

# On a non-vanishing Ext

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ABSTRACT. The existence of valuation domains admitting non-standard uniserial modules for which certain Exts do not vanish was proved in [1] under Jensen's Diamond Principle. In this note, the same is verified using the ZFC axioms alone.

In the construction of large indecomposable divisible modules over certain valuation domains  $R$ , the first author used the property that  $R$  satisfied  $\text{Ext}_R^1(Q, U) \neq 0$ , where  $Q$  stands for the field of quotients of  $R$  (viewed as an  $R$ -module) and  $U$  denotes any uniserial divisible torsion  $R$ -module, for instance, the module  $K = Q/R$ ; see [1]. However, both the existence of such a valuation domain  $R$  and the non-vanishing of Ext were established only under Jensen's Diamond Principle  $\diamond$  (which holds true, e.g., in Gödel's Constructible Universe).

Our present goal is to get rid of the Diamond Principle, that is, to verify in ZFC the existence of valuation domains  $R$  that admit divisible non-standard uniserial modules and also satisfy  $\text{Ext}_R^1(Q, U) \neq 0$  for several uniserial divisible torsion  $R$ -modules  $U$ . (For the proof of Corollary 3, one requires only 6 such  $U$ .)

We start by recalling a few relevant definitions. By a *valuation domain* we mean a commutative domain  $R$  with 1 in which the ideals form a chain under inclusion. A *uniserial*  $R$ -module  $U$  is defined similarly as a module whose submodules form a chain under inclusion.  $K = Q/R$  is a uniserial torsion  $R$ -module, it is *divisible* in the sense that  $rK = K$  holds for all  $0 \neq r \in R$ . A divisible uniserial  $R$ -module is called *standard* if it is an epic image of the uniserial module  $Q$ ; otherwise it is said to be *non-standard*. The existence of valuation domains admitting non-standard uniserials has been established in ZFC; see e.g. [3], [2], and the literature quoted there.

As the  $R$ -module  $Q$  is uniserial, it can be represented as the union of a well-ordered ascending chain of cyclic submodules:

$$R = Rr_0 < Rr_1^{-1} < \dots < Rr_\alpha^{-1} < \dots < \bigcup_{\alpha < \kappa} Rr_\alpha^{-1} = Q \quad (\alpha < \kappa),$$

where  $r_0 = 1$ ,  $r_\alpha \in R$ , and  $\kappa$  denotes an infinite cardinal and also the initial ordinal of the same cardinality. As a consequence,  $K = \bigcup_{\alpha < \kappa} (Rr_\alpha^{-1}/R)$  where  $Rr_\alpha^{-1}/R \cong$

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$R/Rr_\alpha$  are cyclically presented  $R$ -modules. We denote by  $\iota_\alpha^\beta : Rr_\alpha^{-1}/R \rightarrow Rr_\beta^{-1}/R$  the inclusion map for  $\alpha < \beta$ , and may view  $K$  as the direct limit of its submodules  $Rr_\alpha^{-1}/R$  with the monomorphisms  $\iota_\alpha^\beta$  as connecting maps.

A uniserial divisible torsion module  $U$  is a ‘clone’ of  $K$  in the sense of Fuchs-Salce [2], if there are units  $e_\alpha^\beta \in R$  for all pairs  $\alpha < \beta$  ( $< \kappa$ ) such that

$$e_\alpha^\beta e_\beta^\gamma - e_\alpha^\gamma \in Rr_\alpha \quad \text{for all } \alpha < \beta < \gamma < \kappa,$$

and  $U$  is the direct limit of the direct system of the modules  $Rr_\alpha^{-1}/R$  with connecting maps  $\iota_\alpha^\beta e_\alpha^\beta : Rr_\alpha^{-1}/R \rightarrow Rr_\beta^{-1}/R$  ( $\alpha < \beta$ ); i.e. multiplication by  $e_\alpha^\beta$  followed by the inclusion map. It might be helpful to point out that though  $K$  and  $U$  need not be isomorphic, they are ‘piecewise’ isomorphic in the sense that they are unions of isomorphic pieces.

Let  $R$  denote the valuation domain constructed in the paper [1] (see also Fuchs-Salce [2]) that satisfies  $\text{Ext}_R^1(Q, K) \neq 0$  in the constructible universe  $L$ . Moreover, there are clones  $U_n$  of  $K$ , for any integer  $n > 0$ , that satisfy  $\text{Ext}_R^1(Q, U_n) \neq 0$ : for convenience, we let  $K = U_0$ .

This  $R$  has the value group  $\Gamma = \bigoplus_{\alpha < \omega_1} \mathbb{Z}$ , ordered anti-lexicographically, and its quotient field  $Q$  consists of all formal rational functions of  $u^\gamma$  with coefficients in an arbitrarily chosen, but fixed field, where  $u$  is an indeterminate and  $\gamma \in \Gamma$ . It is shown in [2] that such an  $R$  admits divisible non-standard uniserials (i.e. clones of  $K$  non-isomorphic to  $K$ ), and under the additional hypothesis of  $\diamond_{\aleph_1}$ ,  $\text{Ext}_R^1(Q, U_n) \neq 0$  holds; in other words, there is a non-splitting exact sequence

$$0 \rightarrow U_n \rightarrow H_n \xrightarrow{\phi} Q \rightarrow 0.$$

Using the elements  $r_\alpha \in R$  introduced above, for each  $n < \omega$  we define a tree  $T_n$  of length  $\kappa$  whose set of vertices at level  $\alpha$  is given by

$$T_{n\alpha} = \{x \in H_n \mid \phi(x) = r_\alpha^{-1}\}.$$

The partial order  $<_T$  is defined in the following way:  $x <_T y$  in  $T_n$  if and only if, for some  $\alpha < \beta$ , we have  $\phi x = r_\alpha^{-1}$  and  $\phi y = r_\beta^{-1}$  such that

$$x = r_\alpha^{-1} r_\beta y \quad \text{in } H_n,$$

where evidently  $r_\alpha^{-1} r_\beta \in R$ .

Fix an integer  $n > 0$ , and define  $T$  as the union of the trees  $T_0, T_1, \dots, T_n$  with a minimum element  $z$  adjoined. It is straightforward to check that  $(T, <_T)$  is indeed a tree with  $\kappa$  levels, and the inequalities  $\text{Ext}_R^1(Q, U_i) \neq 0$  ( $i = 0, 1, \dots, n$ ) guarantee that  $T$  has no branch of length  $\kappa$ .

We now define a model  $\mathbf{M}$  as follows. Its universe is the union of the universes of  $R, Q, U_i, H_i$  ( $i = 0, \dots, n$ ), and it has the following relations:

- (i) unary relations  $R, Q, U_i, H_i, T$ , and  $S = \{r_\alpha \mid \alpha < \kappa\}$ ,
- (ii) binary relation  $<_T$ , and  $<_S$  (which is the natural well-ordering on  $S$ ),

- (iii) individual constants  $0_R, 0_Q, 0_{U_i}, 1_R$ , and
- (iv) functions: the operations in  $R, Q, U_i, H_i$ , where  $R$  is a domain,  $Q, U_i, H_i$  are  $R$ -modules,  $\phi_i$  is an  $R$ -homomorphism from  $H_i$  onto  $Q$  ( $i = 0, \dots, n$ , and  $\psi : Q \rightarrow K$  with  $\text{Ker } \psi = R$  is the canonical map.

We argue that even though our universe  $V$  does not satisfy  $V=L$ , the class  $L$  does satisfy it, and so in  $L$  we can define the model  $\mathbf{M}$  as above. Let  $\mathbf{T}$  be the first order theory of  $\mathbf{M}$ . So the first order (countable theory)  $\mathbf{T}$  has in  $L$  a model in which

$(^*)_{\mathbf{M}}$  the tree  $(T, <_T)$  with set of levels  $(S, <_S)$  and with the function  $\phi = \cup \phi_i$  giving the levels, as interpreted in  $\mathbf{M}$ , has no full branch (this means that there is no function from  $S$  to  $T$  increasing in the natural sense and inverting  $\phi$ , or any  $\phi_i$ ).

Hence we conclude as in [3] (by making use of Shelah [4]) that there is a model  $\mathbf{M}'$  with those properties in  $V$  (in fact, one of cardinality  $\aleph_1$ ).

Note that  $(S, <_S)$  as interpreted in  $\mathbf{M}'$  is not well ordered, but it is still a linear order of uncountable cofinality (in fact, of cofinality  $\aleph_1$ ), the property  $(^*)_{\mathbf{M}'}$  still holds, and it is a model of  $\mathbf{T}$ . This shows that all relevant properties of  $\mathbf{M}$  in  $L$  hold for  $\mathbf{M}'$  in  $V$ , just as indicated in [3].

It should be pointed out that, as an alternative, instead of using a smaller universe of set theory  $L$ , we could use a generic extension not adding new subsets of the natural numbers (hence essentially not adding new countable first order theories like  $\mathbf{T}$ ).

If we continue with the same argument as in [3], then using [4] we can claim that we have proved in ZFC the following theorem:

**THEOREM 1.** *There exist valuation domains  $R$  admitting non-standard uniserial torsion divisible modules such that  $\text{Ext}_R^1(Q, U_i) \neq 0$  for various clones  $U_i$  of  $K$ .* ■

Hence we derive at once that the following two corollaries are true statements in ZFC; for their proofs we refer to [2].

**COROLLARY 2.** *There exist valuation domains  $R$  such that if  $U, V$  are non-isomorphic clones of  $K$ , then  $\text{Ext}_R^1(U, V)$  satisfies:*

- (i) *it is a divisible mixed  $R$ -module;*
- (ii) *its torsion submodule is uniserial.* ■

More relevant consequences are stated in the following corollaries; they solve Problem 27 stated in [2].

**COROLLARY 3.** *There exist valuation domains admitting indecomposable divisible modules of cardinality larger than any prescribed cardinal.* ■

**COROLLARY 4.** *There exist valuation domains with superdecomposable divisible modules of countable Goldie dimension.* ■

## References

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