \text{rk}_D(f) = \alpha \]

and define

\[ D_\alpha = \{ A \subseteq \kappa : A \in D \quad \text{or} \quad \kappa \setminus A \notin D, \quad \text{and} \quad \text{rk}_{D+({\kappa \setminus A})}(f_\alpha) < \alpha \} \]

This is a \( \kappa \)-complete filter on \( \kappa \) see [Sh 589]. So for some \( D^* \) the set

\( A = \{ \alpha : D_\alpha = D^* \} \)

is unbounded in \( \lambda \). By [Sh 589, \S 4] (alternatively use [Shig, V] on normal filters)

\( (*) \) for \( \alpha < \beta \) from \( A, f_\alpha < D^* f_\beta \) and \( D^* \) is a \( \kappa \)-complete filter on \( \kappa \).

But as \( \mu = \mu^{(\kappa)} \) letting \( \alpha^* = \text{sup}(\text{Rang}(f)) + 1 \) which is \( < \lambda \), so

\( |\alpha^*| < \mu \), there is a family \( \mathcal{P} \subseteq [\alpha^*]^{\kappa} \)

such that for every \( a \in [\alpha^*]^{\kappa} \),

for some \( i(s) < \kappa \) and \( a_i \in \mathcal{P} \) for \( i < i(s) \) we have \( a \subseteq \bigcup_{i < i(s)} a_i \) hence

for every \( \alpha \in A \), for some \( a_\alpha \in \mathcal{P} \) we have

\[ \{ i \in \kappa : f_\alpha(i) \in a_\alpha \} \neq \emptyset \mod D^* \]

So for some \( \alpha^* \) and unbounded \( B \subseteq A \) we have \( \alpha \in B \Rightarrow a_\alpha = \alpha^* \) and moreover for some \( b^* \subseteq \kappa \) we have \( \alpha \in B \Rightarrow b^* = \{ i \in \kappa : f_\alpha(i) \in a^* \} \)

and moreover \( \alpha \in B \Rightarrow f_\alpha \upharpoonright b^* = f^* \). But this contradict \( (*) \).
(3) We can find \( \langle u_{\alpha, \varepsilon} : \varepsilon < \lambda, \alpha < \lambda^+ \rangle \) such that:

(a) \( \langle u_{\alpha, \varepsilon} : \varepsilon < \lambda \rangle \) is \( \subseteq \)-increasing continuous such that \( |u_{\alpha, \varepsilon}| < \lambda \), and \( \cup\{u_{\alpha, \varepsilon} : \varepsilon < \lambda\} = \alpha \).

(b) if \( \alpha < \beta < \lambda^+ \) and \( \alpha \in u_{\beta, \varepsilon} \) then \( u_{\beta, \varepsilon} \cap \alpha = u_{\alpha, \varepsilon} \).

Let \( f_\alpha \in \lambda^\lambda \) be \( f_\alpha(\varepsilon) = \text{otp}(u_{\alpha, \varepsilon}) \), so it is well known that \( f_\alpha/D_\lambda \) is the \( \alpha \)-th function, in particular \( \alpha < \beta \Rightarrow f_\alpha <_{D_\lambda} f_\beta \) where \( D_\lambda \) is the club filter on \( \lambda \); in fact \( \alpha < \beta < \lambda^+ \Rightarrow f_\alpha <_{j^\delta_\lambda} f_\beta \). Choose

\[ \bar{C} = \langle C_\delta : \delta \in S^\lambda_\kappa \rangle, \quad C_\delta \text{ a club of } \delta \text{ of order type } \kappa, \text{ and let } g_\delta \in \kappa^\delta \text{ enumerate } C_\delta, \text{ i.e. } g_\delta(i) \text{ is the } i \text{-th member of } C_\delta \]

For \( \zeta < \lambda \) let \( g^*_\zeta \in \kappa^\lambda \) be constantly \( \zeta \), and let \( g^* \in \lambda^\lambda \) be defined by \( g^*(\delta) = \text{rk}_{j^\delta_\lambda}(g^*_\delta) \)

\[ (\ast)_0 \quad g^* \in \lambda^\lambda \text{ and } \zeta < g^*(\delta) \]

[why? by an assumption]

For \( \alpha < \lambda^+ \) we define \( f^*_\alpha \in \lambda^\lambda \) by:

\[
f^*_\alpha(\varepsilon) = \begin{cases} \text{rk}_{j^\delta_\lambda}(f_\alpha \circ g_\varepsilon) & \text{if } \varepsilon \in S^\alpha_\kappa \\ 0 & \text{if } \varepsilon \in \lambda \setminus S^\alpha_\kappa \end{cases}
\]

Note that \( f_\alpha \circ g_\varepsilon \) is a function from \( \kappa \) to \( \lambda \).

Now

\[ (\ast)_1 \quad f^*_\alpha \in \lambda^\lambda \text{ for } \alpha < \lambda^+ \]

[Why? as \( f_\alpha \circ g_\varepsilon \in \kappa^\lambda \), so by a hypothesis \( \text{rk}_{j^\delta_\lambda}(f_\alpha \circ g_\varepsilon) < \lambda \)]

\[ (\ast)_2 \text{ for } \alpha < \lambda^+ \]

\[ (\ast)_2 \text{ for } \alpha < \lambda^+ \]

\[ (\ast)_3 \text{ for } \alpha < \lambda^+ \]

\[ \delta \in E_\alpha \Rightarrow f^*_\alpha(\delta) < g^*(\delta), \text{ so } f^*_\alpha <_{D_\lambda} g^* \in \lambda^\lambda \]

[why? the first statement by the definition of \( E_\alpha \), of \( f^*_\alpha \) and of \( g^*(\delta) \). The second by the first \((\ast)_0\).]

\[ (\ast)_4 \text{ if } \alpha < \beta < \lambda^+ \text{ then } f^*_\alpha <_{j^\delta_\lambda} f^*_\beta \text{ hence } f^*_\alpha <_{D_\lambda} f^*_\beta \]

[why? the first as \( f_\alpha <_{j^\delta_\lambda} f_\beta \) hence for some \( \varepsilon < \lambda \), we have

\[ \varepsilon < \zeta < \lambda \rightarrow f_\alpha(\zeta) < f_\beta(\zeta) \]

hence \( \delta \in S^\lambda_\kappa \setminus (\varepsilon + 1) \Rightarrow \]

\[ f_\alpha \upharpoonright C_\delta <_{j^\delta_\lambda} f_\beta \upharpoonright C_\delta \Rightarrow f_\alpha \circ g_\delta <_{j^\delta_\lambda} f_\beta \circ g_\delta \Rightarrow \text{rk}_{j^\delta_\lambda}(f_\alpha \circ g_\delta) < \text{rk}_{j^\delta_\lambda}(f_\beta \circ g_\delta) \Rightarrow \]

\[ f^*_\alpha(\delta) < f^*_\beta(\delta) \]

Let \( f^*_{\lambda^+} =: g^* \), so

\[ (\ast) \alpha \leq \lambda^+ \Rightarrow f^*_\alpha \in \lambda^\lambda \text{ and } \alpha < \beta \leq \lambda^+ \Rightarrow f_\alpha <_{D_\lambda} f_\beta \]

This of course suffices by ??.

\[ \text{\(2\) recall } S^\lambda_\kappa = \{ \delta < \lambda : \text{cf}(\delta) = \kappa \}\]
(4) The same proof.

Remark: In the proof of 2.5(2) it is enough that $U_{\mu^+}(\mu) = \mu$ (see [Sh 589]).

[References of the form math.XX/· · · refer to arXiv.org]

REFERENCES


