POLISH ALGEBRAS, SHY FROM FREEDOM SH771

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Abstract. Every Polish group is not free whereas some F_σ group may be free. Also every automorphism group of a structure of cardinality, e.g. $\beth_\omega$ is not free.

This research was partially supported by the Israel Science Foundation.
Publication 771
I would like to thank Alice Leonhardt for the beautiful typing.
Latest Revision - 2012/May/25

§0 Introduction

Our first motivation was the question: can a countable structure have an automorphism group, which a free uncountable group? This is answered negatively in [Sh 744].

This was a well known problem in group theory at least in England (David Evans in a meeting in Durham 1987) and we thank Simon Thomas for telling us about it. Independently in descriptive set theory, Howard and Kechris [BeKe96] ask if there is an uncountable free Polish group, i.e. which is on a complete separable metric space. A related result (before [Sh 744]) was gotten by Solecki [So98], [So99] who proved that the group of automorphisms of a countable structure cannot be an uncountable free abelian group. Having the problem arise independently supported the feeling that it is a natural problem.

The idea of the proof in [Sh 744] was to prove that such a group has some strong algebraic completeness or compactness, more specifically for any sequence $\langle d_n : n < \omega \rangle$ of elements of the group converging to the identity many countable sets of equations are solvable. This is parallel in some sense to Hensel lemma for the p-adics, and seem to me interesting in its own right.

Lecturing in a conference in Rutgers, February 2001, I was asked whether I am really speaking on Polish groups. We then prove the parallel result (i.e. 0.1 below) using a more restrictive condition on the set of equations. Parallel theorems hold, for semi groups and for metric algebras, e.g. with non isolated unit (e is a unit means $\{e\}$ is a subalgebra).

More specifically we prove (see Conclusion 4.3, 4.1(1)).

0.1 Theorem. *1) There is no Polish group which as a group is free and uncountable.*

2) Slightly more generally, assume

- (a) *G is a metric space*
- (b) *G is a group with continuous xy, x^{-1}*
- (c) *G is complete*
- (d) *the density of G is $< |G|$.*

Then G is not free.

0.2 Thesis: If G is a Polish algebra satisfying one of the compactness conditions defined below, then it is in fact large in the sense of lots of sets of equations has a solution.

A reader interested just in this theorem 0.1 can read just §3.

0.3 Question: What are the restrictions on $\mathrm{Aut}(\mathbb{A})$ for uncountable structures $\mathbb{A}$?

We also prove that if $\mathbb{A}$ is a structure of cardinality μ, μ is strong limit of cofinality $\aleph_0$ (e.g. $\beth_\omega$) and the automorphism group of $\mathbb{A}$ is of cardinality $> \mu$, then it is far from being free; this does not follow directly from 0.1(2) as the natural metric considered here does not satisfy all the conditions.

In [Sh 744] this is proved in the special case where $\mathbb{A} = \bigcup\limits_n P_n^{\mathbb{A}}$ satisfying $|P_n^{\mathbb{A}}| < \mu$.

* * *

Note: An arbitrary subgroup e.g. of the symmetric group of size $\aleph_1$ can consistently be made into an automorphism group by Just, Shelah, Thomas [JShT 654]. So $\beth_\omega$ is more interesting from this point of view.

0.4 Question: Is there a model theory of Polish spaces?

Naturally we would like to develop a parallel to classification theory (see [Sh:c]). A natural test problem is to generalize "Morley theorem = Los conjecture". But we only have one model so what is the meaning?
Well, we may change the universe. If we deal with abelian groups (or any variety) it is probably more natural to ask when is such (Borel) algebra free.

0.5 Example: If $\mathbb{P}$ is adding $(2^{\aleph_0})^+$-Cohen subsets of ω then

$$(\mathbb{C})^{\mathbf{V}} \text{ and } (\mathbb{C})^{\mathbf{V}[G]}$$

are both algebraically closed fields of characteristic 0 which are not isomorphic (as they have different cardinalities).

So we restrict ourselves to forcing $\mathbb{P}_1 \lessdot \mathbb{P}_2$ such that

$$(2^{\aleph_0})^{\mathbf{V}[\mathbb{P}_1]} = (2^{\aleph_0})^{\mathbf{V}[\mathbb{P}_2]}$$

and compare the Polish models in $\mathbf{V}^{\mathbb{P}_1}, \mathbf{V}^{\mathbb{P}_2}$. We may restrict our forcing notions to c.c.c. or whatever...

0.6 Example: Under any such interpretation

- (a) $\mathbb{C}$ = the field of complex numbers is categorical
- (b) $\mathbb{R}$ = the field of the reals is not
  (by adding $2^{\aleph_0}$ many Cohen reals).

(Why? Trivially: $\mathbb{R}^{\mathbf{V}[\mathbb{P}_2]}$ is complete in $\mathbf{V}[\mathbb{P}_2]$ while $\mathbb{R}^{\mathbf{V}[\mathbb{P}_1]}$ in $\mathbf{V}^{[\mathbb{P}_2]}$ is not complete, but there are less trivial reasons).

0.7 Conjecture: We have a dychotomy, i.e. either the model is similar to categorical theories, or there are "many complicated models".

0.8 Thesis: Classification theory for such models resemble more the case of $\mathbb{L}_{\omega_1,\omega}$ than the first order.

See [Sh:h]; as support for this thesis we prove:

0.9 Theorem. *There is an F_σ abelian group (i.e. a F_σ-definition, in fact an explicit definition) such that $\mathbf{V} \models$ "G is a free abelian group" iff $\mathbf{V} \models 2^{\aleph_0} < \aleph_{736}$.*

Comments: In the context of the previous theorem we cannot do better than F_σ, but we may hope for some other example which is not a group or categoricity is not because of freeness.

0.10 Conclusion: Freeness (of an F_σ-abelian group) can stop at $\aleph_n$ (any n).

A connection with the model theory is that by Hart-Shelah [HaSh 323] such things can also occur in $\mathbb{L}_{\aleph_1,\aleph_0}$ whereas (by [Sh 87a], [Sh 87b] Theorem) if $\bigwedge\limits_n (2^{\aleph_n} < 2^{\aleph_{n+1}})$ and $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$, categorical in every $\aleph_n$, then ψ is categorical in every λ. See more in [ShVi 648].

The parallels here are still open.

This casts some light on the thesis that non-first order logics are "more distant" from the "so-called" mainstream mathematics.

Lecturing on [Sh 744], in MAMLS meeting (Rutgers, Feb.2001) some asked if this apply to Polish group, (not just automoprhisms of countable structures) and showed it. Subsequently (during Hattingham, Aug. 2001 Euresco conference), telling Sabbagh about the results for Polish groups he wondered whether we get "non $\aleph_0$-freeness", but the results give only non-strongly-$\aleph_1$-freeness. Subsequently his ex-student Anatole Khelif who continues this work and proves non-$\aleph_0$-freeness.

Lately Blass asks on definable abelian subgroups of $\mathbb{Z}^\omega$, answers are derived for this from [Sh 402] and §5. The works originally have a section on stability theory in this context, proving a generalization of the existence of large indiscernibility set. Very lately (Feb 2009) it was moved to [Sh 849], also the parts on "semi-groups", the semi-group of endo-morphism and for metric algebras were removed.

We may be more humble than in 0.4.

0.11 Question: Is there model theory for equational theories, stressing free algebras?

The material in §1 - §5 (except some generalizations) was presented in a course in Rutgers, Sept. - Oct 2001 and I thank the audience for their comments and Haim Horowitz for help in proofreading. We thank the referee for doing much to improve the presentation in particular by "less is better".

0.12 Notation. 1) Let ω denote the set of natural numbers, and let $x < \omega$ mean "x is a natural number".
2) Let a, b, c, d denote members of G (a group).
3) Let $\bar{d}$ denote a finite sequence $\langle d_n : n < n^* \rangle$, and similarly in other cases.
4) Let $k, \ell, m, n, i, j, r, s, t$ denote natural numbers.
5) Let ε, ζ, ξ denote reals > 0.

0.13 Definition. 1) A group word is a sequence $x_1^{n_1} x_2^{n_2}, \dots, x_k^{n_k}$ where the x_ℓ are variables or elements of a group and $n_\ell \in \mathbb{Z}$ for $\ell = 1, \dots, k$.
2) The word is reduced if $n_\ell \neq 0, x_\ell \neq x_{\ell+1}$.
3) The length of a word $w = x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ is $\sum\limits_{\ell=1}^{k} |n_\ell|$.
4) A group term $w(x_1, \dots, x_n)$ is a word of the form $x_{\ell_1}^{n_1} x_{\ell_2}^{n_2} \dots x_{\ell_k}^{n_k}$ with $\ell_i \in \{1, \dots, n\}, n_\ell \in \mathbb{Z}$ (actually $r_\ell \in \{1, \dots, -1\}$ suffice).

§1 METRIC GROUPS AND METRIC MODELS

We first define metric [complete] group, and give a natural major example: automorphism groups. The natural example of metric group is the group of automorphism of a countable structure.

1.1 Definition. $\mathfrak{a} = (G, \mathfrak{d})$ is called a metric group if:

(a) G is a group

(b) G is a metric space for the metric $\mathfrak{d}$

(c) the functions xy, x^{-1} are continuous.

2) Saying $(G, \mathfrak{d})$ is complete, means complete as a metric space.

1.2 Notation: For a metric group $\mathfrak{a}$, the metric is denoted by $\mathfrak{d}_{\mathfrak{a}}$ and the unit is denoted by $e_{\mathfrak{a}}$ and the group by $G_{\mathfrak{a}}$. When no confusion arises "G is a metric group" means $(G, \mathfrak{d}_G)$ is a metric group.

Now we define cases closer to automorphism groups, in those cases the proof is very similar to the one in [Sh 744].

1.3 Definition. 1) G is a specially metric group when:

(α) G is a metric group

(β) for every $\zeta, \varepsilon \in \mathbb{R}^+$ there is $\xi \in \mathbb{R}^+$ such that: if $x_1, x_2, y_1, y_2 \in \{a : \mathfrak{d}_G(a, e_G) < \varepsilon\}$ then $\mathfrak{d}_G(x_1, x_2) < \xi \wedge \mathfrak{d}(y_1, y_2) < \xi$ implies $\mathfrak{d}_G(x_1 y_1, x_2 y_2) < \zeta \wedge \mathfrak{d}(x_1^{-1}, x_2^{-1}) < \zeta$; this is a kind of uniform continuity (inside the ε-neighborhood of e_G; this is harder if we increase ε and/or decrease ζ)

(γ) for arbitrarily small $\zeta \in \mathbb{R}^+$ the set $\{a \in G : \mathfrak{d}(a, e_G) < \zeta\}$ is a subgroup of G.

2) We say $\bar{\zeta} = \langle \zeta_n : n < \omega \rangle$ is strongly O.K. for G if:

(a) $\zeta_n \in \mathbb{R}^+_n$

(b) ζ_n satisfies clause (γ) of part (1), i.e. $\{a \in G : \mathfrak{d}(a, e_G) < \zeta_n\}$ is a subgroup of G

(c) $\zeta_{n+1} \le \zeta_n$ and $0 = \inf\{\zeta_n : n < \omega\}$

(d) if $x_1, x_2, y_1, y_2 \in \{a \in G : \mathfrak{d}_G(a, e_G) < \zeta_0\}$ and $\mathfrak{d}(x_1, x_2) < \zeta_{n+1}, \mathfrak{d}_G(y_1, y_2) < \zeta_{n+1}$ and $r(1), r(2) \in \{1, -1\}$ then $\mathfrak{d}_G(x_1^{r(1)} y_1^{r(2)}, x_2^{r(1)} y_2^{r(2)}) < \zeta_n$.

3) We say G is specially$^+$ metric group if in part (1) we have $(\alpha), (\beta)$ and

$(\gamma)^+$ for every $\zeta \in \mathbb{R}^+$ the set $\{a \in G : \mathfrak{d}(a, e_G) < \zeta\}$ is a subgroup of G.

1.4 Observation. 1) If G is a special metric group then there is a sequence $\bar{\zeta}$ which is strongly O.K. for G.
2) We can in clause (d) of 1.3(2) above omit $r(1), r(2)$ and conclude only "$\mathfrak{d}(x_1 y_1, x_2 y_2) < \zeta_n$ and $\mathfrak{d}(x_1^{-1}, x_2^{-1}) < \zeta_n$. This causes just slight changes in the computations of length in the proof or replacing $\bar{\zeta}$ by a suitable subsequence.
3) Every specially$^+$ metric group is a specially metric group.

Proof. Easy.

1.5 Definition. Assume $\mathbb{A}$ is a countable structure with automorphism group $G =$ Aut$(\mathbb{A})$ and for notational simplicity its set of elements is ω, the set of natural numbers (and, of course, it is infinite, otherwise trivial).

We define a metric $\mathfrak{d} = \mathfrak{d}_{\mathbb{A}} = \mathfrak{d}^{\rm aut}_{\mathbb{A}}$ on G by

$$\mathfrak{d}(f, g) = \sup\{2^{-n} : f(n) \neq g(n) \text{ or } f^{-1}(n) \neq g^{-1}(n)\}.$$

Let $\mathrm{Aut}_{\mathbb{A}} = (\mathrm{Aut}(\mathbb{A}), \mathfrak{d}^{\rm aut}_{\mathbb{A}})$, but we may write $G_{\mathbb{A}}$ or $G^{\rm aut}_{\mathbb{A}}$.

1.6 Claim. *: For $\mathbb{A}$ as above:*
$(\mathrm{Aut}_{\mathbb{A}}, \mathfrak{d}^{\rm aut}_{\mathbb{A}})$ *is a complete separable specially$^+$ metric group.*

Proof. Easy.

§2 Semi metric groups: on automorphism groups of uncountable structures

It seems natural investigating the automorphism groups of a model $\mathbb{A}$ say of cardinality, e.g. $\beth_\omega$, intending to put in a framework where we shall be able to prove it is not a free group of cardinality $> \beth_\omega$. Now choose $\bar{A} = \langle A_n : n < \omega \rangle$ such that, e.g. the universe of $\mathbb{A}$ is $\beth_\omega$ and $A_n = \beth_n$. For such $\bar{A}$ there is a natural metric on $\mathrm{Aut}(\mathbb{A})$ under which it is a complete metric group, <u>but</u> usually of too big density; there is another natural metric on $\mathrm{Aut}(\mathbb{A})$ under which it is a complete metric space with density $\beth_\omega$ <u>but</u> multiplication is not continuous and Cauchi sequences may not converge to any point. To get the desired results we use "indirectly complete-metric" defined in 2.1, which combine the two metrics; in other words we weaken the completeness demand; hopefully this will have other applications as well, e.g. also for Borel groups. We had looked at some generalizations: replacing $\mathrm{Aut}(\mathbb{A})$ by other derived structures and $\mathbb{A}$ a Polish Algebra.

The reader may skip this section if not interested in the results concerning $\beth_\omega$. One way to present what we are doing is

2.1 Definition. We say G is an indirectly complete metric group if:

- (a) G is a group
- (b) G is a metric space under $\mathfrak{d}_G$
- (c) if $\bar{c} = \langle c_n : n < \omega \rangle$ satisfies $\mathfrak{d}_G(c_n, c_{n+1}) < 1/2^n$, <u>then</u> letting $d_n = c_{2n}^{-1} c_{2n+1}$ and $\bar{d} = \langle d_n : n < \omega \rangle$ we have
  - $(*)$ for some metric $\mathfrak{d}' = \mathfrak{d}_{G,\bar{d}}$, under $\mathfrak{d}', G$ is a metric group, and $\bar{d}$ converges to some c under $\mathfrak{d}'$, moreover $(G, \mathfrak{d}')$ is complete.

2.2 Definition. Assume $\mathbb{A}$ is a structure.
1) $\bar{A}$ is an ω-representation of $\mathbb{A}$ if $\bar{A} = \langle A_n : n < \omega \rangle, A_n \subseteq A_{n+1}$ for $n < \omega$ and $\cup\{A_n : n < \omega\}$ is the universe of $\mathbb{A}$.
2) For every ω-representation $\bar{A}$ of $\mathbb{A}$ let $\mathrm{Aut}_{\bar{A}}(\mathbb{A}) = \{f \in \mathrm{Aut}(\mathbb{A})$: for every $n < \omega$ for some $m < \omega$ we have $(\forall x \in A_n)(f(x) \in A_m \ \& \ f^{-1}(x) \in A_m)\}$.
3) If $\bar{A}$ is an ω-representation of $\mathbb{A}$ and $G = \mathrm{Aut}(\mathbb{A})$ <u>then</u> we define $\mathfrak{d} = \mathfrak{d}_{\mathbb{A},\bar{A}} = \mathfrak{d}^{\mathrm{aut}}_{\mathfrak{A},\bar{A}}$, a metric[1] on G by

[1] this is proved in 2.3

$$\begin{aligned}\mathfrak{d}(f,g) = \sup\{2^{-n} :\ & \text{there is } a \in A_n \text{ such that}\\ & \text{for some } (f',g') \in \{(f,g),(f^{-1},g^{-1})\}\\ & \text{one of the following possibilities holds}\\ & (a) \quad \text{for some } m<\omega \text{ we have } f'(a) \in A_m \Leftrightarrow g'(a) \notin A_m,\\ & (b) \quad f'(a) \neq g'(a) \text{ are in } A_n\}.\end{aligned}$$

4) If $\bar{A}$ is an ω-representation of $\mathbb{A}$ and $G = \mathrm{Aut}(\mathbb{A})$ then we define $\mathfrak{d}' = \mathfrak{d}'_{\mathbb{A},\bar{A}}$ a metric on G by $\mathfrak{d}'(f,g) = \sup\{2^{-n} : f \restriction A_n = g \restriction A_n \text{ and } f^{-1} \restriction A_n = g^{-1} \restriction A_n\}$.

2.3 Claim. *Assume $\bar{A}$ is an ω-representation of an infinite structure $\mathbb{A}$.*
1) $(\mathrm{Aut}(\mathbb{A}), \mathfrak{d}_{\mathbb{A},\bar{A}})$ is an indirectly complete metric group with density $\leq 2^{\aleph_0} + \Sigma\{2^{|A_n|} : n < \omega\}$; in fact it is a complete metric space (but in general not a metric group).
2) $(\mathrm{Aut}(\mathbb{A}), \mathfrak{d}'_{\mathbb{A},\bar{A}})$ is a complete metric group of density $\leq \Sigma\{\|\mathbb{A}\|^{|A_n|} : n < \omega\}$; so if each A_n is finite the density is $\leq \aleph_0$.
3) If the universe of $\mathbb{A}$ is ω and $A_n = n = \{0,\dots,n-1\}$ <u>then</u> $\bar{A} = \langle A_n : n < \omega\rangle$ is an ω-representation of $\mathbb{A}$ with each A_n finite and $\mathfrak{d}'_{\mathbb{A},\bar{A}} = \mathfrak{d}^{\mathrm{aut}}_{\mathbb{A}}$ from Definition 1.5(1) and under it $\mathrm{Aut}(\mathbb{A})$ is a complete separable specially metric group.

Proof. 1) Let $\mathfrak{d} = \mathfrak{d}_{\mathbb{A},\bar{A}}$. First we show that

$(*)_1$ $\mathfrak{d}$ is a metric (even ultrametric).
$(*)_2$ $\mathfrak{d}(f^{-1},g^{-1}) = \mathfrak{d}(f,g)$
$(*)_3$ G as a metric space under $\mathfrak{d}_G$ has density $\leq \Sigma\{2^{|A_n|} : n < \omega\}$.

We may wonder whether $(\mathrm{Aut}(\mathbb{A}), \mathfrak{d})$ is complete, i.e. whether every $\mathfrak{d}$-Cauchi sequence $\langle f_n : n < \omega\rangle$ in G, $\mathfrak{d}$-converge to some $f \in G$.

Before answering we prove a weaker substitute (in fact, it is the one we shall use for proving the indirect completeness below).

Now we note that:

$(*)_4$ if $\langle f_n : n < \omega\rangle$ is a $\mathfrak{d}$-Cauchi sequence, <u>then</u> for every $a \in \mathbb{A}$, $\langle f_n(a) : n < \omega\rangle$ is eventually constant and $\langle f_n^{-1}(a) : n < \omega\rangle$ is eventually constant so the limit f is a well defined permutation of $\mathbb{A}$, moreover belongs to $G = \mathrm{Aut}(\mathbb{A})$.

Very nice, <u>but</u> multiplication[2] is not continuous in general for this metric, $\mathfrak{d}$. We also have

- $(*)_5$ $(\mathrm{Aut}(\mathbb{A}), \mathfrak{d})$ is a complete metric space.
  [Why? Let $\bar{f} = \langle f_n : n < \omega \rangle$ be a $\mathfrak{d}$-Cauchi sequence, by $(*)_4$ there is $f \in \mathrm{Aut}(\mathbb{A})$ to which $\bar{f}$ pointwise converges. Now let $n(*) < \omega$ be given so for some $n(1) < \omega$ we have: $n \geq n(1) \Rightarrow \mathfrak{d}(f_n, f_{n(1)}) < 2^{-n(*)}$ and we shall prove $n \geq n(1) \Rightarrow \mathfrak{d}(f, f_n) < 2^{-n(*)}$. So for each $c \in A_{n(*)}$ we shall check clauses (a),(b) in the definition of $\mathfrak{d}$ in Definition 2.2(3). First for every $m < \omega$ we have $n \geq n(1) \Rightarrow f_n(c) \in A_m \equiv f_{n(1)}(c) \in A_m$, but $\langle f_n(c) : n < \omega \rangle$ is eventually constantly $f(c)$ hence $f(c) \in A_m \equiv f_{n(1)}(c) \in A_m$. This takes care of clause (a).
  As for clause (b), similarly $n \geq n(1) \wedge [\{f_n(a), f_{n(1)}(a)\} \subseteq A_n \Rightarrow f_n(a) = f_{n(1)}(a) \rangle$ hence $n \geq n(1) \wedge [\{f(a), f_{n(1)}(a)\} \subseteq A_n \Rightarrow f(a) = f_{n(1)}(a) \rangle$. The same holds for $\langle f_n^{-1} : n < \omega \rangle, f^{-1}$ so we are done.]

But still we have to prove that $(\mathrm{Aut}(\mathbb{A}), \mathfrak{d})$ is an indirectly complete metric group. The only clause left is (c) of Definition 2.1. So assume $g_n \in G, \mathfrak{d}(g_n, g_{n+1}) < 1/2^n$, hence by $(*)_4 + (*)_5$ the sequence $\langle g_n : n < \omega \rangle$ converge to some $g \in G$ by the metric and hence pointwise converge to g. Let $f_n = g_{2n}^{-1} g_{2n+1}$, easily $\langle f_n : n < \omega \rangle$ pointwise converge to $e_G = \mathrm{id}_{\mathbb{A}}$, let $f = e_G$; so it suffices to find a metric $\mathfrak{d}'$ such that $(G, \mathfrak{d}')$ is a complete metric group in which $\langle f_n : n < \omega \rangle$ converge to f. We prove this assuming just

- $\boxtimes$ $\bar{f} = \langle f_n : n < \omega \rangle$ is an ω-sequence of members of G which pointwise converge to $f \in G$.

Let $B_n = B^n_{\bar{f}} = \{a \in \mathbb{A} : a \in A_n$ and for every $m \in (n, \omega)$ we have $f_m(a) = f_n(a)$ & $f_m^{-1}(a) = f_n^{-1}(a)\}$. Clearly $\bar{B} = \langle B^n_{\bar{f}} : n < \omega \rangle$ is an increasing ω-sequence of subsets of $\mathbb{A}$ with union (the universe of) $\mathbb{A}$. Recall that $\mathfrak{d}' =: \mathfrak{d}'_{\mathbb{A}, \bar{B}}$ was defined by

$$\mathfrak{d}'(f, g) = \sup\{2^{-n} : f \restriction B_n \neq g \restriction B_n \text{ or } f^{-1} \restriction B_n \neq g^{-1} \restriction B_n\}.$$

Now by parts (2),(3)

- $(*)_6$ the group G with $\mathfrak{d}'$ is a complete metric group

[2]E.g. let $\mathbb{A}$ be a trivial structure (i.e. with the empty vocabulary so G is the group of permutations of $\mathbb{A}$) and $b_n \neq c_n \in A_{n+1} \backslash A_n$ and $a_n \in A_0$ be pairwise distinct (for $n < \omega$). Let h_k exchange a_n, c_n if $n < k$ and is the identity otherwise. Let f interchange a_n, b_n if $n < \omega$ and is the identity otherwise. Let g_k interchange b_n, c_n if $n < k$ and is the identity otherwise. Now $\langle g_k : k < \omega \rangle$ is a Cauchi sequence and so has a limit g, but $h_k = f g_k f$ and $\langle h_k : k < \omega \rangle$ is not a Cauchi sequence as $\mathfrak{d}(h_{k_1}, h_{k_1}) = 1$ if $k_1 < k_2$ as witnessed by $a = a_{k_1}$.

and obviously

$(*)_7$ $\mathfrak{d}'(f_n, f_m) <$ Max$\{2^{-n}, 2^{-m}\}$ and $\mathfrak{d}'(f_n, f) \leq 2^{-n}$

hence

$(*)_8$ $\langle f_n : n < \omega \rangle$ converge to f by $\mathfrak{d}'$.

So by $(*)_6 + (*)_8$ we are done.
2),3) Left to the reader. $\square_{2.3}$

§3 Compactness of metric algebras

Note that below if $u_n = \{t_n\} = \{n\}$ we may write x_n instead of $\bar{x}_n$, n instead $t \in u_n$ and d_n instead of $\bar{d}_n$. Also 3.1 - 3.6 works for Polish algebras as well.

3.1 The completeness Lemma. *Assume $\mathfrak{a}$ is a Polish group $G = G_{\mathfrak{a}}$ such that*

$\circledast(a)$ *$\langle u_n : n < \omega\rangle$ is a sequence of pairwise disjoint non-empty finite sets*

(b) *$\bar{x}_n = \langle x_t : t \in u_n\rangle$*

(c) *$\bar{\sigma}_n(\bar{x}_{n+1}) = \langle \sigma_{n,t}(\bar{x}_{n+1}) : t \in u_n\rangle$ is a sequence of group terms, possibly with parameters (from $G_{\mathfrak{a}}$) so $\bar{\sigma}_n(\bar{d}) = \langle \sigma_{n,t}(\bar{d}) : t \in u_n\rangle$ for any $\bar{d} = \langle d_s : s \in u_{n+1}\rangle, d_s \in G$*

(d) *$\zeta = \langle \zeta_n : n < \omega\rangle$ is a sequence of positive reals converging to 0*

(e) *$\bar{d}_{n+1} = \langle d_{n+1,t} : t \in u_{n+1}\rangle$ with each $d_{n+1,t}$ an element of G such that if $\bar{d}'_{n+1} = \langle d'_{n+1,t} : t \in u_{n+1}\rangle$ is of distance $< \zeta_{n+1}$ from $\bar{d}_{n+1}$, (that is $d'_{n+1,t} \in \mathrm{Ball}(d_{n+1,t}, \zeta_{n+1})$ for each $t \in u_{n+1}$), <u>then</u> $\bar{\sigma}_n(\bar{d}'_{n+1}) \in \mathrm{Ball}(\bar{d}_n, \zeta_n)$ which means: $t \in u_n \Rightarrow \sigma_{n,t}(\ldots, d'_{n+1,s}, \ldots)_{s \in u_{n+1}} \in \mathrm{Ball}(d_{n,t}, \zeta_n)$*

(f) *for every $n < \omega$ and a positive real ε there is $m > n$ such that*

$(*)_\varepsilon$ *if $d'_{m,t} \in \mathrm{Ball}(d_{m,t}, \zeta_n)$ for every $t \in u_m$ <u>then</u> the distance between $\sigma_n(\sigma_{n-1}(\ldots, \sigma_{m-1}(d'_m), \ldots), \sigma_n(\sigma_{n+1}(\ldots, \sigma_{m-1}(d_m)) \ldots)$ is $< \varepsilon$.*

<u>Then</u> *there are $d^*_{n,t} \in M$ for $n < \omega, t \in u_n$ which solves the set of equations*

$$d^*_{n,t} = \sigma_{n+1}(d^*_{n+1,s})_{s \in u_{n+1}}$$

and satisfies

$$d^*_{n+1,t} \in \mathrm{Ball}_G(d_{n+1,t}, \zeta_n).$$

3.2 Remark. 1) In "special" versions we have $\bar{d}_n = \bar{\sigma}_n(\bar{d}_{n+1})$ (and in [Sh 744] we have $d_n = \sigma_n(d_{n+1})$ but here there is no "the true solution which we perturb".
2) Condition (e) in Lemma 3.1 says that if in large m we perturb $\bar{d}_m$ with error $< \zeta_m$ and compute down by the $\bar{\sigma}$'s we still get a reasonable $\bar{d}_n$ but not necessarily a very good one.

Proof. For every k we shall define $\langle c^k_{n,t} : t \in u_n, n < \omega\rangle$, a sequence of elements of the algebra.

First, if $n \geq k$ let $c^k_{n,t} = d_{n,t}$. Second, we define $\bar{c}^k_n = \langle c^k_{n,t} : t \in u_n \rangle$ by downward induction on $n \leq k$.

$\underline{n = k}$ by the first case.

$\underline{n < k}$ let $\bar{c}^k_n = \bar{\sigma}_n(\bar{c}^k_{n+1})$.

By clauses (e) and (f) we have, respectively

- $(*)_1$ $c^k_{n,t} \in \text{ Ball}(d_{n,t}, \zeta_n)$.
- $(*)_2$ for every positive real $\varepsilon > 0$ and $n < \omega$, there is $m > n$ such that if $k \geq m$ then $t \in u_n \Rightarrow c^k_{n,t} \in \text{ Ball}(c^m_{n,t}, \varepsilon)$.

So, for each n and $t \in u_n$ the sequence $\langle c^k_{n,t} : k < \omega \rangle$ is a Cauchy sequence by $(*)_2$; hence it converges to some $c_{n,t} \in M$. Now

- $\boxtimes$ the sequence $\langle c_{n,t} : n < \omega, t \in u_n \rangle$ forms a solution: for every $n < \omega$ and $t \in u_n$ the equation $c^k_{n,t} = \sigma_{n,t}(\ldots, c^k_{n+1,s}, \ldots)_{s \in u_{n+1}}$ is satisfied whenever $n > k$ hence in the limit $c_{n,t} = \sigma_{n,t}(\ldots, c_{n+1,s}, \ldots)_{s \in u_{n+1}}$. $\square_{3.1}$

Recall about groups

3.3 Fact: A free group is torsion free and the group is not divisible, in fact, every element c has at most one n-th root for each $n = 1, 2, \ldots$ and has no root for every large enough n except when c is the unit.

Recall

3.4 Definition. We say N is a retract of the group (or the algebra) M when $N \subseteq M$ and there is a homomorphism form M onto N which is the identity on N.

3.5 Fact: Every countable subgroup of a free group G is contained in a countable subgroup which is a retract of G.

We now give a criterion to show non-freeness. We could use $\bar{x}$ instead of x, of course.

3.6 Claim. *1)*

- (a) $\mathfrak{a}$ *is a complete metric group, the group is* $G_{\mathfrak{a}}$ *and* $e_* \in G_{\mathfrak{a}}$
- (b) $B \subseteq G_{\mathfrak{a}}$ *is countable with* e_* *belonging to the closure of* $B \backslash \{e_*\}$
- (c) Ξ *is a set of terms of the form* $\sigma(x, \bar{y})$
- (d) *if* $\sigma(x, \bar{y}) \in \Xi$ *and* $\bar{b} \in {}^{\ell g(\bar{y})}B$ *and* $c \in G_{\mathfrak{a}}$ *then* $\{x \in G_{\mathfrak{a}} : c = \sigma(x, \bar{b})\}$ *is finite*

(e) *for every finite* $A \subseteq G_{\mathfrak{a}}$ *and* ζ *a positive real there are a sequence* $\bar{b}$ *from* ${}^{\ell g(\bar{y})}B$ *and term* $\sigma(x,\bar{y}) \in \Xi$ *such that:*

- (α) $\sigma(e_*,\bar{b}) \in \mathrm{Ball}(e_*,\zeta)$
- (β) $\sigma(c,\bar{b}) \notin A$ *for every* $c \in G_{\mathfrak{a}}$.

Then no countable sub-group of $G_{\mathfrak{a}}$ *containing* B *is a retract (in the algebraic sense) of* $G_{\mathfrak{a}}$. *Hence* $G_{\mathfrak{a}}$ *is not free.*
2) We can omit e_*, *i.e. omit clause (b) and the last phrase of clause (a) and change clause (e) to*

(e)′ *for any finite* $A \subseteq G_{\mathfrak{a}}$ *and real* $\zeta > 0$ *and* $d \in G_{\mathfrak{a}}$ *there is a term* $\sigma(x,\bar{y}) \in \Xi$ *and sequence* $\bar{b} \in {}^{\ell g(\bar{y})}B$ *and element* $d' \in G_{\mathfrak{a}}$ *such that*

- (α) $\sigma^{M_{\mathfrak{a}}}(d',\bar{b}) \in \mathrm{Ball}_{\mathfrak{a}}(d,\zeta)$
- (β) *for no* $c \in G_{\mathfrak{a}}$ *do we have* $\sigma^{G_{\mathfrak{a}}}(c,\bar{b}) \in A$.

3.7 Remark. We can similarly phrase sufficient conditions for "$G_{\mathfrak{a}}$ is unstable in $\aleph_0$" [for quantifier free formulas, see [Sh 849]].

Proof. 1) Like the proof of part (2) below except that we add to $(*)$:

(η) $e_n = e_*$.

Also part (1) can be derived from part (2).
2) We rely on 3.1; without loss of generality $|B| = \aleph_0$.

Assume toward contradiction that G is a countable reduct of $G_{\mathfrak{a}}$ which includes B, so we can choose h^*, a homomorphism from $M_{\mathfrak{a}}$ onto G which extends $\mathrm{id}_{G_{\mathfrak{a}}}$. Let $\langle a_n : n < \omega \rangle$ list G. Let $u_n = \{n\}$. We choose $\bar{b}_n, e_n$ and $\sigma_n(x,\bar{y}_n)$ and ζ_n by induction on n such that

$(*)$(α) $\sigma_n(x,\bar{y}_n) \in \Xi$

- (β) $\bar{b}_n$ a sequence from $B \subseteq G \subseteq G_{\mathfrak{a}}$ of length $\ell g(\bar{y}_n)$
- (γ) ζ_n a positive real, $\zeta_{n+1} < \zeta_n/2$
- (δ) $e_n \in G_{\mathfrak{a}}$
- (ε) $\sigma_n(e_{n+1},\bar{b}_n) \in \mathrm{Ball}_{\mathfrak{a}}(e_n,\zeta_n)$, moreover if $c \in \mathrm{Ball}_{\mathfrak{a}}(e_{n+1},\zeta_{n+1})$ then $\sigma_n(c,\bar{b}_n) \in \mathrm{Ball}_{\mathfrak{a}}(e_n,\zeta_n)$

(ζ) if $k < n$ and $c, c' \in \text{Ball}_{\mathfrak{a}}(e_{n+1}, \zeta_{n+1})$ and we define the terms $\sigma_{n+1,\ell}(x)$ for $\ell \leq n+1$, with parameters by downward induction on ℓ as follows $\sigma_{n+1,n+1}(x) = x, \sigma_{n+1,\ell}(x) = \sigma_\ell(\sigma_{n+1,\ell+1}(x), \bar{b}_\ell)$ then the $\mathfrak{d}$-distance between $\sigma_{n+1,k}(c)$ and $\sigma_{n+1,k}(c')$ is $< \zeta_n$.

Let us carry the induction, in stage n we choose e_n, ζ_n and $\sigma_{n-1}(x, \bar{b}_{n-1})$ if $n > 0$.

Case 1: $n = 0$.
This is straightforward.

Case 2: $n = k+1$.
Let D be the set of $\bar{c} = \langle c_m : m \leq k \rangle$ which satisfies

$\boxtimes_k (i)$ $m < k \Rightarrow c_m = \sigma_m(c_{m+1}, h^*(\bar{b}_m))$
(ii) $c_0 = a_k$.

We can prove by induction on $m \leq k$ that $\{c_m : \bar{c} \in D\}$ is finite, and let $A = \{c_k : \bar{c} \in D\}$. By clause $(e)'$ of the assumptions (see 3.6(2)) applied with (A, ζ_k, e_k) here standing for (A, ζ, d) there, there are $r < \omega, \sigma = \sigma(x, y_0, \ldots, y_{r-1}) \in \Xi$ and $\bar{b} \in {}^r(G_{\mathfrak{a}})$ and d' as there. We let $\bar{b}_k = \bar{b}, \sigma_k = \sigma, \bar{y}_k = \langle y_0, \ldots, y_{r-1} \rangle, e_n = d'$.

Lastly, we should choose $\zeta_n \in \mathbb{R}^+$. There are several demands but each holds for every small enough $\zeta > 0$, more exactly one for clause (ε) and for each $m < n$, one for clause (ζ).

Having carried the induction, clearly 3.1 apply hence there is a solution $\langle d^*_n : n < \omega \rangle$, that is $G_{\mathfrak{a}} \models d^*_m = \sigma_m(d^*_{m+1}, \bar{b}_m)$ for $m < \omega$. But h^* is a homomorphism from $G_{\mathfrak{a}}$ into G so $\langle h^*(d^*_n) : n < \omega \rangle$ satisfies all the equations in $\boxtimes_k$ and $h^*(\bar{b}_n) = \bar{b}_n$ as $\bar{b}_n$ is a sequence from $B \subseteq G$ and $h^* \restriction G = \text{id}_G$; hence (recalling clause (e)$'$) by our choice in stage $n = k+1, h^*(d^*_0) \neq a_k$. As this holds for every k and $\{a_k : k < \omega\}$ list the elements of $G_{\mathfrak{a}}$ we are done. $\square_{3.6}$

3.8 Remark. 1) If we phrase algebraic compactness, it is preserved by taking reducts.
2) In a reasonable variant we can replace "B countable" by $\|B\| <$ cov(meagre); we hope to return to this elsewhere.
3) We can change the demand on Ξ: at most one solution in clause (e), A a singleton in clause (f). This suffices for groups.

§4 Conclusions

<u>4.1 Conclusion</u> 1) If $(G, \mathfrak{d})$ is a complete metric group of density $< |G|$, <u>then</u>:

- (a) G is not free,
- (b) if G is $\aleph_1$-free (see 4.2) then for some countable $A \subseteq G$, there is no countable reduct H of G including A, recalling 3.4.

2) It suffices that G is an indirectly complete metric group and as a metric space it is of density $< |G|$.
3) Instead "density" $< |G|$" it is enough to assume that the topology induces by the metric is not discrete.

Recall (see [EM02])

4.2 Definition. A group G is λ-free <u>when</u> every subgroup of G generated by $< \lambda$ elements is a gree group.

Proof. 1), 2) Clause (a) follows from clause (b) by Fact 3.5. Let $\mu = \text{density}(G)$ and $y_i \in G$ be pairwise distinct for $i < \mu^+$. Without loss of generality $y_i \notin \langle \{y_j : j < i\} \rangle_G$. So for some increasing sequence $\langle i_n : n < \omega \rangle$ the sequence $\langle y_{i_n} : n < \omega \rangle$ is a Cauchi sequence.

For part (1), by completeness it converges say to y^*, the convergence is for $\mathfrak{d}$. Now $\langle b_n : n < \omega \rangle =: \langle (y^*)^{-1} y_{i_{2n+1}} : n < \omega \rangle$ converges to e_G, the members are pairwise distinct so without loss of generality $\neq e_G$.

However for part (2) we know that some $(G, \mathfrak{d}')$, i.e. equal to $(G, \mathfrak{d})$ as a group but with a different metric; is a complete metric group with an ω-sequence $\langle b_n : n < \omega \rangle$ of members of $G \backslash \{e_G\}$ converging to e_G. Let $\mathfrak{d}' = \mathfrak{d}$.

Let $\Xi = \{x^m y : m < \omega\}$ and $B = \{b_n : n < \omega\}$ and $e_* = e_G$. Now we shall apply 3.6(1). In the assumptions, clauses (a)-(c) are obvious. As for clause (d) we are using: equations of the form $x^m a' = a''$ has at most one solution in G, see 3.3. We are left with clause (e), so we are given a real $\zeta > 0$ and a finite set $A \subseteq G$. We can choose $b \in ((B \backslash A) \backslash \{e_G\})$ of distance $< \zeta$ from e_G. Let $\sigma(x, y) = x^n y$ and $\bar{b} = \langle b \rangle$ where $n < \omega$ is the minimal $n > 1$ such that $[a \in A \Rightarrow ab^{-1}$ has no n-th root]. This is possible, see Fact 3.3. So first $\sigma(e_G, \bar{b}) = b \in \text{Ball}_{\mathfrak{d}'}(e_G, \zeta)$ as required in subclause (α) of clause (e). Second, if $a \in A$ then ab^{-1} has no n-th root so $c \in G \Rightarrow c^n \neq ab^{-1} \Rightarrow c^n b \neq a$ which means that $c \in G \Rightarrow \sigma(c, \bar{b}) \notin A$ as required in subclause (β) of clause (e) so by 3.6(1) we are done.
3) Choose $\langle y_n : n < \omega \rangle$ converging to some y^* such that $\langle y_n : n < \omega \rangle \hat{\ } \langle y^* \rangle$ is with no repetitions, possible on $(G, \mathfrak{d})$ is not discrete. Now continue as above. $\square_{4.1}$

In particular
4.3 Conclusion: There is no free uncountable Polish group.

4.4 Conclusion: 1) Assume $\mathbb{A}$ is a countable structure. Then $\mathrm{Aut}(\mathbb{A})$, the group of automorphisms of $\mathbb{A}$, is not a free uncountable group, in fact it satisfies the conclusions of 4.1.
2) Assume $\mathbb{A}$ is a structure of cardinality λ and $\lambda = \mu = \beth_\omega$ or more generally $\lambda = \Sigma\{\lambda_n : n < \omega\}, 2^{\lambda_n} < 2^{\lambda_{n+1}}, \mu = \Sigma\{2^{\lambda_n} : n < \omega\} < 2^\lambda$. Then $\mathrm{Aut}(\mathbb{A})$ cannot be free of cardinality $> \mu$, in fact, it satisfies the conclusions of 4.1.

Proof. 1) By 1.6, $\mathrm{Aut}(\mathbb{A})$ is a Polish group and apply 4.3.
2) Without loss of generality the universe of $\mathbb{A}$ is λ, using $\bar{A} = \langle \lambda_n : n < \omega \rangle$ we know by 2.3(1) that $(\mathrm{Aut}(\mathbb{A}), \mathfrak{d}_{\mathbb{A},\bar{A}})$ is an indirectly complete metric group and apply 4.1(2). $\square_{4.4}$

§5 Quite free but not free abelian groups

If uncountable Polish groups are not free, we may look at wider classes: F_σ, Borel analytic, projective, $\mathbf{L}[\mathbb{R}]$.

5.1 Question: 1) Is the freeness of a reasonably definable abelian group absolute?
2) For which cardinals λ does λ-freeness imply freeness (or λ^+-freeness) for nicely definable abelian groups, in particular for $\lambda = \aleph_\omega$?
3) Similarly for other varieties (or any case when "free" is definable, say for any universal Horn theory).

This is connected also to [Sh 402] whose original aim was a question of Marker "are there non-free Whitehead Borel Abelian groups". But already in [Sh 402] it seems to me the basic question is to clarify freeness in such groups; that is, question 5.1 above.
Blass asked about definable subgroups of $\mathbb{Z}^\omega$ (see question 5.10): by [Sh 402] and the construction here we quite resolve this.

Recall that [Sh 402] analyze $\aleph_1$-free abelian groups which are Σ^1_1 or so. A natural dividing line was suggested; the complicated half was proved to be not Whitehead, and at least for me is an analog to not $\aleph_0$-stable. The low half is $\aleph_2$-free. So under CH we were done, but what if $2^{\aleph_0} > \aleph_1$? Are they also free? This was left open by [Sh 402].

We shed some light by giving an example (an F_σ one) showing that the non-CH case in [Sh 402] is a real problem. This resolves the original problem:

$\boxplus$ (a) it is consistent that there are non-free Borel Whitehead groups
(b) it is consitent that there are no non-free Borel Whitehead groups.

[Why? Clause (a) is derived in 5.13 whereas by [Sh 402] we have clause (b), in fact derive it from CH.]

But what about the further question, e.g. 5.1(2)? The examples seem to indicate (at least to me) that the picture in [Sh 87a], [Sh 87b] is the right one here, connecting theories of $\psi \in \mathbb{L}_{\omega_1,\omega}$ with Σ^1_1-models. Also related are [EM2], [MkSh 366] on almost freeness for varieties, and see [EM02] on abelian groups. In particular we conjecture "every $\aleph_\omega$-free Borel group is free".

We shall use freely the well known theorem saying

$\boxtimes$ a subgroup of a free abelian group is a free abelian group.

5.2 Definition. For $k(*) < \omega$ we define an abelian group $G = G_{k(*)}$, it is generated by $\{x_{m,\bar\eta,\nu} : m \le k(*)$ and $\nu \in {}^{\omega>}2$ and $\bar\eta = \langle \eta_\ell : \ell \le k(*), \ell \ne m\rangle$ where $\eta_\ell \in {}^\omega 2\} \cup \{y_{\bar\eta,n} : n < \omega$ and $\bar\eta = \langle \eta_\ell : \ell \le k(*)\rangle$ where $\eta_\ell \in {}^\omega 2\}$ freely except the equations:

$$\begin{aligned}\boxtimes_{\bar\eta,n}\quad & (n!)y_{\bar\eta,n+1} = y_{\bar\eta,n} + \\ & \sum\{x_{m,\bar\eta_m,\nu} : m \le k(*) \text{ and } \bar\eta_m = \bar\eta \restriction \{m' \le k(*) : m' \ne m\} \text{ and} \\ & \qquad \nu = \eta_m \restriction n\}.\end{aligned}$$

(Note that if $m_1 < m_2 \le k(*)$ then $\bar\eta_{m_1} \ne \bar\eta_{m_2}$ having different index sets).

Explanation. A canonical example of a non-free group is $(\mathbb{Q}, +)$. Other examples are related to it after we divide by something. The y's here play that role of providing (hidden) copies of $\mathbb{Q}$. What about x's? For each $\bar\eta \in {}^{k(*)+1}({}^\omega 2)$ we use $m \le k(*)$ to give $\langle y_{\bar\eta,n} : n < \omega\rangle, k(*) + 1$ "chances", "opportunities" to avoid having $(\mathbb{Q}, +)$ as a quotient, one for each cardinal $\le \aleph_{k(*)}$. More specifically, if $H \subseteq G$ is the subgroup which is generated by $X = \{x_{m,\bar\eta,\nu} : m \ne m(*)$ and $\bar\eta$ is a function from $\{\ell \le k(*) : \ell \ne m\}$ to ${}^\omega 2$ and $\nu \in {}^{\omega>}2\}$, still in G/H the $\{y_{\bar\eta,n} : n < \omega\}$ does not generate a copy of $\mathbb{Q}$, as witnessed by $\{x_{m(*),\bar\eta_{m(*)},\eta_{m(*)}\restriction n} : n < \omega\}$.

5.3 Claim. *The abelian group $G_{k(*)}$ is a Borel group, even an F_σ-one, that is the set of elements and the graphs of $+$ and the function $x \mapsto -x$ (i.e. $\{(x, y, z) : G_{k(*)} \models$ "$x + y = z$"$\}$ hence also $\{(x, -x) : x \in G_{k(*)}\}$) are F_σ-sets; hence Borel.*

Proof. Let cd be a one-to-one function from the set of finite sequences of natural numbers onto the set of natural numbers and we define:

$\odot_1$ (a) $\mathrm{cd}(x_{m,\bar\eta,\nu}) = \langle \mathrm{cd}(\langle m, \mathrm{cd}(\nu), \dots, \eta_\ell(i), \dots\rangle_{\ell \le k(*), \ell \ne m} : i < \omega\rangle$ so it $\in {}^\omega\omega$ and let $\mathscr{X}_0 = \{\mathrm{cd}(x_{n,\bar\eta,\nu}) : (n, \bar\eta, \nu)$ as in Definition 5.2$\}$

(b) $\mathrm{cd}(y_{\bar\eta,n}) = \langle \mathrm{cd}(n, \dots, \eta_\ell(i) \dots)_{\ell \le k(*)} : i < \omega\rangle$ and $\mathscr{X}_1 = \{\mathrm{cd}(y_{\bar\eta,n}) : (\bar\eta, n)$ as in Definition 5.2$\}$

(c) for a sequence $\bar a = \langle a_\ell : \ell < n\rangle$ of integers let $\mathrm{abs}(\bar a) = \langle |a_\ell| : \ell < n\rangle$ and $\mathrm{sign}(\bar a) = \langle \mathrm{sign}(a_\ell): \ell < n\rangle$, where $\mathrm{sign}(a_\ell)$ is 0,1,2 if a_ℓ is negative, zero, positive respectively.

Clearly $\mathscr{X}_0, \mathscr{X}_1$ are disjoint. We say ν represents $x \in G_{k(*)}$ as witnessed by $\langle (z_\ell, a_\ell, m) : \ell < n\rangle$ <u>when</u>:

$\odot_2$ (a) $G \models x = \sum\limits_{\ell<n} a_\ell z_\ell$,

(b) $z_\ell \in \{x_{n,\bar\eta,\nu} : (n,\bar\eta,\nu)$ as in Definition 5.2$\} \cup \{y_{\bar\eta,m'} : (\bar\eta,m')$ as in Definition 5.2$\}$

(c) $\langle z_\ell : \ell < n\rangle$ is without repetitions

(d) $\langle \mathrm{cd}(z_\ell) \restriction m : \ell < n\rangle$ are pairwise distinct

(e) if $\langle (z'_\ell, a'_\ell, m') : \ell < n'\rangle$ satisfies clauses (a)-(d), then $m \le m'$

(f) if $n = 0$ then $m = 0$

(g) $\mathrm{cd}(z_0) <_{\mathrm{lex}} \mathrm{cd}(z_1) <_{\mathrm{lex}} \dots$ where $<_{\mathrm{lex}}$ is the lexicographic order on ${}^\omega\omega$

(h) $\nu = \langle \mathrm{cd}(\langle n\rangle\hat{}\ \mathrm{sign}(\bar a)\hat{}\ \mathrm{abs}(\bar a)\hat{}\langle \mathrm{cd}(z_\ell)(i) : \ell < n\rangle) : i < \omega\rangle$ where $\bar a = \langle a_\ell : \ell < n\rangle$, of course.

Now for $n < \omega, \bar a = \langle a_\ell : \ell < n\rangle \in {}^n(\mathbb{Z}\backslash\{0\}), i < \omega$ and $\bar\varrho = \langle \varrho_\ell : \ell < n\rangle \in {}^n({}^i\omega)$ is $<_{\mathrm{lex}}$-increasing hence without repetitions (and if $n = 0$ we let $i = 0$) we let

$$\begin{aligned} Z_{\bar a,\bar\varrho} = \{\nu :& \nu \text{ represent some } x \in G_{k(*)} \text{ as witnessed by} \\ & \langle (z_\ell, a_\ell, i) : \ell < n\rangle \text{ and } \mathrm{cd}(z_\ell) \restriction i = \varrho_\ell \text{ for } \ell < n\}. \end{aligned}$$

Let $\mathscr{Y}$ be the set of such pairs $(\bar a, \bar\varrho)$

$(*)_1$ $\langle Z_{\bar a,\bar\varrho} : (\bar a,\bar\varrho) \in \mathscr{Y}\rangle$ is a sequence of pairwise disjoint closed subsets of ${}^\omega\omega$

$(*)_2$ every member of G is represented by one and only one member of $Z := \cup\{Z_{\bar a,\bar\varrho} : (\bar a,\bar\varrho) \in \mathscr{Y}\}$.

[Why? For any $i < n$ clearly $\{x_{m,\bar\eta,\nu} : (m,\bar\eta,\nu)$ as in Definition 5.2$\} \cup \{y_{\bar\eta,i} : \bar\eta \in {}^{k(*)+1}({}^\omega 2)\}$ generates freely a subgroup $G'_{k(*),i}$ of $G_{k(*)}$.
Why? E.g. let G^* be a vector space over the field $\mathbb{Q}$ with basis $\{x^*_{m,\bar\eta,\nu} : (m,\bar\eta,\nu)$ as in Definition 5.2$\} \cup \{y^*_{\bar\eta,i} : \bar\eta \in {}^{k(*)+1}({}^\omega 2)\}$. We define a mapping h into G^* from the set of generators $\{x_{m,\bar\eta,\nu} : (m,\bar\eta,\nu)$ as in Definition 5.2$\} \cup \{y_{\bar\eta,j} : \bar\eta \in {}^{k(*)+1}({}^\omega 2)$ and $j < \omega\}$ as follows:

(a) $h(x_{m,\bar\eta,\nu}) = x^*_{m,\bar\eta,\nu}$

(b) $h(y_{\bar\eta,i}) = y^*_{\bar\eta,i}$

(c) we choose $h(y_{\bar\eta,j})$ for $j \in (i,\omega)$ by induction on j such that h maps $\boxtimes_{\bar\eta,j-1}$ to an equation satisfied in G^*

(d) we choose $h(y_{\bar\eta,j})$ for $j \in [0,i)$ by downward induction on j such that h maps $\boxtimes_{\bar\eta,j+1}$ to an equation satisfied in G^*.

So as h maps the set of generators of $G_{k(*)}$ into G^* preserving the equations in 5.2 we can extend h to a homomorphism from $G_{k(*)}$ into G^*.

Now the set which generates $G'_{k(*),i}$ is mapped in a one-to-one way, onto a basis of G^*, hence it generates $G'_{k(*),i}$ freely, as stated above.

Inspecting 5.2 clearly the quotient $G_{k(*)}/G'_{k(*),i}$ is torsion. The rest should be clear, too.]

- $(*)_3$ $\mathscr{U} = \{(\nu_1,\nu_2,\nu_3) : \nu_\ell$ represent $x_\ell \in G_{k(*)}$ for $\ell = 1,2,3$ and $G_{k(*)} \models$ "$x_1 + x_2 = x_3$"$\}$ is the graph of a three-place function
- $(*)_4$ for any $(\bar a_\ell, \bar\varrho_\ell) \in \mathscr{Y}$ for $\ell = 1,2,3$ the set $\{(\nu_1,\nu_2,\nu_3) \in \mathscr{U} : \nu_\ell \in Z_{\bar a_\ell,\bar\varrho_\ell}$ for $\ell = 1,2,3\}$ is a closed set.

Clearly we are done. $\square_{5.3}$

As a warm up we note:

5.4 Claim. *$G_{k(*)}$ is an $\aleph_1$-free abelian group.*

Proof. Let $U \subseteq {}^\omega 2$ be countable (and infinite) and define G'_U like G restricting ourselves to $\eta_\ell \in U$; by the Löwenheim-Skolem-Tarski argument it suffices to prove that G'_U is a free abelian group. List ${}^{k(*)+1}U$ without repetitions as $\langle \bar\eta_t : t < \omega\rangle$, and by induction on $t < \omega$ choose $s_t < \omega$ such that $[r < t \ \ \& \ \ \bar\eta_r \restriction k(*) = \bar\eta_t \restriction k(*) \Rightarrow \emptyset = \{\eta_{t,k(*)} \restriction \ell : \ell \in [s_t,\omega)\} \cap \{\eta_{r,k(*)} \restriction \ell : \ell \in [s_r,\omega)\}]$.
Let

$$Y_1 = \{x_{m,\bar\eta,\nu} : m < k(*), \bar\eta \in {}^{k(*)+1\setminus\{m\}}U \text{ and } \nu \in {}^{\omega>}2\}$$

$$Y_2 = \Big\{x_{m,\bar\eta,\nu} : m = k(*), \bar\eta \in {}^{k(*)}U \text{ and for no } t < \omega \text{ do we have } \bar\eta = \bar\eta_t \restriction k(*) \ \ \& \ \ \nu \in \{\eta_{t,k(*)} \restriction \ell : s_t \le \ell < \omega\}\Big\}$$

$$Y_3 = \{y_{\bar\eta_t,n} : t < \omega \text{ and } n \in [s_t,\omega)\}.$$

Now

- $(*)_1$ $Y_1 \cup Y_2 \cup Y_3$ generates G'_U.

[Why? Let G' be the subgroup of G'_U which $Y_1 \cup Y_2 \cup Y_3$ generates. First we prove by induction on $n < \omega$ that for $\bar\eta \in {}^{k(*)}U$ and $\nu \in {}^n 2$ we have $x_{k(*),\bar\eta,\nu} \in G'$. If

$x_{k(*),\bar{\eta},\nu} \in Y_2$ this is clear; otherwise, by the definition of Y_2 for some $\ell < \omega$ and $t < \omega$ such that $\ell \geq s_t$ we have $\bar{\eta} = \bar{\eta}_t \restriction k(*), \nu = \eta_{t,k(*)} \restriction \ell$.

Now

(a) $y_{\bar{\eta}_t,\ell+1}, y_{\bar{\eta}_t,\ell}$ are in $Y_3 \subseteq G'$.

Hence by the equation $\boxtimes_{\bar{\eta},n}$ in Definition 5.2, clearly $x_{k(*),\bar{\eta},\nu} \in G'$. So as $Y_1 \subseteq G' \subseteq G'_U$, all the generators of the form $x_{m,\bar{\eta},\nu}$ with each $\eta_\ell \in U$ are in G'. Also we have

(b) $x_{m,\bar{\eta}_t \restriction \{i \leq k(*), i \neq m\},\nu}$ belong to $Y_1 \subseteq G'$ if $m < k(*)$.

Now for each $t < \omega$ we prove that all the generators $y_{\bar{\eta}_t,n}$ are in G'. If $n \geq s_t$ then clearly $y_{\bar{\eta}_t,n} \in Y_3 \subseteq G'$. So it suffices to prove this for $n \leq s_t$ by downward induction on n; for $n = s_t$ by an earlier sentence, for $n < s_t$ by $\boxtimes_{\bar{\eta},n}$. The other generators are in this subgroup so we are done.]

$(*)_2$ $Y_1 \cup Y_2 \cup Y_3$ generates G'_U freely.

[Why? Translate the equations.

Alternatively, let $\bar{z} = \langle z_\alpha : \alpha < \alpha(*) \rangle$ list the set $\{x_{m,\bar{\eta},\nu} : (m, \bar{\eta}, \nu)$ as in Definition 5.2 but $\bar{\eta} \in {}^{k(*)+1}U\} \cup \{y_{\bar{\eta},n} : \bar{\eta} \in {}^{k(*)+1}U$ and $n < \omega\}$ which generates G'_U and $\bar{z}$ lists it without repetitions such that for some increasing continuous sequence $\langle \alpha_i : i \leq \omega + \omega \rangle$ we have $\alpha_0 = 0, \alpha_{\omega+\omega} = \alpha(*)$ and

(a) $\{z_\alpha : \alpha < \alpha_1\} = Y_1 \cup Y_2 \cup Y_3$

(b) $\{z_\alpha : \alpha \in [\alpha_{1+n}, \alpha_{1+n+1})\} = \{x_{k(*),\bar{\eta},\nu} \in G'_U : x_{k(*),\bar{\eta},\nu} \notin Y_2$ and $\ell g(\nu) = n\}$

(c) $\{z_\alpha : \alpha \in [\alpha_{\omega+r}, \alpha_{\omega+r+1})\} = \{y_{\bar{\eta}_t,n} : t < \omega, n < s_t$ and $r = s_t - n\}$.

Now the proof above shows that:

⊛ there is a one-to-one function from the set Ξ of equations defining G'_U onto $[\alpha_1, \alpha_{\omega+\omega})$ such that:
if the equation φ is mapped to the ordinal α then: if z_β appears in the equation then $\beta \leq \alpha$ and z_α appears in the equation and its coefficient is 1 or -1.

This clearly suffices.] $\square_{5.4}$

Now systematically

5.5 Definition. 1) For $U \subseteq {}^\omega 2$ let G_U be the subgroup of G generated by $Y_U = \{y_{\bar\eta,n} : \bar\eta \in {}^{k(*)+1}(U)$ and $n < \omega\} \cup \{x_{m,\bar\eta,\nu} : m \le k(*)$ and $\bar\eta \in {}^{(k(*)+1)\backslash\{m\}}(U)$ and $\nu \in {}^{\omega>}2\}$. Let G^+_U be the divisible hull of G_U and $G^+ = G^+_{({}^\omega 2)}$.
2) For $U \subseteq {}^\omega 2$ and finite $u \subseteq {}^\omega 2$ let $G_{U,u}$ be the subgroup[3] of G generated by $\cup\{G_{U \cup (u\backslash\{\eta\})} : \eta \in u\}$; and for $\bar\eta \in {}^{k(*)+1\ge}U$ let $G_{U,\bar\eta}$ be the subgroup of G generated by $\cup\{G_{U\cup\{\eta_k : k<\ell g(\bar\eta) \text{ and } k \ne \ell\}} : \ell < \ell g(\bar\eta)\}$.
3) For $U \subseteq {}^\omega 2$ let $\Xi_U = \{$the equation $\boxtimes_{\bar\eta,n} : \bar\eta \in {}^{k(*)+1}U$ and $n < \omega\}$. Let $\Xi_{U,u} = \cup\{\Xi_{U\cup(u\backslash\{\beta\})} : \beta \in u\}$.

5.6 Claim. *0) If $U_1 \subseteq U_2 \subseteq {}^\omega 2$ <u>then</u> $G^+_{U_1} \subseteq G^+_{U_2} \subseteq G^+$.*
1) For any $n(*) < \omega$, the abelian group G^+_U (which is a vector space over $\mathbb{Q}$), has the basis $Y^{n(*)}_U := \{y_{\bar\eta,n(*)} : \bar\eta \in {}^{k(*)+1}(U)\} \cup \{x_{m,\bar\eta,\nu} : m \le k(*), \bar\eta \in {}^{(k(*)+1)\backslash\{m\}}(U)$ and $\nu \in {}^{\omega>}2\}$.
2) For $U \subseteq {}^\omega 2$ the abelian group G_U is generated by Y_U freely (as an abelian group) except the set Ξ_U of equations.
3) If $U_m \subseteq {}^\omega 2$ for $m < m(*)$ <u>then</u> the subgroup $G_{U_0} + \ldots + G_{U_{m(*)-1}}$ of G is generated by $Y_{U_0} \cup Y_{U_1} \cup \ldots \cup Y_{U_{m(*)-1}}$ freely (as an abelian group) except the equations in $\Xi_{U_0} \cup \Xi_{U_1} \cup \ldots \cup \Xi_{U_{m(*)-1}}$. Also $G_{U_0} + \ldots + G_{U_{m(*)-1}}$ is a direct summand of G when

> $\circledast$ *if $\eta_0, \ldots, \eta_{k(*)} \in \cup\{U_m : m < m(*)\}$ such that*
> $(\forall \ell \le k(*))(\exists m < m(*))[\{\eta_0, \ldots, \eta_{k(*)}\}\backslash\{\eta_\ell\} \subseteq U_m)$
> <u>then</u> *for some $m < m(*)$ we have $\{\eta_0, \ldots, \eta_{k(*)}\} \subseteq U_m$.*

4) If $U_\ell = U\backslash U'_\ell$ for $\ell < m(*) \le k(*)$ and $\langle U'_\ell : \ell < m(*)\rangle$ are pairwise disjoint then $\circledast$ holds.
5) $G_{U,u} \subseteq G_{U\cup u}$ if $U \subseteq {}^\omega 2$ and $u \subseteq {}^\omega 2\backslash U$; moreover $G_{U,u} \subseteq_{\rm pr} G_{U\cup u} \subseteq_{\rm pr} G$.
6) If $\langle U_\alpha : \alpha < \alpha(*)\rangle$ is $\subseteq$-increasing continuous <u>then</u> also $\langle G_{U_\alpha} : \alpha < \alpha(*)\rangle$ is $\subseteq$-increasing continuous.
7) If $U_1 \subseteq U_2 \subseteq U \subseteq {}^\omega 2$ and $u \subseteq {}^\omega 2\backslash U$ is finite, $|u| < k(*)$ and $U_2\backslash U_1 = \{\eta\}$ and $v = u \cup \{\eta\}$ <u>then</u> $(G_{U,u} + G_{U_2\cup u})/(G_{U,u} + G_{U_1\cup u})$ is isomorphic to $G_{U_1\cup v}/G_{U_1,v}$.
8) If $U \subseteq {}^\omega 2$ and $u \subseteq {}^\omega 2\backslash U$ has $\le k(*)$ members <u>then</u> $(G_{U,u} + G_u)/G_{U,u}$ is isomorphic to $G_u/G_{\emptyset,u}$.

Proof. 0) Obvious.
1) See the proof of $(*)_2$ inside the proof of 5.3.
2),3),4) Follows.

[3]note that if $u = \{\eta\}$ then $G_{U,u} = G_U$

5) Note that without loss of generality $|u| \geq 2$. First, $G_{U,u} \subseteq G_{U \cup u}$ follows by the definition. Second, we deal with proving $G_{U,u} \subseteq_{\rm pr} G_{U \cup u}$. So let $|u| = m(*)+1$ and $\langle \eta_\ell : \ell \leq m(*) \rangle$ list u, necessarily with no repetitions and let $U_\ell = U \cup (u \backslash \{\eta_\ell\})$ (so $G_{U,u} = G_{U_0} + \ldots + G_{U_{m(*)}}$) and assume $z \in G_{U \cup u}, a \in \mathbb{Z} \backslash \{0\}$ and az belongs to $G_{U_0} + \ldots + G_{U_{m(*)}}$ so it has the form $\Sigma\{b_i x_{m_i,\bar\eta_i,\nu_i} : i < i(*)\} + \Sigma\{c_j y_{\bar\rho_j,n_j} : j < j(*)\}$ with $b_i, c_j \in \mathbb{Z}$ and $\bar\eta_i, \bar\rho_j$ are (finite) sequences of members of $U_{\ell(i)}, U_{k(j)}$ respectively and are as required in Definition 5.2 where $\ell(i), k(j) \leq m(*)$.

Now similarly as $z \in G_{U \cup u}$, we can find $z = \Sigma\{b'_i x_{m'_i,\bar\eta'_i,\nu'_i} : i < i'(*)\} + \Sigma\{c'_j y_{\bar\rho'_j,n'_j} : j < j'(*)\}$.

By the equations in Definition 5.2 without loss of generality for some $n(*)$ we have: $j < j(*) \Rightarrow n_j = n(*)$ and $j < j'(*) \Rightarrow n'_j = n(*)$. Also without loss of generality each of the sequences $\langle (m_i, \bar\eta_i, \nu_i) : i < i(*) \rangle, \langle \bar\rho_j : j < j(*) \rangle$ is with no repetitions, and also in $\langle (m'_i, \bar\eta'_i, \nu'_i) : i < i'(*) \rangle, \langle \bar\rho'_j : j < j'(*) \rangle$ there is no repetition (for $\langle \bar\rho_j : j < j(*) \rangle$ and $\langle \bar\rho'_j : j < j'(*) \rangle$ we use $n_j = n(*), n'_j = n(*)$).

Together

$\circledast$ $\Sigma\{b_i x_{m_i,\bar\eta_i,\nu_i} : i < i(*)\} + \Sigma\{c_j y_{\bar\rho_j,n(*)} : j < j(*)\} = \Sigma\{ab'_i x_{m'_i,\bar\eta'_i,\nu_i} : i < i'(*)\} + \Sigma\{ac'_j y_{\bar\rho'_j,n(*)} : j < j'(*)\}$.

Now this equation holds in $G_{U \cup u}$ hence is G and even in G^+. By part (1) and the "no repetitions", after possible permuting we get $i(*) = i'(*), j(*) = j'(*), (m_i, \bar\eta_i, \nu_i) = (m'_i, \bar\eta'_i, \nu'_i), b_i = ab'_i$ for $i < i(*), \bar\rho_j = \bar\rho'_j$ for $j < j(*), c_j = ac'_j$ for $j < j(*)$. But this proves that $\{x_{m'_i,\bar\eta'_i,\nu_i} : i < i'(*)\} \cup \{y_{\bar\rho'_j,n(*)} : j < j'(*)\} \subseteq G_{U,u}$ hence $z \in G_{U,u}$ as required.

Third, the proof of $G_{U \cup u} \subseteq_{\rm pr} G$ is similar.

6) Easy.

7) Clearly $U_1 \cup v = U_2 \cup u$ hence $G_{U_1 \cup u} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u}$ hence $G_{U,u} + G_{U_1 \cup u}$ is a subgroup of $G_{U,u} + G_{U_2 \cup u}$, so the first quotient makes sense.

Hence by an isomorphism theorem $(G_{U,u} + G_{U_2 \cup u})/(G_{U,u} + G_{U_1 \cup u})$ is isomorphic to $G_{U_2 \cup u}/(G_{U_2 \cup u} \cap (G_{U,u} + G_{U_1,u}))$. Now $G_{U_1,v} \subseteq G_{U_1 \cup v} = G_{U_2 \cup u}$ and $G_{U_1,v} = \Sigma\{G_{U_1 \cup (v \backslash \{\nu\})} : \nu \in v\} = \Sigma\{G_{U_1 \cup (v \backslash \{\nu\})} : \nu \in u\} + G_{U_1 \cup (v \backslash \{\eta\})} \subseteq G_{U,u} + G_{U_1 \cup u}$. Together $G_{U_1,v}$ is included in their intersection, i.e. $G_{U_2 \cup u} \cap (G_{U,u} + G_{U_1 \cup u})$ include $G_{U_1,v}$ and using part (1) both has the same divisible hull inside G^+. But $G_{U_1,v}$ is a pure subgroup of G by part (5) hence of $G_{U_1 \cup v}$. Hence necessarily $G_{U_2 \cup u} \cap (G_{U,u} + G_{U_1 \cup u}) = G_{U_1,v}$, so as $G_{U_2 \cup u} = G_{U_1 \cup v}$ we are done.

8) The proof is similar to the proof of part (7). Note that $G_{U,u} \subseteq G_{U,u} + G_u$ hence the first quotient makes sense. So by an isomorphism theorem $(G_{U,u} + G_u)/G_{U,u}$ is isomorphic to $G_u/(G_{U,u} \cap G_u)$. Now $G_{U,u} \cap G_u$ includes $G_{\emptyset,u}$ and using part (1) both has the same divisible hull inside G^+. But $G_{\emptyset,u}$ is a pure subgroup of G_u by part (5). So necessarily $G_{U,u} \cap G_u = G_{\emptyset,u}$, so $G_u/(G_{U,u} \cap G_u) = G_u/G_{\emptyset,u}$, so we are done. $\square_{5.6}$

Discussion: For the reader's convenience we write what the group $G_{k(*)}$ is for the case $k(*) = 0$. So, omitting constant indexes and replacing sequences of length one by the unique entry we get that it is generated by $y_{\eta,n}$ (for $\eta \in {}^\omega 2, n < \omega$) and x_ν (for $\nu \in {}^{\omega>}2$) freely as an abelian group except the equations $(n!)y_{\eta,n+1} = y_{\eta,n} + x_{\eta\restriction n}$. Note that if K is the countable subgroup generated by $\{x_\nu : \nu \in {}^{\omega>}2\}$ then G/K is a divisible group of cardinality continuum hence G is not free. So G is $\aleph_1$-free but not free.

Now we have the main proof

5.7 Main Claim. *1) The abelian group $G_{U\cup u}/G_{U,u}$ is free if $U \subseteq {}^\omega 2, u \subseteq {}^\omega 2\backslash U$ and $1 \le |u| \le k \le k(*)$ and $|U| \le \aleph_{k(*)-k}$.*
2) If $U \subseteq {}^\omega 2$ and $|U| \le \aleph_{k(*)}$, then G_U is free.

Proof. 1) We prove this by induction on $|U|$; without loss of generality $|u| = k$ as also $k' = |u|$ satisfies the requirements.

Case 1: U is countable.

So let $\{\nu^*_\ell : \ell < k\}$ list u be with no repetitions, now if $k = 0$, i.e. $u = \emptyset$ then $G_{U\cup u} = G_U = G_{U,u}$ so the conclusion is trivial. Hence we assume $u \ne \emptyset$, and let $u_\ell := u\backslash\{\nu^*_\ell\}$ for $\ell < k$.

Let $\langle \bar\eta_t : t < t^* \le \omega\rangle$ list with no repetitions the set $\Lambda_{U,u} := \{\bar\eta \in {}^{k(*)+1}(U \cup u)$: for no $\ell < k$ does $\bar\eta \in {}^{k(*)+1}(U \cup u_\ell)\}$. Now comes a crucial point: let $t < t^*$, for each $\ell < k$ for some $r_{t,\ell} \le k(*)$ we have $\eta_{t,r_{t,\ell}} = \nu^*_\ell$ by the definition of $\Lambda_{U,u}$, so $|\{r_{t,\ell} : \ell < k\}| = k < k(*)+1$ hence for some $m_t \le k(*)$ we have $\ell < k \Rightarrow r_{t,\ell} \ne m_t$ so for each $\ell < k$ the sequence $\bar\eta_t \restriction (k(*)+1\backslash\{m_t\})$ is not from $\{\langle \rho_s : s \le k(*)$ and $s \ne m_t\rangle : \rho_s \in (U \cup u_\ell)$ for every $s \le k(*)$ such that $s \ne m_t\}$.

For each $t < t^*$ we define $S(t) = \{m \le k(*) : \{\eta_{t,s} : s \le k(*) \ \ \& \ \ s \ne m\}$ is included in $U \cup u_\ell$ for no $\ell < k\}$. So $m_t \in S(t) \subseteq \{0,\dots,k(*)\}$ and $m \in S(t) \Rightarrow \bar\eta_t \restriction \{j \le k(*) : j \ne m\} \notin (U \cup u_\ell)$ for every $\ell < k$. For $m \le k(*)$ let $\bar\eta'_{t,m} := \bar\eta_t \restriction \{j \le k(*) : j \ne m\}$ and $\bar\eta'_t := \bar\eta'_{t,m_t}$. Now we can choose $s_t < \omega$ by induction on t such that

- $(*)$ if $t_1 < t, m \le k(*)$ and $\bar\eta'_{t_1,m} = \bar\eta'_{t,m}$, then $\eta_{t,m} \restriction s_t \notin \{\eta_{t_1,m} \restriction \ell : \ell < \omega\}$.

Let $Y^* = \{x_{m,\bar\eta,\nu} \in G_{U\cup u} : x_{m,\bar\eta,\nu} \notin G_{U\cup u_\ell}$ for $\ell < k\} \cup \{y_{\bar\eta,n} \in G_{U\cup u} : y_{\bar\eta,n} \notin G_{U\cup u_\ell}$ for $\ell < k\}$.
Let

$$Y_1 = \{x_{m,\bar\eta,\nu} \in Y^*\text{: for no } t < t^* \text{ do we have } m = m_t \ \ \& \ \ \bar\eta = \bar\eta'_t\}.$$

$$\begin{aligned} Y_2 = \{x_{m,\bar\eta,\nu} \in Y^* : & x_{m,\bar\eta,\nu} \notin Y_1 \text{ but for no} \\ & t < t^* \text{ do we have } m = m_t \ \& \ \bar\eta = \bar\eta'_t \ \& \\ & \eta_{t,m_t} \restriction s_t \trianglelefteq \nu \triangleleft \eta_{t,m_t}\} \end{aligned}$$

$$Y_3 = \{y_{\bar\eta,n} : y_{\bar\eta,n} \in Y^* \text{ and } n \in [s_t,\omega) \text{ for the } t < t^* \text{ such that } \bar\eta = \bar\eta_t\}.$$

Now the desired conclusion follows from

$(*)_1$ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U\cup u}/G_{U,u}$

$(*)_2$ $\{y + G_{U,u} : y \in Y_1 \cup Y_2 \cup Y_3\}$ generates $G_{U\cup u}/G_{U,u}$ freely.

Proof of $(*)_1$. It suffices to check that all the generators of $G_{U\cup u}$ belong to $G'_{U\cup u} =: \langle Y_1 \cup Y_2 \cup Y_3 \cup G_{U,u}\rangle_G$.

First consider $x = x_{m,\bar\eta,\nu}$ where $\eta \in {}^{k(*)+1\setminus\{m\}}(U \cup u), m \le k(*)$ and $\nu \in {}^n2$ for some $n < \omega$. If $x \notin Y^*$ then $x \in G_{U,u_\ell}$ for some $\ell < k$ but $G_{U\cup u_\ell} \subseteq G_{U,u} \subseteq G'_{U\cup u}$ so we are done, hence assume $x \in Y^*$. If $x \in Y_1 \cup Y_2 \cup Y_3$ we are done so assume $x \notin Y_1 \cup Y_2 \cup Y_3$. As $x \notin Y_1$ for some $t < t^*$ we have $m = m_t$ & $\bar\eta = \eta'_t$. As $x \notin Y_2$, clearly for some t as above we have $\eta_{t,m_t} \restriction s_t \trianglelefteq \nu \triangleleft \eta_{t,m_t}$ let $n = \ell g(\nu)$ so $s_t \le n$. Hence by Definition 5.2 the equation $\boxtimes_{\bar\eta_t,n}$ from Definition 5.2 holds, now $y_{\bar\eta_t,n}, y_{\bar\eta_t,n+1} \in Y_3 \subseteq G'_{U\cup u}$. So in order to deduce from the equation that $x = x_{m,\bar\eta,\nu} = x_{m_t,\bar\eta'_t,\eta_{t,m_t}\restriction n}$ belongs to $G'_{U\cup u}$, it suffices to show that $x_{j,\bar\eta'_{t,j},\eta_{t,j}\restriction n} \in G'_{U\cup u}$ for each $j \le k(*), j \ne m_t$. But each such $x_{j,\bar\eta'_{t,j},\eta_{t,j}\restriction n}$ belong to $G'_{U\cup u}$ as it belongs to $Y_1 \cup Y_2$.
[Why? Otherwise necessarily for some $r < t^*$ we have $j = m_r, \bar\eta'_{t,j} = \bar\eta'_{r,m_r}$ and $\eta_{r,m_r} \restriction s_r \trianglelefteq \eta_{t,j} \restriction n \triangleleft \eta_{r,m_r}$ so $n \ge s_r$ and as said above $n \ge s_t$. Clearly $r \ne t$ as $m_r = j \ne m_t$, now as $\bar\eta'_{t,m_r} = \bar\eta'_{r,m_r}$ and $\bar\eta_t \ne \bar\eta_r$ (as $t \ne r$) clearly $\eta_{t,m_r} \ne \eta_{r,m_r}$. Also $\neg(r < t)$ by $(*)$ above applied with r, t here standing for t_1, t there as $\eta_{r,m_r} \restriction s_r \trianglelefteq \eta_{t,j} \restriction n \triangleleft \eta_{r,m_r}$. Lastly for if $t < r$, again $(*)$ applied with t, r here standing for t_1, t there as $n \ge m_t$ gives contradiction.]
So indeed $x \in G'_{U\cup u}$.

Second consider $y = y_{\bar\eta,n} \in G_{U\cup u}$, if $y \notin Y^*$ then $y \in G_{U,u} \subseteq G'_{U\cup u}$, so assume $y \in Y^*$. If $y \in Y_3$ we are done, so assume $y \notin Y_3$, so for some $t, \bar\eta = \bar\eta_t$ and $n < s_t$. We prove by downward induction on $s \le s_t$ that $y_{\bar\eta,s} \in G'_{U\cup u}$, this clearly suffices. For $s = s_t$ we have $y_{\bar\eta,s} \in Y_3 \subseteq G'_{U\cup u}$; and if $y_{\bar\eta,s+1} \in G'_{U\cup u}$ use the equation $\boxtimes_{\bar\eta_t,s}$ from 5.2, in the equation $y_{\bar\eta,s+1} \in G'_{U\cup u}$ and the x's appearing in the equation belong to $G'_{U\cup u}$ by the earlier part of the proof (of $(*)_1$) so necessarily $y_{\bar\eta,s} \in G'_{U\cup u}$, so we are done.

Proof of $(*)_2$. We rewrite the equations in the new variables recalling that $G_{U\cup u}$ is generated by the relevant variables freely except the equations of $\boxtimes_{\bar\eta,n}$ from Definition 5.2. That is, let $\langle z_\alpha : \alpha < \alpha(*)\rangle$ list Y^* without repetitions such that for some $\le$-increasing continuous sequence $\langle \alpha_i : i \le \omega+\omega\rangle$ of ordinals we have:

- (a) $\alpha_0 = 0, \alpha_{\omega+\omega} = \alpha(*)$
- (b) $\{z_\alpha : \alpha < \alpha_1\} = Y_1 \cup Y_2 \cup Y_3$
- (c) $\{z_\alpha : \alpha \in [\alpha_{1+n}, \alpha_{1+n+1})\} = \{x_{m,\bar\eta,\nu} \in Y^* : x_{\bar m,\bar\eta,\nu} \notin Y_1 \cup Y_2$ and ν is of lenth $n\}\}$
- (d) $\{z_\alpha : \alpha \in [\alpha_{\omega+n}, \alpha_{\omega+r+1})\} = \{y_{\bar\eta,n} \in Y^* : y_{\bar\eta,n} \notin Y_3$ and $r = s_t - n\}$.

Clearly

- $\circledast$ there is a one-to-one function from the set $\Xi_{U\cup u} \setminus \cup\{\Xi_{U\cup u\setminus\{\nu\}} : \nu \in u\}$ into $[\alpha_1, \alpha_{\omega+\omega})$ such that:
  if the equation φ is mapped to the ordinal α then if z_β appears in the equation then $\beta < \alpha$ and z_α appears in the equation with coefficient one or minus one.

This clearly suffices.

<u>Case 2</u>: U is uncountable.

As $\aleph_1 \le |U| \le \aleph_{k(*)-k}$, necessarily $k < k(*)$.

Let $U = \{\rho_\alpha : \alpha < \mu\}$ where $\mu = |U|$, list U with no repetitions. Now for each $\alpha \le |U|$ let $U_\alpha := \{\rho_\beta : \beta < \alpha\}, u_\alpha = u \cup \{\rho_\alpha\}$. Now

- $\odot_1$ $\langle (G_{U,u} + G_{U_\alpha \cup u})/G_{U,u} : \alpha < |U|\rangle$ is an increasing continuous sequence of subgroups of $G/G_{U,u}$
  [Why? By 5.6(6).]
- $\odot_2$ $G_{U,u} + G_{U_0\cup u}/G_{U,u}$ is free.
  [Why? This is $(G_{U,u} + G_{\emptyset\cup u})/G_{U,u} = (G_{U,u} + G_u)/G_{U,u}$ which by 5.6(8) is isomorphic to $G_u/G_{\emptyset,u}$ which is free by Case 1.]

Hence it suffices to prove that for each $\alpha < |U|$ the group $(G_{U,u} + G_{U_{\alpha+1}\cup u})/(G_{U,u} + G_{U_\alpha\cup u})$ is free. But easily

- $\odot_3$ this group is isomorphic to $G_{U_\alpha \cup u_\alpha}/G_{U_\alpha,u_\alpha}$.
  [Why? By 5.6(7) with $U_\alpha, U_{\alpha+1}, U, \rho_\alpha, u$ here standing for U_1, U_2, U, η, u there.]
- $\odot_4$ $G_{U_\alpha\cup u_\alpha}/G_{U_\alpha,u_\alpha}$ is free.
  [Why? By the induction hypothesis, as $\aleph_0 + |U_\alpha| < |U| \le \aleph_{k(*)-(k+1)}$ and $|u_\alpha| = k+1 \le k(*)$.]

2) If $k(*) = 0$ just use 5.4, so assume $k(*) \geq 1$. Now the proof is similar to (but easier than) the proof of case (2) inside the proof of part (1) above.

$\square_{5.7}$

5.8 Claim. *If $U \subseteq {}^\omega 2$ and $|U| \geq \aleph_{k(*)+1}$ <u>then</u> G_U is not free.*

Proof. Assume toward contradiction that G_U is free and let χ be large enough; for notational simplicity assume $|U| = \aleph_{k(*)+1}$. O.K. as a subgroup of a free abelian group is a free abelian group. We choose N_ℓ by downward induction on $\ell \leq k(*)$ such that

(a) N_ℓ is an elementary submodel[4] of $(\mathscr{H}(\chi), \in, <^*_\chi)$

(b) $\|N_\ell\| = |N_\ell \cap \aleph_{k(*)}| = \aleph_\ell$ and $\aleph_\ell + 1 \subseteq N_\ell$

(c) $G_U \in N_\ell$ and $N_{\ell+1}, \ldots, N_{k(*)} \in N_\ell$.

Let $G_\ell = G_U \cap N_\ell$, a subgroup of G_U. Now

$(*)_0$ $G_U/(\Sigma\{G_\ell : \ell \leq k(*)\})$ is a free (abelian) group.
[Easy. Let $Z \subset G_U$ be the set of free generators. Without loss of generality $Z \in N_1$. Then $Z_\ell = Z \cap N_\ell$ is the set of free generators of $G_U \cap N_\ell = G_\ell$. $\bigcup\limits_\ell Z_\ell$ is the set of free generators of ΣG_ℓ. So $G_U/\Sigma G_\ell$ is free.]

Now

$(*)_1$ letting U^1_ℓ be U for $\ell = k(*) + 1$ and $\bigcap\limits_{m=\ell}^{k(*)} (N_m \cap U)$ for $\ell \leq k(*)$; we have: U^1_ℓ has cardinality $\aleph_\ell$ for $\ell \leq k(*) + 1$
[Why? By downward induction on ℓ. For $\ell = k(*) + 1$ this holds by an assumption. For $\ell = k(*)$ this holds by clause (b). For $\ell < k(*)$ this holds by the choice of N_ℓ as the set $\bigcap\limits_{m=\ell+1}^{k(*)} (N_m \cap U)$ has cardinality $\aleph_{\ell+1} \geq \aleph_\ell$ and belong to N_ℓ and clause (b) above.]

$(*)_2$ $U^2_\ell =: U^1_{\ell+1} \backslash (N_\ell \cap U)$ has cardinality $\aleph_{\ell+1}$ for $\ell \leq k(*)$
[Why? As $|U^1_{\ell+1}| = \aleph_{\ell+1} > \aleph_\ell = \|N_\ell\| \geq |N_\ell \cap U|$.]

[4] $\mathscr{H}(\chi)$ is $\{x$: the transitive closure of x has cardinality $< \chi\}$ and $<^*_\chi$ is a well ordering of $\mathscr{H}(\chi)$

$(*)_3$ for $m < \ell \le k(*)$ the set $U^3_{m,\ell} =: U^2_\ell \cap \bigcap\limits_{r=m}^{\ell-1} N_r$ has cardinality $\aleph_m$
[Why? By downward induction on m. For $m = \ell - 1$ as $U^2_\ell \in N_m$ and $|U^2_\ell| = \aleph_{\ell+1}$ and clause (b). For $m < \ell$ similarly.]

Now for $\ell = 0$ choose $\eta^*_\ell \in U^2_\ell$, possible by $(*)_2$ above. Then for $\ell > 0, \ell \le k(*)$ choose $\eta^*_\ell \in U^3_{0,\ell}$. This is possible by $(*)_3$. So clearly

$(*)_4$ $\eta^*_\ell \in U$ and $\eta^*_\ell \in N_m \cap U \Leftrightarrow \ell \ne m$ for $\ell, m \le k(*)$.
[Why? If $\ell = 0$, then by its choice, $\eta^*_\ell \in U^2_\ell$, hence by the definition of U^2_ℓ in $(*)_2$ we have $\eta^*_\ell \notin N_\ell$, and $\eta^*_\ell \in U^1_{\ell+1}$ hence $\eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)}$ by $(*)_1$ so $(*)_4$ holds for $\ell = 0$. If $\ell > 0$ then by its choice, $\eta^*_\ell \in U^3_{0,\ell}$ but $U^3_{m,\ell} \subseteq U^2_\ell$ by $(*)_3$ so $\eta^*_\ell \in U^2_\ell$ hence as before $\eta^*_\ell \in N_{\ell+1} \cap \ldots \cap N_{k(*)}$ and $\eta^*_\ell \notin N_\ell$. Also by $(*)_3$ we have $\eta^*_\ell \in \bigcap\limits_{r=0}^{\ell-1} N_\ell$ so $(*)_4$ really holds.]

Let $\bar\eta^* = \langle \eta^*_\ell : \ell \le k(*)\rangle$ and let G' be the subgroup of G_U generated by $\{x_{m,\bar\eta,\nu} : m \le k(*)$ and $\bar\eta \in {}^{k(*)+1\setminus\{m\}}U$ and $\nu \in {}^{\omega>}2\} \cup \{y_{\bar\eta,n} : \bar\eta \in {}^{k(*)+1}U$ but $\bar\eta \ne \bar\eta^*$ and $n < \omega\}$. Easily $G_\ell \subseteq G'$ recalling $G_\ell = N_\ell \cap G_U$ hence $\Sigma\{G_\ell : \ell \le k(*)\} \subseteq G'$, but $y_{\bar\eta^*,0} \notin G'$ hence

$(*)_5$ $y_{\bar\eta^*,0} \notin \sum\{G_\ell : \ell \le k(*)\}$.

But for every n

$(*)_6$ $\bar n! y_{\bar\eta^*,n+1} - y_{\bar\eta^*,n} = \Sigma\{x_{m,\bar\eta^*\restriction(k(*))+1\setminus\{m\}),\eta^*_m\restriction n} : m \le k(*)\} \in \Sigma\{G_\ell : \ell \le k(*)\}$.
[Why? $x_{m,\bar\eta^*\restriction(k(*))+1\setminus\{m\}),\eta^*_m\restriction n} \in G_m$ as $\bar\eta^* \restriction (k(*)) + 1\setminus\{m\}) \in N_m$ by $(*)_4$.]

We can conclude that in $G_U/\sum\{G_\ell : \ell \le k(*)\}$, the element $y_{\bar\eta^*,0} + \sum\{G_\ell : \ell \le k(*)\}$ is not zero (by $(*)_5$) but is divisible by every natural number by $(*)_6$. This contradicts $(*)_0$ so we are done. $\square_{5.8}$

5.9 Conclusion. $G_{k(*)}$ is a Borel and even F_σ abelian group which is $\aleph_{k(*)+1}$-free <u>but</u> if $2^{\aleph_0} \ge \aleph_{k(*)+1}$ is not free and even not $\aleph_{k(*)+2}$-free.

Proof. $G_{k(*)}$ is Borel and F_σ by 5.3, it is $\aleph_{k(*)+1}$-free by 5.7 and if $2^{\aleph_0} \ge \aleph_{k(*)+1}$ it is not $\aleph_{k(*)+2}$-free by 5.8. $\square_{5.9}$

Blass asks
5.10 Question: Suppose (a) + (b) below, does it follow that forcing with $\mathbb{Q}$ add reals?

- (a) G is a Borel definition of an uncountable abelian subgroup of ${}^\omega\mathbb{Z}$ (the Specker group) which is not free
- (b) the forcing $\mathbb{Q}$ satisfies $\Vdash_{\mathbb{Q}}$ "$G^{\mathbf{V}}$ is free".

Now
5.11 Fact: For any Borel abelian group G: if CH, then the answer to 5.10 is yes. If not CH then for some Borel, Abelian group the answer is not for $\mathbb{Q} =$ Levy$(\aleph_1, 2^{\aleph_0})$.

Proof. First, assume CH holds and G is as in (a) of 5.10; (or just defined absolutely enough such that $G^{\mathbf{V}}$ is a subgroup of $G^{\mathbf{V}^{\mathbb{Q}}}$ for any forcing notion $\mathbb{Q}$ and is still not free). Then by [Sh 402] the group $G^{\mathbf{V}}$ is non-free in some strong way such that no forcing not collapsing $2^{\aleph_0}$ to $\aleph_0$ can make it free (that is, for some countable $G_0 \subseteq G^{\mathbf{V}}, G^{\mathbf{V}}/G_0$ contains the direct sum of $2^{\aleph_0}$ finite rank non-free abelian groups).
This is a strong yes answer.
Second, assume if $2^{\aleph_0} > \aleph_1$ we can find such group: for $k(*) \geq 1$, our $G_{k(*)}$ if $\aleph_1 < \aleph_{k(*)+1} \leq 2^{\aleph_0}$, see below, provide a strong negative answer. So together this gives answers to a question of Blass. $\square_{5.11}$

5.12 Corollary. *1) The group $G_{k(*)}$ is embeddable into ${}^\omega\mathbb{Z}$, even purely.*
2) Hence forcing which does not add bounded subsets to $\aleph_{k(*)}$ can make it free (i.e. Levy$(\aleph_\ell, 2^{\aleph_0})$ *if $\ell \leq k(*)$ while if our universe satisfies $2^{\aleph_0} > \aleph_{k(*)}$ it is not free there).*

Proof. 1) For every $n < \omega$ we define a function f_n from Y to $G_{k(*)}$ where Y is the set of generators of $G_{k(*)}$, i.e.

$$Y = \{y_{\bar\eta,n+1} : n < \omega, \bar\eta \in {}^{k(*)+1}({}^\omega 2)\} \cup \{x_{m,\bar\eta_m,\nu} : m \leq k(*),$$
$$\bar\eta_m \in {}^{\{\ell:\ell \leq k(*), \ell \neq m\}}({}^\omega 2) \text{ and } \nu \in {}^{\omega>}2\}.$$

First define a function h_n: for $\eta \in {}^{\omega\geq}2, h_n(\eta)$ is a sequence of length $\ell g(\eta)$ and

$$(h_n(\eta))(\ell) = \begin{cases} \eta(\ell) & \text{if } \ell < n \ \& \ \ell < \ell g(\eta) \\ 0 & \text{if } \ell \geq n \ \& \ \ell < \ell g(\eta) \end{cases}.$$

For $\bar\eta = \langle \eta_\ell : \ell \in u\rangle \in {}^u({}^{\omega\geq}2)$ we let $h_n(\bar\eta) = \langle h_n(\eta_\ell) : \ell \in u\rangle$.
Lastly, let

$$f_n(y_{\bar\eta,n+1}) = y_{h_n(\bar\eta),n+1}$$

$$f_n(x_{m,\bar\eta_m,\nu}) = x_{m,h_n(\bar\eta_m),h_n(\nu)}.$$

Does f_n induce a homomorphism from $G_{k(*)}$ into $G_{k(*)}$? For this it is enough to check that for every one of the relations from Definition 5.2, its f_n-image is satisfied in $G_{k(*)}$, but this is obvious as it is mapped to another one of the equations in the definition of $G_{k(*)}$: the equation in $\boxtimes_{\bar\eta,m}$ is mapped to the equation in $\boxtimes_{h_n(\bar\eta),m}$.

So f_n extends to an endomorphism $\hat f_n$ of $G_{k(*)}$. Easily

⊛ if $L \subseteq G_{k(*)}$ is a finite rank subgroup (so free) then for n large enough $\hat f_n \restriction L$ is one to one.

Now the range of $\hat f_n$ is clearly countable hence free, and is infinite; let it be $\bigoplus\limits_{\ell<\omega} \mathbb{Z} z_{n,\ell}$. Hence for some homomorphisms $g_{n,\ell}$ from $\mathrm{Rang}(f_n)$ to $\mathbb{Z}$ for $\ell < \omega$ we have

$$\begin{aligned} z \in \ \mathrm{Rang}(\hat f_n) \Rightarrow & z = \Sigma\{g_{n,\ell}(z) z_{n,\ell} : \ell < \omega\} \\ & \text{where } g_{n,\ell}(z) = 0 \text{ for every } \ell \text{ large enough} \end{aligned}$$

Let $f_{n,\ell} = g_{n,\ell} \circ \hat f_n \in \ \mathrm{Hom}(G_{k(*)}, \mathbb{Z})$. Those homomorphisms give, by renaming the $f_{n,\ell}$'s, an embedding of $G_{k(*)}$ into ${}^\omega\mathbb{Z}$. Looking at the construction, it is a pure one.
2) By 5.7. $\square_{5.12}$

5.13 Claim. *Assume* MA $+2^{\aleph_0} > \aleph_2$.

If $k(*) > 2$ *and* $2^{\aleph_0} \geq \aleph_{k(*)+1}$ *<u>then</u>* $G = G_{k(*)}$ *is a non-free Whitehead Borel (abelian) group.*

Recall

5.14 Definition. An abelian group G is called a Whitehead group <u>when</u>: if H is an abelian group and g is a homomorphism from H onto G with kernal isomorphic to $\mathbb{Z}$, then the kernel is a direct summand of H (equivalently, g is invertible).

Proof. By 5.3 we know that $G_{k(*)}$ is a Borel group and by 5.8 it is not free. Let $\langle \eta_\alpha : \alpha < 2^{\aleph_0} \rangle$ list ${}^\omega 2$ with no repetitions and $\mathscr{U}_\alpha = \{\beta : \beta < \alpha\}$.

So $\langle \mathscr{U}_\alpha : \alpha < 2^{\aleph_0} \rangle$ be $\subseteq$-increasing continuous with union ${}^\omega 2$ such that $\mathscr{U}_0 = \emptyset, |\mathscr{U}_\alpha| \leq |\alpha|$; and let $H_\alpha := G_{\mathscr{U}_\alpha}$, see Definition 5.5(1). So $\langle H_\alpha : \alpha < 2^{\aleph_0} \rangle$ is a $\subseteq$-increasing continuous sequence of subgroups of G with union G. For $\alpha < 2^{\aleph_0}$, letting $u_\alpha = \{\alpha\}$ recalling Definition 5.5 we have $G_{\mathscr{U}_\alpha \cup u_\alpha} = G_{\mathscr{U}_{\alpha+1}} = H_{\alpha+1}$ and $G_{\mathscr{U}_\alpha, u_\alpha} = G_{\mathscr{U}_\alpha} = H_\alpha$, hence $H_{\alpha+1}/H_\alpha = G_{\mathscr{U}_\alpha \cup u_\alpha}/G_{\mathscr{U}_\alpha, u_\alpha}$ and by 5.7(1) the latter group is $\aleph_2$-free so $H_{\alpha+1}/H_\alpha$ is $\aleph_2$-free. As MA holds and $|H_{\alpha+1}/H_\alpha| < 2^{\aleph_0}$ and $H_{\alpha+1}/H_\alpha$ is $\aleph_2$-free we know that it is a Whitehead group.

As H_α is $\subseteq$-increasing continuous, $H_0 = \{0\}$ and each $H_{\alpha+1}/H_\alpha$ is a Whitehead group, it follows that $\cup\{H_\alpha : \alpha < 2^{\aleph_0}\}$ is a Whitehead group, which means that G is as required. $\square_{5.13}$

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