A PARTITION RELATION USING STRONGLY COMPACT CARDINALS

SH761

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Abstract. If $\kappa$ is strongly compact and $\lambda > \kappa$ and $\lambda$ is regular (or alternatively $\text{cf}(\lambda) \geq \kappa$), then $(2^{<\lambda})^+ \rightarrow (\lambda + \zeta)^2$ holds for $\zeta, \theta < \kappa$.

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The aim of this paper is to prove the following theorem.

**0.1 Theorem.** If $\kappa$ is a strongly compact cardinal, $\lambda > \kappa$ is regular and $\zeta, \theta < \kappa$ then the partition relation $(2^{<\lambda})^+ \rightarrow (\lambda + \zeta)^2_\theta$ holds.

**0.2 Theorem.** In Theorem 0.1. Instead $\lambda$ regular, $\text{cf}(\lambda) > \kappa$ suffices.

We notice that our argument is valid in the case $\kappa = \omega$. As for the history of the problem we point out that Hajnal proved in an unpublished work, that $(2^\omega)^+ \rightarrow (\omega_1 + n)^2_2$ holds for every $n < \omega$. Then it was showed in [Sh 26, §6] that for $\kappa > \omega$ regular and $2^{|\alpha|} < \kappa$, the relation $(2^{<\kappa})^+ \rightarrow (\kappa+\alpha)^2_2$ is true. More recently Baumgartner, Hajnal, and Todorčević in [BHT93] extended this to the case when the number of colors is arbitrary finite. Earlier by [Sh 424], we have $(2^{<\lambda})^+ \rightarrow (\lambda \times m)^2_n$ for $n$ large enough (this was complimentary to the main result there that $\aleph_0 < \lambda = \lambda^{<\lambda} + 2^\lambda$ arbitrarily large does not imply $2^\lambda \rightarrow (\lambda \times \omega)^2_2$). Subsequently [BHT93] improves $n$. We hope that the way the strong compactness was used will be useful elsewhere; see [Sh 666] for a discussion of a possible consistency of failure. I also thank Peter Komjath for improving the presentation.

**Notation.** If $S$ is a set, $\kappa$ a cardinal then $[S]^\kappa = \{a \subseteq S : |a| = \kappa\}$, $[S]^{<\kappa} = \{a \subseteq S : |a| < \kappa\}$. If $D$ is some filter over a set $S$ then $X \in D^+$ denotes that $S \setminus X \notin D$ and $X \subseteq S$. If $\kappa < \mu$ are regular cardinals then $S^\kappa_\mu = \{\alpha < \mu : \text{cf}(\alpha) = \kappa\}$, a stationary set. The notation $A = \{x_\alpha : \alpha < \gamma\}$, etc., means that $A$ is enumerated increasingly.
§1 The case of $\lambda$ regular

1.1 Lemma. Assume $\mu = \mu^\theta$. Assume that $D$ is a normal filter on $\mu^+$ and $A^* \in D^+$ satisfies $\delta \in A^* \Rightarrow \text{cf}(\delta) > \theta$, and $F'$ is a function with domain $[A^*]^2$ and range of cardinality $\theta$. Then there are a normal filter $D_0$ on $\mu^+$ extending $D$, $A_0 \in D_0$ with $A_0 \subseteq A^*$ and $C_0 \subseteq \text{Rang}(F')$ satisfying $\text{Rang}(F' | [A_0]^2) = C_0$ such that: if $X \in D_0$ then $\text{Rang}(F' | [X]^2) \supseteq C_0$.

We first prove a claim

1.2 Claim. Assume $\mu = \mu^\theta$ and $F' : [S^*]^2 \rightarrow C_*$, $|C_*| \leq \theta$, $D$ is a normal filter on $\mu^+$, $S^* \subseteq \mu^+$ belongs to $D^+$ and $\delta \in S^* \Rightarrow \text{cf}(\delta) > \theta$. There is a set $A \in D^+$ such that $A \subseteq S^*$ and some $C \subseteq C_0$ satisfying $\text{Rang}(F' | [A]^2) = C$ and: if $f : A \rightarrow \mu^+$ is a regressive function, then for some $\alpha < \mu^+$ we have $\text{Rang}(F' | [f^{-1}(\alpha)]^2) = C$ and $f^{-1}(\alpha)$ is a subset of $\mu^+$ from $D^+$.

Proof. Toward contradiction assume that no such sets $A, C$ exist. We build a tree $T$ as follows. Every node $t$ of the tree will be of the form

$$t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle$$

for some ordinal $\varepsilon = \varepsilon(t)$ where $\langle A_\alpha : \alpha \leq \varepsilon \rangle$ is a decreasing, continuous sequence of subsets of $\mu^+$; for every $\alpha < \varepsilon$, $f_\alpha$ is a regressive function on $A_\alpha$; and $\langle i_\alpha : \alpha < \varepsilon \rangle$ is a sequence of distinct elements of $C_*$. It will always be true that if $t < T t'$, then each of the three sequences of $t'$ extend the corresponding one of $t$.

To start, we make the node $t$ with $\varepsilon(t) = 0$, $A_0 = S^*$ the root of the tree.

At limit levels we extend (the obvious way) all cofinal branches to a node.

If we are given an element $t = \langle \langle A_\alpha : \alpha \leq \varepsilon \rangle, \langle f_\alpha : \alpha < \varepsilon \rangle, \langle i_\alpha : \alpha < \varepsilon \rangle \rangle$ of the tree and the set $A_\varepsilon$ is $= 0 \mod D$ then we leave $t$ as a terminal node. Otherwise, let $C = C_t = \text{Rang}(F' | [A_\varepsilon]^2)$ and notice that by hypothesis, toward contradiction, the pair $A_\varepsilon, C_t$ cannot be as required in the Claim. There is, therefore, a regressive function $f = f_t$ with domain $A_\varepsilon$, such that for every $x < \mu^+$ the set $\text{Rang}(F' | [f^{-1}(x)]^2)$ is a proper subset of $C_t$ or $f^{-1}(x)$ is $= 0 \mod D$ subset of $\mu^+$. We make the immediate extensions of $t$ the sequences of the form $a t_x = \langle \langle A_\alpha : \alpha \leq \varepsilon + 1 \rangle, \langle f_\alpha : \alpha < \varepsilon + 1 \rangle, \langle i_\alpha : \alpha < \varepsilon + 1 \rangle \rangle$ where $A_{\varepsilon+1} = f^{-1}(x)$, $f_\alpha = f_t$ and $i_\varepsilon \in C_t$ is some colour value such that: if $A_{\varepsilon+1} \neq 0 \mod D$ then $i_\varepsilon$ is not in the range of $F' | [A_\varepsilon]^2$.
Having constructed the tree observe that every element \( x \in S^* \subseteq \mu^+ \) belongs to a set \( A^{t(x)} \) for some (unique) terminal node \( t(x) \) of \( T \). Also, \( \varepsilon(x) < \theta^+ (\leq \mu^+) \) holds by the selection of the \( i_\beta \)'s as \( \{i^{t(x)}_\alpha : \alpha < \varepsilon(x)\} \) is a sequence of members of \( C_* \) with no repetitions while \( C_* \), the set of colours, has \( \leq \theta \) members. For some set \( S \subseteq S^* \) of ordinals \( x < \mu^+ \) which belong to \( D^+ \) (by the normality of \( D \)) the value of \( \varepsilon(x) \) is the same, say \( \varepsilon \). For \( x \in S \) we let \( g_\alpha(x) = f^{t(x)}_\alpha(x) \) where \( f^{t(x)}_\alpha \) is the \( \alpha \)-th regressive function in the node \( t(x) \in T \). Again, by \( \mu^\mu = \mu \) & \( \forall \alpha \in S \) \([\text{cf}(\alpha) > \theta] \) we have that \( (\forall x \in S') (\forall \alpha < \varepsilon) g_\alpha(x) = \beta_\alpha \) holds for some sequence \( \langle \beta_\alpha : \alpha < \varepsilon \rangle \) and subset \( S' \subseteq S \) from \( D^+ \). But then we get that the set \( S' \) satisfies \( x, y \in S' \Rightarrow (A^{t(x)}_\alpha, f^{t(x)}_\alpha, i^{t(x)}_\alpha) \) = \( (A^{t(y)}_\alpha, f^{t(y)}_\alpha, i^{t(y)}_\alpha) \) for every \( \alpha < \varepsilon \); we can prove this by induction on \( \alpha \). We can then prove that \( A^{t(x)}_\varepsilon = A^{t(y)}_\varepsilon \) for \( x, y \in S' \). We can conclude that \( x, y \in S' \Rightarrow t(x) = t(y) \), so \( S' \subseteq A^{t(t)}_\varepsilon \) for some terminal node \( t \), but this latter set is in \( D^+ \), a contradiction.

\[ \square_{1.2} \]

**Proof of Lemma 1.1.** Apply Claim 1.2 with \( S^* = A^* \) to get corresponding \((C, A)\). Define the ideal \( I \) as follows. For \( X \subseteq \mu^+ \) we let \( X \in I \) if there are a member \( E \) of \( D \) and a regressive function \( f : X \cap A \to \mu^+ \) such that every \( \text{Rang}(F^i | [f^{-1}(\alpha)]^2) \) is a proper subset of \( C \) or \( f^{-1}(\alpha) \) is \( a = \theta \) mod \( D \) subset of \( \mu^+ \).

Now:

**1.3 Claim.** \( I \) is a normal ideal on \( \mu^+ \) (and \( A^* = \mu^+ \mod I ) \).

**Proof.** Straightforward.

Set \( D_0 \) to be the dual filter of \( I \), let \( A_0 = A \) and let \( C_0 = C \); by 1.2 we are done. \[ \square_{1.1} \]

**1.4 Remark.** 1) If Lemma 1.1 holds for some \( D_0, A_0, C_0 \) then it holds for \( D_1, A_1, C_0 \) when the normal filter \( D_1 \) extends \( D_0 \), and \( A_1 \in D_1 \) satisfies \( A_1 \subseteq A_0 \).

2) If \( D_0, A_0, C_0 \) satisfy Lemma 1.5, and \( X \in D^+_0 \) then \( X \) contains a homogeneous set of order type \( \lambda + 1 \) of color \( \xi \) for every \( \xi \in C_0 \).

3) Lemma 1.1 is closely related to the proof in [Sh 26].

**Proof of Theorem 0.1.** Let \( \mu = 2^{\lt \lambda} \), and \( F : [\mu^+]^2 \to \theta \) be a colouring; we apply 1.1 for \( A^* = S^\mu_* \), \( (F = F, \theta = \theta, \mu = \mu) \) and \( D \) the club filter. We shall write \( F(\alpha, \beta) \) for \( F(\{\alpha, \beta\}) \) and 0 for \( F(\alpha, \alpha) \).

We fix \( A_0, D_0, C_0 \) which we get by 1.1.
1.5 Lemma. Almost every $\delta \in A_0$: (i.e. for all but a set $= \emptyset \mod D_0$) satisfies the following: if $s \in [A_0 \cap \delta]^{<\lambda}$ and $\{z_\alpha : \alpha < \gamma\} \subset A_0 \cap [\delta, \mu^+)$ with $\gamma < \kappa$ then there is $\{y_\alpha : \alpha < \gamma\} \subset A_0 \cap (\sup(s), \delta)$ such that:

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ (for $x \in s, \alpha < \gamma$);
(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ (for $\alpha < \beta < \gamma$).

Proof. By simple reflection (using the regularity of $\lambda$).

1.6 Lemma. There is $A'_0 \subseteq A_0, A'_0 \in D_0$ such that: if $\delta \in A'_0, s \in [\delta]^{<\lambda}$ and $\xi \in C_0$, then there exists a $\delta_1 \in A_0, \delta < \delta_1$ such that

(a) $F(x, \delta) = F(x, \delta_1)$ (for $x \in s$);
(b) $F(\delta, \delta_1) = \xi$.

Proof. Otherwise, there is some $X \subseteq A_0, X \in D_0^+$ such that for every $\delta \in X$ there are $s(\delta) \in [\delta]^{<\lambda}$ and $\xi(\delta) \in C_0$ such that there is no $\delta_1 > \delta$ satisfying (a) and (b). By normality and $\mu = \mu^{<\lambda}$ we can assume that $s(\delta) = s$ and $\xi(\delta) = \xi$ holds for $\delta \in X$. By Lemma 1.1, that is the choice of $(A_0, D_0, C_0)$, there must exist $\delta < \delta_1$ in $X$ with $F(\delta, \delta_1) = \xi$ and this is a contradiction. $\Box_{1.6}$

Continuation of the proof of Theorem 0.1. Let $A'_0 \subseteq A_0$ satisfy Lemmas 1.1 and 1.6 and pick some $\delta_1 \in A'_0$ and then let $T = A'_0 \setminus (\delta_1 + 1)$.

1.7 Lemma. There exists a function $G : T \times T \rightarrow C_0$ such that: if $s \in [\delta_1]^{<\lambda}, \gamma < \kappa$, and $Z = \{z_\alpha : \alpha < \gamma\} \subset T$ then there is $\{y_\alpha : \alpha < \gamma\} \subset (\sup(s), \delta_1)$ such that

(a) $F(x, y_\alpha) = F(x, z_\alpha)$ (for $x \in s, \alpha < \gamma$);
(b) $F(y_\alpha, y_\beta) = F(z_\alpha, z_\beta)$ (for $\alpha < \beta < \gamma$);
(c) $F(y_\alpha, z_\beta) = G(z_\alpha, z_\beta)$ (for $\alpha, \beta < \gamma$).

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1. In fact, if $A_1^+ \in D_0^+$ then for some $A'_0 \subseteq A_1 \cap A_0, A_1 \setminus A'_0 = \emptyset \mod D_0$ and the conclusion holds for every $\delta \in A'_0$. 

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Proof. As $\kappa$ is strongly compact, it suffices to show that for every $Z \in [T]^{<\kappa}$ there exists a function $G : Z \times Z \to \theta$ as required. Clauses (a) and (b) are obvious by Lemma 1.5, and it is clear that, if we fix $Z$, then for every $s \in [\delta_1]^{<\lambda}$ there is an appropriate $G : Z \times Z \to \theta$. We show that there is some $G : Z \times Z \to \theta$ that works for every $s$. Assume otherwise, that is, for every $G : Z \times Z \to \theta$ there is some $s_G \in [\delta_1]^{<\lambda}$ such that $G$ is not appropriate for $s_G$. Notice that the number of these functions $G$ is less than $\kappa$. Then no $G$ could be right for $s = \cup\{s_G : G$ a function from $Z \times Z$ to $\theta\} \in [\delta_1]^{<\lambda}$, a contradiction. 

Continuation of the proof of Theorem 0.1. We now apply Lemma 1.1 to the colouring $\bar{G}\{x, y\} = \bar{G}(x, y) = \langle F(x, y), G(x, y) \rangle$ for $x < y$ in $T$ and $0$ otherwise, and the filter $D_0$ and the set $T$ and get the normal filter $D_1 \supseteq D_0$, the set $A_1 \subseteq T \subseteq A_0$ such that $A_1 \in D_1$ and the colour set $C_1 \subseteq \theta \times \theta$. Notice that actually $C_1 \subseteq C_0 \times C_0$.

We can also apply Lemmas 1.5 and 1.6 and get some set $A'_1 \subseteq A_1$.

1.8 Lemma. There is a set $a \in [A'_1]^{<\kappa}$ such that for every decomposition $a = \cup\{a_\xi : \xi \in C_1\}$ there is some $\xi \in C_1$ such that

(a) for every $\xi \in C_1$ there is an $\bar{\xi}$-homogeneous subset for the colouring $\bar{G}$ of order type $\zeta$ in $a_\xi$,

(b) similarly for every $\xi \in C_0$ and $F$.

Proof. This follows from the strong compactness of $\kappa$ as $A'_1$ itself has this partition property (or more details in 2.8). 

Continuation of the Proof of 0.1. Fix a set $a$ as in 1.8.

We now describe the construction of the required homogeneous subset. Let $\delta_2 \in A'_1$ be some element with $\delta_2 > \sup(a)$. For $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ let $a_\bar{\xi}$ be the following set:

$$a_\bar{\xi} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$ 

By Lemma 1.8, there is some $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ for which the statement in 1.8 above is true and necessarily (as $a \cup \{\delta_2\} \subseteq A'_1 \subseteq A_0$ and $a_\bar{\xi} \neq \emptyset$) we have $\xi_1, \xi_2 \in C_0$. Select some $b \subseteq a_\bar{\xi}$, otp$(b) = \zeta$ such that $F$ is constantly $\xi_2$ on $b$; this is possible by clause (b) of 1.8. This set $b$ will be the $\zeta$ part of our homogeneous set of ordinals of order type $\lambda + \zeta$, so we will have to construct a set of order type $\lambda$ below $b$. By
induction on $\alpha$ we will choose $x_\alpha$ such that the set $\{x_\alpha : \alpha < \lambda\} \subseteq \delta_1$ satisfies the following conditions:

$(\ast)_1 F(x_\beta, x_\alpha) = \xi_2$ (for $\beta < \alpha$),

$(\ast)_2 F(x_\alpha, b \cup \{\delta_2\}) = \xi_2$, i.e. $F(x_\alpha, y) = \xi_2$ when $y \in b \cup \{\delta_2\}$.

Assume that we have reached step $\alpha$, that is, we are given the set of ordinals with $\{x_\beta : \beta < \alpha\} \subseteq \delta_1$ and call this set $s$. Applying Lemma 1.6 for $A_1, A'_1, \delta_2$ and $s \cup b$ and the colouring $\bar{G}$ here standing for $A_0, A'_0, \delta, s$ and the colouring $F$ there (that is the choice of $A'_1$) we get that there exists some $\delta_3 > \delta_2$ (standing for $\delta_1$ there) such that

$(i) \quad \delta_3 \in A_1$

$(ii) \quad \bar{G}(x, \delta_3) = \bar{G}(x, \delta_2)$ for $x \in s \cup b$

$(iii) \quad \bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2),$

hence:

$(\ast)_3 F(x_\beta, \delta_3) = \xi_2$ (for $\beta < \alpha$).

[Why? As $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ by (ii) and the choice of $\bar{G}$ and $F(x_\beta, \delta_2) = \xi_2$ by $(\ast)_2$ from the induction hypothesis.]

$(\ast)_4 G(b \cup \{\delta_2\}, \delta_3) = \xi_2$, i.e. $G(y, \delta_3) = \xi_2$ when $y \in b \cup \{\delta_2\}$.

[Why? If $y \in b$ then by (ii) and the definition of $\bar{G}$ we have $G(y, \delta_3) = G(y, \delta_2)$, but $b \subseteq a_\xi$ so by the choice of $a_\xi$ we have $G(y, \delta_2) = \xi_2$. For $y = \delta_2$ use clause (iii) that is $(\xi_1, \xi_2) = \bar{G}(\delta_2, \delta_3) = (F(\delta_2, \delta_3), G(\delta_2, \delta_3))$.]

By the choice of $G$ this implies that there is some $x_\alpha$ as required; that is by the choice of $\bar{G}$ (see Lemma 1.7), applied to $Z = \{z_i : i < \gamma\}$ enumerating the set $b \cup \{\delta_2, \delta_3\}$ and $s$ as above, we get $\{y_i : i < \gamma\}$, now necessarily $\delta_3 = z_{\gamma - 1}$, and we can choose $y_{\gamma - 1}$ as $x_\alpha$. \[\square_{1, 1}\]
§2 The case of $\lambda$ singular

We prove version 0.2 of the main theorem.

Proof of Theorem 0.2. Let $\sigma = \text{cf}(\lambda)$. Let $\lambda = \sum_{\varepsilon < \sigma} \lambda_\varepsilon$ with $\lambda_\varepsilon > \sigma \geq \kappa > \theta$ strictly increasing. Let $\mu_\varepsilon = 2^{\lambda_\varepsilon}$ and $\mu = \sum \{ \mu_\varepsilon : \varepsilon < \sigma \} = 2^{<\lambda}$. We also fix $F : [\mu^+]^2 \rightarrow \theta$.

2.1 Claim. For some $\bar{C}$ we have:

(a) $\mathcal{C} = \langle \mathcal{C}_\alpha : \alpha \in S \rangle$
(b) $S \subseteq \mu^+$, $\mathcal{C}_\delta \subseteq \delta$
(c) $\text{otp}(\mathcal{C}_\delta) \leq \sigma$
(d) $S^* = \{ \delta < \lambda : \text{otp}(\mathcal{C}_\delta) = \sigma \}$ is stationary
(e) $\mathcal{C}_\delta$ unbounded in $\delta$ if $\text{otp}(\mathcal{C}_\delta) = \sigma$
(f) $\alpha \in \mathcal{C}_\delta \Rightarrow \alpha \in S$ & $\mathcal{C}_\alpha = \mathcal{C}_\delta \cap \alpha$.

□2.1

Proof. By [Sh 420, §1] as $\sigma^+ < \mu^+$, $\sigma = \text{cf}(\sigma)$.

Continuation of the proof of 0.2: Let $D_0, A_0, C_0$ be as given by Lemma 1.1 with the club filter of $\mu^+, S^*$ (from clause (d) of 2.1 above) here standing for $D, A^*$ there so $A_0 \subseteq S^*$.

Notation: $\varepsilon(\alpha) = \text{otp}(C_\alpha)$.

2.2 Claim. Let $\chi > 2^\mu$, $<^\chi$ a well ordering of $\mathcal{H}(\chi)$). For any $x \in \mathcal{H}(\chi)$ we can find $\mathfrak{B} = \langle \mathfrak{B}_\alpha : \alpha < \lambda \rangle$ such that:

(a) $\mathfrak{B}_\alpha \prec (\mathcal{H}(\chi), \in, <^\chi)$
(b) $\lambda, \mu, F, (\lambda_\varepsilon : \varepsilon < \sigma), \mathcal{C}, A_0, C_0, D_0$ belong to $\mathfrak{B}_\alpha$
(c) $\langle \mathfrak{B}_\beta : \beta < \alpha \rangle \in \mathfrak{B}_\alpha$ if $\alpha \notin S^*$
(d) $\| \mathfrak{B}_\beta \| = \mu_{\varepsilon(\beta)}$ and $[\mathfrak{B}_\beta]^{\lambda_{\varepsilon(\beta)}} \subseteq \mathfrak{B}_\beta$ and $\mu_{\varepsilon(\beta)} + 1 \subseteq \mathfrak{B}_\beta$ (actually follows)
(e) $\mathfrak{B}_\alpha = \cup \{ \mathfrak{B}_\beta : \beta \in \mathcal{C}_\alpha \}$ if $\alpha \in S^*$.

Proof. Straightforward.

2.3 Observation. 1) We have $\varepsilon(\alpha) < \varepsilon(\beta)$ and $\mathfrak{B}_\alpha \in \mathfrak{B}_\beta$ and $\mathfrak{B}_\alpha \prec \mathfrak{B}_\beta$ if $\alpha \in \mathcal{C}_\beta$. 
2.4 Claim. There is a set $A'_0 \subseteq A_0$ such that

(α) $A'_0 \in D_0$ and $\alpha < \delta \in A'_0 \Rightarrow \sup(\mathfrak{B}_\alpha \cap \mu^+) < \delta$

(β) if $\xi \in C_0$ and $\delta \in A'_0$ and $s \in \cup\{[\delta \cap \mathfrak{B}_\alpha]^{<\lambda_\alpha} : \alpha \in \mathfrak{C}_\delta\}$, then there is $\delta_1 \in A_0$ such that $\delta < \delta_1$ and

(a) $F(x, \delta) = F(x, \delta_1)$ for $x \in s$

(b) $F(\delta, \delta_1) = \xi$.

Proof. Requirement (α) holds for all but a non stationary set of $\delta \in A_0$. Requirement (β) is proved as in 1.6. \(\square_{2.4}\)

Now fix $A'_0 \subseteq A_0$ as in 2.4, and fix $\delta_1 \in A'_1$ and let $T = A'_0 \setminus (\delta_1 + 1)$. Recall $\delta_1 \in A'_0 \subseteq S^* = \{\delta : \text{otp}(\mathfrak{C}_\delta) = \sigma, \delta = \sup(\mathfrak{C}_\delta)\} \subseteq \{\delta < \mu^+ : \text{cf}(\delta) = \sigma\}$.

2.5 Claim. There is a function $G_\varepsilon : T \times T \to C_0$ such that:

\(\square\) if $s \in [\delta \cap \mathfrak{B}_\alpha]^{<\lambda_\varepsilon}$ and $\varepsilon = \varepsilon(\alpha)$ and $\alpha \in \mathfrak{C}_\delta$, and $\gamma < \kappa$ and $Z = \{z_\beta : \beta < \gamma\} \subseteq \delta \cap \mathfrak{B}_\alpha = \mu^+ \cap \mathfrak{B}_\alpha$ such that:

(a) $F(x, y_\beta) = F(x, z_\beta)$ for $x \in s, \beta < \delta$

(b) $F(\delta, \delta_1) = \xi$

(c) $F(z_{\beta_1}, y_{\beta_2}) = G(y_{\beta_1}, y_{\beta_2})$ for $\beta_1, \beta_2 < \delta$

(d) $y_0 > \sup(s)$.

Proof. Like 1.7.

2.6 Claim. There exists a function $G : T \times T \to C_0$ such that if $s \in [T]^{<\kappa}$, then for arbitrarily large $\varepsilon < \sigma$ we have $G \upharpoonright (s \times s) = G_\varepsilon \upharpoonright (s \times s)$.

Proof. Let $D^*$ be a uniform $\kappa$-complete ultrafilter on $\sigma$ and define $G$ by $G(\alpha, \beta)$ is the unique $\xi \in C_0$ such that $\{\varepsilon < \sigma : G_\varepsilon(\alpha, \beta) = \xi\} \in D^*$. \(\square_{2.6}\)

Continuation of the Proof of 0.2. Now we apply Lemma 1.1 to the colouring $\bar{G}$ where $\bar{G}\{x, y\} = \bar{G}(x, y) = (F(x, y), G(x, y))$ for $x < y$ in $T$ and zero otherwise and the filter $D_0$ and the set $T$. We get a normal filter $D_1$ and a set $A_1 \subseteq T \subseteq A'_0$ and a set of colours $C_1$. As $A_1 \subseteq A_0$ necessarily $C_1 \subseteq C_0 \times C_0$. 
2.7 Claim. There is $A'_1 \subseteq A_1$ such that:

$A_1 \setminus A'_1 = \emptyset \mod D_1$

$(\beta)$ if $\delta \in A'_1, \alpha \in C_{\delta}$ and $s \in [\delta \cap \mathfrak{B}_\alpha]^{\leq \lambda(\alpha)}$ and $\bar{\xi} \in C_1$, then for some $\delta_*$ we have $\delta < \delta_* \in A_1$ and

$(a)$ $\bar{G}(x, \delta) = \bar{G}(x, \delta_1)$ for every $x \in s$

$(b)$ $\bar{G}(\delta, \delta_*) = \bar{\xi}$.

Proof. Like the proof of 1.6 \hfill $\square_{2.7}$

2.8 Claim. There is a set $a \in [A'_1]^< \kappa$ such that:

$\square$ for every decomposition of $a$ as $\cup \{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ there is $\bar{\xi} \in C_1$ such that

$(\alpha)$ for every $\bar{\xi} \in C_1$ there is $b \subseteq a_{\bar{\xi}}$ of order type $\zeta$ such that $\bar{G} \upharpoonright [b]^2$ is constantly $\bar{\varepsilon}$

$(\beta)$ for every $\varepsilon \in C_0$ there is $b \subseteq a_{\bar{\xi}}$ of order type $\zeta$ such that $\bar{F} \upharpoonright [b]^2$ is constantly $\varepsilon$.

Proof. The claim holds since $A'_1$ has this property and $\kappa$ is strongly compact. If $A'_1 = \cup \{a_{\bar{\xi}} : \bar{\xi} \in C_1\}$ for some $\bar{\xi}, a_{\bar{\xi}} \in D_1^+$ hence clause $(\alpha)$ holds by the choice of $D_1, C_1$; and clause $(\beta)$ holds as $D_1^+ \subseteq D_0^+$ (as $D_0 \subseteq D_1$) and the choice of $D_0, C_0$. \hfill $\square_{2.8}$

Continuation of the proof of 0.2. Now choose $\delta_2 \in A'_1$ such that $\delta_2 > \sup(a)$ and for $\bar{\xi} = (\xi_1, \xi_2) \in C_1 \subseteq \theta \times \theta$ define $a_{\bar{\xi}}$ as

$$\bar{a}_{\bar{\xi}} = \{x \in a : \bar{G}(x, \delta_2) = \bar{\xi}\}.$$ 

Clearly $\langle a_{\bar{\xi}} : \bar{\xi} \in C_1 \rangle$ is a decomposition of $a$ and so there is $\bar{\xi} = (\xi_1, \xi_2) \in C_1$ as guaranteed by $\square$ of 2.8. In particular, there is $b \subseteq a_{\bar{\xi}}$ of order type $\zeta$ such that $\bar{F} \upharpoonright [b]^2$ is constantly $\xi_2$ (note that $(\xi_1, \xi_2) \in C_1 \subseteq C_0 \times C_0$ so $\xi_2 \in C_0$). Now let $E = \{\varepsilon < \sigma : G_{\varepsilon}(\alpha, \delta_2) = G(\alpha, \delta_2) \text{ for every } \alpha \in b\}$. By the definition of $G$ this is an unbounded subset of $\sigma$ and clearly

$(*)$ if $\varepsilon \in E$ and $\alpha \in b$ then $G_{\varepsilon}(\alpha, \delta_2) = G(\alpha, \delta_2) = (\xi_1, \xi_2)$. 


For $\alpha < \lambda$ let $\Upsilon(\alpha) = \text{Min}\{\varepsilon \in E : \alpha < \lambda_\varepsilon\}$ and let $C_{\delta_1} = \{\gamma(\Upsilon) : \Upsilon < \sigma\}_\prec.

Now we try to choose by induction on $\alpha < \lambda$ an element $x_\alpha$ satisfying

$$(*)_0 \quad x_\alpha < \delta_1 \text{ and moreover } x_\alpha \in \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}, \text{ and } \beta < \alpha \Rightarrow x_\beta < x_\alpha$$

$$(*)_1 \quad F(x_\beta, x_\alpha) = \xi_2 \text{ for } \beta < \alpha$$

$$(*)_2 \quad F(x_\alpha, \beta) = \xi_2 \text{ for } \beta \in b \cup \{\delta_2\}.$$ 

At step $\alpha$, by $2.7$, that is by the choice of $A'_1$ applying clause $(\beta)$ there with

$\{x_\beta : \beta < \alpha\} \cup b, \delta_2, \xi$ here standing for $s, \delta, \xi$ there, we can find $\delta_3$ satisfying the requirement there on $\delta_1$, so

$$(i) \quad \delta_2 < \delta_3 \in A_1$$

$$(ii) \quad \bar{G}(x, \delta_3) = \bar{G}(x, \delta_2) \text{ for } x \in s \cup b$$

$$(iii) \quad \bar{G}(\delta_2, \delta_3) = (\xi_1, \xi_2).$$

Now

$$(*)_3 \quad F(x_\beta, \delta_3) = \xi_2 \text{ for } \beta < \alpha.$$ 

[Why? By (ii) we have $\bar{G}(x_\beta, \delta_3) = \bar{G}(x_\beta, \delta_2)$ hence $F(x_\beta, \delta_3) = F(x_\beta, \delta_2)$ but the latter by $(*)_2$ is equal to $\xi_2.$]

$$(*)_4 \quad G(\beta, \delta_3) = \xi_2 \text{ for } \beta \in b$$

[Why? By (ii) and as $\beta \in b \Rightarrow \bar{G}(\beta, \delta_2) = (\xi_1, \xi_2) \Rightarrow G(\beta, \delta_2) = \xi_2).$]

$$(*)_5 \quad G(\delta_2, \delta_3) = \xi_2$$

[Why? By clause (iii).]

$$(*)_6 \quad \{x_\beta : \beta < \alpha\} \text{ is a subset of } \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}.$$ 

Let $\langle y_i : i < \zeta + 2 \rangle$ list $b \cup \{\delta_2, \delta_3\}$ increasing order.

Now we use the choice of $G_{\Upsilon(\alpha)}$ to choose an increasing sequence $\langle z_i : i < \zeta + 2 \rangle$ in $\delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$, $z_0 > x_\beta$ for $\beta < \alpha$ such that $F(z_i, y_j) = G(y_i, y_j)$ for $i, j < \zeta + 2$ and $F(x_\beta, z_i) = F(x_\beta, y_i)$ for $i < \zeta + 2$. Let $x_\alpha = z_{\zeta + 1}$ so $x_\alpha = \delta_1 \cap \mathfrak{B}_{\gamma(\Upsilon(\alpha))}$ is $> x_\beta$ for $\beta < \alpha$.

Also $x_\alpha$ satisfies $(*)_0$ of the recursive definition. Now $\beta < \alpha \Rightarrow F(x_\beta, x_\alpha) = F(x_\beta, z_{\zeta + 1}) = F(x_\beta, y_{\zeta + 1}) = F(x_\beta, \delta_3)$ which is $\xi_2$ by $(*)_3$ above, so for our choice of $x_\alpha$, $(*)_1$ holds. Next if $\beta \in b \cup \{\delta_2\}$ then $F(x_\alpha, x_\beta) = F(x_\beta, z_{\zeta + 1}) = G(x_\beta, \delta_3)$ which is $\xi_2$ by $(*)_4$ or $(*)_5$. So $x_\alpha$ is as required. \(\Box_{0.2}\)
REFERENCES.


