NOWHERE PRECIPITOUSNESS OF THE NON-STATIONARY IDEAL OVER $\mathcal{P}_\kappa\lambda$

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Abstract. We prove that if λ is a strong limit singular cardinal and κ a regular uncountable cardinal $< \lambda$, then $NS_{\kappa\lambda}$, the non-stationary ideal over $\mathcal{P}_\kappa\lambda$, is nowhere precipitous. We also show that under the same hypothesis every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.

§1. Introduction

Throughout this paper we let κ denote an uncountable regular cardinal and λ a cardinal $\geq \kappa$. Let $NS_{\kappa\lambda}$ denote the non-stationary ideal over $\mathcal{P}_\kappa\lambda$. $NS_{\kappa\lambda}$ is the minimal κ-complete normal ideal over $\mathcal{P}_\kappa\lambda$. If X is a stationary subset of $\mathcal{P}_\kappa\lambda$, then $NS_{\kappa\lambda}|X$ denotes the κ-complete normal ideal generated by the members of $NS_{\kappa\lambda}$ and $\mathcal{P}_\kappa\lambda - X$. We refer the reader to Kanamori [6, Section 25] for basic facts about the combinatorics of $\mathcal{P}_\kappa\lambda$.

Large cardinal properties of ideals have been investigated by various authors. One of the problems studied by these set theorists was to determine which large cardinal properties can $NS_{\kappa\lambda}$ or $NS_{\kappa\lambda}|X$ bear for various κ, λ and $X \subseteq \mathcal{P}_\kappa\lambda$. In the course of this investigation, special interest has been paid to two large cardinal properties, namely precipitousness and saturation.

If $NS_{\kappa\lambda}|X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$, then we say that $NS_{\kappa\lambda}$ is *nowhere precipitous*. In [8] Matsubara and Shioya proved that if λ is a strong limit singular cardinal and $\operatorname{cf}\lambda < \kappa$, then $NS_{\kappa\lambda}$ is nowhere precipitous. In §2 we extend this result by showing that $NS_{\kappa\lambda}$ is nowhere precipitous if λ is a strong limit singular cardinal.

In [10] Menas conjectured the following:

Menas' Conjecture. *Every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

This conjecture implies that $NS_{\kappa\lambda}|X$ cannot be $\lambda^{<\kappa}$-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$. By the work of several set theorists we know that Menas' Conjecture is independent of ZFC. One of the most striking results concerning this conjecture is the following theorem of Gitik [4].

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Gitik's Theorem. *Suppose that κ is a supercompact cardinal and $\lambda > \kappa$. Then there is a p.o. $\mathbb{P}$ that preserves cardinals $\geq \kappa$ such that $\Vdash_{\mathbb{P}}$ "$\exists X$ (X is a stationary subset of $\mathcal{P}_\kappa\lambda \wedge X$ cannot be partitioned into κ^+ disjoint stationary sets)".*

In §2 we also show that if λ is a strong limit singular cardinal, then every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets. Gitik [4] mentions that GCH fails in his model of a "non-splittable" stationary subset of $\mathcal{P}_\kappa\lambda$. Our result shows that GCH *must* fail in such a model of a non-splittable stationary subset of $\mathcal{P}_\kappa\lambda$ if λ is singular.

We often consider the poset $\mathbb{P}_I$ of I-positive subsets of $\mathcal{P}_\kappa\lambda$ i.e. subsets of $\mathcal{P}_\kappa\lambda$ not belonging to I, ordered by

$$X \leq_{\mathbb{P}_I} Y \Longleftrightarrow X \subseteq Y.$$

We say that an ideal I is "proper" if $\mathbb{P}_I$ is a proper poset. In [9] Matsubara proved the following result:

Proposition. *Let δ be a cardinal $\geq 2^{2^{2^\lambda}}$. If there is a "proper" λ^+-complete normal ideal over $\mathcal{P}_{\lambda^+}\delta$ then $NS_{\aleph_1\lambda}$ is precipitous.*

It is not known whether $NS_{\kappa\lambda}$ can be precipitous for singular λ. In [1] it is conjectured that $NS_{\kappa\lambda}$ cannot be precipitous if λ is singular. Therefore it is interesting to ask the following question:

Question. *Can $\mathcal{P}_\kappa\lambda$ bear a "proper" κ-complete normal ideal where κ is the successor cardinal of a singular cardinal?*

In §3 we give a negative answer to this question.

§2. On $NS_{\kappa\lambda}$ for strong limit singular λ

We first state our main results.

Theorem 1. *If λ is a strong limit singular cardinal, then $NS_{\kappa\lambda}$ is nowhere precipitous.*

Theorem 2. *If λ is a strong limit singular cardinal, then every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

One of the key ingredients of our proof of the main results is Lemma 3. Part (ii) of Lemma 3 was proved in Matsubara [7]. Part (i) appeared in Matsubara-Shioya [8]. For the proof of Part (ii) we refer the reader to Kanamori [6, page 345]. However we will present the proof of (i) because the idea of this proof will be used later.

Lemma 3. *If $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$, then*
(i) *$NS_{\kappa\lambda}$ is nowhere precipitous*
(ii) *every stationary subset of $\mathcal{P}_\kappa\lambda$ can be partitioned into $\lambda^{<\kappa}$ disjoint stationary sets.*

Before we present the proof of part (i), we make some comments concerning this lemma. First note that the hypothesis of our lemma is satisfied if λ is a strong limit cardinal with $\mathrm{cf}\,\lambda < \kappa$. Secondly under this hypothesis every unbounded subset of $\mathcal{P}_\kappa\lambda$ must have a size of 2^λ. We also note that Lemma 3 can be generalized in the following manner:

For an ideal I over some set A, we let $\mathrm{non}(I) = \min\{|X| \mid X \subseteq A,\ X \notin I\}$ and $\mathrm{cof}(I) = \min\{|J| \mid J \subseteq I, \forall X \in I, \exists Y \in J\,(X \subseteq Y)\}$. The proof of Lemma 3 actually shows that if $\mathrm{non}(I) = \mathrm{cof}(I)$ then I is nowhere precipitous (i.e. for every I-positive X, $I|X$ is not precipitous) and every I-positive subset X of A can be partitioned into $\mathrm{non}(I)$ many disjoint I-positive sets.

Proof of Lemma 3 (i). For I an ideal over $\mathcal{P}_\kappa\lambda$, let $G(I)$ denote the following game between two players, Nonempty and Empty: Nonempty and Empty alternately choose I-positive sets X_n, $Y_n \subseteq \mathcal{P}_\kappa\lambda$ respectively so that $X_n \supseteq Y_n \supseteq X_{n+1}$ for $n = 1, 2, \ldots$. After ω moves, Empty wins $G(I)$ if $\bigcap_{n\in\omega-\{0\}} X_n = \emptyset$. See [3] for a proof of the following characterization.

Proposition. *I is nowhere precipitous if and only if Empty has a winning strategy in $G(I)$.*

Let $\langle f_\alpha \mid \alpha < 2^\lambda\rangle$ enumerate functions from $\lambda^{<\omega}$ into $\mathcal{P}_\kappa\lambda$. For a function $f : \lambda^{<\omega} \to \mathcal{P}_\kappa\lambda$, we let $C(f) = \{s \in \mathcal{P}_\kappa\lambda \mid \bigcup f'' s^{<\omega} \subseteq s\}$. For $X \subseteq \mathcal{P}_\kappa\lambda$, X is stationary if and only if $C(f_\alpha) \cap X \neq \emptyset$ for every $\alpha < 2^\lambda$.

We now describe Empty's strategy in $G(NS_{\kappa\lambda})$ using the hypothesis $2^{<\kappa} < \lambda^{<\kappa} = 2^\lambda$. Suppose that X_1 is Nonempty's first move. Choose $\langle s^1_\alpha \mid \alpha < 2^\lambda\rangle$, a sequence of elements of X_1 by induction on α in the following manner: Let s^1_0 be any element of $X_1 \cap C(f_0)$. Suppose we have $\langle s^1_\alpha \mid \alpha < \beta\rangle$ for some $\beta < 2^\lambda$. Since $\{s^1_\alpha \mid \alpha < \beta\}$ is a non-stationary, in fact bounded, subset of $\mathcal{P}_\kappa\lambda$, $X_1 - \{s^1_\alpha \mid \alpha < \beta\}$ is stationary. Pick an element from $(X_1 - \{s^1_\alpha \mid \alpha < \beta\}) \cap C(f_\beta)$ and call it s^1_β. Let Empty play $Y_1 = \{s^1_\alpha \mid \alpha < 2^\lambda\}$. It is easy to see that Y_1 is a stationary subset of $\mathcal{P}_\kappa\lambda$. Inductively suppose Nonempty plays his $n+1$-st move X_{n+1} immediately following Empty's n-th move $Y_n = \{s^n_\alpha \mid \alpha < 2^\lambda\}$. Choose $\langle s^{n+1}_\alpha \mid \alpha < 2^\lambda\rangle$, a sequence from X_{n+1} in the following manner: Let s^{n+1}_0 be any element of $(X_{n+1} - \{s^n_0\}) \cap C(f_0)$. Suppose we have $\langle s^{n+1}_\alpha \mid \alpha < \beta\rangle$, for some $\beta < 2^\lambda$. Pick an element of the stationary set $(X_{n+1} \cap C(f_\beta)) - (\{s^{n+1}_\alpha \mid \alpha < \beta\} \cap \{s^n_\alpha \mid \alpha \leq \beta\})$ and call it s^{n+1}_β. Let Empty play $Y_{n+1} = \{s^{n+1}_\alpha \mid \alpha < 2^\lambda\}$. This defines a strategy for Empty.

Claim. *The strategy described above is a winning strategy for Empty.*

Proof of Claim. Suppose $X_1, Y_1, X_2, Y_2, \ldots$ is a run of the game $G(NS_{\kappa\lambda})$ where Empty followed the above strategy. We want to show that $\bigcap_{n\in\omega-\{0\}} Y_n = \emptyset$. Suppose otherwise. Let t be an element of $\bigcap_{n\in\omega-\{0\}} Y_n$. Then for each $m \in \omega - \{0\}$, there is a unique ordinal $\alpha_m < 2^\lambda$ such that $s^m_{\alpha_m} = t$. But by the way the s^n_αs are chosen, $s^0_{\alpha_0} = s^1_{\alpha_1} = s^2_{\alpha_2} = \cdots$ implies $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$. This is impossible. Thus we must have $\bigcap_{n\in\omega-\{0\}} Y_n = \emptyset$. $\square$

End of proof of Lemma 3 (i). $\square$

We now prove Theorem 2 using Lemma 3 and Theorem 1.

Proof of Theorem 2. Let λ be a strong limit singular cardinal. If $\mathrm{cf}\,\lambda < \kappa$ then by Lemma 3 (ii), we are done. So assume $\mathrm{cf}\,\lambda \geq \kappa$. In this case we have $\lambda^{<\kappa} = \lambda$. Therefore it is enough to show that $NS_{\kappa\lambda}|X$ is not λ-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$. But this is a consequence of $NS_{\kappa\lambda}$ being nowhere precipitous. In fact we know that $NS_{\kappa\lambda}|X$ cannot be λ^+-saturated for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$. $\square$

We need some preparation to present the proof of Theorem 1. Let λ be a strong limit singular cardinal and κ be a regular uncountable cardinal $< \lambda$. If $\mathrm{cf}\,\lambda < \kappa$ then by Lemma 3 we conclude that $NS_{\kappa\lambda}$ is nowhere precipitous.

From now on let us assume that λ is a strong limit cardinal with $\kappa \leq \operatorname{cf}\lambda < \lambda$. Let $\langle \lambda_\alpha \mid \alpha < \operatorname{cf}\lambda \rangle$ be a continuous increasing sequence of strong limit singular cardinals converging to λ with $\lambda_0 > \operatorname{cf}\lambda$. The following lemma is another key ingredient of our proof.

Lemma 4. *For every $X \subseteq \mathcal{P}_\kappa\lambda$, if for each $\alpha < \operatorname{cf}\lambda$ with $\operatorname{cf}\alpha < \kappa$, $|\{t \in X \mid \sup(t) = \lambda_\alpha\}| < 2^{\lambda_\alpha}$, then X is non-stationary.*

Proof of Lemma 4. Since $\{t \in X \mid \sup(t) \notin t\}$ is a club subset of $\mathcal{P}_\kappa\lambda$, without loss of generality we may assume that $\sup(t) \notin t$ for every t in X. For each $\alpha < \operatorname{cf}\lambda$ with $\operatorname{cf}\alpha < \kappa$, we let $X_\alpha = \{t \in X \mid \sup(t) = \lambda_\alpha\}$. We need the following fact from pcf theory by S. Shelah.

Fact. *There is a club subset $C \subseteq \operatorname{cf}\lambda$ such that $\operatorname{pp}(\lambda_\alpha) = 2^{\lambda_\alpha}$ for every $\alpha \in C$.*

The proof of the above fact can be obtained from 5.15 of [12] or by combining Conclusion XI 5.13 [11, page 414], Corollary VIII 1.6(2) [11, page 321], and Conclusion II 5.7 [11, page 94]. [12] contains updates and corrections to [11]. The reader can look at Holz-Steffens-Weitz [5] for the pcf theory, particularly Theorem 9.1.3 [5, page 271].

For each $\alpha \in C$ with $\operatorname{cf}\alpha < \kappa$, let a_α be a set of regular cardinals cofinal in λ_α such that

(a) every member of a_α is above $\operatorname{cf}\lambda$
(b) $|a_\alpha| = \operatorname{cf}\lambda_\alpha$, and
(c) $\exists \delta_\alpha > |X_\alpha|\ [\delta_\alpha \in \operatorname{pcf}(a_\alpha)]$

Let $a = \bigcup\{a_\alpha \mid \alpha \in C \wedge \operatorname{cf}\alpha < \kappa\}$. Let $\langle f_\beta \mid \beta < \lambda \rangle$ enumerate all of the members of $\{f \mid f$ is a function, domain(f) is a bounded subset of λ, and f is regressive i.e. $f(\gamma) < \gamma$ for every $\gamma \in \operatorname{domain}(f)\}$.

For each $t \in \mathcal{P}_\kappa\lambda$ we define $g_t \in \prod a$ by letting $g_t(\sigma) = \sup\{f_\beta(\sigma) + 1 \mid \beta \in t \wedge \sigma \in \operatorname{dom}(f_\beta)\}$, if $\sigma \in \bigcup_{\beta \in t} \operatorname{domain}(f_\beta)$, and $g_t(\sigma) = 0$ otherwise. Note that $|t| < \kappa \leq \operatorname{cf}\lambda < \min(a)$ guarantees $g_t \in \prod a$. Now by (c) in the definition of a_αs and the fact that $\{g_t \upharpoonright a_\alpha \mid t \in X_\alpha\}$ is a subset of $\prod a_\alpha$ of cardinality $\leq |X_\alpha| < \delta_\alpha \in \operatorname{pcf}(a_\alpha)$, there is some $h_\alpha \in \prod a_\alpha$ such that $\forall t \in X_\alpha\ [g_t \upharpoonright a_\alpha <_{J_{<\delta_\alpha}(a_\alpha)} h_\alpha]$. (For the definition of $J_{<\delta_\alpha}(a_\alpha)$, we refer the reader to section 3.4 of [5].) Therefore

$$\forall t \in X_\alpha\ \exists \sigma \in a_\alpha\ [g_t(\sigma) < h_\alpha(\sigma)] \tag{1}$$

holds. As $\min(a) > \operatorname{cf}\lambda$ and $a = \bigcup\{a_\alpha \mid \alpha \in C \wedge \operatorname{cf}\alpha < \kappa\}$, there is $h \in \prod a$ such that $h_\alpha < h \upharpoonright a_\alpha$ for every $\alpha \in C$ with $\operatorname{cf}\alpha < \kappa$.

Let $W = \{t \in \mathcal{P}_\kappa\lambda \mid$ (i) for some $\alpha \in C$ $\sup(t) = \lambda_\alpha$ with $\operatorname{cf}\alpha < \kappa$, and (ii) if $\delta \in t$ then for some $\beta \in t$, $h \upharpoonright (a \cap \delta) = f_\beta\}$. Note that W is a club subset of $\mathcal{P}_\kappa\lambda$.

Claim. *$X \cap W = \emptyset$.*

Proof of Claim. Suppose otherwise, say $t \in X \cap W$. By (i) in the definition of W, $t \in X_\alpha$ for some $\alpha \in C$ with $\operatorname{cf}\alpha < \kappa$. By (1) we have

$$\exists \sigma \in a_\alpha\ [g_t(\sigma) < h_\alpha(\sigma)]. \tag{2}$$

Since $\sup(t) = \lambda_\alpha$, there must be some $\delta \in t$ such that $\delta > \sigma$. Now by (ii) in the definition of W, $h \upharpoonright (a \cap \delta) = f_\beta$ for some $\beta \in t$. Since $\sigma \in a \cap \delta$, $h(\sigma) = f_\beta(\sigma)$. By the definition of g_t we have $f_\beta(\sigma) < g_t(\sigma)$. From $h_\alpha < h \upharpoonright a_\alpha$, we know $h_\alpha(\sigma) < h(\sigma)$. Therefore we have $h_\alpha(\sigma) < g_t(\sigma)$ contradicting (2). □

End of proof of Lemma 4. $\square$

For each $\alpha < \operatorname{cf}\lambda$ with $\operatorname{cf}\alpha < \kappa$, let us fix a sequence $\langle f^\alpha_\xi \mid \xi < 2^{\lambda_\alpha}\rangle$ that enumerates members of $\{f \mid f$ is a function such that $\operatorname{domain}(f) \subseteq \lambda_\alpha^{<\omega}$ and $\operatorname{range}(f) \subseteq \lambda_\alpha\}$. Furthermore for each function f with $\operatorname{domain}(f) \subseteq \lambda_\alpha^{<\omega}$ and $\operatorname{range}(f) \subseteq \lambda_\alpha$, we let $C_\alpha[f] = \{t \in \mathcal{P}_\kappa\lambda \mid t^{<\omega} \subseteq \operatorname{domain}(f)$, $\sup(t) = \lambda_\alpha$, and t is closed under $f\}$. We need the following lemma to present the proof of Theorem 1.

Lemma 5. *Suppose X is a stationary subset of $\mathcal{P}_\kappa\lambda$. For every $Y \subseteq \{s \in \mathcal{P}_\kappa\lambda \mid s\cap\kappa \in \kappa\}$, if for each $\alpha < \operatorname{cf}\lambda$ with $\operatorname{cf}\alpha < \kappa$ the following condition $(*)$ holds, then Y is stationary.*

$$(*) \qquad \forall\xi < 2^{\lambda_\alpha}\ \left(|C_\alpha[f^\alpha_\xi] \cap X| = 2^{\lambda_\alpha} \longrightarrow C_\alpha[f^\alpha_\xi] \cap Y \neq \emptyset\right)$$

Proof of Lemma 5. Since $s \cap \kappa \in \kappa$ for every $s \in Y$, to show that Y is stationary it is enough to show that $Y \cap C[g] \neq \emptyset$ for every function $g : \lambda^{<\omega} \to \lambda$ where $C[g]$ denotes the set $\{t \in \mathcal{P}_\kappa\lambda \mid g''t^{<\omega} \subseteq t\}$. For the proof of this fact, we refer the reader to Foreman-Magidor-Shelah [2, Lemma 0]. Let us fix a function $g : \lambda^{<\omega} \to \lambda$. Now we let $E = \{\alpha < \operatorname{cf}\lambda \mid \operatorname{cf}\alpha < \kappa\}$ and for each $\alpha \in E$ we let $W_\alpha = \{s \in \mathcal{P}_\kappa\lambda \mid \sup(s) = \lambda_\alpha \wedge \lambda_\alpha \notin s\}$. Note that $\bigcup_{\alpha\in E} W_\alpha$ is a club subset of $\mathcal{P}_\kappa\lambda$. For each $\alpha \in E$, we let g_α denote $g \cap (\lambda_\alpha^{<\omega} \times \lambda_\alpha)$. Now partition E into two sets E^+ and E^- where

$$E^+ = \{\alpha \in E \mid |C_\alpha[g_\alpha] \cap X| = 2^{\lambda_\alpha}\} \quad \text{and}$$
$$E^- = \{\alpha \in E \mid |C_\alpha[g_\alpha] \cap X| < 2^{\lambda_\alpha}\}.$$

We need the following:

Claim. *$X \cap \bigcup\{W_\alpha \mid \alpha \in E^-\}$ is non-stationary.*

Proof. It is enough to show that $Z = C[g] \cap X \cap \bigcup\{W_\alpha \mid \alpha \in E^-\}$ is non-stationary. Note that for each $\alpha \in E^+$, $Z \cap W_\alpha = \emptyset$ and for each $\alpha \in E^-$, $Z \cap W_\alpha \subseteq C_\alpha[g_\alpha] \cap X$. Therefore $|Z \cap W_\alpha| < 2^{\lambda_\alpha}$ for every $\alpha \in E$. Hence, by Lemma 4, we conclude that Z is non-stationary. $\square$

From Claim we know that $X \cap \bigcup\{W_\alpha \mid \alpha \in E^+\}$ is stationary. Pick an element α^* from E^+. Consider the partial function g_{α^*} $(= g \cap (\lambda_{\alpha^*}^{<\omega} \times \lambda_{\alpha^*}))$. Let $\xi^* < 2^{\lambda_{\alpha^*}}$ be such that $f^{\alpha^*}_{\xi^*} = g_{\alpha^*}$. Since $\alpha^* \in E^+$, we have $|C_{\alpha^*}[g_{\alpha^*}] \cap X| = 2^{\lambda_{\alpha^*}}$. Since $f^{\alpha^*}_{\xi^*} = g_{\alpha^*}$ and Y satisfies condition $(*)$, we know that $C_{\alpha^*}[g_{\alpha^*}] \cap Y \neq \emptyset$. Therefore $C[g] \cap Y \neq \emptyset$ showing that Y is stationary.

End of proof of Lemma 5. $\square$

Finally we are ready to complete the proof of Theorem 1. To present a winning strategy for Empty in the game $G(NS_{\kappa\lambda})$, we introduce some new types of games. For each $\alpha \in E = \{\alpha < \operatorname{cf}\lambda \mid \operatorname{cf}\alpha < \kappa\}$, we define the game G_α between Nonempty and Empty as follows: Nonempty and Empty alternately choose sets $X_n, Y_n \subseteq W_\alpha = \{s \in \mathcal{P}_\kappa\lambda \mid \sup(s) = \lambda_\alpha \notin s\}$ respectively so that $X_n \supseteq Y_n \supseteq X_{n+1}$ and $\forall\xi < 2^{\lambda_\alpha}$ $(|C_\alpha[f^\alpha_\xi] \cap X_n| = 2^{\lambda_\alpha} \longrightarrow C_\alpha[f^\alpha_\xi] \cap Y_n \neq \emptyset)$ for $n = 1, 2, \ldots$. Empty wins G_α iff $\bigcap_{n\in\omega-\{0\}} Y_n = \emptyset$.

By the same argument as the proof of Lemma 3 (i), we know that Empty has a winning strategy, say τ_α, in the game G_α for each $\alpha \in E$. Now we show how to combine the strategies τ_αs to produce a winning strategy for Empty in $G(NS_{\kappa\lambda})$. Suppose X_1 is Nonempty's first move in $G(NS_{\kappa\lambda})$. We let $X_1^* = X_1 \cap \{s \in \mathcal{P}_\kappa\lambda \mid s \cap \kappa \in \kappa\} \cap \bigcup\{W_\alpha \mid \alpha \in E\}$. Since $\{s \in \mathcal{P}_\kappa\lambda \mid s \cap \kappa \in \kappa\} \cap \bigcup\{W_\alpha \mid \alpha \in E\}$ is a club subset of $\mathcal{P}_\kappa\lambda$, X_1^* is stationary in $\mathcal{P}_\kappa\lambda$. For each $\alpha \in E$, we simulate a run of the game G_α as follows: Let us pretend that Nonempty's first move in G_α is $X_1^* \cap W_\alpha$. Let Empty play her strategy τ_α, so Empty's first move is $\tau_\alpha(\langle X_1^* \cap W_\alpha\rangle)$. Now in the game $G(NS_{\kappa\lambda})$, let Empty play $Y_1 = \bigcup\{\tau_\alpha(\langle X_1^* \cap W_\alpha\rangle) \mid \alpha \in E\}$. Lemma 5 guarantees that Y_1 is stationary in $\mathcal{P}_\kappa\lambda$. In general if $\langle X_1^*, Y_1, X_2, Y_2, \ldots, X_n\rangle$ is a run of $G(NS_{\kappa\lambda})$ up to Nonempty's n-th move, then we let Empty play $Y_n = \bigcup\{\tau_\alpha(\langle X_1^* \cap W_\alpha, X_2 \cap W_\alpha, \ldots, X_n \cap W_\alpha\rangle) \mid \alpha \in E\}$. Once again we know Y_n is a stationary subset of X_n. For each $\alpha \in E$, since τ_α is a winning strategy in G_α we have

$$\bigcap_{n\in\omega-\{0\}} \tau_\alpha(\langle X_1^* \cap W_\alpha, X_2 \cap W_\alpha, \ldots, X_n \cap W_\alpha\rangle) = \emptyset.$$

Because the W_αs are pairwise disjoint, we conclude that $\bigcap_{n\in\omega-\{0\}} Y_n = \emptyset$. Therefore we have a winning strategy for Empty in the game $G(NS_{\kappa\lambda})$. This proves that $NS_{\kappa\lambda}$ is nowhere precipitous for every strong limit singular λ.

End of proof of Theorem 1. □

§3. On "proper" ideals over $\mathcal{P}_\kappa\lambda$

First we define that we mean by a "proper" ideal.

Definition. An ideal I over a set A is a "proper" ideal if the corresponding p.o. $\mathbb{P}_I$ is proper (in the sense of proper forcing).

We refer the reader to Shelah [13] for the background of properness.

As we mentioned in §1, we are interested in the question of whether it is possible to have a κ-complete normal "proper" ideal over $\mathcal{P}_\kappa\lambda$ where κ is the successor of some singular cardinal. We give a negative answer to this question. Here we present a more general result.

Theorem 6. (i) *Suppose I is a κ-complete normal ideal over κ. If $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \delta\} \notin I$ for some cardinal δ satisfying $\delta^+ < \kappa$, then I is not "proper".*

(ii) *Suppose I is a κ-complete normal ideal over $\mathcal{P}_\kappa\lambda$. If $\{s \in \mathcal{P}_\kappa\lambda \mid \operatorname{cf}(s\cap\kappa) = \delta\} \notin I$ for some cardinal δ satisfying $\delta^+ < \kappa$, then I is not "proper".*

Note that if κ is the successor cardinal of a singular cardinal, then every κ-complete normal ideal over $\mathcal{P}_\kappa\lambda$ satisfies the hypothesis of (ii).

Proof of Theorem 6. Since the proof of (ii) is identical to that of (i), we only present the proof of (i).

Let I and δ be as in the hypothesis of (i). First note that if $\delta = \aleph_0$ then the set $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \delta\}$ forces "$\operatorname{cf}\kappa = \aleph_0$" showing $\mathbb{P}_I$ cannot be proper. Therefore we may assume that δ is uncountable.

We need the following claim:

Claim 1. *There are a stationary subset E of $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \aleph_0\}$ and an I-positive subset X of $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \delta\}$ such that $E \cap \alpha$ is non-stationary for every α in X.*

Proof. Let $\{E_\gamma \mid \gamma < \delta^+\}$ be a family of pairwise disjoint stationary subsets of $\{\alpha < \kappa \mid \operatorname{cf}\alpha = \aleph_0\}$. For each $\alpha < \kappa$ with $\operatorname{cf}\alpha = \delta$, there must be a club subset of

α with cardinality δ. Therefore for such an ordinal α, there is some $\gamma_\alpha < \delta^+$ such that $E_{\gamma_\alpha} \cap \alpha$ is non-stationary. By the κ-completeness of I, there is some $\gamma^* < \delta^+$ such that $X = \{\alpha < \kappa \mid \mathrm{cf}\,\alpha = \delta \wedge \gamma_\alpha = \gamma^*\} \notin I$. If we let $E = E_{\gamma^*}$, then $E \cap \alpha$ is non-stationary for every α in X. $\square$

For each α from X, let c_α be a club subset of α with $c_\alpha \cap E = \emptyset$. Let $\vec{C}$ denote $\langle c_\alpha \mid \alpha \in X\rangle$. Let χ be a large enough regular cardinal. Assume that N is a countable elementary substructure of $\langle H(\chi), \epsilon\rangle$ satisfying $\{I, E, X, \vec{C}\} \subseteq N$ and $\sup(N \cap \kappa) \in E$.

We are ready to show that I is not "proper".

Claim 2. *If Y is a subset of X such that $Y \notin I$ (therefore $Y \in \mathbb{P}_I$ and $Y \leq X$), then Y is not $(N, \mathbb{P}_I)$-generic.*

Claim 2 implies that $\mathbb{P}_I$ is not proper.

Proof of Claim 2. Suppose otherwise. Assume that there exists $Y \leq X$ such that Y is $(N, \mathbb{P}_I)$-generic.

For each $\alpha < \kappa$ we define a function $f_\alpha : X \to \kappa$ by $f_\alpha(\gamma) = \mathrm{Min}(c_\gamma - \alpha)$ if $\gamma > \alpha$, and $f_\alpha(\gamma) = 0$ otherwise. It is clear that $f_\alpha \in N$ for each $\alpha \in N \cap \kappa$.

For each $\alpha \leq \beta < \kappa$, we let $T^\alpha_\beta = \{\gamma \in X \mid f_\alpha(\gamma) = \beta\}$. For each fixed $\alpha < \kappa$, using the normality of I, we see that $\{T^\alpha_\beta \mid \alpha \leq \beta < \kappa, T^\alpha_\beta \notin I\}$ is a maximal antichain below X in $\mathbb{P}_I$. Let $\vec{T}^\alpha = \langle T^\alpha_\beta \mid \alpha \leq \beta < \kappa, T^\alpha_\beta \notin I\rangle$. It is clear that $\vec{T}^\alpha \in N$ for $\alpha \in N \cap \kappa$.

Since Y is $(N, \mathbb{P}_I)$-generic, for $\alpha \in N \cap \kappa$ $\{T^\alpha_\beta \mid \alpha \leq \beta \wedge \beta \in N \cap \kappa \wedge T^\alpha_\beta \notin I\}$ is predense below Y in $\mathbb{P}_I$. So we must have $Y - \bigcup\{T^\alpha_\beta \mid \alpha \leq \beta \wedge \beta \in N \cap \kappa \wedge T^\alpha_\beta \notin I\} \in I$ for each $\alpha \in N \cap \kappa$. Let $Y_\alpha = Y - \bigcup\{T^\alpha_\beta \mid \alpha \leq \beta \wedge \beta \in N \cap \kappa \wedge T^\alpha_\beta \notin I\}$. We have $\bigcup_{\alpha \in N\cap\kappa} Y_\alpha \in I$. This implies $Y - \bigcup_{\alpha \in N \cap \kappa} Y_\alpha \notin I$. Let γ^* be an element of $Y - \bigcup_{\alpha \in N\cap\kappa} Y_\alpha$ with $\gamma^* > \sup(N \cap \kappa)$. Note that $\gamma^* \in Y - Y_\alpha$ for each $\alpha \in N \cap \kappa$. Hence if $\alpha \in N \cap \kappa$, then there exists $\beta_\alpha \in N \cap \kappa$ such that $\gamma^* \in T^\alpha_{\beta_\alpha}$. Thus $f_\alpha(\gamma^*) = \beta_\alpha \in N \cap \kappa$ for each $\alpha \in N \cap \kappa$. This means that $\mathrm{Min}(c_{\gamma^*} - \alpha) \in N \cap \kappa$ for each $\alpha \in N \cap \kappa$, showing $c_{\gamma^*} \cap N$ is unbounded in $\sup(N \cap \kappa)$.

Since $\sup(N \cap \kappa) < \gamma^*$, we must have $\sup(N \cap \kappa) \in c_{\gamma^*}$. But this implies $\sup(N \cap \kappa) \in c_{\gamma^*} \cap E$ which contradicts $c_\alpha \cap E = \emptyset$ for each $\alpha \in X$ and $\gamma^* \in Y \subseteq X$. This contradiction shows that Y cannot be $(N, \mathbb{P}_I)$-generic. $\square$

End of proof of Theorem 6. $\square$

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