# NONREFLECTING STATIONARY SETS IN $\mathcal{P}_\kappa\lambda$

SAHARON SHELAH AND MASAHIRO SHIOYA

ABSTRACT. Let $\kappa$ be a regular uncountable cardinal and $\lambda \geq \kappa^+$. The principle of Stationary Reflection in $\mathcal{P}_\kappa\lambda$ has been successful in settling problems of infinitary combinatorics in the case $\kappa = \omega_1$. For $\kappa \geq \omega_2$ the principle is known to fail if $\lambda$ is large enough. In this paper the principle is shown to fail for every $\lambda \geq \kappa^+$.

## 1. INTRODUCTION

In [6] Foreman, Magidor and Shelah introduced the following principle for a cardinal $\lambda \geq \omega_2$: If $S$ is a stationary set in $\mathcal{P}_{\omega_1}\lambda$, $S \cap \mathcal{P}_{\omega_1} A$ is stationary in $\mathcal{P}_{\omega_1} A$ for some $\omega_1 \subset A \subset \lambda$ of size $\omega_1$. Let us call the principle Stationary Reflection in $\mathcal{P}_{\omega_1}\lambda$. It follows from Martin's Maximum [6] and holds after a supercompact cardinal is Lévy-collapsed to $\omega_2$ [2]. For recent applications of reflection principles for stationary sets in $\mathcal{P}_{\omega_1}\lambda$, see e.g. [3, 14, 16, 17].

What if $\omega_1$ is replaced by a higher regular cardinal? Feng and Magidor [4] proved that Stationary Reflection in $\mathcal{P}_{\omega_2}\lambda$ fails if $\lambda$ is large enough. Their argument shows in effect that Stationary Reflection in $\mathcal{P}_\kappa\lambda$ for some large enough $\lambda$ implies that the club filter on $\kappa$ is presaturated (see also [2]). It is known that the club filter on a successor cardinal $\geq \omega_2$ cannot be presaturated [10].

Extending the Feng–Magidor result, Foreman and Magidor [5] proved in effect that Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails if $\kappa$ is regular $\geq \omega_2$ and $\lambda$ is large enough. More precisely

**Theorem 1.** *Let $\kappa$ be regular $\geq \omega_2$. Then Stationary Reflection in $\mathcal{P}_\kappa\lambda$ fails for every $\lambda \geq 2^{\kappa^+}$.*

We include a proof of Theorem 1 in §4. A further example of nonreflection, which is based on PCF Theory [11] can be found in [12].

This paper shows that for $\kappa \geq \omega_2$ Stationary Reflection in $\mathcal{P}_\kappa \lambda$ fails *everywhere*:

**Theorem 2.** *Let $\kappa$ be regular $\geq \omega_2$. Then Stationary Reflection in $\mathcal{P}_\kappa \lambda$ fails for every $\lambda \geq \kappa^+$.*

In §3 we prove Theorem 2 in much greater generality.

## 2. Preliminaries

For background material we refer the reader to [7]. Throughout the paper, we use $\kappa, \lambda, \mu$ to denote an infinite cardinal. We write $S_\lambda^\kappa$ for $\{\gamma < \lambda : \operatorname{cf} \gamma = \kappa\}$, and $[\lambda]^\mu$ for $\{x \subset \lambda : |x| = \mu\}$.

Let $A$ be a set of ordinals. The set of limit points of $A$ is denoted $\lim A$. It is easy to see $|\lim A| \leq |A|$. $A$ is called $\sigma$-closed if $\gamma \in A$ for every $\gamma \in \lim A$ of cofinality $\omega$.

Let $\kappa$ be regular, $\omega_1 \leq \kappa \leq \mu \leq \lambda$ and $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa \lambda$. We write $C(f)$ for $\{x \in \mathcal{P}_\kappa \lambda : f``[x]^{<\omega} \subset \mathcal{P}(x)\}$. For $x \in \mathcal{P}_\kappa \lambda$ the smallest superset of $x$ in $C(f)$ is denoted $\operatorname{cl}_f x$. It is well-known that for every club $C \subset \mathcal{P}_\kappa \lambda$ there is $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa \lambda$ with $C(f) \subset C$.

Stationary Reflection in $\mathcal{P}_\kappa \lambda$ states that if $S$ is a stationary set in $\mathcal{P}_\kappa \lambda$, $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size $\kappa$. Let $S$ be a stationary set in $\mathcal{P}_\kappa \lambda$. $S$ is called nonreflecting if it witnesses the failure of Stationary Reflection, i.e. $S \cap \mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for every $\kappa \subset A \subset \lambda$ of size $\kappa$. More generally $S$ is called $\mu$-nonreflecting if $S \cap \mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for every $\mu \subset A \subset \lambda$ of size $\mu$. If $S$ is a $\mu$-nonreflecting stationary set in $\mathcal{P}_\kappa \mu^+$ and $\mu^+ \leq \lambda$, $\{x \in \mathcal{P}_\kappa \lambda : x \cap \mu^+ \in S\}$ is (easily seen to be) a $\mu$-nonreflecting stationary set. In particular Stationary Reflection in $\mathcal{P}_\kappa \lambda$ fails for every $\lambda \geq \kappa^+$ iff Stationary Reflection in $\mathcal{P}_\kappa \kappa^+$ fails.

## 3. Main Theorem

This section is devoted to the main Theorem 3 and its corollaries. We prove Theorem 3 using ideas from Nonstructure Theory [13]. Similar ideas can be found in the proof of Diamond for $\mathcal{P}_\kappa \lambda$ [10, 15].

**Theorem 3.** *Let $\kappa$ be regular $\geq \omega_2$ and $\mu$ a cardinal $\geq \kappa$. Assume there are $\{c_\xi : \xi < \mu\} \subset \mathcal{P}_\kappa \mu$ and a stationary $T \subset \mathcal{P}_\kappa \mu$ of size $\mu$ such that if $z \in T$ and $b \in [z]^\omega$, there is $\xi \in z$ with $b \subset c_\xi$. Then $\mathcal{P}_\kappa \lambda$ has a $\mu$-nonreflecting stationary subset for every $\lambda \geq \mu^+$.*

*Proof.* It suffices to give a $\mu$-nonreflecting stationary set in $\mathcal{P}_\kappa\mu^+$.

Let $\{c_\xi : \xi < \mu\}$ and $T$ be as above. By Solovay's theorem we have a partition of $S_{\mu^+}^\omega$ into $\mu$ disjoint stationary sets $\{S_z : z \in T\}$. For $\mu \leq \gamma < \mu^+$ fix a bijection $\pi_\gamma : \mu \to \gamma$.

Set $S = \{x \in \mathcal{P}_\kappa\mu^+ : \forall \gamma \in x - \mu(\pi_\gamma\text{``}(x\cap\mu) \subset x) \wedge x\cap\mu \in T \wedge \sup x \in S_{x\cap\mu}\}$.

**Claim.** *$S$ is stationary in $\mathcal{P}_\kappa\mu^+$.*

*Proof.* Since $\{x \in \mathcal{P}_\kappa\mu^+ : \forall \gamma \in x - \mu(\pi_\gamma\text{``}(x\cap\mu) \subset x)\}$ is club, it suffices to show that $\{x \in \mathcal{P}_\kappa\mu^+ : x\cap\mu \in T \wedge \sup x \in S_{x\cap\mu}\}$ is stationary.

Fix $f : [\mu^+]^{<\omega} \to \mathcal{P}_\kappa\mu^+$. For $z \in T$ consider the following game $\mathcal{G}(z)$ of length $\omega$ between two players $I$ and $II$:

In round $n$ $I$ chooses $\mu \leq \gamma_n < \mu^+$. Then $II$ chooses $x_n \in C(f)$ with $\gamma_n < \sup x_n$. We further require $\sup x_n < \gamma_{n+1}$ and $x_n \subset x_{n+1}$. Finally $II$ wins just in case $x_n \cap \mu = z$ for every $n < \omega$.

Set $T' = \{z \in T : II \text{ has no winning strategy in } \mathcal{G}(z)\}$.

**Subclaim.** *$T'$ is nonstationary in $\mathcal{P}_\kappa\mu$.*

*Proof.* Suppose otherwise. Note that the game $\mathcal{G}(z)$ is closed for $II$, hence determined. Hence for $z \in T'$ we have a winning strategy $\sigma_z$ for $I$ in $\mathcal{G}(z)$. Set $D = \{\delta < \mu^+ : f\text{``}[\delta]^{<\omega} \subset \mathcal{P}_\kappa\delta\}$, which is club. By induction on $n < \omega$ we define $\beta_n \in S_{\mu^+}^\omega \cap D$ and $x_n^z$ for $z \in T'$ so that $\langle x_n^z : n < \omega \rangle$ is a play of $II$ in $\mathcal{G}(z)$ against $\sigma_z$ and $\sup x_n^z = \beta_n$ for every $z \in T'$ as follows:

Assume we have $\beta_i$ and $\{x_i^z : z \in T'\}$ for $i < n$ as above. Since $|T'| \leq |T| = \mu$, we have $\sup_{z\in T'} \sigma_z(\langle x_i^z : i < n\rangle) < \beta_n \in S_{\mu^+}^\omega \cap D$. Then $\beta_{n-1} = \sup x_{n-1}^z < \sigma_z(\langle x_i^z : i < n\rangle) < \beta_n$ for every $z \in T'$.

Fix $z \in T'$. Since $\sup x_{n-1}^z < \beta_n \in S_{\mu^+}^\omega \cap D$, $C_n^z = \{x \in \mathcal{P}_\kappa\beta_n \cap C(f) : x_{n-1}^z \subset x \wedge \sup x = \beta_n\}$ is club in $\mathcal{P}_\kappa\beta_n$. Let $x_n^z$ be $\pi_{\beta_n}\text{``}z$ if $\pi_{\beta_n}\text{``}z \in C_n^z$, otherwise an element of $C_n^z$.

Set $\beta = \sup_{n<\omega} \beta_n$. Then $\mu \leq \sup_{z\in T'} \sigma_z(\emptyset) < \beta_0 < \beta$. Since $\beta_n \in S_{\mu^+}^\omega \cap D$ for every $n < \omega$, $C = \{x \in \mathcal{P}_\kappa\beta \cap C(f) : \forall n < \omega$ $(\pi_{\beta_n}\text{``}(x\cap\mu) = x\cap\beta_n \wedge \sup(x\cap\beta_n) = \beta_n)\}$ is club in $\mathcal{P}_\kappa\beta$. Since $T'$ is stationary in $\mathcal{P}_\kappa\mu$, we can take $x \in C$ so that $x\cap\mu \in T'$.

Set $z = x\cap\mu \in T'$. Since $x \in C$, we see by induction on $n < \omega$ that $\pi_{\beta_n}\text{``}z = \pi_{\beta_n}\text{``}(x\cap\mu) = x\cap\beta_n \in C_n^z$ and $x_n^z = x\cap\beta_n$. Hence $x_n^z \cap \mu = x\cap\mu = z$ for every $n < \omega$. Thus $II$ wins in $\mathcal{G}(z)$ against $\sigma_z$ with the play $\langle x_n^z : n < \omega \rangle$. This contradicts that $\sigma_z$ is a winning strategy for $I$ in $\mathcal{G}(z)$, as desired. $\square$

Fix $z \in T - T'$ and a winning strategy $\tau$ for $II$ in $\mathcal{G}(z)$. Since $S_z$ is stationary in $\mu^+$, we have $\mu < \gamma \in S_z$ such that $\sup \tau(s) < \gamma$ for

every $s \in \gamma^{<\omega}$. Since $\mathrm{cf}\,\gamma = \omega$, we have $\gamma_n$ inductively so that $\gamma_0 = \mu$, $\sup \tau(\langle \gamma_i : i < n \rangle) < \gamma_n$ and $\sup_{n<\omega} \gamma_n = \gamma$. Then $\langle \gamma_n : n < \omega \rangle$ is a play of $I$ in $\mathcal{G}(z)$ against $\tau$.

For $n < \omega$ set $x_n = \tau(\langle \gamma_i : i \leq n \rangle)$. Then $II$ wins in $\mathcal{G}(z)$ with the play $\langle x_n : n < \omega \rangle$. Hence $\{x_n : n < \omega\} \subset C(f)$ is increasing, $x_n \cap \mu = z$ and $\gamma_n < \sup x_n < \gamma_{n+1}$ for every $n < \omega$. Set $x = \bigcup_{n<\omega} x_n$. Then $x \in C(f)$, $x \cap \mu = z \in T$ and $\sup x = \sup_{n<\omega} \sup x_n = \sup_{n<\omega} \gamma_n = \gamma \in S_z = S_{x \cap \mu}$, as desired. $\square$

**Claim.** $S$ *is* $\mu$-*nonreflecting.*

*Proof.* Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\mu \subset A \subset \mu^+$ of size $\mu$. Then $\{x \in \mathcal{P}_\kappa A : \forall \gamma \in x - \mu (\pi_\gamma "(x \cap \mu) \subset x)\}$ is unbounded in $\mathcal{P}_\kappa A$. Hence $\gamma = \pi_\gamma "\mu = \pi_\gamma "(A \cap \mu) \subset A$ for every $\gamma \in A - \mu$. Thus $A = \delta$ for some $\mu \leq \delta < \mu^+$.

**Subclaim.** $\mathrm{cf}\,\delta < \kappa$.

*Proof.* Since $\{x \in \mathcal{P}_\kappa \delta : \pi_\delta "(x \cap \mu) = x\}$ is club, $S' = \{x \in S \cap \mathcal{P}_\kappa \delta : \pi_\delta "(x \cap \mu) = x\}$ is stationary in $\mathcal{P}_\kappa \delta$. Fix $x \in S'$. Since $\sup x \in S_{x \cap \mu} \subset S_{\mu^+}^\omega$, we have $b_x \in [x]^\omega$ with $\sup b_x = \sup x$. Since $\pi_\delta^{-1} "b_x \in [x \cap \mu]^\omega$ and $x \cap \mu \in T$, we have $\xi \in x \cap \mu$ with $\pi_\delta^{-1} "b_x \subset c_\xi$.

Now we have $\xi^* < \mu$ and a stationary $S^* \subset S'$ such that $b_x \subset \pi_\delta "c_{\xi^*}$ for every $x \in S^*$. Since $S^*$ is unbounded in $\mathcal{P}_\kappa \delta$, $\delta = \sup_{x \in S^*} \sup x = \sup_{x \in S^*} \sup b_x \leq \sup \pi_\delta "c_{\xi^*} \leq \delta$. Hence $\delta = \sup \pi_\delta "c_{\xi^*}$ has cofinality $< \kappa$. $\square$

Thus $\{x \in S \cap \mathcal{P}_\kappa \delta : \sup x = \delta\}$ is stationary in $\mathcal{P}_\kappa \delta$. Take $x, y$ from this set so that $x \cap \mu \neq y \cap \mu$. Then $\delta = \sup x = \sup y \in S_{x \cap \mu} \cap S_{y \cap \mu}$. This contradicts $S_{x \cap \mu} \cap S_{y \cap \mu} = \emptyset$, as desired. $\square$

Therefore $\mathcal{P}_\kappa \mu^+$ has a $\mu$-nonreflecting stationary subset. $\square$

Now Theorem 2 follows from Theorem 3 with $\mu = \kappa$: It is easy to check that the hypothesis of Theorem 3 is satisfied with $c_\xi = \xi$ for $\xi < \kappa$ and $T = S_\kappa^{\omega_1}$.

Theorem 3 with $\mu = \kappa^+$ yields the following

**Corollary 1.** *Let* $\kappa$ *be regular* $\geq \omega_2$. *Then* $\mathcal{P}_\kappa \lambda$ *has a* $\kappa^+$-*nonreflecting stationary subset for every* $\lambda \geq \kappa^{++}$.

*Proof.* It suffices to check that the hypothesis of Theorem 3 is satisfied.

For $\kappa \leq \gamma < \kappa^+$ we have a club $T_\gamma \subset \mathcal{P}_\kappa \gamma$ of size $\kappa$. List the set $\bigcup_{\kappa \leq \gamma < \kappa^+} T_\gamma$ as $\{c_\xi : \xi < \kappa^+\}$. Then $D = \{\delta < \kappa^+ : \bigcup_{\kappa \leq \gamma < \delta} T_\gamma = \{c_\xi : \xi < \delta\}\}$ is club. Set $T = \{z \in \bigcup_{\kappa \leq \gamma < \kappa^+} T_\gamma : \forall b \in [z]^\omega \exists \xi \in z(b \subset c_\xi)\}$. Then $|T| \leq \kappa^+$. We show that $T$ is stationary in $\mathcal{P}_\kappa \kappa^+$.

Fix $f : [\kappa^+]^{<\omega} \to \mathcal{P}_\kappa \kappa^+$. We have $\delta \in S^\kappa_{\kappa^+} \cap D$ with $f``[\delta]^{<\omega} \subset \mathcal{P}_\kappa \delta$. Then $T_\delta \cap C(f)$ is club in $\mathcal{P}_\kappa \delta$. Moreover $\{c_\xi : \xi < \delta\} = \bigcup_{\kappa \leq \gamma < \delta} T_\gamma$ is unbounded in $\mathcal{P}_\kappa \delta$. Hence we can build an increasing sequence $\{z_\alpha : \alpha < \omega_1\} \subset T_\delta \cap C(f)$ so that $z_\alpha \subset c_\xi$ for some $\xi \in z_{\alpha+1}$. Then $\bigcup_{\alpha < \omega_1} z_\alpha \in T \cap C(f)$, as desired. $\qquad\square$

If $\mathrm{cf}\,\mu < \kappa$, $\mathcal{P}_\kappa \mu$ has no stationary subset of size $\mu$. So Theorem 3 has nothing to say in this case. It has something to say, however, about a question of [8]:

**Corollary 2.** *Let $\kappa$ be regular $\geq \omega_2$ and $\mu^{<\kappa} = \mu$. Then $\mathcal{P}_\kappa \lambda$ has a $\mu$-nonreflecting stationary subset for every $\lambda \geq \mu^+$.*

*Proof.* Since $\mu^{<\kappa} = \mu$, we can list $\mathcal{P}_\kappa \mu$ as $\{c_\xi : \xi < \mu\}$. Then $T = \{z \in \mathcal{P}_\kappa \mu : \forall b \in [z]^\omega \exists \xi \in z (b \subset c_\xi)\}$ is stationary:

Fix $f : [\mu]^{<\omega} \to \mathcal{P}_\kappa \mu$. Build an increasing sequence $\{z_\alpha : \alpha < \omega_1\} \subset C(f)$ so that $z_\alpha = c_\xi$ for some $\xi \in z_{\alpha+1}$. Then $\bigcup_{\alpha < \omega_1} z_\alpha \in T \cap C(f)$, as desired. $\qquad\square$

## 4. Proof of Theorem 1

This section presents the Foreman–Magidor example of a nonreflecting stationary set in $\mathcal{P}_\kappa \lambda$ as we understand it. Although the construction seems to work only for $\lambda \geq 2^{\kappa^+}$, the example has the virtue that the intersection with $\{x \in \mathcal{P}_\kappa \lambda : \mathrm{cf}(x \cap \kappa) = \omega\}$ is stationary [5]. In contrast our example of Theorem 2 is (easily seen to be) a subset of $\{x \in \mathcal{P}_\kappa \lambda : \mathrm{cf}(x \cap \kappa) > \omega = \mathrm{cf} \sup(x \cap \kappa^+)\}$.

Two Subclaims below are proved using ideas to show that Chang's Conjecture holds after a measurable cardinal is Lévy-collapsed to $\omega_2$ [9] and that $\mathcal{P}_\kappa \kappa^+$ has a club subset of size $\leq (\kappa^+)^{\omega_1}$ [1] respectively.

*Proof of Theorem 1.* Since $\lambda \geq 2^{\kappa^+}$, we can list (possibly with repetition) the functions $: \kappa^+ \to \mathcal{P}_\kappa \kappa$ as $\{g_\xi : \xi < \lambda\}$. For $\kappa \leq \gamma < \kappa^+$ fix a bijection $\pi_\gamma : \kappa \to \gamma$. Define $h : \kappa \times (\kappa^+ - \kappa) \to \mathcal{P}_\kappa \kappa^+$ by $h(\alpha, \beta) = \lim \pi_\beta``\alpha$. Then $D = \{x \in \mathcal{P}_\kappa \lambda : \forall \xi \in x (g_\xi``(x \cap \kappa^+) \subset \mathcal{P}(x)) \wedge \forall \gamma \in x \cap (\kappa^+ - \kappa)(\pi_\gamma``(x \cap \kappa) = x \cap \gamma) \wedge h``((x \cap \kappa) \times (x \cap (\kappa^+ - \kappa))) \subset \mathcal{P}(x)\}$ is club.

Set $S = \{x \in \mathcal{P}_\kappa \lambda : \{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is nonstationary in $\kappa^+\}$.

**Claim.** *$S$ is stationary in $\mathcal{P}_\kappa \lambda$.*

*Proof.* Suppose otherwise. By induction on $n < \omega$ we build $f_n : [\lambda]^{<\omega} \to \mathcal{P}_\kappa \lambda$ and $\xi_n : [\lambda]^{<\omega} \to \lambda$ as follows:

Since $S$ is nonstationary, we have $f_0$ with $C(f_0) \subset D - S$. Assume next we have $f_n$. Define $\xi_n$ and $f_{n+1}$ by $g_{\xi_n(a)}(\gamma) = \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa$

and $f_{n+1}(a) = f_n(a) \cup \{\xi_n(a)\}$. Finally define $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ by $f(a) = \bigcup_{n<\omega} f_n(a)$.

**Subclaim.** *If $x \in C(f)$, $\{\sup(z \cap \kappa^+) : x \subset z \in C(f) \wedge z \cap \kappa = x \cap \kappa\}$ is unbounded in $\kappa^+$.*

*Proof.* Fix $\alpha < \kappa^+$. Since $x \in C(f) \subset C(f_0) \subset \mathcal{P}_\kappa\lambda - S$, $\{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is stationary in $\kappa^+$. Hence we have $x \subset y \in D$ such that $y \cap \kappa = x \cap \kappa$ and $\alpha < \sup(y \cap \kappa^+)$. Fix $\alpha < \gamma \in y \cap \kappa^+$. Then $z = \bigcup\{\mathrm{cl}_{f_n}(a \cup \{\gamma\}) : n < \omega \wedge a \in [x]^{<\omega}\}$ witnesses the Subclaim:

Since $\gamma \in z$, $\alpha < \gamma \leq \sup(z \cap \kappa^+)$. By the definition of $f$, it is easy to check $x \subset z \in C(f)$. To see $z \cap \kappa \subset x \cap \kappa$, fix $\beta \in z \cap \kappa$. Then $\beta \in \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa = g_{\xi_n(a)}(\gamma)$ for some $n < \omega$ and $a \in [x]^{<\omega}$. Since $x \in C(f)$ and $a \in [x]^{<\omega}$, $\xi_n(a) \in f(a) \subset x \subset y$. Since $\xi_n(a), \gamma \in y \in D$, $\beta \in g_{\xi_n(a)}(\gamma) \subset y \cap \kappa = x \cap \kappa$, as desired. $\square$

For $i = 0, 1$ build an increasing sequence $\{x_\xi^i : \xi < \omega_1\} \subset C(f)$ so that $x_\xi^i \cap \kappa = x_0^0 \cap \kappa \in S_\kappa^{\omega_1}$, $\kappa < \sup(x_\xi^0 \cap \kappa^+) \leq \sup(x_\xi^1 \cap \kappa^+) < \sup(x_{\xi+1}^0 \cap \kappa^+)$ but $x_0^0 \cap \kappa^+$ is not an initial segment of $x_1^0 \cap \kappa^+$ as follows:

First we have $x_0^0 \in C(f)$ such that $x_0^0 \cap \kappa \in S_\kappa^{\omega_1}$ and $\kappa < \sup(x_0^0 \cap \kappa^+)$. By the Subclaim we can take $x_1^0$ from $X = \{z \in C(f) : x_0^0 \subset z \wedge z \cap \kappa = x_0^0 \cap \kappa\}$ so that $\sup(x_1^0 \cap \kappa^+)$ is the $\kappa$-th element of $\{\sup(z \cap \kappa^+) : z \in X\}$. Since $x_1^0 \cap \kappa^+$ has $< \kappa$ initial segments, we have $x_0^1 \in X$ such that $\sup(x_0^1 \cap \kappa^+) < \sup(x_1^0 \cap \kappa^+)$ but $x_0^1 \cap \kappa^+$ is not an initial segment of $x_1^0 \cap \kappa^+$, as required above. The rest of the construction using the Subclaim is routine.

Set $x^i = \bigcup_{\xi<\omega_1} x_\xi^i$. Since $\{x_\xi^i : \xi < \omega_1\} \subset C(f)$ is increasing and $\kappa \geq \omega_2$, $x^i \in C(f)$. Since $x_\xi^i, x^i \in C(f) \subset D$, $x_\xi^i \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$: $x_\xi^i \cap \gamma = \pi_\gamma``(x_\xi^i \cap \kappa) = \pi_\gamma``(x_0^0 \cap \kappa) = \pi_\gamma``(x^i \cap \kappa) = x^i \cap \gamma$ for every $\gamma \in x_\xi^i \cap (\kappa^+ - \kappa)$. By the construction of $x_\xi^i$'s, $\sup(x^0 \cap \kappa^+) = \sup_{\xi<\omega_1}\sup(x_\xi^0 \cap \kappa^+) = \sup_{\xi<\omega_1}\sup(x_\xi^1 \cap \kappa^+) = \sup(x^1 \cap \kappa^+) \in S_{\kappa^+}^{\omega_1}$.

**Subclaim.** $x^i \cap \kappa^+$ *is $\sigma$-closed.*

*Proof.* Fix $\gamma \in \lim(x^i \cap \kappa^+)$ of cofinality $\omega$. Then we have $b \subset x^i \cap \kappa^+$ of order type $\omega$ with $\sup b = \gamma$. Since $\kappa < \sup(x^i \cap \kappa^+) \in S_{\kappa^+}^{\omega_1}$, we have $b \subset \beta \in x^i \cap (\kappa^+ - \kappa)$. Since $\beta \in x^i \in D$, $\pi_\beta^{-1}``(x^i \cap \beta) = x^i \cap \kappa = x_0^0 \cap \kappa \in S_\kappa^{\omega_1}$. Since $\pi_\beta^{-1}``b \in [\pi_\beta^{-1}``(x^i \cap \beta)]^\omega$, we have $\pi_\beta^{-1}``b \subset \alpha \in x^i \cap \kappa$. Hence $b \subset \pi_\beta``\alpha$. Since $\alpha, \beta \in x^i \in D$, $\gamma = \sup b \in \lim \pi_\beta``\alpha = h(\alpha, \beta) \subset x^i$, as desired. $\square$

Thus we have $\sup(x_1^0 \cap \kappa^+) < \gamma \in x^0 \cap x^1 \cap \kappa^+$. Since $\gamma \in x^i \in D$, $x^0 \cap \gamma = \pi_\gamma``(x^0 \cap \kappa) = \pi_\gamma``(x_0^0 \cap \kappa) = \pi_\gamma``(x^1 \cap \kappa) = x^1 \cap \gamma$. This

contradicts that $x_\xi^i \cap \kappa^+$ is an initial segment of $x^i \cap \kappa^+$ but $x_0^1 \cap \kappa^+$ is not an initial segment of $x_1^0 \cap \kappa^+$, as desired. $\square$

**Claim.** *$S$ is nonreflecting.*

*Proof.* Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size $\kappa$. Fix a bijection $\pi : \kappa \to A$. Then $\{\gamma < \kappa : \pi``\gamma \in S\}$ is stationary. Since $\{\gamma < \kappa : (\pi``\gamma) \cap \kappa = \gamma\}$ is club, their intersection $T$ is stationary. Since $\{y \in \mathcal{P}_\kappa \lambda : \pi``(y \cap \kappa) \subset y \in D\}$ is club, $\{y \in \mathcal{P}_\kappa \lambda : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary. Hence $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary in $\kappa^+$.

Since $|T| = \kappa$, we have $\gamma \in T$ such that $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa = \gamma\}$ is stationary in $\kappa^+$. Note that $(\pi``\gamma) \cap \kappa = \gamma$ by $\gamma \in T$. Hence $\{\sup(y \cap \kappa^+) : \pi``\gamma \subset y \in D \wedge y \cap \kappa = (\pi``\gamma) \cap \kappa\}$ is stationary in $\kappa^+$. But $\pi``\gamma \in S$ by $\gamma \in T$. Contradiction. $\square$

Therefore Stationary Reflection in $\mathcal{P}_\kappa \lambda$ fails. $\square$

Finally we remark that the same proof goes through even if "non-stationary" is replaced by "bounded" in the definition of $S$.

## References

[1] J. Baumgartner, *On the size of closed unbounded sets*, Ann. Pure Appl. Logic 54 (1991) 195–227.

[2] M. Bekkali, Topics in Set Theory, Lecture Notes in Math. 1476, Springer, Berlin, 1991.

[3] J. Cummings, M. Foreman and M. Magidor, *Squares, scales and stationary reflection*, J. Math. Logic 1 (2001) 35–98.

[4] Q. Feng and M. Magidor, *On reflection of stationary sets*, Fund. Math. 140 (1992) 175–181.

[5] M. Foreman and M. Magidor, *Large cardinals and definable counterexamples to the continuum hypothesis*, Ann. Pure Appl. Logic 76 (1995) 47–97.

[6] M. Foreman, M. Magidor and S. Shelah, *Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part I*, Ann. Math. 127 (1988) 1–47.

[7] A. Kanamori, The Higher Infinite, Springer Monogr. Math., Springer, Berlin, 2003.

[8] Y. Matsubara, *Stationary preserving ideals over $\mathcal{P}_\kappa \lambda$*, J. Math. Soc. Japan 55 (2003) 827–835.

[9] S. Shelah, Proper Forcing, Lecture Notes in Math. 940, Springer, Berlin, 1982.

[10] ———, Around Classification Theory of Models, Lecture Notes in Math. 1182, Springer, Berlin, 1986.

[11] ———, Cardinal Arithmetic, Oxford Logic Guides 29, Oxford Univ. Press, New York, 1994.

[12] ———, *Existence of almost free abelian groups and reflection of stationary set*, Math. Japon. 45 (1997) 1–14.

[13] ———, Nonstructure Theory, Oxford Logic Guides, Oxford Univ. Press, New York, to be published.

8

[14] _____, *Reflection implies the SCH*, preprint.

[15] M. Shioya, *Splitting $\mathcal{P}_\kappa\lambda$ into maximally many stationary sets*, Israel J. Math. 114 (1999) 347–357.

[16] _____, *Stationary reflection and the club filter*, preprint.

[17] W. Woodin, The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal, de Gruyter Ser. Log. Appl. 1, Walter de Gruyter, Berlin, 1999.

INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, 91904 ISRAEL.

*E-mail address*: shelah@math.huji.ac.il

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, TSUKUBA, 305-8571 JAPAN.

*E-mail address*: shioya@math.tsukuba.ac.jp