Kulikov's problem on universal torsion-free abelian groups

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Abstract. Let T be an abelian group and λ an uncountable regular cardinal. We consider the question of whether there is a λ-universal group G^* among all torsion-free abelian groups G of cardinality less than or equal to λ satisfying $\text{Ext}(G,T)=0$. Here G^* is said to be λ-universal for T if, whenever a torsion-free abelian group G of cardinality less than or equal to λ satisfies $\text{Ext}(G,T)=0$, then there is an embedding of G into G^*. For large classes of abelian groups T and cardinals λ it is shown that the answer is consistently no. In particular, for T torsion, this solves a problem of Kulikov.

1. Introduction

Given a class $\mathfrak{C}$ of objects and a property P it is natural to ask for universal objects in $\mathfrak{C}$ with respect to P. A universal object with respect to P is an object $C \in \mathfrak{C}$ satisfying P such that every other object of the class $\mathfrak{C}$ that satisfies P can be identified with a subobject of C. The existence of universal objects clearly simplifies the theorie of the objects satisfying P since very often properties of objects are inherited by subobjects. Thus, there is a distinguished object in $\mathfrak{C}$ with respect to P that is a representative among all the objects of $\mathfrak{C}$ satisfying P.

On the other hand, if there exist no universal objects with respect to P, this indicates that the objects satisfying P have a complicated structure. Since the definition of universal objects is "universal" the question of their existence appears in almost every field of mathematics. e.g. group theory, theory of ordered structures, Banach-spaces etc.

In the present paper we focus on the theory of abelian groups and define our class $\mathfrak{C} = \mathcal{TF}_\lambda$ to be the class of all torsion-free abelian groups G of cardinality (rank) less than or equal to λ, where λ is a fixed cardinal. The property $P = P_T$ is related to a fixed torsion abelian group T and a group $G \in \mathcal{TF}_\lambda$ satisfies P_T if and only if $\text{Ext}(G,T)=0$, where $\text{Ext}(-,T)$ denotes the first derived functor of the functor $\text{Hom}(-,T)$. It was Kulikov [**KN**, Question 1.66] who first asked whether or not there exist universal groups in $\mathcal{TF}_\lambda$ for all (uncountable) cardinals λ and torsion abelian groups T. Clearly, if the group T is cotorsion, hence satisfies $\text{Ext}(\mathbb{Q},T)=0$, then there is always (for every λ) a universal group in $\mathcal{TF}_\lambda$, namely

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the torsion-free divisible group of rank λ. Moreover, since every free abelian group F satisfies $\mathrm{Ext}(F,T)=0$, one has to consider torsion-free groups of cardinality less than or equal to a fixed cardinal rather than searching for universal objects among all torsion-free abelian groups with P_T. As was mentioned earlier, the existence of universal objects in $\mathcal{TF}_\lambda$ with respect to P_T sheds some light on the complexity of the structure of objects in $\mathcal{TF}_\lambda$. This has also consequences to more complicated theories. We shall discuss cotorsion theories as an example.

Cotorsion theories for abelian groups have been introduced by Salce in 1979 [**S**]. Following his notation we call a pair $(\mathcal{F},\mathcal{C})$ a cotorsion theory if $\mathcal{F}$ and $\mathcal{C}$ are classes of abelian groups which are maximal with respect to the property that $\mathrm{Ext}(F,C)=0$ for all $F\in\mathcal{F}$, $C\in\mathcal{C}$.

Salce [**S**] has shown that every cotorsion theory is cogenerated by a class of torsion and torsion–free groups where $(\mathcal{F},\mathcal{C})$ is said to be cogenerated by the class $\mathcal{A}$ if $\mathcal{C}=\mathcal{A}^\perp=\{X\in\mathrm{Mod}\text{–}\mathbb{Z}\mid \mathrm{Ext}(A,X)=0 \textit{ for all } A\in\mathcal{A}\}$ and $\mathcal{F}={}^\perp(\mathcal{A}^\perp)=\{Y\in\mathrm{Mod}\text{–}\mathbb{Z}\mid \mathrm{Ext}(Y,X)=0 \textit{ for all } X\in\mathcal{A}^\perp\}$. Examples for cotorsion theories are: $(\mathcal{L},\mathrm{Mod}\text{–}\mathbb{Z})=({}^\perp(\mathbb{Z}^\perp),\mathbb{Z}^\perp)$ where $\mathcal{L}$ is the class of all free groups, and the classical one $(\mathcal{TF},\mathcal{CO})=({}^\perp(\mathbb{Q}^\perp),\mathbb{Q}^\perp)$ where $\mathcal{TF}$ is the class of all torsion–free groups and $\mathcal{CO}$ is the class of all (classical) cotorsion groups. In view of the last example the classes $\mathcal{F}$ and $\mathcal{C}$ of a cotorsion theory $(\mathcal{F},\mathcal{C})$ are said to be the torsion–free class and the cotorsion class of this cotorsion theory.

If we restrict our attention to cotorsion classes cogenerated by a single torsion-free group G, then ordering these classes by inclusion, we obviously have that $\mathbb{Z}^\perp$ is maximal and $\mathbb{Q}^\perp$ is minimal among these classes. Moreover, in [**GSW**] Göbel, Wallutis and the first author have shown that any partially ordered set can be embedded into the lattice of all cotorsion classes. Hence there is no hope at all to characterize these classes. But if we restrict to torsion-free groups in $\mathcal{TF}_\lambda$, then the existence of a universal group can be helpful. If $G\in\mathcal{TF}_\lambda$ and $T\in G^\perp$ is "as complicated as possible", then the universal object C related to P_T satisfies $C^\perp\subseteq G^\perp$ and hence we obtain new information about $G^\perp$. This is just one example where universal objects could be helpful.

To the authors's knowledge there is no published literature on Kulikov's problem except for [**St**], where the second author proved that in Gödel's universe ($V=L$) for every cardinal λ and torsion abelian group T there exists a universal group $G\in\mathcal{TF}_\lambda$ if T has only finitely many non-trivial bounded p-components. Moreover, if λ is finite, then this is an if and only if result already in ZFC. Thus, in $V=L$, the class $\mathcal{TF}_\lambda$ behaves well with respect to the property T_P for a large class of torsion abelian groups T.

In this paper we prove that the result from [**St**] is not provable in ZFC. We show that it is consistent with ZFC and GCH that for every abelian group T (not necessarily torsion) and every uncountable regular cardinal λ there is a cardinal $\kappa>\lambda$ such that the class $\mathcal{TF}_\kappa$ has no universal object with respect to P_T. Moreover, we prove that for torsion abelian groups T of cardinality less than or equal to $\aleph_1$ (also more general situations are considered) there is no uncountable cardinal λ such that $\mathcal{TF}_\lambda$ has universal groups with respect to the property P_T. This answers Kulikov's problem consistently in the negative.

All groups under consideration are abelian. The notations are standard and for unexplained notions in abelian group theory and set theory we refer to [**Fu**] and [**EM**], [**J**]. For uniformization see [**K**] or [**S2**].

2. λ-universal groups

In this section we introduce the notions of λ-universal groups for a given group T and obtain some basic properties. We are mainly interested in the case when our group T is torsion but for the sake of generality we let T be an arbitrary (abelian) group. By $\mathcal{TF}$ we denote the class of all torsion-free groups. For a cardinal λ we denote by $\mathcal{TF}_\lambda(T)$ the class of all torsion-free groups G of rank less than or equal to λ such that $\mathrm{Ext}(G,T)=0$, i.e. $\mathcal{TF}_\lambda(T)=\{G\in\mathcal{TF}:\mathrm{Ext}(G,T)=0 \text{ and } \mathrm{rk}(G)\leq\lambda\}$. Moreover, we let $\mathcal{TF}(T)=\bigcup\limits_{\lambda}\mathcal{TF}_\lambda(T)=\{G\in\mathcal{TF}:\mathrm{Ext}(G,T)=0\}$ be the class of all torsion-free groups G satisfying $\mathrm{Ext}(G,T)=0$.

Definition 2.1. *Let T be a group and λ a cardinal. A torsion-free group G of rank less than or equal to λ is called* λ-universal for T *if $G\in\mathcal{TF}_\lambda(T)$ and for every $H\in\mathcal{TF}_\lambda(T)$ there is an embedding $i:H\longrightarrow G$ of H into G.*

Note that in Definition 2.1 for λ infinite we may replace the rank of G by its cardinality. Kulikov [**KN**, Question 1.66] asked the following question:

Question 2.2. *Let T be a torsion group and λ an uncountable cardinal. Is there always a λ-universal group for T?*

Let us first mention that a positive consistency result was obtained by the second author in [**St**]. Moreover, the case of finite λ was considered.

Lemma 2.3 ([**St**]). *Let T be a torsion group and n a strictly positive integer. Then there exists an n-universal group G for T if and only if T has only finitely many non-trivial bounded p-components. In this case, G is completely decomposable.*

Lemma 2.4 ([**St**], $V=L$). *If T is a torsion group with only finitely many non-trivial bounded p-components and λ is a cardinal, then there is a λ-universal group for T which is completely decomposable.*

We therefore shall restrict ourselves to uncountable (regular) cardinals in most of the results. Let us begin with some basic observations and recall that a basic subgroup of a torsion group T is a direct sum B of cyclic groups which is pure in T and has divisible quotient T/B.

Lemma 2.5. *Let T be a torsion group, B a basic subgroup of T and λ a cardinal. A torsion-free group G is λ-universal for T if and only if G is λ-universal for B.*

Proof. The proof follows easily since for any torsion-free group H we have $\mathrm{Ext}(H,T)=0$ if and only if $\mathrm{Ext}(H,B)=0$ (see for example [**St**, Lemma 1.2]). □

Lemma 2.6. *Let T be a group and $T=D\oplus R$ where D is divisible and R is reduced. If λ is a cardinal, then a torsion-free group G is λ-universal for T if and only if G is λ-universal for R.*

Proof. The proof is straightforward since for any torsion-free group H we have $\mathrm{Ext}(H,T)=\mathrm{Ext}(H,D)\oplus\mathrm{Ext}(H,R)$ and $\mathrm{Ext}(H,D)=0$ since D is divisible. □

Thus it is enough to consider reduced groups. Moreover, among the reduced ones we only have to deal with groups that are not cotorsion. Recall that a group T is called *cotorsion* if $\mathrm{Ext}(\mathbb{Q}, T) = 0$ which is equivalent to $\mathcal{TF}(T) = \mathcal{TF}$.

LEMMA 2.7. *Let T be a cotorsion group. Then there is a λ-universal group for every cardinal λ.*

PROOF. If T is cotorsion, then for every cardinal λ we have $\bigoplus_{\lambda} \mathbb{Q} \in \mathcal{TF}_{\lambda}(T)$. Since every torsion-free group of rank less than or equal to λ can be embedded into its divisible hull it follows that $\bigoplus_{\lambda} \mathbb{Q}$ is λ-universal for T. □

The following lemma shows that it makes sense to restrict ourselves to (torsion) groups T and cardinals λ such that $\lambda \geq |T|$.

LEMMA 2.8 ([**St**]). *Let G be any group and T a torsion group. Then* $\mathrm{Ext}(G, T) = 0$ *if and only if* $\mathrm{Ext}(G, T') = 0$ *for all pure subgroups T' of T such that $|T'| \leq |G|$.*

We shall even assume that $\lambda > |T|$.

3. (T, λ, γ)-suitable groups

In what follows let $\lambda > \gamma$ be fixed infinite regular cardinals unless otherwise stated.

DEFINITION 3.1. *Let T be a group of cardinality less than λ. A group G is called (T, λ, γ)-suitable if the following conditions are satisfied:*

- (i) $\mathrm{Ext}(G, T) \neq 0$;
- (ii) *There are free groups F, F_i $(i \leq \gamma)$ such that*
  - (a) *the F_i's $(i \leq \gamma)$ form an increasing chain such that $F_{\gamma} = \bigcup_{i<\gamma} F_i \subseteq F$;*
  - (b) $\mathrm{rk}(F_i) \leq |i| + \aleph_0$;
  - (c) *F/F_i is free for all $i < \gamma$;*
  - (d) $F/F_{\gamma} \cong G$.

Our first lemma shows that there is always a (T, λ, ω)-suitable group for non-trivial (not cotorsion) T.

LEMMA 3.2. *Let T be a group of cardinality less than λ and G a countable group such that* $\mathrm{Ext}(G, T) \neq 0$. *Then G is (T, λ, ω)-suitable. In particular, if T is not cotorsion then there is a (T, λ, γ)-suitable group G.*

PROOF. Let T and G be as stated. Choose a free resolution

$$0 \to K \xrightarrow{id} F \to G \to 0$$

of G. Without loss of generality we may assume that K and F are of countable rank. Choose elements $e_i \in K$ $(i < \omega)$ such that $K = \bigoplus_{i<\omega} \mathbb{Z}e_i$. Put $F_n = \bigoplus_{i \leq n} \mathbb{Z}e_i \subseteq F$ for $n < \omega$. Then each F_n is a direct summand of F and hence G is (T, λ, ω)-suitable. If T is not cotorsion, then $\mathrm{Ext}(\mathbb{Q}, T) \neq 0$ and hence the above arguments show that $\mathbb{Q}$ is (T, λ, ω)-suitable. □

The next results show the existence of (T, λ, γ)-suitable groups for uncountable γ under certain assumptions. Recall that a group G is called *almost-free* if all its subgroups of smaller cardinality are free.

LEMMA 3.3. *Let T be a group of cardinality less than λ and G an almost-free group of cardinality γ such that* $\mathrm{Ext}(G,T)\neq 0$. *Then G is (T,λ,γ)-suitable.*

PROOF. Take a γ-filtration $G=\bigcup_{\alpha<\gamma} G_\alpha$ of G such that each G_α is free. By [**EM**, Lemma XII.1.4] there is a free resolution associated with this filtration. This is to say there are free groups $K=\bigoplus_{\alpha<\gamma} F_\alpha$ and $K=\bigoplus_{\alpha<\gamma} K_\alpha$ such that the short sequences

$$0\longrightarrow K\longrightarrow F\longrightarrow G\longrightarrow 0$$

and

$$0\longrightarrow \bigoplus_{\alpha<\beta} K_\alpha\longrightarrow \bigoplus_{\alpha<\beta} F_\alpha\longrightarrow G_\beta\longrightarrow 0$$

are exact for all $\beta<\gamma$. Since each G_β is free it follows that G is (T,λ,γ)-suitable. □

LEMMA 3.4. *Let T be a group and H an epimorphic image of T. If G is (H,λ,γ)-suitable then G is (T,λ,γ)-suitable.*

PROOF. The claim follows immediately noting that $\mathrm{Ext}(G,T)=0$ implies $\mathrm{Ext}(G,H)=0$. □

PROPOSITION 3.5. *Let $S\subseteq\gamma$ be stationary non-reflecting such that* $\mathrm{cf}(\alpha)=\omega$ *for all $\alpha\in S$ and assume that $\diamondsuit_S$ holds. Let T be a group which is not cotorsion and has an epimorphic image of size less than or equal to γ that is not cotorsion. Then there exists a strongly γ-free torsion-free group G of size γ which is (T,λ,γ)-suitable. In particular, this holds if T is torsion or T itself is of cardinality less than or equal to γ.*

PROOF. Let B be the epimorphic image of T of size less than or equal to γ which is not cotorsion. Then Lemma 3.4 shows that it is enough to construct a (B,λ,γ)-suitable group. Therefore, we may assume without loss of generality that T has cardinality less than or equal to γ. Since T is not cotorsion there exists by Lemma 3.2 a countable torsion-free group R which is (T,λ,ω)-suitable. Let $\lambda_n\geq\aleph_0$ ($n\in\omega$) be cardinals. As in [**EM**, Corollary VII.1.2] there exist free abelian groups $K\subseteq F$ such that $K=\bigcup_{n\in\omega} K_n$ and F/K_n is free for all $n\in\omega$, K_0 is free of rank λ_0 and K_{n+1}/K_n is free of rank λ_{n+1}. Moreover, $\mathrm{Ext}(F/K,T)\neq 0$ since F/K is isomorphic to R. As in the proof of [**EM**, VII.1.4] we can construct a torsion-free group G of cardinality γ which has a γ-filtration $G=\bigcup_{\alpha<\gamma} G_\alpha$ satisfying the following for all $\alpha<\beta<\gamma$:

(i) G_α is free of rank $|\alpha|+\aleph_0$;
(ii) if α is a limit ordinal, then $G_\alpha=\bigcup_{\delta<\alpha} G_\delta$;
(iii) if $\alpha\notin S$, then G_β/G_α is free of rank $|\beta|+\aleph_0$;
(iv) if $\alpha\in S$, then $\mathrm{Ext}(G_\beta/G_\alpha,T)\neq 0$.

Since $\diamondsuit_S$ holds it follows that $\mathrm{Ext}(G,T)\neq 0$ (see e.g. [**EM**, XII.1.15]), hence G is (T,λ,γ)-suitable. Finally, if T is torsion then we choose a basic subgroup B' of T and a countable unbounded direct summand B of B' which is therefore not cotorsion. Note that B exists since T is not cotorsion. It is well-known that B is an epimorphic image of T (see [**Fu**, Theorem 36.1]), hence Lemma 3.4 shows that it is enough to construct a (B,λ,γ)-suitable group. □

4. The uniformization

From now on let S be a stationary subset of λ consisting of limit ordinals of cofinality γ. To prove our next theorem we shall use a construction for modules which was almost identically developed in [**ES**]. Thus we shall not give all the proofs but for the convenience of the reader we shall recall the basic definitions as well as the construction and the main properties of the constructed module (group).

Definition 4.1. *A* ladder system $\bar{\eta}$ *on* S *is a family of functions* $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ *such that* $\eta_\delta : \gamma \to \delta$ *is strictly increasing with* $\sup(\mathrm{rg}(\eta_\delta)) = \delta$*, where* $\mathrm{rg}(\eta_\delta)$ *denotes the range of* η_δ*. We call the ladder system* tree-like *if for all* $\delta, \nu \in S$ *and every* $\alpha, \beta \in \gamma$*,* $\eta_\delta(\alpha) = \eta_\nu(\beta)$ *implies* $\alpha = \beta$ *and* $\eta_\delta(\rho) = \eta_\nu(\rho)$ *for all* $\rho \leq \alpha$*.*

For a ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ on S we can form a *tree* $B_{\bar{\eta}} \subseteq {}^{\leq \gamma}\lambda$ of height γ in the following way: Let $B_{\bar{\eta}} = \{\eta_\delta \restriction_\alpha : \delta \in S, \alpha \leq \mathrm{l}(\eta_\delta)\}$, where $\mathrm{l}(\eta_\delta)$ denotes the length of η_δ, i.e. $l(\eta_\delta) = \sup\{\eta_\delta(\alpha) : \alpha \in \mathrm{dom}(\eta_\delta)\}$. Note that $B_{\bar{\eta}}$ is partially ordered by defining $\eta \leq \nu$ if and only if $\eta = \nu \restriction_{\mathrm{l}(\eta)}$.
From this tree we now build a group. Let T be a group and let G be (T, λ, γ)-suitable. Fix a chain $\langle F_\alpha : \alpha \leq \gamma \rangle$ for G as in Definition 3.1. For each $\eta \in B_{\bar{\eta}}$ we let $H_\eta = F_{\mathrm{l}(\eta)}$ and if $\eta \leq \nu \in B_{\bar{\eta}}$ then let $i_{\eta,\nu}$ be the inclusion map of H_η into H_ν. Finally, let $H_{\bar{\eta}}$ be the direct limit of $(H_\eta, i_{\eta,\nu} : \eta \leq \nu \in B_{\bar{\eta}})$. More precisely, $H_{\bar{\eta}}$ equals $\bigoplus\{H_\eta : \eta \in B_{\bar{\eta}}\}/K$ where K is the subgroup generated by all elements of the form $x_\eta - y_\nu$ where $y_\nu \in H_\nu, x_\eta \in H_\eta$, $\eta \leq \nu$ and $i_{\eta,\nu}(y_\nu) = x_\eta$. Canonically we can embed H_η into $H_{\bar{\eta}}$ and we shall therefore regard H_η as a submodule of $H_{\bar{\eta}}$ in the sequel.

Definition 4.2. *Let* κ *be an uncountable regular cardinal. The tree* $B_{\bar{\eta}}$ *is called* κ-free *if for every* $F \subseteq S$ *such that* $|F| < \kappa$ *there is a function* $\Psi : F \to \gamma$ *such that*

$$\{\{\eta_\delta \restriction_\alpha : \Psi(\delta) < \alpha \leq \gamma\} : \delta \in F\}$$

is a family of pairwise disjoint sets. The ladder system $\bar{\eta}$ *is called* κ-free *if* $B_{\bar{\eta}}$ *is* κ*-free.*

We now state some properties of the constructed group $H_{\bar{\eta}}$.

Lemma 4.3. *Let* κ *be an uncountable regular cardinal. If* $B_{\bar{\eta}}$ *is* κ*-free, then* $H_{\bar{\eta}}$ *is a* κ*-free group.*

Proof. See [**ES**, Lemma 1.4]. □

Lemma 4.4. *Suppose that* S *is non-reflecting. Then* $H_{\bar{\eta}}$ *is not free.*

Proof. See [**ES**, Lemma 1.5]. Since S is non-reflecting $\bar{\eta}$ is λ-free and it is easy to see that for all $\delta \in S$ there exists $\nu \geq \delta$ such that for all $\mu < \gamma$, $\eta_\nu \restriction_\mu \in \{\eta_\alpha \restriction_\mu : \alpha < \delta\}$. □

We recall the definition of μ-uniformization for a ladder system $\bar{\eta}$ and a cardinal μ.

Definition 4.5. *If* μ *is a cardinal and* $\bar{\eta}$ *is a ladder system on* S *we say that* $\bar{\eta}$ *has* μ-uniformization *if for every family* $\{c_\delta : \delta \in S\}$*, where* $c_\delta : \mathrm{rg}(\eta_\delta) \to \mu$*, there exists* $\Psi : \lambda \to \mu$ *and* $\Psi^* : S \to \mu$ *such that for all* $\delta \in S$*,* $\Psi(\eta_\delta(\alpha)) = c_\delta(\eta_\delta(\alpha))$ *whenever* $\Psi^*(\delta) \leq \alpha < \gamma$*.*

THEOREM 4.6. *Let T be a group of cardinality less than λ and G a group which is (T,λ,γ)-suitable. Moreover, assume that S is non-reflecting and $\bar\eta$ is a tree-like ladder system on S that has $2^{(|T|^\gamma)}$-uniformization. Then there exists a torsion-free group H of size λ such that*

(i) *H has a λ-filtration $\langle \bar H_\alpha : \alpha < \lambda\rangle$;*
(ii) *if $\alpha \in S$, then $\bar H_{\alpha+1}/\bar H_\alpha \cong G$;*
(iii) *if $\alpha \notin S$, then $\bar H_\beta/\bar H_\alpha$ is free for all $\alpha \le \beta$;*
(iv) $\mathrm{Ext}(H,T) = 0$.

In fact, H satisfies $\mathrm{Ext}(H,W) = 0$ for all groups W of cardinality less than or equal to $|T|$.

PROOF. Let T, S, $\bar\eta$ and G be as stated and choose the group $H_{\bar\eta} = \bigoplus\{H_\eta : \eta \in B_{\bar\eta}\}/K$ as constructed above. Note that $\bar\eta$ is λ-free since S is non-reflecting. Then $H_{\bar\eta}$ is almost-free but not free by Lemma 4.3 and Lemma 4.4. Moreover, [**ES**, Theorem 1.7] shows that $H_{\bar\eta}$ satisfies $\mathrm{Ext}(H_{\bar\eta},W) = 0$ for every group W of size less than or equal to $|T|$. Finally, it is easy to see that $H_{\bar\eta}$ has a λ-filtration as stated letting $\bar H_\alpha = \bigcup\{(H_\eta + K)/K : \eta \in B_{\bar\eta}, \sup(\mathrm{rg}(\eta)) < \alpha\}$. □

It is our aim to apply Theorem 4.6 to ladder systems which have μ-uniformization for all $\mu < \lambda$. In this case Theorem 4.6 is applicable to a lot of regular cardinals $\lambda > \gamma$ if T is small in some sense (e.g. $|T| \le \gamma$). For instance, if λ is strongly inaccessible or $\lambda = \kappa^+$ with κ singular and $\mathrm{cf}(\kappa) = \gamma$, then certainly $\lambda > 2^{|T|^\gamma}$ (even $\kappa > 2^{(|T|^\gamma)}$) and hence uniformization holds. But problems arise when $\lambda = \kappa^+$ and κ is regular. We shall show that we can improve Theorem 4.6 in this case if $\diamondsuit_\gamma$ holds and the ladder system is tree-like.

DEFINITION 4.7. *Let $\lambda = \kappa^+$ and $\mathrm{cf}(\kappa) = \kappa$. A ladder system $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ has* strong κ-uniformization *if for every system $\bar P = \langle P_\alpha : \alpha < \lambda\rangle$ such that*

(i) *$\emptyset \ne P_\delta \subseteq \{f | f : \mathrm{rg}(\eta_\delta) \to \delta \cap \kappa\}$ if $\delta \in S$;*
(ii) *if $\alpha = \eta_\delta(i)$ for some $\delta \in S$ and $i < \gamma$, then $P_{\eta_\delta(i)} = \{f \restriction_{\mathrm{rg}(\eta_\delta\restriction_{(i+1)})} | f \in P_\delta\}$;*
(iii) *if $\delta \in S$ and $i < \gamma$ is a limit ordinal, then for every increasing sequence $\langle f_j : j < i\rangle$, $f_j \in P_{\eta_\delta(j)}$ there exists $f_i \in P_{\eta_\delta(i)}$ which extends the union $\bigcup\limits_{j<i} f_j$.*

there exists a function $f : \lambda \to \kappa$ such that for all $\delta \in S$, $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$.

PROPOSITION 4.8. *Let $\lambda = \gamma^+$, γ regular and let $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ be a tree-like ladder system on S such that $\bar\eta$ has γ-uniformization and $\diamondsuit_\gamma$ holds. Then $\bar\eta$ has strong γ-uniformization.*

PROOF. Let $\bar P$ be given as stated and let J be stationary in γ such that $\diamondsuit_\gamma(J)$ holds. For simplicity we shall identify J with γ. Thus there exists a system of diamond functions $\bar h = \langle h_\delta : \delta \to \gamma : \delta < \gamma\rangle$ such that for every function $h : \gamma \to \gamma$ the set $\{\delta < \gamma : h \restriction_\delta = h_\delta\}$ is stationary in γ. For each $\delta \in S$ we define for $i < \gamma$

$$h_i^\delta : \mathrm{rg}(\eta_\delta \restriction_i) \to \gamma \text{ via } \eta_\delta(j) \mapsto h_i(j).$$

Moreover, we put for $\delta \in S$, $E_\delta = \{i < \gamma : h_i^\delta \subseteq f \text{ for some } f \in P_\delta\}$ and let $g_i^\delta \in P_\delta$ be such that $g_i^\delta \restriction_{\mathrm{rg}(\eta_\delta\restriction_i)} = h_i^\delta$ for $i \in E_\delta$. Note that $E_\delta \ne \emptyset$ since $\diamondsuit_\gamma$ holds. Define

$f_\delta : \mathrm{rg}(\eta_\delta) \to H(\gamma)$ for $\delta \in S$ as follows

$$f_\delta(\eta_\delta(i)) = \left\langle g_j^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(i+1)})} : j \leq i, j \in E_\delta \right\rangle.$$

Here $H(\gamma)$ denotes the class of sets hereditarily of cardinality $< \gamma$. Note that $H(\gamma)$ has size $\leq \gamma$. By the γ-uniformization of $\bar{\eta}$ we can find $F : \lambda \to H(\gamma)$ such that for all $\delta \in S$ there exists $\alpha_\delta < \gamma$ such that for all $\alpha_\delta \leq i < \gamma$ we have $f_\delta(\eta_\delta(i)) = F(\eta_\delta(i))$. For $i < \gamma$ let

$$F(\eta_\delta(i)) = \left\langle G_j^{\eta_\delta(i)} : j \leq i, j \in E_{\eta_\delta(i)} \right\rangle$$

for some $E_{\eta_\delta(i)} \subseteq \gamma$. Note, that $F(\eta_\delta(i))$ does not depend on δ and that F is well-defined since the ladder system is tree-like.

We now define $f : \lambda \to \gamma$ by defining $f : \mathrm{rg}(\eta_\delta) \to \gamma$ for every $\delta \in S$. Clearly this is enough. In order to define f on $\mathrm{rg}(\eta_\delta)$ for fixed $\delta \in S$ we use induction on $i < \gamma$. For $i = 0$ choose any member $u \in P_{\eta_\delta(0)}$ and put $f(\eta_\delta(0)) = u(\eta_\delta(0))$. Now assume that $f(\eta_\delta(j))$ has been defined for $j < i$ and $\delta \in S$ such that for $j < i$, $f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(j+1)})} \in P_{\eta_\delta(j)}$. Put $\bar{f}_\delta = \{f(\eta_\delta(j)) : j < i\}$ and let

$$J_i^\delta = \{j \in E_{\eta_\delta(i)} : G_j^{\eta_\delta(i)} \restriction_{(\mathrm{rg}(\eta_\delta \restriction_i))} \subseteq \bar{f}_\delta\}.$$

If $j_i^\delta = \min(J_i^\delta)$ exists then let $f(\eta_\delta(i)) = G_{j_i^\delta}^{\eta_\delta(i)}(\eta_\delta(i))$. If $\min(J_i^\delta)$ does not exist, then we distinguish between two cases: if i is a limit ordinal, then (iii) from Definition 4.7 implies that there is $f_i \in P_{\eta_\delta(i)}$ which extends $\bigcup_{j<i} f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(j+1)})}$. If i is a successor ordinal, then (ii) from Definition 4.7 ensures that there is $f_i \in P_{\eta_\delta(i)}$ extending $f \restriction_{\mathrm{rg}(\eta_\delta \restriction_i)}$. In both cases put $f(\eta_\delta(i)) = f_i(\eta_\delta(i))$. Note that f is well-defined since the ladder system is tree-like, hence $\min(J_i^\delta)$ exists if and only if $\min(J_j^\nu)$ exists for $\eta_\delta(i) = \eta_\nu(j)$ $(\delta, \nu \in S,\ i, j < \gamma)$. It remains to check that $f \restriction_{\mathrm{rg}(\eta_\delta)} \in P_\delta$. By the uniformization we have for $\delta \in S$ that for $i \geq \alpha_\delta$, $G_j^{\eta_\delta(i)} = g_j^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(i+1)})}$ for all $j \leq i$, $j \in E_{\eta_\delta(i)}$. We define

$$h : \gamma \to \gamma \text{ via } h(j) = f(\eta_\delta(j)).$$

By $\diamondsuit_\gamma$ there exists $\beta_\delta \geq \alpha_\delta$ such that $h \restriction_{\beta_\delta} = h_{\beta_\delta}$ and hence

$$f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{\beta_\delta})} = h_{\beta_\delta}^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{\beta_\delta})} = h_{\beta_\delta}^\delta.$$

Thus in the construction $j_{\beta_\delta}^\delta$ existed since for example $\beta_\delta \in J_{\beta_\delta}^\delta$. Therefore, by definition of f, we have

$$f \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(\beta_\delta+1)})} = g_{\beta_\delta}^\delta \restriction_{\mathrm{rg}(\eta_\delta \restriction_{(\beta_\delta+1)})}.$$

By induction on $i \geq \beta_\delta$ we can now show that $f \restriction_{\mathrm{rg}(\eta_\delta)} = g_{\beta_\delta}^\delta \in P_\delta$ and this finishes the proof. □

We can now improve Theorem 4.6.

THEOREM 4.9. *Let T be a group of cardinality less than λ and G a group which is (T, λ, γ)-suitable. Moreover, assume that $\lambda = \gamma^+$, S is non-reflecting and $\bar{\eta}$ is a tree-like ladder system on S that has γ-uniformization. If $\diamondsuit_\gamma$ holds, then there exists a torsion-free group H of size λ such that*

(i) *H has a λ-filtration $\langle \bar{H}_\alpha : \alpha < \lambda \rangle$;*
(ii) *if $\alpha \in S$, then $\bar{H}_{\alpha+1}/\bar{H}_\alpha \cong G$;*
(iii) *if $\alpha \notin S$, then $\bar{H}_\beta/\bar{H}_\alpha$ is free for all $\alpha \leq \beta$;*

(iv) $\mathrm{Ext}(H,T)=0$.

In fact, H satisfies $\mathrm{Ext}(H,W)=0$ *for all groups W of cardinality less than or equal to $|T|$.*

PROOF. The proof is almost identical with the proof of [**ES**, Proposition 1.8] but for the convinience of the reader we state it briefly pointing out the major changes. Let S, $\bar{\eta}$ and G be as stated and choose the group $H_{\bar{\eta}}=\bigoplus\{H_\eta : \eta \in B_{\bar{\eta}}\}/K$ as constructed above. As in the proof of Theorem 4.6 $H_{\bar{\eta}}$ is an almost-free non-free torsion-free group of size λ which has the desired λ-filtration. It remains to show that $\mathrm{Ext}(H_{\bar{\eta}},K)=0$ for all groups K of size less than or equal to the size of T. Let K be such a group and choose a short exact sequence

$$(*) \qquad 0 \to K \xrightarrow{id} N \xrightarrow{\pi} H_{\bar{\eta}} \to 0.$$

We have to show that $(*)$ splits, i.e. we have to find a splitting map $\varphi : H_{\bar{\eta}} \to N$ such that $\pi\circ\varphi = id\restriction_{H_{\bar{\eta}}}$. Choose any set function $u : H_{\bar{\eta}} \to N$ such that $\pi \circ u = id\restriction_{H_{\bar{\eta}}}$. As in the proof of [**ES**, Proposition 1.8] the splitting maps φ of π are in one-one correspondence with set mappings $h : H_{\bar{\eta}} \to N$ such that $h(0)=0$ and for all $x,y\in H_{\bar{\eta}}$ and $z \in \mathbb{Z}$

(i) $zh(x)-h(zx)=zu(x)-u(zx)$ and
(ii) $h(x)+h(y)-h(x+y)=u(x)+u(y)-u(x+y)$

holds. For a subgroup H of $H_{\bar{\eta}}$ we denote by $\mathrm{Trans}(H,K)$ the set of all set mappings $h : H \to N$ satisfying conditions (i) and (ii) from above for all $x,y\in H$ and $z\in\mathbb{Z}$. Thus $(*)$ splits if and only if $\mathrm{Trans}(H_{\bar{\eta}},K)$ is non-empty.

It is now easy to see that we can identify the elements of $\mathrm{Trans}(H_\eta,K)$ with functions from $F_{\mathrm{l}(\eta)}$ to K for each $\eta\in B_{\bar{\eta}}$. Remember that $\langle F_\alpha : \alpha \le \gamma\rangle$ was the chain we fixed as in Definition 3.1 for G. For $\delta\in S$, $i<\gamma$ and $h\in \mathrm{Trans}(H_{\eta_\delta\restriction_i},K)$ let $\mathrm{seq}(h)$ be defined as follows:

$$\mathrm{seq}(h) : \mathrm{rg}(\eta_\delta\restriction_i) \to K \text{ via } \eta_\delta(j)\mapsto h\restriction_{F_{\mathrm{l}(\eta_\delta\restriction_j)}} \quad (j<i).$$

For $\delta\in S$ let $P_\delta=\{\mathrm{seq}(h) : h\in\mathrm{Trans}(H_{\eta_\delta},K)\}$ and for $i<\gamma$ put $P_{\eta_\delta(i)}=\{\mathrm{seq}(h) : h\in \mathrm{Trans}(H_{\eta_\delta\restriction_{(i+1)}},K)\}$. Let P_α be trivial if it has not been defined yet ($\alpha<\lambda$). By Proposition 4.8 the ladder system $\bar{\eta}$ has strong γ-uniformization and it is easy to check that the system $\bar{P}=\langle P_\alpha : \alpha\in\lambda\rangle$ satisfies the conditions of Definition 4.7 since F_n and F_n/F_m are free for $m<n\le\gamma$. Thus there exists a function $f:\lambda\to K$ such that for all $\delta\in S$, $f\restriction_{\mathrm{rg}(\eta_\delta)}\in P_\delta$ since K is of size less than or equal to γ. We now define $h : H_{\bar{\eta}}\to K$ by putting $h\restriction_{H_{\eta_\delta}} = f\restriction_{\mathrm{rg}(\eta_\delta)}$ and clearly h is well-defined and belongs to $\mathrm{Trans}(H_{\bar{\eta}},K)$ and therefore $(*)$ splits. $\square$

5. The Forcing Theorem

Before we state the main theorem of this section let us describe our strategy in order to make the statement of the main theorem plausible. Using class forcing we will construct a model of ZFC satisfying GCH in which for every regular cardinal λ there exists a sequence of stationary non-reflecting subsets S_α of λ of length λ^+ on which we have "enough" uniformization for some ladder system. Using this and the existence of (T,λ,γ)-suitable groups (for some particular γ) we can then construct, for a given torsion group T, a sequence of torsion-free groups G_α ($\alpha<\lambda^+$) of cardinality λ satisfying $\mathrm{Ext}(G_\alpha,T)=0$. These G_α will have λ-filtrations $\langle G_{\alpha,\delta} : \delta<\lambda\rangle$ whose successive quotients satisfy $\mathrm{Ext}(G_{\alpha,\delta+1}/G_{\alpha,\delta},T)\neq$

0 for $\delta \in S_\alpha$. However, we will be able to show that all these groups together do not fit into a single group $G \in \mathcal{TF}_\lambda(T)$ via embedding since this would force $\mathrm{Ext}(G,T) \neq 0$. Thus there can not be any λ-universal group for T.

THEOREM 5.1. *Let V be a model of ZFC in which the generalized continuum hypothesis GCH holds. Then for some class forcing $\mathcal{P}$ not collapsing cardinals and preserving GCH the following is true in $V^{\mathcal{P}}$:*

If $\lambda > \gamma$ are infinite regular cardinals such that $\gamma = \mathrm{cf}(\mu)$ if $\lambda = \mu^+$ then

(i) *there is a normal ideal $J = J^\lambda_\gamma$ on λ;*
(ii) *there is a stationary subset $S = S^\lambda_\gamma$ of λ such that $S \in J$;*
(iii) *if $\delta \in S$, then $\mathrm{cf}(\delta) = \gamma$;*
(iv) *S is non-reflecting, i.e. $S \cap \alpha$ is not stationary in α for every $\alpha < \lambda$;*
(v) *if $S' \subseteq S$ is stationary in λ, then there is a stationary $S^* \in J$ such that $S^* \subseteq S'$;*
(vi) *if $S' \subseteq S$ and $S' \notin J$, then $\diamondsuit_{S'}$ holds;*
(vii) *if $S' \subseteq S$ is stationary and $S' \in J$, then there exists a tree-like ladder system on S' which has μ-uniformization for all $\mu < \kappa$ if $\lambda = \kappa^+$ and κ is singular, and for all $\mu < \lambda$ otherwise;*
(viii) *there are $S_\epsilon = S^\lambda_{\gamma,\epsilon} \in J$ for $\epsilon < \lambda^+$ such that*
    (a) *if $\eta < \epsilon < \lambda^+$, then $S_\epsilon \backslash S_\eta$ is bounded;*
    (b) *if $\epsilon < \lambda^+$, then $S_{\epsilon+1} \backslash S_\epsilon$ is stationary;*
    (c) *$J = \{S' \subseteq S : \exists \epsilon < \lambda^+ \forall \epsilon < \nu < \lambda^+ \ S' \backslash S_\nu$ is not stationary $\}$.*

Moreover, if $\lambda = \mathrm{cf}(\lambda) > \gamma$, then there is a stationary $S^* = S^{\lambda,*}_\gamma$ such that

(1) *if $\alpha \in S^*$, then $\mathrm{cf}(\alpha) = \gamma$;*
(2) *S^* is non-reflecting;*
(3) *$\diamondsuit_{S^*}$ holds.*

The proof of Theorem 5.1 will be divided into several steps. First we deal with each regular λ seperately and then use Easton-support iteration to put the forcings together. We will assume a knwoledge of forcing and our notation follows that of [**J**] with the exception that $p \leq q$ means that the condition q is stronger than the condition p. Let λ be a cardinal. Recall that a poset P is called λ-*complete* if for every $\kappa < \lambda$, every ascending chain

$$p_0 \leq p_1 \leq \cdots \leq p_\alpha \quad (\alpha < \kappa)$$

has an upper bound. Moreover, P is said to be λ-*strategically complete* if Player I has a winning strategy in the following game of length κ for every $\kappa < \lambda$. Players I and II alternately choose an ascending sequence

$$p_0 \leq p_1 \leq \cdots \leq p_\alpha \quad (\alpha < \kappa)$$

of elements of P, where Player I chooses at the even ordinals; Player I wins if and only if at each stage there is a legal move and the whole sequence, $\langle p_\alpha : \alpha < \kappa \rangle$ has an upper bound (see also [**S3**, Definition A1.1]). Note that, if P is λ-strategically complete and G is generic over P, then $V[G]$ has no new functions from κ into V for all $\kappa < \lambda$, hence cardinals $\leq \lambda$ and their cofinalities are preserved.

PROPOSITION 5.2. *Let λ be a regular cardinal and assume $\lambda^{<\lambda} = \lambda$. For any regular $\kappa < \lambda$, there exists a poset $\mathbb{Q}$ of cardinality $\leq \lambda$ which is λ-strategically complete (and hence preserves all cardinals and preserves cofinalities $\leq \lambda$) and is* *such that, for G generic over P, in $V[G]$ there exists a non-reflecting stationary and co-stationary subset S of λ such that every member of E has cofinality κ. (Here, co-stationary means that the set $\lambda \backslash S$ is also stationary).*

Proof. The proof is similar to the proof of [**ES**, Lemma 2.3] but for the convenience of the reader we state it briefly. We let Q be the set of all functions $q : \alpha \to 2 = \{0, 1\}$ $(\alpha < \lambda)$ such that $q(\mu) = 1$ implies that $\mathrm{cf}(\mu) = \kappa$ and such that for all limits $\delta \le \alpha$, the intersection of $q^{-1}[1]$ with δ is not stationary in δ. Then, for G generic over P,

$$S = \bigcup\{q^{-1}[1] : q \in G\}$$

will be the desired set. We have to prove that S is stationary and co-stationary in λ. Hence, assume that q forces f is the name of a continuous increasing function $\bar{f} : \lambda \to \lambda$; choose an ascending chain

$$q_o \le q_1 \le \cdots \le q_\alpha \quad (\alpha < \kappa)$$

such that for each α there exist $\beta_\alpha, \gamma_\alpha$ such that $q_\alpha \Vdash \bar{f}(\beta_\alpha) = \gamma_\alpha$ and

$$\mathrm{dom}(q_\alpha) \ge \gamma_\alpha > \mathrm{dom}(q_\mu)$$

for all $\mu < \alpha$. Let $\delta = \sup\{\gamma_\alpha : \alpha < \kappa\} = \sup\{\mathrm{dom}(q_\alpha) : \alpha < \kappa\}$ and let

$$q_i = \bigcup\{q_\alpha : \alpha < \kappa\} \cup \{(\delta, i)\},$$

for $i = 0, 1$. Then $q_i \in Q$ $(i = 0, 1)$ since $q_i^{-1}[1]$ is not stationary in δ, because δ has cofinality κ. Moreover, $q_1 \Vdash \delta \in \mathrm{rg}(f) \cap S$ and $q_0 \Vdash \delta \in \mathrm{rg}(f) \cap (\lambda \backslash S)$.
Since Q has cardinality $\le \lambda$, it preserves cardinals $> \lambda$. To show that all cardinals $\le \lambda$ are preserved (and their cofinalities), it suffices to prove that Q is λ-strategically complete. Let $\tau < \lambda$ be a limit ordinal. Let Player I choose q_α for even α such that $\mathrm{dom}(q_\alpha)$ is a successor ordinal, say $\delta_\alpha + 1$, and $q_\alpha(\delta_\alpha) = 0$. Moreover, at limit ordinals α he chooses q_α to have domain $= \sup\{\delta_\beta : \beta < \alpha\} + 1$. Then $q = \bigcup\{q_\alpha : \alpha < \mu\}$ is a member of Q because $\{\delta_\alpha : \alpha < \mu, \alpha \text{ even }\}$ is a cub in $\mathrm{dom}(q)$ which misses $q^{-1}[1]$. This is a winning strategy for Player I and thus Q is λ-strategically complete. □

The next proposition is a collection of results from [**S2**], [**S3**] and [**S4**] (see also [**S5**]).

Proposition 5.3. *Let $\lambda > \gamma$ be regular infinite cardinals. Moreover, assume $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and let S be a non-reflecting, stationary and co-stationary subset of λ such that each member of S has cofinality γ. Furthermore, let $\gamma = \mathrm{cf}(\kappa)$ if $\lambda = \kappa^+$. Then there exists a poset P of cardinality $\le \lambda^+$ which is λ-strategically complete, satisfies the λ^+ chain condition, adds no new sequences of length $< \lambda$ and has the following properties:*

- (i) *S is non-reflecting, stationary and co-stationary in λ in V^P;*
- (ii) *if λ is inaccessible, then every ladder system on S has μ-uniformization for all $\mu < \lambda$; in particular, there exists a tree-like ladder system on S;*
- (iii) *if $\aleph_2 \le \lambda = \kappa^+$ and κ is regular, then every ladder system on S has μ-uniformization for all $\mu < \lambda$; in particular, there exists a tree-like ladder system on S;*
- (iv) *if $\lambda = \aleph_1$, then there is a tree-like ladder system on S which has μ-uniformization for all $\mu < \lambda$;*

(v) *if* $\lambda = \kappa^+$ *and* κ *is singular, then there is a tree-like ladder system on* S *which has* μ*-uniformization for all* $\mu < \kappa$*.*

Proof. For λ inaccessible the proof is contained in [**S3**, Case A] and also for the case of $\lambda = \kappa^+$, κ regular (see [**S3**, Case B]). For $\lambda = \aleph_1$ see [**S2**, V 1.7] and for $\lambda = \kappa^+$, κ singular see [**S4**, 2.10, 2.12]. Moreover, simpler versions with less complicated and comprehensive proofs can be found in [**S1**] for all cases if we drop the requirements "for every ladder system..." which is in fact not really needed for our purposes. Finally, let us remark that the co-stationarity is only needed for λ being the successor of a regular cardinal or inaccessible. $\square$

Theorem 5.4. *Let* λ *be a regular cardinal such that* $\lambda^{<\lambda} = \lambda$ *and* $2^\lambda = \lambda^+$*. Then there is a poset* P *of cardinality* $\leq \lambda^+$ *satisfying the* λ^+*-chain condition which is* λ*-strategically complete and adds no new sequences of length* $< \lambda$ *such that in* V^P *for every regular* $\gamma < \lambda$ *with* $\gamma = \mathrm{cf}(\kappa)$ *if* $\lambda = \kappa^+$ *the statements (i) to (viii) and (1) to (3) of Theorem 5.1 hold.*

Before we prove Theorem 5.4 let us show why this is enough to prove Theorem 5.1.

Proof. (of Theorem 5.1)
We start with a model V of ZFC satisfying the generalized continuum hypothesis GCH. For any ordinal α let $P_\alpha = \left\langle P_j, \dot{Q}_i : j \leq \alpha, i < \alpha \right\rangle$ be an iteration with Easton support; i.e. we take direct limits when $\aleph_\alpha$ is regular and inverse limits elsewhere or equivalently we have bounded support below inaccessibles and full support below non-inaccessibles. For any ordinal i, let $\dot{Q}_i$ be the forcing notion in V^{P_i} described in Theorem 5.4 for $\lambda = \aleph_i$ if $\aleph_i$ is regular and let it be 0 elsewhere. Let P be the direct limit of the P_α ($\alpha \in \mathrm{ORD}$). We claim that P has the desired properties. The proof is very similar to the proof of [**ES**, Theorem 2.1] and hence we will only state the main ingredients which are needed.

(i) For every κ and Easton support iteration $\left\langle P_j, \dot{Q}_i : \kappa \leq j \leq \alpha, \kappa \leq i < \alpha \right\rangle$, if each $\dot{Q}_i$ is κ-strategically complete, then so is P_α.
(ii) $P = P_\alpha * P_{\geq\alpha}$, where, in V^{P_α}, $P_{\geq\alpha}$ is the direct limit of P^α_β ($\beta \in \mathrm{ORD}$), with P^α_β the Easton support iteration $\left\langle P^\alpha_j, \dot{Q}^\alpha_i : j \leq \beta, i < \beta \right\rangle$ where $\dot{Q}^\alpha_i = \dot{Q}_{\alpha+i}$.
(iii) $|P_n| = 1$ (for $n \in \omega$); if $\aleph_\delta$ is singular, $|P_\delta| \leq \aleph_\delta$ and $|P_\delta| \leq \aleph_{\delta+1}$ if $\aleph_\delta$ is regular, hence inaccessible.
(iv) $P_{\geq\alpha}$ is $\aleph_\alpha$-strategically complete, and $P_{\geq\alpha+1}$ is even $\aleph_{\alpha+n}$- strategically complete for all $n \in \omega$.

By construction (i) to (viii) and (1) to (3) of Theorem 5.1 are now satisfied in V^P. Note, that stationarity is preserved in the iteration because $P_{\geq\alpha}$ is $\aleph_\alpha$-strategically complete. It remains to prove that V^P is a model of ZFC satisfying GCH and preserving cofinalities (and hence cardinals). This follows very similar as in the proof of [**ES**, Theorem 2.1] and hence we will omit the proof here and leave it as an exercise to the reader. $\square$

It remains to prove Theorem 5.4.

Proof. (of Theorem 5.4) The proof follows from the results in [**S3**] and [**S4**] but for the convenience of the reader we shall give some details. If $\lambda = \kappa^+$ is a successor cardinal, then we are easily done since there is only one γ, namely $\gamma = \mathrm{cf}(\kappa)$ under consideration. We choose P to be the two step iterated forcing of the two forcings from Proposition 5.2 and from Proposition 5.3 with $\gamma = \mathrm{cf}(\kappa)$. Moreover, we may assume that P also forces the sets $S^* = S_\gamma^{\lambda,*}$ satisfying Theorem 5.1 (1) to (3) by an initial forcing. Note that the assumptions on λ in Theorem 5.4 are satisfied by [**HSW**, Ex 12, page 70]. If λ is inaccessible, then it is more complicated since we have to deal with all regular $\gamma < \lambda$. But this was already done in [**S3**, Case B] where a stronger version of Proposition 5.3 was shown. It was proved that there is even a forcing notion P such that for all regular $\gamma < \lambda$ and given non-reflecting stationary, co-stationary subsets S_γ of λ consisting of ordinals $\alpha \in S_\gamma$ of cofinality γ, every ladder system on S_γ has μ-uniformization for all $\mu < \lambda$. Using this stronger result and again forcing the sets $S_\gamma^{\lambda,*}$ satisfying Theorem 5.1 (1) to (3) it remains to show that we can define the ideal $J = J_\gamma^\lambda$ satisfying Theorem 5.1 (vi) and (viii) (point (vii) of Theorem 5.1 is clear).

Our forcing P (from [**S3**] and [**S4**]) is the result of a $(< \lambda)$-support iteration of length λ^+, say $\left\langle P_i, \dot{Q}_j : i \leq \lambda^+, j < \lambda^+ \right\rangle$. Let us assume that P_γ forces the set $S = S_\gamma^\lambda$ and the tree-like ladder system $\bar{\eta}$ on it. In V^{P_γ} there exists a sequence $\left\langle S_\epsilon = S_{\gamma,\epsilon}^\lambda : \epsilon < \lambda^+ \right\rangle$ such that

(i) $S_\epsilon \subseteq S$;
(ii) $\eta < \epsilon < \lambda^+$ implies $S_\epsilon \backslash S_\eta$ is bounded;
(iii) $\epsilon < \lambda^+$ implies $S_{\epsilon+1} \backslash S_\epsilon$ is staionary.

Now we define $J = J_\gamma^\lambda$ as $J = \{S' \subseteq S : \exists \epsilon < \lambda^+ \forall \epsilon < \nu < \lambda^+ \ S' \backslash S_\nu$ is not stationary $\}$. For each $i < \lambda^+$, $\dot{Q}_i$ forces μ_i-uniformization for the ladder system $(\dot{A}_i, \bar{\dot{f}}_i)$ where $\bar{\dot{f}}_i = \left\langle \dot{f}_\delta^i : \delta \in \dot{S}_i \right\rangle$. Here $\dot{S}_i$ and $\dot{f}_\delta^i$ are P_i-names for a member of ${}^{A_\delta}(\mu_i)$. Thus we obtain $\Vdash$ " $\dot{S}_i \subseteq \dot{S}$ is stationary and there is $\epsilon < \lambda^+$ such that $(\forall \epsilon < \eta < \lambda^+)(\dot{S}_i \cap \dot{S}_\eta \backslash \dot{S}_\epsilon$ is not stationary ". A condition in $\dot{Q}_i$ is for instance given by $g : \alpha \to \mu_i$ such that $\delta \in \dot{S}_i$, $\delta \leq \alpha$ implies $f_\delta^i \subseteq^* g$.

It remains to show that $\diamondsuit_{S'}$ holds for $S' \notin J$. Choose $i < \lambda^+$ such that S' comes from V^{P_i}. For some $j \in (i, \lambda^+)$, Q_j is adding λ cohen reals and we can interprete it as adding a diamond sequence $\langle \rho_\epsilon : \epsilon \in S' \rangle$ by initial segments. Trivially, in $V^{P_{j+1}}$, $\diamondsuit_{S'}$ holds and we may work in $V^{P_{j+1}}$ now. For χ large enough we can find for every $x \in H(\chi(\lambda))$ an increasing continuous sequence $\bar{N} = \langle N_i : i < \lambda \rangle$ of elementary submodels of $H(\chi(\lambda), \epsilon, <^*)$ of cardinality less than λ such that $x \in N_0$, $S' \in N_0$ and $\bar{N} \restriction_{(i+1)} \in N_{i+1}$ for all $i < \lambda$. Let $E = \{\delta < \lambda : N_\delta \cap \lambda = \delta\}$ which is a cub in λ. Thus, for $\delta \in E$, for every $p \in (P/P_{j+1}) \cap N_\delta$ there is a condition $p \leq q \in P/P_{j+1}$ which is $(N_\delta, P/P_{j+1})$ generic and forces a value to $G \cap N_\delta$. It is known that we can now replace the diamond sequence on S' which we have in $V^{P/P_{j+1}}$ by one that is preserved by forcing with P/P_{j+1} since P/P_{j+1} adds no new subsets of λ of length less than λ and by the strategically completeness. This finishes the proof. □

6. Application to Kulikov's question

In this final section we show that the answer to Kulikov's question is consistently no for large classes of groups (not necessarily torsion) and cardinals λ.

Definition 6.1. *Let T be a group of cardinality less than λ and let H be a torsion-free group of cardinality λ. Moreover, let $\{H_\alpha : \alpha < \lambda\}$ be a λ-filtration of H. Then*

$$S_\gamma^\lambda[H,T] = \{\delta \in S : \mathrm{Ext}(H_{\delta+1}/H_\delta, T) \neq 0\}.$$

Let us remark that the definition obviously depends on the given filtration. We could make it independent by defining an equivalence relation and letting $\Gamma_H^S(T) = \{E \subseteq S :$ there exists a stationary set $R \subseteq S$ such that $E \cap R = S_\gamma^\lambda[H,T] \cap R\}$. But since we don't need this and since the filtration under consideration shall always be clear from the context we avoid additional notations.

Theorem 6.2. *Let T be a group of cardinality less than λ and let H be a torsion-free group of size λ. Let $S = S_\gamma^\lambda[H,T]$. If $\diamondsuit_S$ holds, then* $\mathrm{Ext}(H,T) \neq 0$.

Proof. The proof is standard and can be found for example in [**EM**, Theorem XII.1.15]. □

We shall now work in the model $V^{\mathcal{P}}$ obtained in Theorem 5.1. Thus all results shall be consistency results with ZFC and GCH. The symbol $(V^{\mathcal{P}})$ indicates that the statement holds in our model $V^{\mathcal{P}}$.

Theorem 6.3 $(V^{\mathcal{P}})$. *Assume that T is a group of cardinality less than λ and G is a torsion-free group which is (T,λ,γ)-suitable with $\gamma < \lambda$ regular, $\gamma = \mathrm{cf}(\mu)$ if $\lambda = \mu^+$. If $\lambda > 2^{(|T|^\gamma)}$ or μ is regular, then there is no λ-universal group for T.*

Proof. Assume that there is a λ-universal group G^* for T. If $\lambda > 2^{(|T|^\gamma)}$, then for $\epsilon < \lambda^+$ we apply Theorem 4.6 to $S_{\gamma,\epsilon}^\lambda$, T, G and the tree-like ladder system $\bar\eta_\epsilon$ on $S_{\gamma,\epsilon}^\lambda$ which exists by Theorem 5.1 (vii). Note that no $S_{\gamma,\epsilon}^\lambda$ reflects in any $\alpha < \lambda$ and that $\bar\eta_\epsilon$ has $2^{(|T|^\gamma)}$-uniformization since $\lambda > 2^{(|T|^\gamma)}$ (and $\mu > 2^{(|T|^\gamma)}$ if $\lambda = \mu^+$, μ singular). If μ is regular, then we may apply Theorem 4.9 instead of Theorem 4.6. Note that $\diamondsuit_\mu$ holds in $V^{\mathcal{P}}$ by Theorem 5.1 (1) to (3). Hence for each $\epsilon < \lambda^+$ we obtain a torsion-free group $H_\epsilon = \bigcup_{\alpha\in\lambda} H_{\epsilon,\alpha}$ satisfying $\mathrm{Ext}(H_\epsilon, T) = 0$ and $\mathrm{Ext}(H_{\epsilon,\alpha+1}/H_{\epsilon,\alpha}, T) \neq 0$ for all $\alpha \in S_\gamma^\lambda[H_\epsilon, T] = S_{\gamma,\epsilon}^\lambda$. By universality of G^* there exist embeddings $i_\epsilon : H_\epsilon \longrightarrow G^*$. We claim that for each $\epsilon < \lambda^+$ we have $S_{\gamma,\epsilon}^\lambda[H_\epsilon,T] \subseteq S_\gamma^\lambda[G^*,T]$ modulo a non-stationary set. To see this choose a λ-filtration $G^* = \bigcup_{\alpha\in\lambda} G_\alpha$ of G^* such that $\mathrm{Ext}(G_{\alpha+1}/G_\alpha, T) = 0$ if and only if for some $\beta > \alpha$ we have $\mathrm{Ext}(G_\beta/G_\alpha, T) = 0$. Fix $\epsilon < \lambda^+$, then there is a cub $C_\epsilon \subseteq \lambda$ such that for all $\alpha \in C_\epsilon$ we have $H_{\epsilon,\alpha} = G_\alpha \cap H_\epsilon$. Thus for $\alpha < \beta \in C_\epsilon$ it follows that $H_{\epsilon,\beta}/H_{\epsilon,\alpha} = (G_\beta \cap H_\epsilon)/(G_\alpha \cap H_\epsilon) \subseteq G_\beta/G_\alpha$ and hence $\mathrm{Ext}(H_{\epsilon,\beta}/H_{\epsilon,\alpha}, T) \neq 0$ implies $\mathrm{Ext}(G_\beta/G_\alpha, T) \neq 0$. Therefore also $\mathrm{Ext}(G_{\alpha+1}/G_\alpha, T) \neq 0$ and $C_\epsilon \subseteq S_\gamma^\lambda[G^*,T]$. Thus by the definition of the normal ideal J (see Theorem 5.1 point (viii)) we have $\bar S = S_\gamma^\lambda[G^*,T] \notin J$ and therefore $\diamondsuit_{\bar S}$ holds by Theorem 5.1 point (vi). Hence $\mathrm{Ext}(G^*,T) \neq 0$ by Theorem 6.2 - a contradiction. □

Before we prove some corollaries let us remark that in $V^{\mathcal{P}}$ the general continuum hypothesis GCH holds. Hence, for a group T we have $2^{(|T|^\gamma)} \leq max\{\gamma^{++}, |T|^{++}\}$ which the reader should keep in mind.

Corollary 6.4 $(V^{\mathcal{P}})$. *Let T be a group of cardinality less than λ which is not cotorsion. If λ is strongly inaccessible, then there is no λ-universal group for T.*

Proof. Since λ is strongly inaccessible it is a limit ordinal and we may choose $\gamma = \omega$. Moreover, for every $\alpha < \lambda$ we have $2^\alpha < \lambda$, hence $\lambda > 2^{(|T|^\omega)}$. Lemma 3.2 implies that there is a (T, λ, ω)-suitable group for T and hence Theorem 6.3 shows that there is no λ-universal group for T. □

Corollary 6.5 ($V^{\mathcal{P}}$). *Let T be a group such that $|T|^+ < \lambda$ which is not cotorsion. If $\lambda = \mu^+$ and $\mathrm{cf}(\mu) = \omega$, then there is no λ-universal group for T.*

Proof. Since $\mathrm{cf}(\mu) = \omega$, we have $\lambda > 2^{(|T|^\omega)}$ and hence we may choose $\gamma = \omega$ and apply Theorem 6.3 to see that there is no λ-universal group for T. Note, that there is a (T, λ, ω)-suitable group by Lemma 3.2. □

Corollary 6.6 ($V^{\mathcal{P}}$). *Let T be a group of cardinality less than λ which is not cotorsion and let $\lambda > 2^{(|T|^\omega)}$ be regular. Moreover, let T have an epimorphic image of size less than or equal to $\mathrm{cf}(\mu)$ if $\lambda = \mu^+$ (e.g. T torsion, T mixed splitting or $|T| \leq \mathrm{cf}(\mu)$). If $2^{(|T|^{\mathrm{cf}(\mu)})} < \lambda$ then there exists no λ-universal group for T.*

Proof. We choose $\gamma = \omega$ if λ is a limit ordinal and $\gamma = \mathrm{cf}(\mu)$ if $\lambda = \mu^+$. By Theorem 5.1 (1) to (3) there exists a stationary non-reflecting set $S \subseteq \gamma$ consisting of limit ordinals of cofinality ω such that $\diamondsuit_S$ holds. Let H be the epimorphic image of T as stated. By assumption we may apply Proposition 3.5 to S, H and λ, γ to obtain a (T, λ, γ)-suitable group G. Since $\lambda > 2^{(|T|^\gamma)}$ we apply Theorem 6.3 to see that there is no λ-universal group for T. □

Corollary 6.7. *Let T be a group which is not cotorsion and λ a cardinal. Then there exists a regular uncountable cardinal $\delta > \lambda$ such that there exists no δ-universal group for T.*

Corollary 6.8 ($V^{\mathcal{P}}$). *Let T be a torsion group of regular cardinality which is not cotorsion. If $\lambda > |T|$ is regular, then there is no λ-universal group for T.*

Proof. Let T and λ be as stated. If λ is strongly inaccessible , then Corollary 6.4 gives the claim. Hence assume that $\lambda = \kappa^+$. Thus $|T| \leq \kappa$ and by Proposition 3.5 there exists a $(T, \lambda, \mathrm{cf}(\kappa))$-suitable group G for T. If κ is regular then Theorem 6.3 shows that there is no λ-universal group for T. If κ is singular, then $\kappa > (|T|^+)^+$ since $|T|$ is regular. Hence $\lambda > 2^{(|T|^{\mathrm{cf}(\kappa)})}$ and again Theorem 6.3 gives the claim. □

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