Abstract. Under certain cardinal arithmetic assumptions, we prove that for every large enough regular \( \lambda \) cardinal, for many regular \( \kappa < \lambda \), many stationary subsets of \( \lambda \) concentrating on cofinality \( \kappa \) has the “middle diamond”. In particular, we have the middle diamond on \( \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \). This is a strong negation of uniformization.

(2018.10.18) We have added some details to Case 2 in the proof of 1.11. Works continuing it are [Sh 898] and [Sh 1028]. We thank Ari Brodski and Adi Jarden for their helpful comments.

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§0 Introduction

[We state concise versions of the main results: provably in ZFC there are many cases of the middle diamond, for many triples \((\lambda, \Upsilon, \kappa)\) where \(\kappa < \lambda\) are regular, \(2 \leq \Upsilon < \lambda\) we can give for any \(\delta \in S^\lambda_\kappa\) a function \(f_\delta\) from some subset \(C_\delta\) of \(\delta\) of order type \(\kappa\) unbounded in \(\delta\) into \(\Upsilon\) such that any \(f : \lambda \to \Upsilon\) extends \(f_\delta\) for stationarily many \(\delta\)'s. We also review some history.]

§1 Middle Diamond: sufficient conditions

[We define families of relevant sequences \(\bar{C} = \langle C_\delta : \delta \in S \rangle\) and properties needed for phrasing and proving our results (see Definition 1.5) and give easy implications; (later in 1.22, 1.25 we define other versions of middle diamond). Then (in 1.10) prove a version of the middle diamond for \((\lambda, \Upsilon, \theta, \kappa)\) guessing only the colour of \(f \upharpoonright C_\delta\) when \(\lambda = \text{cf}(2^\mu)\), the number of possible colours, \(\theta\), is not too large and \(\kappa\) belongs to a not so small set of regular cardinals \(\kappa < \mu\) (e.g., every large enough regular \(\kappa < \beth_\omega\) if \(\beth_\omega \leq \mu\)). The demand on \(\kappa\) says \(2^\mu\) is not the number of \(\kappa\)-branches of a tree with \(< 2^\mu\) nodes. We then prove that a requirement on \((\mu, \theta)\) is reasonable (see 1.11, 1.9, 1.13). In 1.20 we get the version of middle diamond giving \(f \upharpoonright C_\delta\) from the apparently weaker one guessing only the colour of \(f \upharpoonright C_\delta\). We quote various needed results and definitions in 1.3, 1.24.]

§2 Upgrading to larger cardinals

[Here we try to get such results for “many” large enough regular \(\lambda\). We succeed to get it for every large enough not strongly inaccessible cardinal. We prove upgrading: i.e., give sufficient conditions for deducing middle diamond on \(\lambda_2\) from ones on \(\lambda_1 < \lambda_2\) (see 2.1) and 2.3 (many disjoint ones), 2.10, 2.11). We deduce (in 2.4) results of the form “if \(\lambda\) is large enough satisfying a weak requirement, then it has middle diamond for quite many \(\kappa, \theta\)”, using upgrading claims on the results of §1 and generalization of [EK] (see 2.5, 2.6, 2.7). We deal with such theorem more than necessary just for the upgrading.]
§0 Introduction

Using the diamond (discovered by Jensen [J]) we can construct many objects; wonderful but there is a price - we must assume $V = L$ or at least instances of G.C.H. (see Gregory [Gr], Shelah [Sh 108]). In fact $\lambda = \mu^+ = 2^\mu > \beth_\omega \Rightarrow \diamondsuit_{\delta < \lambda : \text{cf}(\delta) = \kappa}$ for every regular large enough $\kappa$ depending on $\mu$ (see [Sh 460] and citations there).

Naturally we would like to get the benefit without paying the price. A way to do this is to use the weak diamond (see [DvSh 65], [Sh:f, AP, §1], [Sh 208], [Sh 638] and lately [Sh 755], see on the definable version in [Sh:f, AP, §3], [Sh 576]).

In the diamond principle for a stationary $S \subseteq \lambda$, for $\delta \in S$ we guess $\eta \upharpoonright \delta$ for $\eta \in \lambda^2$.

In the weak diamond, for stationary $S \subseteq \lambda$ for $\delta \in S$ we do not guess $\eta \upharpoonright \delta$ for $\eta \in \lambda^2$, we just guess whether it satisfies a property - is black or white, i.e., the value of $F_\eta(\eta \upharpoonright \delta) \in \{0, 1\}$ for a pre-given $F$. Many times it is just enough in constructions using diamond to “avoid undesirable company”: whatever you will do I will do otherwise. This is, of course, a considerably weaker principle, but for $\aleph_1$ it follows from CH (which does not imply $\diamondsuit_{\aleph_1}$) and even WCH (i.e. $2^{\aleph_0} < 2^{\aleph_1}$). In fact, it holds for many cardinals: $2^\lambda > 2^{<\lambda} = 2^\theta$ suffices. Having $F$ with two colours (= values) seemed essential. In [Sh 638], we get more colours for $\aleph_1$ if the club filter on $\omega_1$ is $\aleph_2$-saturated but this is a heavy assumption. In [Sh 755], if $\lambda = \lambda^{<\lambda} = 2^\mu$, it is proved that for “many” regular $\kappa < \lambda$, if $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ is stationary then we have the weak diamond for $S$. Now what does “many” mean? For example, if $\lambda \geq \beth_\omega$, “many” include every large enough regular $\kappa < \beth_\omega$; note that it is for “every stationary $S$” unlike the situation, e.g. for $\aleph_1$ under CH (see [Sh:f, V]).

We get there more cases if the colouring $F$ on $\delta^2$ depends only on $\eta \upharpoonright C_\delta$, where $C_\delta \subseteq \delta = \text{sup}(C_\delta)$ and $C_\delta$ is “small”.

Here we get more than two colours and moreover: we guess $\eta \upharpoonright C_\delta, C_\delta \subseteq \delta = \text{sup}(C_\delta)$ and so can choose $\langle M_\delta : \delta \in S \rangle$ such that $M_\delta$ is a model with universe $C_\delta$ and relations only (and e.g. vocabulary $\subseteq \mathcal{H}(\aleph_0)$) such that if $M$ is a model with universe $\lambda$ and relations only (and vocabulary $\subseteq \mathcal{H}(\aleph_0)$) then for stationarily many $\delta \in S, M \upharpoonright C_\delta$ is $M_\delta$. Of course, if $\langle \exists \mu < \lambda \rangle [2^\mu > \lambda]$ then we cannot have $C_\delta = \delta$ so this is a very reasonable restriction. (It would be even better to have $M_\delta \prec M$ below but we do not know). Also for “almost” all $\lambda$ large enough the result holds for many regular $\kappa$.

The motivation has been trying to get better principles provably in ZFC.

We use partial squares and $I[\lambda]$, see [Sh 108], [AShS 221], [Sh 237e], [Sh 420, §1] and [DjSh 562]. We prove (in fact this holds for quite “many” stationary $S \subseteq S^\lambda_\kappa$).

0.1 Theorem. For every $\mu$ strong limit of uncountable cofinality, for every regular $\lambda > \mu$ which is not strongly inaccessible, for every large enough regular $\kappa < \mu$, for every $\chi < \mu, S^\lambda_\kappa$ has the $\chi$-middle diamond (called later “$\lambda$ has the $(\kappa, \chi^+)$-MD$_2$-
property”), which means:

\[ \Box_{\lambda,\kappa} \text{ for some stationary } S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \text{ for every relational vocabulary } \tau \text{ of cardinality } \chi (\text{so } \tau \text{ has no function symbols) we can find } \bar{M} = \langle M_{\delta} : \delta \in S \rangle \text{ which is a } \tau\text{-MD-sequence} \]

(a) \( M_{\delta} \) is a \( \tau \)-model

(b) the universe of \( M_{\delta} \) is an unbounded subset of \( \delta \) of order type \( \chi \times \kappa \) including \( \chi \)

(c) if \( M \) is a \( \tau \)-model with universe \( \lambda \) then for stationarily many \( \delta \in S \) we have \( M_{\delta} \subseteq M \).

It seems that this is stronger in sufficient measure from the weak diamond to justify this name; there are many properties between the weak diamond and the property from 0.1 and we have to put the line somewhere. We shall use the adjective middle diamond to describe even cases with colouring with more than two values (justified by 1.20).

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0.2 Theorem. In 0.1 we can have also \( \bar{f} = \langle f_{\delta} : \delta \in S \rangle \) where \( f_{\delta} : u_{\delta} \rightarrow \delta, u_{\delta} \subseteq \delta, \lambda|u_{\delta}| = \lambda \) such that:

(d) if \( M \) is a \( \tau \)-model with universe \( \lambda \) and \( f : \lambda \rightarrow \lambda \) and \( u \subseteq \lambda, \lambda|u| = \lambda, \) then for stationarily many \( \delta \in S \) we have \( M_{\delta} \subseteq M \lor (f_{\delta} = f \upharpoonright u \subseteq f) \lor u_{\delta} = u \).

Proof of 0.1 and 0.2. By 2.9(1) if \( \lambda > \mu \) is regular not strongly inaccessible, then for every large enough regular \( \kappa < \mu \) the following holds (see Definition 1.19(4)):

\[(*)_{\kappa} \text{ if } \theta < \mu \text{ then some } (\lambda, \kappa, \kappa^{+})\text{-parameter } \bar{C} \text{ has the } \theta\text{-Md property; let } \bar{C} = \langle C_{\delta} : \delta \in S \rangle.\]

Now without loss of generality \( \delta \in S \Rightarrow (\chi \times \theta)^{\omega}|\delta, \) and let \( C_{\delta}' = \{ \gamma : \gamma < \chi \text{ or } (\exists \beta \in C_{\delta})(\beta \leq \gamma < \beta + \chi) \}. \)

Lastly, let \( \theta = 2^{\lambda^{+}+|\tau|} \) and by 1.20 we get the conclusion of 0.1.

As for 0.2, Clause (d) is easy by having a pairwise disjoint sequence of suitable \( \langle S_{i} : i < \lambda \rangle \) which holds by 2.9(2).
Note that having $M_\delta$’s is deduced in 1.20 from “$C$ has the $\theta$-Md-property” with $\theta$ large enough. Note that 0.1 has obvious applications on various constructions, e.g., construction of abelian groups $G$ with $\text{Ext}(G, K) \neq \emptyset$.

Note that in 0.1, 2.9 instead of using “large enough $\kappa < \mu \ldots$”, $\mu$ strong limit and quoting [Sh 460] we can specifically write what we use on $\kappa$ (and on $\theta$). Also in 0.1, if we omit $\text{cf}(\mu) > \aleph_0$ (but still $\mu$ is uncountable) the case $\lambda$ successor is not affected, and in the case $\mu = \beth_\delta$ when $\omega^2$ divides $\delta$, the statement instead “every large enough $\kappa$” can be replaced by “for arbitrarily large regular $\kappa < \mu$”, or even: “for arbitrarily large strong limit $\mu' < \mu$, for every large enough regular $\kappa < \mu'$”. A consequence is (see more on this principle in [Sh 667]).

0.3 Conclusion. 1) If $\lambda, \kappa, \chi, S, \langle M_\delta : \delta \in S \rangle$ are as in 0.1 and $\gamma < \kappa \lor \gamma < \chi$, then we can find $\eta = \langle \eta_\delta : \delta \in S \rangle$, $\eta_\delta$ is an increasing sequence of length $\kappa$ of ordinals $< \delta$ with limit $\delta$ such that:

$$\square \text{ if } F : \lambda \rightarrow \gamma, \text{ then for stationarily many } \delta \in S \text{ we have } i < \kappa \Rightarrow F(\eta_\delta(2i)) = F(\eta_\delta(2i + 1)).$$

2) In fact it is enough to find a $(\lambda, \kappa, \chi)$-Md-parameter $\langle C_\delta : \delta \in S \rangle$ which has the $(\gamma, 2^{\lceil \gamma \rceil^+})$-MD-property (see Definition 1.6(1)).

**Proof.** 1) By (2).

2) Let $\tau = \{P_\zeta : \zeta < |\gamma|^+\}, p_i$ a monadic predicate. Let $\langle M_\delta : \delta \in S \rangle$ be as in 0.1 for $\tau$. Let $F : \lambda \rightarrow \gamma$ and we let $M$ be the following $\tau$-model with universe $\lambda : P^M_\zeta = \{\alpha : \zeta = \text{pr}(\zeta_0, \zeta_1), \zeta_0 < \zeta_1 < |\gamma|^+ \text{ and } F(\alpha + \zeta_0) = F(\alpha + \zeta_1), \alpha \in M_\delta \Rightarrow \alpha + |\gamma|^+ < \text{Min}(M_\delta \setminus \alpha)\}$. Without loss of generality $\text{otp}(|M_\delta|) = \kappa, \delta \in S \Rightarrow |\gamma|^+ / \delta$. Let $\langle \eta_\delta(2i), \eta_\delta(2i + 1) \rangle = (\alpha_i^0 + \zeta_2^0, \alpha_i^0 + \zeta_2^0)$ when $\alpha_i^0$ is the $i$-th member of $M_\delta, \alpha_i^0 \in P$.

Note that if a natural quite weak conjecture on pcf holds, then we can replace $\beth_\omega$ by $\aleph_\omega$ (and similar changes) in most places but still need $\lambda > \lambda_1 > \lambda_2 > 2^{\lambda_2} > 2^{\lambda_0} \geq 2^{\aleph_\omega}$ in the stepping-up lemma; e.g. for every regular large enough $\lambda$, for some $n$, we have middle diamond with $\lambda_0$ colours on $S^\lambda_{8_n}$ - is helpful for abelian group theory. (Under weak conditions we get this for $\aleph_1$, see [Sh:F611].)

Even without this conjecture it is quite hard to have the negation of this statement as it restricts $2^\mu$ and $\text{cf}(2^\mu)$ for every $\mu$!
0.4 Definition. 1) Let “$S \subseteq \lambda$ has square” mean that $\lambda$ is regular uncountable, $S$ a stationary subset of $\lambda$ and: for some club $E$ of $\lambda$ and $S' \subseteq S \subseteq \lambda$ there is a witness $\bar{C} = \langle C_\delta : \delta \in S' \rangle$ which means:

(*) $\bar{C} = \langle C_\delta : \delta \in S' \rangle$ has the square property which means

(a) $C_\delta \subseteq \delta$ is closed,
(b) $\delta \in S'$ limit $\Rightarrow \delta = \sup(C_\delta),$
(c) $\alpha \in C_\delta \Rightarrow C_\alpha = \alpha \cap C_\delta$ and
(d) $\delta \in S \Rightarrow \otp(C_\delta) < \delta.$

1A) We say “$S \subseteq \lambda$ has a strict square” if we add

(e) $\delta \in S \cap E \Rightarrow \otp(C_\delta) = \cf(\delta).$

2) We say “$S \subseteq \lambda$ has a ($\leq \delta^*$) square” if we replace (d) above by

(d') $\delta \in S \Rightarrow \otp(C_\delta) \leq \delta^*.$

If we replace square by ($\leq \delta^*$)-square this means we add $\otp(C_\alpha) \leq \delta^*.$

3) We say ($\leq \kappa, \Theta$)-square if above the demand $\alpha \in C_\delta \Rightarrow C_\alpha = \alpha \cap C_\delta$ is restricted to the case $\cf(\alpha) \in \Theta.$

4) $S_\kappa^\lambda = \{ \delta < \lambda : \cf(\delta) = \kappa \}.$

On $I[\lambda]$ see [Sh 420, §1].

0.5 Definition. 1) For $\lambda$ regular uncountable let $I[\lambda]$ be the family of sets $S \subseteq \lambda$ which have a witness $(E, \bar{P})$ for $S \in I[\lambda]$, which means

(*) $E$ is a club of $\lambda$, $\bar{P} = \langle P_\alpha : \alpha < \lambda \rangle$, $P_\alpha \subseteq P(\alpha), |P_\alpha| < \lambda$, and for every $\delta \in E \cap S$ there is an unbounded subset $C$ of $\delta$ of order type $\delta^*$ such that $\alpha \in C \Rightarrow C \cap \alpha \in P_\alpha.$

2) Equivalently there is a pair $(E, \bar{a})$, $E$ a club of $\lambda$, $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle, a_\alpha \subseteq \alpha, \beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$ and $\delta \in E \cap S \Rightarrow \delta = \sup(a_\delta) > \otp(a_\delta)$ (or even $\delta = \sup(a_\delta)$, $\otp(a_\delta) = \cf(\delta) < \delta$).

0.6 Notation. 1) Let $\lambda, \mu, \kappa, \chi, \theta, \sigma$ denote cardinals, $\Theta$ a set of cardinals, let $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, i, j, \Upsilon$ denote ordinals but $\Upsilon \geq 2$ and $\theta \geq 2$ and $\delta$ is a limit ordinal if not said otherwise.

Clearly 1.10 and some results later continue Engelking Karlowic [EK] which continue works on density of some topological spaces, see more history in [Sh 420].
1.1 Definition. For a regular uncountable \( \lambda \), an ordinal \( \Upsilon \geq 2 \) and cardinal \( \theta \geq 2 \)

1. We say \( S \subseteq \lambda \) is \((\Upsilon, \theta)\)-small if it is \( F-(\Upsilon, \theta)\)-small for some \((\lambda, \Upsilon, \theta)\)-colouring \( F \), also called a \( \theta \)-colouring of \( ^{>\Upsilon} \lambda \), that is: a function \( F \) from \( ^{>\Upsilon} \lambda \) to \( \theta \) where

2. We say \( S \subseteq \lambda \) is \( F - (\Upsilon, \theta)\)-small if \( (F \) is a \( \theta \)-colouring of \( ^{>\Upsilon} \lambda \) and)

\[(*)_{F, S}^\theta \]  for every \( \bar{c} \in S \theta \) for some \( \eta \in ^{\Upsilon} \lambda \) the set \( \{ \delta \in S : F(\eta \upharpoonright \delta) = c_\delta \} \) is not stationary.

3. Let \( D^{md}_{\lambda, \Upsilon, \theta} = \{ A \subseteq \lambda : \lambda \setminus A \) is \((\Upsilon, \theta)\)-small\}, we call it the \((\Upsilon, \theta)\)-middle diamond filter on \( \lambda \). If \( \theta = 2 \) we call this weak diamond filter and may write \( D^{wd}_{\lambda, \Upsilon} \).

4. We omit \( \Upsilon \) if \( \Upsilon = \theta \), writing \( \theta \) instead of \((\Upsilon, \theta)\). We may omit \( \theta \) from “\( S \) is \( F - \theta \)-small”, and \((*)_{F, S}^\theta \) when clear from the context.

1.2 Claim. 1) If \( \lambda = \text{cf}(\lambda) > \aleph_0, \lambda \geq \Upsilon \geq 2 \) and \( \lambda \geq \theta \) then \( D^{md}_{\lambda, \Upsilon, \theta} \) is a normal filter on \( \lambda \).

2) Assume \( S \subseteq \lambda, 2 \leq \Upsilon_\ell \leq \lambda, 2 \leq \theta \) for \( \ell = 1, 2 \). Then \( S \) is \((\Upsilon_1, \theta)\)-small iff \( S \) is \((\Upsilon_2, \theta)\)-small

3) In fact \( \Upsilon_\ell \in [2, 2^{<\lambda}] \) is enough.

Proof. Straightforward.

Recall

1.3 Definition. 1) Let \( \chi^{<\sigma} = \text{Min}\{ |\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\sigma \text{ and every } A \in [\chi]^\sigma \text{ is included in the union of } < \sigma \text{ members of } \mathcal{P} \} \).

2) Let \( \chi^{[\sigma]} = \text{Min}\{ |\mathcal{P}| : \mathcal{P} \subseteq [\chi]^\sigma \text{ and every } A \in [\chi]^\sigma \text{ is equal to the union of } < \sigma \text{ members of } \mathcal{P} \} \).

3) \( \chi^{<\sigma}_{tr} = \sup\{ |\text{lim}_\sigma(t)| : t \) is a tree with \( \leq \chi \) nodes and \( \sigma \) levels\}.

4) For \( J \) an ideal on \( \theta \) let \( U_J(\mu) = \text{Min}\{ |\mathcal{P}| : \mathcal{P} \subseteq [\mu]^\theta \text{ and for every } f : \text{Dom}(J) \to \mu \text{ for some } u \in \mathcal{P} \text{ we have } \{ i \in \text{Dom}(J) : f(i) \in u \} \neq \emptyset \text{ mod } J \} \); we say an ideal \( J \) is an ideal on a set \( \text{Dom}(J) \) of cardinality \( \theta \). If \( J = [\theta]^{<\theta} \) we may write \( \theta \) instead of \( J \).

\( ^{1} \)so \( \Upsilon \) seems immaterial, but below when using \( C \) this is not so (see Definition 1.5)
1.4 Remark. 1) On $\chi^{<\sigma>}$, see [Sh 589], on $\chi^{<\sigma>}$, see [Sh 430], on $\chi^{<\sigma>}, \chi^{[\sigma]}$ see there and in [Sh 460] but no real knowledge of these references is assumed here.
2) Note that when $\Upsilon = \lambda$ only $F \upharpoonright \{ \delta \in \delta \in S \}$ is relevant in Definition 1.1 because for any $\eta \in \lambda$ for some club $E$ of $\lambda$ we have $\delta \in E \Rightarrow \eta \upharpoonright \delta \in \delta \delta$. So we can restrict the colourings to this set.

Below we may usually restrict ourselves to $\Upsilon = \lambda$. But if $\mu < \lambda = \text{cf}(\lambda), 2^\mu = 2^\lambda$ even for $\theta = \Upsilon = 2$, the set $\lambda$ is small (as noted by Uri Abraham, see [DvSh 65]).

1.5 Definition. 1) We say $\bar{C}$ is a $\lambda$-Md-parameter if:
(a) $\lambda$ is regular uncountable,
(b) $S$ is a stationary subset of $\lambda$,
(c) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ and $\lambda \subseteq \delta = \text{sup}(C_\delta)$.

1A) We say $\bar{C}$ is a ($\lambda, \kappa, \chi$)-Md-parameter if in addition $(\forall \delta \in S)(\forall \alpha \in C_\delta)[c_\delta(\delta) = \kappa \wedge |C_\delta \cap \alpha| < \chi]$.
2) We say that $F$ is a ($\bar{C}, \Upsilon, \theta$)-colouring if: $\theta \geq 2, \Upsilon \geq 2, \bar{C} = \langle C_\delta : \delta \in S \rangle$ is a $\lambda$-Md-parameter and $F$ is a function from $\lambda \Upsilon$ to $\theta$ such that:
- if $\delta \in S$ and $\eta_0, \eta_1 \in \delta \Upsilon$ and $\eta_0 \upharpoonright C_\delta = \eta_1 \upharpoonright C_\delta$ then $F(\eta_0) = F(\eta_1)$.
2A) If $\Upsilon = \theta$ we may omit it and write ($\bar{C}, \theta$)-colouring.
2B) In part (2) we can replace $F$ by $F = \langle F_\delta : \delta \in S \rangle$ where $F_\delta : \langle C_\delta \rangle \Upsilon \theta$ such that $\eta \in \delta \Upsilon \wedge \delta \in S \rightarrow F(\eta) = F_\delta(\eta \upharpoonright C_\delta)$. If $\Upsilon = \lambda$ we may use $F_\delta : \langle C_\delta \rangle \delta \theta$ (as we can ignore a non-stationary $S' \subseteq S$); so abusing notation we may write $F(\eta \upharpoonright C_\delta)$. Similarly if $\text{cf}(\Upsilon) = \lambda, \langle \Upsilon_\alpha : \alpha < \lambda \rangle$ is increasing continuous with limit $\Upsilon$ we may restrict ourselves to $\langle C_\delta \rangle \langle \Upsilon_\delta \rangle$.
3) Assume $F$ is a ($\bar{C}, \Upsilon, \theta$)-colouring, where $\bar{C}$ is a $\lambda$-Md-parameter.

We say $\bar{c} \in \delta \theta$ (or $\bar{c} \in \lambda \theta$) is an $F$-middle diamond sequence, or an $F$-Md-sequence if:

\[ (*) \text{ for every } \eta \in \lambda \Upsilon, \text{ the set } \{ \delta \in S : F(\eta \upharpoonright \delta) = c_\delta \} \text{ is a stationary subset of } \lambda. \]

We also may say that $\bar{c}$ is an ($F, S$)-Md-sequence or an ($F, \bar{C}$)-Md-sequence; note that $\Upsilon$ can be reconstructed from $F$.

3A) We say $\bar{c} \in \delta \theta$ is a $D - F$-Md-sequence or $D - (F, \bar{C})$-Md-sequence if $D$ is a filter on $\lambda$ (no harm to assume that $S$ belongs to $D$, or at least $S \in D^+$) and

\[ (**) \text{ for every } \eta \in \lambda \Upsilon \text{ we have } \{ \delta \in S : F(\eta \upharpoonright \delta) = c_\delta \} \neq \emptyset \text{ mod } D. \]

\[ ^2 \text{we can omit the requirement } \delta = \text{sup}(C_\delta) \text{ but then we trivially have middle diamond if } \lambda = \lambda^{<\kappa} \text{ when } \kappa = \cup\{ |C_\delta| : \delta \in S \} \]
4) We say that $\bar{C}$ is a good $(\lambda, \lambda_1)$-Md-parameter, if $\bar{C}$ is a $\lambda$-Md-parameter and for every $\alpha < \lambda$ we have $\lambda_1 > |\{C_\delta \cap \alpha : \delta \in S \& \alpha \in C_\delta\}|$. If $\lambda_1 = \lambda$ we may write “a good $\lambda$-Md-parameter”. Similarly for a $(\lambda, \lambda_1, \kappa, \chi)$-good-Md-parameter and a $(\lambda, \kappa, \chi)$-good-Md-parameter (i.e. adding that $\bar{C}$ is a $(\lambda, \kappa, \chi)$-Md-parameter.

4A) We say that $\bar{C}$ is a weakly good $\lambda$-Md-parameter if it is a $\lambda$-Md-parameter and for every $\alpha < \lambda$ we have $\lambda > |\{C_\delta \cap \alpha : \delta \in S \& \alpha \in \text{nacc}(C_\delta)\}|$. Similarly we define a weakly good $(\lambda, \kappa, \chi)$-Md-parameter and a weakly good $(\lambda, \lambda_1, \kappa, \chi)$-Md-parameter.

4B) We say $\bar{C}$ is a good $^+ - \lambda$-Md-parameter if $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a $\lambda$-Md-parameter and $\alpha \in \bar{C}_{\delta_1} \cap C_{\delta_2} \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$. Similarly we define a good $^+ - (\lambda, \kappa, \chi)$-Md-parameter.

1.6 Definition. 1) We say that $\bar{C}$ (or $(\lambda, \bar{C})$) has the $(D, \Upsilon, \theta)$-Md-property if

(a) $\bar{C}$ is a $\lambda$-Md-parameter

(b) $D$ is a filter on $\lambda$

(c) if $F$ is a $(\bar{C}, \Upsilon, \theta)$-colouring, then there is a $D - F$-Md-sequence, see Definition 1.5, part (3A).

We may omit $D$ if it is the club filter on $\lambda$.

2) We say that $\bar{C}$ has the $(D, \theta)$-Md-property as in part (1) only that $F$ is a $(\bar{C}, \theta, \theta)$-colouring. We may omit $D$ if $D$ is the club filter. We may omit $\theta$ if $\theta = 2$.

3) We say that $S \subseteq \lambda$ has the $(\Upsilon, \theta)$-Md-property when some $\bar{C} = \langle C_\delta : \delta \in S \rangle$ has, and then we say the $\theta$-Md-property if $\Upsilon = \theta$.

In Definition 1.5(2),(2A) we replace “colouring” by “colouring$_2$” if we have all $\eta : \omega \rightarrow \Upsilon$ (or $\eta : \omega \rightarrow (C_\delta) \rightarrow T)$; and let “colouring$_1$” be the original notion (so in Definition 1.5(3),(3A) we have new meaning when we use such $F$’s).

4) We write $\text{Md}_2$-property if we use colouring$_2$ $F$ and we write $\text{Md}_1$-property for the Md-property.

1.7 Remark. We may consider replacing $C_\delta$ by $(C_\delta, D_\delta)$, $D_\delta$ a filter on $C_\delta$, that is $F_\delta(\eta)$ would depend on $(\eta | C_\delta)/D_\delta$ only. For the time being it does not make a real difference.
1.8 Claim. 1) The restrictions in 1.5(2B) are O.K. That is in 1.5(2B), the first sentence is obvious; for the second sentence (if  \( \Upsilon = \lambda \)) recall that for every \( \eta \in \lambda \Upsilon \), for some club \( E \) of \( \lambda, \delta \in E \Rightarrow \eta \upharpoonright \delta \in \delta ; \) and similarly in the third sentence.
2) In Definition 1.5(4), if \( \lambda_1 > \lambda \) then goodness holds trivially.
3) If \( C = \langle C_\delta : \delta \in S \rangle \) is a weakly good \((\lambda, \kappa, \chi)\)-parameter, then for some club \( E \) of \( \lambda \) there is \( C' = \langle C'_\delta : \delta \in S \cap E \rangle \) which is a good \((\lambda, \kappa, \chi)\)-parameter.
4) If \( C = \langle C_\delta : \delta \in S \rangle \) has the \((D, \Upsilon, \theta)\)-Md-property, and \( C' = \langle C'_\delta : \delta \in S \rangle, C'_\delta \subseteq C_\delta \), then also \( C'' \) has the \((D, \Upsilon, \theta)\)-Md-property.

Proof. 1),2) Easy.
3) Note that the \( C'_\delta \)'s are not required to be a club of \( \delta \), see [Sh 420, \S 1] but we elaborate. Let \( \langle A_\beta : \beta < \lambda \rangle \) list without repetition \( \{C_\delta \cap \alpha : \delta \in S \text{ and } \alpha \in \text{nacc}(C_\delta)\} \) and be such that \( A_\beta = A_{\gamma} \cap \alpha \Rightarrow A_\beta \in \{A_\varepsilon : \varepsilon \leq \gamma\} \). Let \( E = \{\delta < \lambda : \beta < \delta \text{ and } \alpha \in S \text{ and } \beta \in \text{nacc}(C_\alpha) \} \) then \( C_\alpha \cap \beta \in \{A_\gamma : \gamma < \delta\} \), clearly \( E \) is a club of \( \lambda \).
Lastly, for \( \delta \in S \cap E \) let \( C'_\delta = \{\gamma < \delta : \text{for some } \alpha \in \text{nacc}(C_\delta) \text{ we have } C_\delta \cap \alpha = A_\gamma\} \). Now \( \langle C'_\delta : \delta \in S \cap E \rangle \) is as required.
4) Should be clear. \( \square_{1.8} \)

From the following definition the case \( \text{Sep}(\mu, \theta) \) is used in the main claim below.

1.9 Definition. 1) \( \text{Sep}(\chi, \mu, \theta_0, \theta, \Upsilon) \) means that for some \( \bar{f} \):
   
   \( (a) \; \bar{f} = \langle f_\varepsilon : \varepsilon < \chi \rangle \)
   \( (b) \; f_\varepsilon \) is a function from \( \mu(\theta_0) \) to \( \theta \)
   \( (c) \) for every \( \varphi \in \chi^\theta \) the set \( \{\nu \in \mu(\theta_0) : \text{for every } \varepsilon < \chi \text{ we have } f_\varepsilon(\nu) \neq \varphi(\varepsilon)\} \) has cardinality \( < \Upsilon \).

2) We may omit \( \theta_0 \) if \( \theta_0 = \theta \). We write \( \text{Sep}(\mu, \theta) \) if for some \( \Upsilon = \text{cf}(\Upsilon) \leq 2^\mu \) we have \( \text{Sep}(\mu, \mu, \theta, \theta, \Upsilon) \) and \( \text{Sep}(< \mu, \theta) \) if for some \( \Upsilon = \text{cf}(\Upsilon) \leq 2^\mu \) and \( \sigma < \mu \) we have \( \text{Sep}(\sigma, \mu, \theta, \theta, \Upsilon) \).

1.10 The Main Claim. Assume

\( (a) \; \lambda = \text{cf}(2^\mu), \) \( D \) is a \( \mu^+ \)-complete filter on \( \lambda \) extending the club filter
\( (b) \; \kappa = \text{cf}(\kappa) < \chi \leq \lambda \)
\( (c) \; \bar{C} = \langle C_\delta : \delta \in S \rangle \) is a \((\lambda, \kappa, \chi)\)-good-Md-parameter, or just weakly good or just \((\langle C_\delta, C'_\delta : \delta \in S \rangle \) is a good \((\lambda, \kappa, \chi)\)-Md-\( \theta \)-parameter and \( S \in D \)
\( (d) \; 2^{<\chi} \leq 2^\mu \) and \( \theta \leq \mu \)
\( (e) \; \alpha < 2^\mu \Rightarrow |\alpha|^{<\kappa > \theta} < 2^\mu \)
\( (f) \; \text{Sep}(\mu, \theta). \)
Then \( \bar{C} \) has the \((D, 2^{\mu}, \theta)\)-Md-property (recall that this means that the number of colours is \( \theta \) not just 2).

We may wonder if clause (f) of the assumption is reasonable; the following Claim gives some sufficient conditions for clause (f) of 1.10 to hold.

1.11 Claim. Clause (f) of 1.10 holds, i.e., \( \text{Sep}(\mu, \theta) \) holds, if at least one of the following holds:

(a) \( \mu = \mu^\theta \)
(b) \( U_\theta(\mu) = \mu + 2^\theta \leq \mu \)
(c) \( U_J(\mu) = \mu \) where for some \( \sigma \) we have \( J = [\sigma]^{\theta} \), \( \theta \leq \sigma, \sigma^\theta \leq \mu \)
(d) \( \mu \) is a strong limit of cofinality \( \neq \theta = \text{cf}(\theta) < \mu \)
(e) \( \mu \geq \bigtriangleup_\omega(\theta) \).

We shall prove this later.

1.12 Remark. 1) Note that \( \lambda = \lambda^{<\lambda} = \chi, \kappa = \text{cf}(\kappa) < \lambda, S = S^\lambda_\kappa, C_\delta = \delta \) is a particular case (of interest) of the assumptions (a)-(b) of 1.10 and that \( \langle C_\delta : \delta \in S^\lambda_\kappa \rangle \) is a \((\lambda, \kappa, \lambda)\)-good\(^+\) Md-parameter.
2) We can replace the requirement “\( S \subseteq S^\lambda_\kappa \)” by (implicit in (c) of 1.10)

\((***) [\delta \in S \Rightarrow \delta \text{ is a regular cardinal}]. \)

Then clause (e) of 1.10 is replaced by “\( \delta \in S \Rightarrow \delta^{<\delta^{>\mu}} < 2^\mu \)”.
3) We may consider finding a guessing sequence \( \bar{c} \) simultaneously for many colourings. In the notation of [Sh:f, AP, §1] this is the same as having large \( \mu(0) \). We may reconsider the problem of \( \lambda^+\)-entangled linear ordering where \( \lambda = \lambda^{\aleph_0} \), see [Sh 462].
4) For \( \theta = 2 \), (i.e. weak diamond) the proof below can be simplified: we can forget \( \mathcal{F} \) and choose \( g_\delta \in H^\mu\{\varrho_\eta : \eta \in \mathcal{P}_\delta\} \) such that for some \( i < \mu \) the sequence \( \langle g_\delta(i) : \delta \in S \rangle \) is a weak diamond sequence for \( \mathcal{F} \).
5) Note that the “minimal” natural choices for getting \( \theta\)-MD property are \( \kappa = \aleph_0, \theta = 2^{\aleph_0}, \mu = 2^\theta, \lambda = \text{cf}(2^\mu) \).
6) Note that in the main claim (1.10) if \( 2^\mu \) is regular, \( \Upsilon = 2^\mu \) is allowed.

Proof of 1.10. In clause (c) we can assume \( \langle (C_\delta, C'_\delta) : \delta \in S \rangle \) is a good \((\lambda, \kappa, \chi)\)-Md\(^+\)-parameter (in the other cases we choose \( C'_\delta = \text{nacc}(C_\delta) \) and use 1.8(3)).
Let \( \mathbf{F} \) be a given \((\bar{C}, 2^\mu, \theta)\)-colouring.
By assumption (f) we have Sep(µ, θ) which means (see Definition 1.9(2)) that for some Υ = cf(Υ) ≤ 2^{µ} we have Sep(µ, µ, θ, Υ).

Let \( \mathcal{F} = \{f_{\xi} : \xi < µ\} \) where \( \mathcal{F} \) exemplify Sep(µ, µ, θ, Υ), see Definition 1.9(1) and for \( \rho \in \mu^\theta \) let \( \text{Sol}_{\rho} := \{\nu \in \mu^\theta : \text{for every } \varepsilon < \mu \text{ we have } \rho(\varepsilon) \neq f_\varepsilon(\nu)\} \) where Sol stands for solutions, so by clause (c) of the Definition 1.9(1) of Sep it follows that

\[
(*) \rho \in \mu^\theta \Rightarrow |\text{Sol}_{\rho}| < \Upsilon.
\]

Let \( h \) be an increasing continuous function from \( \lambda \) into \( 2^\mu \) with unbounded range; if \( \lambda \) is regular, \( h = \text{id}_\lambda \) is O.K. Recall that \( \lambda = \text{cf}(2^\mu) \geq 2^{<\chi} \). Let \( \text{cd} \) be a one-to-one function from \( \mu(2^\mu) \) onto \( 2^\mu \) such that

\[
\alpha = \text{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu \rangle) \Rightarrow \alpha \geq \sup\{\alpha_\varepsilon : \varepsilon < \mu\}
\]

and let the function \( \text{cd}_i : 2^\mu \to 2^\mu \) for \( i < \mu \) be such that \( \text{cd}_i(\text{cd}(\langle \alpha_\varepsilon : \varepsilon < \mu \rangle)) = \alpha_i \).

Let \( \mathcal{C} = \{C_\delta \cap \alpha : \delta \in \Sigma C_\delta\} \) (recall that trivially \( |\mathcal{C}| \leq \lambda \leq 2^\mu \)) and let \( T = \{\eta : \text{for some } C \in \mathcal{C} \text{ (hence } C \text{ is of cardinality } < \chi) \text{, we have } \eta \in C(2^\mu)\} \). By assumptions (c) + (d) clearly \( T \) is of cardinality \( \leq 2^\mu \) but \( |T| \geq 2^\mu \).

Let \( \mathcal{P}_\delta \) be a subset of \( \text{otp}(C_\delta) \)-branches and \( \text{cf}(\text{otp}(C_\delta)) = \kappa \) and set of nodes \( \subseteq T_{<\delta} \) hence with \( < 2^\mu \) nodes hence by assumption (e)

\[
(**) |\mathcal{P}_\delta| < 2^\mu.
\]

For each \( \eta \in \mathcal{P}_\delta \) and \( \varepsilon < \mu \) we define \( \nu_{\eta,\varepsilon} \in C_\delta(\eta) \) by \( \nu_{\eta,\varepsilon}(\alpha) = \text{cd}_\varepsilon(\eta(\alpha)) \) for \( \alpha \in C_\delta \). Now for \( \eta \in \mathcal{P}_\delta \), clearly \( \rho_\eta = (\text{F}(\nu_{\eta,\varepsilon}) : \varepsilon < \mu) \) belongs to \( \mu^\theta \). Clearly \( \{\rho_\eta : \eta \in \mathcal{P}_\delta\} \) is a subset of \( \mu^\theta \) of cardinality \( \leq |\mathcal{P}_\delta| \) which as said above is \( < 2^\mu \).}

**Proof:** Let \( \vartheta^* \in \mu^\theta \) and let \( \text{Sol}_{\vartheta^*} \) from the beginning of the proof and remember (*) above, so \( \eta \in \mathcal{P}_\delta \Rightarrow \text{Sol}_{\rho_\eta} \) has cardinality \( \Upsilon = \text{cf}(\Upsilon) \leq 2^\mu \). Hence we can find \( g^*_{\varepsilon} \in \mu^\theta \setminus \text{Sol}_{\rho_\eta} \).

[Why? if \( \Upsilon < 2^\mu \) then \( |\cup \{\text{Sol}_{\rho_\eta} : \eta \in \mathcal{P}_\delta\}| \leq \Sigma |\text{Sol}_{\rho_\eta} : \eta \in \mathcal{P}_\delta| \leq \Upsilon \times |\mathcal{P}_\delta| < 2^\mu = \mu^\theta \) and if \( \Upsilon = 2^\mu \) then \( 2^\mu \) is regular, \( |\mathcal{P}_\delta| < 2^\mu, \eta \in \mathcal{P}_\delta \Rightarrow |\text{Sol}_{\rho_\eta}| < 2^\mu \) and again \( |\cup \{\text{Sol}_{\rho_\eta} : \eta \in \mathcal{P}_\delta\}| < 2^\mu = \mu^\theta \).]

Let \( \varepsilon < \mu \). Recall that \( g^*_{\varepsilon} \in \mu^\theta \) for \( \delta \in \Sigma \) and \( f_\varepsilon \) from the list of \( \mathcal{F} \) is a function from \( \mu^\theta \) to \( \theta \) so \( f_\varepsilon(g^*_{\varepsilon}) < \theta \). Hence we can consider the sequence \( \tilde{c}_\varepsilon = \{f_\varepsilon(g^*_{\varepsilon}) : \delta \in \Sigma\} \) as a candidate for being an \( \text{F-Md-sequence} \). If one of them is, we are done. So
assume towards a contradiction that for each \( \varepsilon < \mu \) there is a sequence \( \eta_\varepsilon \in \lambda(2^\mu) \) that exemplifies the failure of \( \varepsilon^* \) to be an \( F \)-Md-sequence, so there is a \( E_\varepsilon \in D \) such that

\[ \exists \delta \in S \cap E_\varepsilon \Rightarrow F(\eta_\varepsilon \upharpoonright C_\delta) \neq f_\varepsilon(\varrho_\delta^*) \].

Define \( \eta^* \in \lambda(2^\mu) \) by \( \eta^*(\alpha) = \text{cd}(\{\eta_\varepsilon(\alpha) : \varepsilon < \mu\}) \). Now as \( \lambda \) is regular uncountable it follows by choice of \( h \) that \( E =: \{\delta < \lambda : \text{for every } \alpha < \delta \text{ we have } \eta^*(\alpha) < h(\delta) \} \) and if \( \delta' \in S, \alpha \in C_{\delta'} \cap \delta \) and \( C'' = C_{\delta'} \cap \alpha \) then \( \eta^* \upharpoonright C'' \in T_{<\delta} \) is a club of \( \lambda \) (see the choice of \( T \) and \( T_{<\delta} \), and recall as well that by assumption (c) the sequence \( \langle (C_\delta, C'_\delta) : \delta \in S \rangle \) is good, see Definition 1.5(4) + (4D)).

By clause (a) in the assumption of our claim 1.10, the filter \( D \) includes the clubs of \( \lambda \) so clearly \( E \in D \); also \( D \) is \( \mu^+ \)-complete hence \( E^* =: \cap\{E_\varepsilon : \varepsilon < \mu\} \cap E \) belongs to \( D \). Now choose \( \delta \in E^* \cap S \), fixed for the rest of the proof; clearly \( \eta^* \upharpoonright C_\delta \in \mathcal{P}_\delta \); just check the definitions of \( \mathcal{P}_\delta \) and \( E, E^* \). Let \( \varepsilon < \mu \). Now recall that \( \nu_{\eta^* \upharpoonright C_\delta, \varepsilon} \) is the function from \( C_\delta \) to \( h(\delta) \) defined by

\[ \nu_{\eta^* \upharpoonright C_\delta, \varepsilon}(\alpha) = \text{cd}_\varepsilon(\eta^*(\alpha)). \]

But by our choice of \( \eta^* \) clearly \( \alpha \in C_\delta \Rightarrow \text{cd}_\varepsilon(\eta^*(\alpha)) = \eta_\varepsilon(\alpha) \), so

\[ \alpha \in C_\delta \Rightarrow \nu_{\eta^* \upharpoonright C_\delta, \varepsilon}(\alpha) = \eta_\varepsilon(\alpha) \quad \text{so} \quad \nu_{\eta^* \upharpoonright C_\delta, \varepsilon} = \eta_\varepsilon \upharpoonright C_\delta. \]

Hence \( F(\nu_{\eta^* \upharpoonright C_\delta, \varepsilon}) = F(\eta_\varepsilon \upharpoonright C_\delta) \). As \( \delta \in E^* \subseteq E_\varepsilon \) clearly \( F(\eta_\varepsilon \upharpoonright C_\delta) \neq f_\varepsilon(\varrho_\delta^*) \) and as \( \eta^* \upharpoonright C_\delta \in \mathcal{P}_\delta \) clearly \( \rho_{\eta^* \upharpoonright C_\delta} \in \mu\theta \) is well defined. Now easily \( \rho_{\eta^* \upharpoonright C_\delta}(\varepsilon) = F(\nu_{\eta^* \upharpoonright C_\delta, \varepsilon}) \) by the definition of \( \rho_{\eta^* \upharpoonright C_\delta} \), so we have \( \rho_{\eta^* \upharpoonright C_\delta}(\varepsilon) \neq f_\varepsilon(\varrho_\delta^*) \).

As this holds for every \( \varepsilon < \mu \) it follows by the definition of \( \text{Sol}_{\rho_{\eta^* \upharpoonright C_\delta}} \) that \( \varrho_\delta^* \in \text{Sol}_{\rho_{\eta^* \upharpoonright C_\delta}} \). But \( \eta^* \upharpoon-right C_\delta \in \mathcal{P}_\delta \) was proved above so \( \rho_{\eta^* \upharpoonright C_\delta} \in \{\rho_\eta : \eta \in \mathcal{P}_\delta\} \) hence \( \varrho_\delta^* \in \text{Sol}_{\rho_{\eta^* \upharpoonright C_\delta}} \subseteq \cup \{\text{Sol}_{\rho_\nu} : \nu \in \mathcal{P}_\delta\} \) whereas \( \varrho_\delta^* \) has been chosen outside this set, contradiction. \( \square \) 1.10

We return to looking at \( \text{Sep} \) (see Definition 1.9) which appears in assumption (f) of 1.10. There are obvious monotonicity properties:

**Claim.** If \( \text{Sep}(\chi, \mu, \theta_0, \theta_1, \Upsilon) \) holds and \( \chi' \geq \chi, (\theta_0')^{\mu'} \leq (\theta_0)^{\mu}, \theta_1' \leq \theta_1 \) and \( \Upsilon' \geq \Upsilon \), then \( \text{Sep}(\chi', \mu', \theta_0', \theta_1', \Upsilon') \) holds.

**Proof.** Let \( h \) be a one-to-one function from \( \mu'(\theta_0') \) into \( \mu(\theta_0) \). Let \( \bar{f} = \{f_\varepsilon : \varepsilon < \chi\} \) exemplify \( \text{Sep}(\chi, \mu, \theta_0, \theta_1, \Upsilon) \). For \( \varepsilon < \chi' \) we define \( f'_\varepsilon : \mu'(\theta_0') \to \theta_1' \) by: \( f'_\varepsilon(\nu) = \cdots \)
\[ f_\varepsilon(h(\nu)) \] if \( \varepsilon < \chi \) \& \( f_\varepsilon(h(\nu)) < \theta_1' \) and \( f_\varepsilon'(\nu) = 0 \) otherwise. Let \( g' \in (\chi')(\theta_1') \) and let \( A_{\theta_1}' = \{ \nu \in (\mu')(\theta_0') : \) for every \( \varepsilon < \chi' \) we have \( f_\varepsilon'(\nu) \neq g'(\varepsilon) \}. \) Let \( \bar{\rho} = g' \upharpoonright \chi \) so \( \bar{\rho} \in \chi(\theta_1') \subseteq \chi(\theta_1) \), hence by the choice of \( f_\varepsilon : \varepsilon < \chi \) the set \( A_{\theta_0} = \{ \nu \in \mu(\theta_0) : \) for every \( \varepsilon < \chi \) we have \( f_\varepsilon(\nu) \neq g(\varepsilon) \} \) has cardinality \( \leq 2^\mu \). If \( \nu \in A_{\theta_1}' \) then \( \nu \in \mu(\theta_0) \) and \( h(\nu) \in \mu(\theta_0) \), and for \( \varepsilon < \chi' \) we have \( f_\varepsilon'(\nu) \neq g'(\varepsilon) \) hence for \( \varepsilon < \chi(\leq \chi') \) we have either \( f_\varepsilon(h(\nu)) < \theta_1' \) \& \( g(\varepsilon) = g'(\varepsilon) \neq f_\varepsilon'(\nu) = f_\varepsilon(h(\nu)) \) or \( f_\varepsilon(h(\nu)) \geq \theta_1' \) \& \( g(\varepsilon) = g'(\varepsilon) < \theta_1' \) so \( f_\varepsilon(h(\nu)) \neq g(\varepsilon) \). Hence \( \nu \in A_{\theta_1}' \Rightarrow h(\nu) \in A_{\theta_0} \) and as \( h \) is a one-to-one function we have \( |A_{\theta_1}'| \leq |A_{\theta_0}| < \Upsilon \leq \Upsilon' \) and we are done. \( \square \)

To show that 1.10 has reasonable assumptions we should still prove 1.11.

**Proof of 1.11.** Our claim gives sufficient conditions for \( \text{Sep}(\mu, \theta) \), i.e. \( \text{Sep}(\mu, \mu, \theta, \theta, \Upsilon) \) for some \( \Upsilon = \text{cf}(\Upsilon) \leq 2^\mu \). Clearly if \( \Upsilon < 2^\mu \) we can waive "\( \Upsilon \) regular" as \( \Upsilon^+ \) can serve as well. Let \( \theta_1 = \theta \).

**Case 1:** \( \mu = \mu^\theta, \Upsilon = \theta, \theta_0 \in [\theta, \mu) \) and we shall prove \( \text{Sep}(\mu, \mu, \theta_0, \theta_1, \Upsilon) \). Let

\[ \mathcal{F} = \left\{ f : f \text{ is a function with domain } \mu(\theta_0) \text{ and} \right. \]

\[ \text{for some } u \in [\mu]^\theta \text{ and sequence } \bar{\rho} = \langle \rho_i : i < \theta \rangle, \]

\[ \text{with no repetition, } \rho_i \in \mu(\theta_0), \text{ we have} \]

\[ (\forall \nu \in \mu(\theta_0)[\rho_i \subseteq \nu \Rightarrow f(\nu) = i] \text{ and} \]

\[ (\forall \nu \in \mu(\theta_0)[\left( \bigwedge_{i < \theta} (\rho_i \not\subseteq \nu) \right) \Rightarrow f(\nu) = 0] \right\}. \]

We write \( f = f_{\bar{u}, \bar{\rho}} \), if \( u, \bar{\rho} \) witness that \( f \in \mathcal{F} \) as above.

Recalling \( \mu = \mu^\theta \), clearly \( |\mathcal{F}| = \mu \). Let \( \mathcal{F} = \{ f_\varepsilon : \varepsilon < \mu \} \) and we let \( \bar{\mathcal{F}} = \{ f_\varepsilon : \varepsilon < \mu \} \). Clearly clauses (a),(b) of Definition 1.9 (with \( \mu, \mu, \theta_0, \theta_1 = \theta \) here standing for \( \chi, \mu, \theta, \theta \) there) holds and let us check clause (c). So suppose \( \bar{\rho} \in \mu(\theta) \) and let \( R = R_{\bar{\rho}} := \{ \nu \in \mu(\theta_0) : \) for every \( \varepsilon < \mu \) we have \( f_\varepsilon(\nu) \neq g(\varepsilon) \}. \) We have to prove \( |R| < \theta \) (as we have chosen \( \Upsilon = \theta \)).

Towards contradiction, assume that \( R \subseteq \mu(\theta_0) \) has cardinality \( \geq \theta \) and choose \( R' \subseteq R \) of cardinality \( \theta \). Hence we can find \( u \in [\mu]^{\theta_0} \) such that \( \langle \nu \upharpoonright u : \nu \in R' \rangle \) is without repetitions.

Let \( \{ \nu_i : i < \theta \} \) list \( R' \) without repetitions and let \( \rho_i = : \nu_i \upharpoonright u \). Now let \( \bar{\rho} = \langle \rho_i : i < \theta \rangle \), so \( f_{\bar{u}, \bar{\rho}} \) is well defined and belong to \( \mathcal{F} \). Hence for some \( \zeta < \mu \) we have \( f_{\bar{u}, \bar{\rho}} = f_\zeta \). Now for each \( i < \theta, \nu_i \in R' \not\subseteq R \) hence by the definition of \( R, (\forall \varepsilon < \mu)(f_\varepsilon(\nu_i) \neq g(\varepsilon)) \) in particular for \( \varepsilon = \zeta \) we get \( f_\zeta(\nu_i) \neq g(\zeta) \). But the
choice of \( \zeta \), \( f_\zeta(\nu_i) = f_{\mu, \rho}(\nu_i) \) and by the definition of \( f_{\mu, \rho} \) recalling \( \nu_i \upharpoonright u = \rho_i \) we have \( f_{\mu, \rho}(\nu_i) = i \), so \( i = f_\zeta(\nu_i) \neq \varnothing(\zeta) \). This holds for every \( i < \theta \) whereas \( \varnothing \in \mu^\theta \), contradiction.

**Case 2:** \( 2^\theta \leq \mu \) and \( U_\theta[\mu] = \mu \).

Let \( \Upsilon = (2^\theta)^+ \) or just \( (2^\theta)^+ \) hence \( \Upsilon = \cf(\Upsilon) \leq \mu^+ \leq 2^\mu \).

Let \( \{u_i : i < \mu\} \subseteq [\mu]^\theta \) exemplify \( U_\theta[\mu] = \mu \). Define \( \mathcal{F} \) as in case 1 except that for notational simplicity \( \theta_0 = \theta \) and we restrict ourselves in the definition of \( \mathcal{F} \) to \( u \in \bigcup\{\mathcal{P}(u_i) : i < \mu\} \). Clearly \( |\mathcal{F}| \leq \mu \) and choose \( \bar{f} = \langle f_\varepsilon : \varepsilon < \mu \rangle \) as there.

Assume that \( \varnothing \in \mu^\theta \) and \( R = R_\varnothing \subseteq \mu^\theta \) is defined as in Case 1, and toward contradiction assume that \( |R| \geq (2^\theta)^+ \), hence we can find \( \rho^*, \langle (\alpha_\zeta, \rho_\zeta) : \zeta < \theta \rangle \) such that:

\[
\begin{align*}
(\ast) \quad (a) & \quad \rho^*, \rho_\zeta \in R \\
(b) & \quad \rho_\zeta \upharpoonright \{\alpha_\varepsilon : \varepsilon < \zeta\} = \rho^* \upharpoonright \{\alpha_\varepsilon : \varepsilon < \zeta\} \\
(c) & \quad \rho_\zeta(\alpha_\zeta) \neq \rho^*(\alpha_\zeta).
\end{align*}
\]

[Why? I had thought it should be clear but we elaborate. Let \( \chi_* = (2^\mu)^+ \) hence \( \theta^\lambda, R, \bar{f} \in \mathcal{H}(\chi)_* \).

Now by induction on \( \zeta < \theta \) we choose \( M_\zeta \):

\[
\begin{align*}
(\ast)' \quad (a) & \quad M_\zeta \prec (\mathcal{H}(\chi), \in) \\
(b) & \quad M_\zeta \text{ has cardinality } 2^{[\zeta]} + \aleph_0 \\
(c) & \quad \mu, \theta, \bar{f}, \varnothing \text{ belongs to } M_\zeta \\
(d) & \quad |M_\zeta|_0^{[\zeta]} \subseteq M_\zeta \\
(e) & \quad \text{if } \varepsilon < \zeta \text{ then } M_\varepsilon < M_\zeta \text{ and } M_\varepsilon \in M_\zeta.
\end{align*}
\]

Clearly possible and the cardinality of \( M := \bigcup\{M_\alpha : \alpha < \theta\} \) is \( \Sigma(2^\mu + \aleph_0) = 2^{\theta^\mu} \).

Now choose \( \rho^* \in R \setminus M \), possible because \( |M| = 2^{\theta^\mu} < (2^{\theta^\mu})^+ \leq |R| \). Next we try to choose a pair \( (\rho_\zeta, \alpha_\zeta) \) by induction on \( \zeta < \theta \) such that:

\[
\begin{align*}
\bullet_1 & \quad |M_\zeta|_0^{[\zeta]} \subseteq M_\zeta \\
\bullet_2 & \quad \rho_\zeta, \alpha_\zeta \in M_{\zeta+1} \\
\bullet_3 & \quad \text{if } \varepsilon < \zeta \text{ then } \rho_\varepsilon(\alpha_\varepsilon) \neq \rho^*(\alpha_\varepsilon) \\
\bullet_4 & \quad \rho^*(\alpha_\zeta) \neq \rho_\zeta(\alpha_\zeta).
\end{align*}
\]

Why can we carry the induction on \( \zeta \)? Arriving to \( \zeta \) the set \( X_\zeta = \{(\rho_\varepsilon, \alpha_\varepsilon) : \varepsilon < \zeta\} \) is a subset of \( M_\zeta \) of cardinality \( \leq [\zeta] \) hence it belongs to \( M_\zeta \). Consider the set \( \mathcal{X}_\zeta = \{\rho \in R : \text{ for every pair } (\rho, \alpha) \text{ from } \lambda, \rho(\alpha_\varepsilon) \neq \rho_\varepsilon(\alpha_\varepsilon)\} \).

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Now obviously this set belongs to $M_\zeta$, also $\rho^* \in \mathcal{Y}_\zeta$ by the induction hypothesis on $\zeta$. But $\rho^*$ being not in $M$ hence not in $M_\zeta$; hence $|\mathcal{Y}_\zeta| > 1$ (and much more hence we can choose $\rho' \neq \rho'' \in \mathcal{Y}$ in $M_\zeta$. As $\rho' \neq \rho''$ are members of $\mathcal{Y}_\zeta$ and $\mathcal{Y}_\zeta \subseteq R \subseteq M\theta$, there is $\alpha < \mu$ such that $\rho'(\alpha) \neq \rho''(\alpha)$ and as $\rho', \rho'' \in M_\zeta$, without loss of generality $\alpha \in M_\zeta$. As $\rho'(\alpha) \neq \rho''(\alpha)$ necessarily for some $\rho \in \{\rho', \rho''\}$ we have $\rho^*(\alpha) \neq \rho(\alpha)$. So we can choose $(\rho, \rho^*) = (\rho, \alpha)$.

Having carried the induction clearly $\rho^*, \langle (\rho, \alpha^*_z) : \zeta < \theta \rangle$ are as required.

Clearly $\langle \alpha^*_z : \zeta < \theta \rangle$ is with no repetitions.

So by the choice of $\{u_i : i < \mu\}$ we can choose $U_\theta[\mu] = \mu$, i.e., the definition of $U_\theta[\mu]$, for some $i < \mu$ the set $u_i \cap \{\alpha^*_z : \zeta < \theta\}$ has cardinality $\theta$, call it $u$. So $\{\rho \mid u : \rho \in R\}$ has cardinality $\geq \theta$ and we can continue as in Case 1. If we wish to use $\theta_0 \in [\theta, \mu]$ instead of $\theta_0 = \theta$ above let $p$ be a pairing function on $\mu$. Without loss of generality each $u_i$ is closed under $p$ and its inverses, and $f^*_{u_\rho}$ is defined iff for some $\varepsilon < \mu$ we have $u_\varepsilon \subseteq u_\varepsilon$ and $i < \theta \Rightarrow \text{Rang}(\rho_i) \subseteq u_\varepsilon$. In the end choose $i$ such that $u_i \cap \{\text{pr}(\rho, \rho^*_\alpha) : \zeta < \theta\}$ has cardinality $\theta$.

**Case 3:** $U_J[\mu] = \mu$ where $J = [\sigma]^{<\theta}$ for $\sigma$ such that $\sigma^\theta \leq \mu, \sigma \geq \theta$.

Let $\Upsilon = [2^{<\sigma}]^+$. The proof is similar to Case 2, only find now $\rho^*$ and $\langle (\alpha^*_z, \rho^*_z) : \zeta < \sigma \rangle$ and choose $i < \mu$ such that $u_i \cap \{\text{pr}(\rho, \rho^*_\alpha) : \zeta < \sigma\}$ has cardinality $\theta$.

**Case 4:** $\mu > \theta \neq \text{cf}(\mu)$ and $\mu$ is strong limit.

Follows by case 2.

**Case 5:** $\mu \geq \aleph_\omega(\theta)$.

By [Sh 460] we can find a regular $\sigma < \aleph_\omega(\theta)$ which is $> \theta$ and such that $U_\sigma[\mu] = \mu$, so Case 2 applies.

Note that in case 2, we can replace $2^\theta \leq \mu$ by: there is $\mathcal{F} \subseteq \theta^\theta$ of cardinality $\leq \mu$ such that for every $A \in [\theta]^\theta$ for some $f \in \mathcal{F}$ we have $\theta = \text{Rang}(f \upharpoonright A)$. The proof is obvious. Similarly in case 3. \hfill $\Box_{1.11}$

There are some obvious monotonicity properties regarding the $\text{Md}$-property.

**1.14 Claim.** 1) If $(\lambda, \bar{C})$ has the $\theta$-$\text{Md}$-property (see Definition 1.6(1)) and $\bar{C} = \langle C_\delta : \delta \in S \rangle$ and $C_\delta$ is an unbounded subset of $C_\delta$ for $\delta \in S$ then $(\lambda, \langle C_\delta : \delta \in S \rangle)$ has a $\theta$-$\text{Md}$ property. We can also increase $S$ and decrease $\theta$; we can deal with the $(\Upsilon, \theta)$-$\text{-Md}$-property.

2) Assume $\lambda = \text{cf}(\lambda) > \aleph_0, S \subseteq \{\delta : \delta < \lambda, \text{cf}(\delta) = \kappa\}$ is stationary and $\kappa < \chi \leq \lambda$.

Then some $\langle C_\delta : \delta \in S \rangle$ is a $(\lambda, \kappa, \chi)$-$\text{-Md}$-parameter.

**Proof.** 1) Obvious.

2) Easy (note: goodness is not required).

\hfill $\Box_{1.14}$
We may consider the following generalization.

1.15 Definition. 1) We say $\bar{C}$ is a $\lambda$-Md$_3$-parameter if:

(a) $\lambda$ is regular uncountable
(b) $S$ is a stationary subset of $\lambda$ (consisting of limit ordinals)
(c) $\bar{C} = \langle C_\delta, C'_\delta, C_{\delta, \varepsilon} : \delta \in S, \varepsilon < \text{cf}(\delta) \rangle$
(d) $C_\delta \subseteq \delta = \sup(C_\delta)$ and $C'_\delta \subseteq \delta = \sup(C'_\delta)$, \(\text{otp}(C'_\delta) = \kappa\) and $C_\delta = \bigcup\{C_{\delta, \varepsilon} : \varepsilon < \text{cf}(\delta)\}$
(e) $\zeta < \text{cf}(\delta) \Rightarrow \sup(\bigcup_{\varepsilon < \zeta} C_{\delta, \varepsilon}) < \delta$ and $C_{\delta, \varepsilon}$ increase with $\varepsilon$

2) We say that a $\lambda$-Md$_3$-parameter $\bar{C}$ is good if $\alpha < \lambda \Rightarrow \lambda > |\{C_{\delta, \zeta} : \delta \in S, \zeta < \text{cf}(\delta)\}|$ and the $\zeta$-th member of $C'_\delta$ is $\leq \alpha$.
3) The $(D, \Upsilon, \theta)$-Md$_3$-property for $\bar{C}$ means the $(D, \Upsilon, \theta)$-Md-property for $\langle C_\delta : \delta \in S \rangle$, similarly for the other variants of this property (so the $(D, \Upsilon, \theta)$-Md$_3$-property for $\lambda$ for a stationary $S \subseteq S^\lambda_\kappa$ is the same as the $(D, \Upsilon, \theta)$-Md$_3$-property).
4) If $\{C_{\delta, \varepsilon} : \varepsilon < \text{cf}(\delta)\} = \{C_\delta \cap \alpha : \alpha \in \text{nacc}(C_\delta)\}$ we may write just $\langle C_\delta : \delta \in S \rangle$ (so Md$_3$ is like Md in this case).

We may consider the following variant.

1.16 Claim. Assume

(a) $\lambda$ is regular uncountable, $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ is stationary and $\chi < \lambda$
(b) $\mathcal{B}$ is a model with universe $\lambda$ such that $A \in [\lambda]^{<\chi} \Rightarrow \text{cl}_{\mathcal{B}}(A) \in [\lambda]^{<\chi}$; we may allow infinitary functions provided that

(*)$\mathcal{B}, \kappa$ if $\langle A_i : i < \kappa \rangle$ is increasing, $A_i \in [\lambda]^{<\chi}$ then $\text{cl}_{\mathcal{B}}(\bigcup_{i < \kappa} A_i) = \bigcup_{i < \kappa} \text{cl}_{\mathcal{B}}(A_i)$

(c) $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a good $(\lambda, \kappa, \kappa^+)$.Md$_3$-parameter
(d) $S^* = \{\delta \in S : \text{for every } \alpha < \delta, \text{cl}_{\mathcal{B}}(\alpha) \subseteq \delta\}$
(e) $C_{\delta, \varepsilon}^* = \alpha_{\delta, \varepsilon} \cap \text{cl}_{\mathcal{B}}(C_\delta \cap \alpha_{\delta, \varepsilon})$ for $\delta \in S^*$ and $\varepsilon < \kappa$ where $\alpha_{\delta, \varepsilon} \subseteq C_\delta$ is increasing with $\varepsilon$ with limit $\delta$ and $C_{\delta, \varepsilon}^* = \bigcup\{C_{\delta, \varepsilon}^* : \varepsilon < \text{cf}(\delta)\}, C_\delta = \{\alpha_{\delta, \varepsilon} : \varepsilon < \text{cf}(\delta)\}$.

---

3 if $(C_\delta : \delta \in S)$ is a good $(\lambda, \kappa, \chi)$-Md$_3$-parameter $C'_\delta \subseteq \delta = \sup(C'_\delta)$, \(\text{otp}(C'_\delta) = \kappa\)$ and we let $C_{\delta, \varepsilon} = \{\alpha \in C_\delta : \text{otp}(\alpha \cap C'_\delta) \leq \varepsilon + 1\}$ for $\varepsilon < \kappa$ then $(C_\delta, C_{\delta, \varepsilon} : \delta \in S, \varepsilon < \kappa)$ is a $\lambda$-Md$_1$-parameter.
Then \( \langle C^*_\delta, C_\delta, C^*_\delta, \varepsilon : \delta \in S^*, \varepsilon < \text{cf}(\delta) \rangle \) is a good \((S, \kappa, \chi)\)-Md\(_3\)-parameter (and \(S \setminus S^*\) is non-stationary); if \( \chi \leq \kappa \) then \( \bigcup_{\varepsilon < \kappa} C^*_\delta, \varepsilon = c^\ell_{\text{cf}}(C_\delta) \).

**Proof.** Easy.

1.17 Claim. 1) Assume \( \bar{C} = \langle C_\delta, C'_\delta, C^*_\delta, \varepsilon : \delta \in S, \varepsilon < \text{cf}(\delta) \rangle \) is a good \((\lambda, \kappa, \chi)\)-Md\(_3\)-parameter. Then claim 1.10 holds (i.e. we get there the \((D, \theta)\)-Md\(_3\)-property). 2) In part (1) hence in 1.10, we can weaken clause (e) to

\[(e)' \mbox{ (if } \lambda = 2^\mu \mbox{ we can use } h = \text{id}_\lambda) : \mbox{ there is } h : \lambda \to 2^\mu \mbox{ increasing continuous with limit } 2^\mu \mbox{ and there is a set } T = \{\eta_\alpha : \alpha < 2^\mu\} \mbox{ and function } g : \lambda \to \lambda \mbox{ such that if } \eta \in \lambda(2^\mu) \mbox{ then for stationary many } \delta \in S \mbox{ we have}
\[(a) \ \eta \upharpoonright C_\delta \in \{\eta_\alpha : \alpha < h(g(\delta))\}
\[(b) \ \zeta < \text{cf}(\delta) \implies \eta \upharpoonright \bigcup_{\varepsilon < \zeta} C_\delta, \varepsilon \in \{\eta_\beta : \beta < \delta\}.

**Proof.** Same proof as 1.10.

1.18 Claim. In the proof of 1.10 instead of \( f : \lambda \to \Upsilon \) and \( f_\delta : C_\delta \to \theta \) we can use \( f : [\lambda]^{<\aleph_0} \to \Upsilon \) and \( f_\delta : [C_\delta]^{<\aleph_0} \to \theta \) (so get the \( \theta \)-MD\(_\ell\)-property for \( \ell = 0, 1, 2 \)).

**Proof.** Let \( cd_* \) be a one-to-one function from \( \omega > \lambda \) into \( \lambda \) and without loss of generality \( \alpha_\ell \leq cd_* (\langle \alpha_0, \alpha_1, \ldots, \alpha_{n-1} \rangle) < \text{Max}(\{\omega_\pi(0) + \ldots \omega_\pi(n-1) : \pi \mbox{ a permutation of } n\}) \). We are given \( \bar{C} = \langle C_\delta : \delta \in S \rangle \), without loss of generality every \( \delta \in S \) is infinite and divisible by \( \delta \).

Let \( C_{1,\delta} = \{cd_*(\eta) : \eta \in \omega > (C_\delta)\} \) and \( C'_{1,\delta} = nacc(C_\delta) \).

1.19 Definition. 1) For a set \( A \) of ordinals and an ordinal \( \alpha \) let \( A* \alpha = \{\beta : \beta < \alpha \} \) or for some \( \gamma \in A \) we have \( \beta \in [\gamma, \gamma + \alpha) \).
2) The pair \((\lambda, \bar{C})\) or just \( \bar{C} \) has the \((\chi, \theta)\)-MD\(_0\)-property (or use MD instead of MD\(_0\)) if we can find \( \langle f_\delta : \delta \in S \rangle \) such that:

\[(a) \ f_\delta \mbox{ is a function from } C'_\delta = C_\delta * \alpha^*_\delta \mbox{ to } \theta \mbox{ for some } \alpha^*_\delta < \chi
\[(b) \mbox{ if } f \mbox{ is a function from } \lambda \mbox{ to } \theta \mbox{ and } \alpha < \chi \mbox{ is non-zero then for stationary many } \delta \in S \mbox{ we have } f_\delta \subseteq f \mbox{ and } \alpha^*_\delta = \alpha.
If we omit $\chi$ we mean $\chi = 2$ so $\alpha^*_\delta = 1$ for every $\delta \in S$.  
3) $(D, \chi, \theta)$-MD$_2$-property means: if $\tau$ is a relational vocabulary of cardinality $< \theta$, then we can find $\langle M_\delta : \delta \in S \rangle$ such that:

(a) $M_\delta$ is a $\tau$-model with universe $C'_\delta = C_\delta \ast \alpha^*_\delta$ for some $\alpha^*_\delta < \chi$

(b) if $M$ is a $\tau$-model with universe $\lambda$ and $\alpha < \chi$ then the set $A_{M, \alpha} = \{ \delta \in S : M \upharpoonright C_\delta = M_\delta$ and $\alpha^*_\delta = \alpha \}$ is stationary and even belongs to $D^+$.

If we omit $\chi$ we mean $\chi = 2$ so $\alpha^*_\delta = 1$ for every $\delta \in S$. Similarly with $D$.
4) We say $(\lambda, S)$ has the $(\chi, \theta)$-MD$_r$-property if for some $\bar{C} = \langle C_\delta : \delta \in S \rangle$ the pair $(\lambda, \bar{C})$ has the $(\chi, \theta)$-MD$_r$-property. We say that $\lambda$ has the $(\kappa, \chi, \theta)$-property if $(\lambda, S^K_\kappa)$ has the $(\chi, \theta)$-MD$_r$-property.

The following claim shows the close affinity of the properties stated in theorem 0.1 in the introduction and the $\theta$-Md-property with which we deal in this section here. This claim is needed to deduce Theorem 0.1 from 1.10.

**1.20 Claim.** 1) Assume $(\lambda, \bar{C})$ has the $\theta$-Md-property where $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is a $(\lambda, \kappa, \chi)$-Md-parameter. If $\kappa < \chi$, $\theta = \theta^{< \chi}$ then $(\lambda, \bar{C})$ has the $(\chi, \theta)$-MD$_0$-property.

2) Assume in addition that $\bar{C}$ is a good$^+$ $(\lambda, \kappa, \chi)$-Md-parameter then $(\lambda, \bar{C})$ has the $(D, \chi, \theta)$-MD$_2$-property.

3) In part (2) we can allow $\tau$ to have function symbols but have:

(a)$'$ $M_\delta$ is a $\tau$-model with universe $\supseteq C'_\delta = C_\delta \ast \alpha^*_\delta$ for some $\alpha^*_\delta < \chi$

(b)$'$ if $M$ is a $\tau$-model with universe $\lambda$ and $\alpha < \lambda$ then the following set belongs to $D^+$

$$A'_{M, \alpha} = \{ \delta \in S : \alpha^*_\delta = \alpha \text{ and } (M \upharpoonright c^M(C'_\delta), e)_{e \in C'_\delta} \cong (M_\delta, e)_{e \in C'_\delta} \}.$$

4) In part (2) we can replace good$^+$ by good if we add

$\square$ each $C_\delta$ is closed under $pr$, where $pr$ is a fixed pairing function on $\lambda$.

**Proof.** The proof consists of a translation of one version of middle diamond to another (note: “$\tau$ is a relational” is a strong restriction).

1) First fix $\alpha^* < \chi$ and deal with $\langle C''_\delta : \delta \in S \rangle$, $C''_\delta = C'_\delta \setminus [0, \alpha^*)$ recalling $C'_\delta = C_\delta \ast \alpha^*$. Let $cd$ be a one-to-one function from $x > \theta$ onto $\theta$. Define $\langle F_\delta : \delta \in S \rangle$ as follows: if $f \in (C_\delta)_\theta$ then $F_\delta(f) = cd(f(\beta_{\delta, \varepsilon}) : \varepsilon < \text{otp}(C_\delta))$ where $\langle \beta_{\delta, \varepsilon} : \varepsilon < \text{otp}(C_\delta) \rangle$ list $C_\delta$ in increasing order so $F_\delta$ is a function from $(C_\delta)_\theta$ into $\theta$. As we assume “$(\lambda, \bar{C})$
has the \(\theta\)-Md-property”, we can apply its definition, hence for \(\bar{F} = \langle F_\delta : \delta \in S \rangle\) there is a diamond sequence \(\langle c_\delta : \delta \in S \rangle\).

We now define \(f'_\delta : C'_\delta \to \theta\) as follows: if \(\beta_0\) is a member of \(C_\delta\) and \(\alpha \in [\beta_0, \beta_0 + \alpha^*]\) and \(\alpha < \text{Min}(C'_\delta \setminus (\beta_0 + 1))\) and \(c_\delta = \text{cd}(\langle \xi_{\delta,\varepsilon} : \varepsilon < \text{otp}(C_\delta) \rangle\) and \(\xi_{\delta,\varepsilon} = \text{cd}(\langle \zeta_{\delta,\varepsilon,i} : i < \alpha^* \rangle)\) then \(f'_\delta(\alpha) = \xi_{\delta,\varepsilon,i,\alpha^*} - \beta_0\); otherwise \(f'_\delta(\alpha) = 0\). So \(f'_\delta : C'_\delta \to \theta\). Why is \(\langle f'_\delta : \delta \in S \rangle\) as required for \(\langle C'_\delta : \delta \in S \rangle\) (and \(\alpha^*\))? Assume that \(f : \lambda \to \theta\), so we can define \(f' : \lambda \to \theta\) by \(f'(\alpha) = \text{cd}(\langle f(\alpha + \varepsilon) : \varepsilon < \alpha^* \rangle)\), hence for stationarily many \(\delta \in S\) we have \(c_\delta = F_\delta(f' \upharpoonright C_\delta)\). Now we check that \(f \upharpoonright C'_\delta\) is equal to \(f'_\delta\) defined above.

Let \(\xi_{\delta,\varepsilon,i} = f'(\beta_{\delta,\varepsilon})\) and \(\xi_{\delta,\varepsilon+i} = f(\beta_{\delta,\varepsilon} + i)\) for \(i < \alpha^*, \varepsilon < \text{otp}(C_\delta)\) so \(\xi_{\delta,\varepsilon} = f'(\beta_{\delta,\varepsilon}) = \text{cd}(\langle \zeta_{\delta,\varepsilon,i} : i < \alpha^* \rangle)\), \(c_\delta = F_\delta(f' \upharpoonright C_\delta) = \text{cd}(\langle \xi_{\delta,\varepsilon} : \varepsilon < \text{otp}(C_\delta) \rangle)\). Hence by the definition of \(f'_\delta\) we have: if \(\beta = \beta_{\delta,\varepsilon} \in C_\delta, \alpha \in [\beta, \beta + \varepsilon)\) and \(\alpha < \text{Min}(C_\delta \setminus (\beta + 1))\) then \(f'_\delta(\alpha) = \xi_{\delta,\varepsilon,\alpha-\beta} = f(\beta_{\delta,\varepsilon} + (\alpha - \beta)) = f(\beta + (\alpha - \beta)) = \beta\) as required.

To have “every \(\alpha^* < \chi^*\)” (and guess also \(f \upharpoonright (C'_\delta \setminus C''_\delta) \subseteq f \upharpoonright [0, \alpha^*)\)) we just have to partition \(S\) to \(\langle S_\alpha : \alpha < \chi^* \rangle\) such that each \(\bar{C} \upharpoonright S_\alpha\) has the \(\theta\)-Md-property which we can do by 1.23(3) below (or work a little more above, i.e., \(f'(\alpha) = \text{cd}(\langle f(\alpha + \varepsilon) : \varepsilon < \alpha^* \rangle)\)).

2), 3), 4) Left to the reader. \(\square_{1.20}\)

1.21 Claim. 1) Assume \(\lambda = \text{cf}(\lambda) > \kappa = \text{cf}(\kappa), S \subseteq S^\lambda_{\kappa}\) and

(a) \(\bar{C}^\ell\) is a \(\lambda\)-Md-parameter for \(\ell = 1, 2\)

(b) \(\bar{C}^1 = \langle C_\delta : \delta \in S \rangle\) has the \((\Upsilon, \theta)\)-Md-property

(c) \(A_\alpha \in [\alpha]^{<\chi} \& \Upsilon \upharpoonright |A_\alpha| = \Upsilon\) for \(\alpha < \lambda\) and \(\delta \in S \implies C_\delta^2 = \cup\{A_\alpha : \alpha \in C_\delta\}\).

Then \(\bar{C}^2\) has the \((\Upsilon, \theta)\)-Md-property.

2) If \(\bar{C}\) is a \((\lambda, \kappa, \chi^+\)\)-Md-parameter with the \((\Upsilon, \theta)\)-Md-property, \(\Upsilon^\chi = \Upsilon\) and \(C'_\delta = \{\alpha + i : \alpha \in C_\delta \vee \alpha = 0 \text{ and } i < \chi\}\), then \(\bar{C}' = \langle C'_\delta : \delta \in \text{Dom}(\bar{C}) \rangle\) is a \((\lambda, \kappa, \chi^+)\)-Md-parameter with the \(\theta\)-Md-property.

Proof. Easy by coding (as in the proof of 1.20).

1.22 Definition. Assume \(\lambda > \kappa\) are regular and \(\bar{C} = \langle C_\delta : \delta \in S \rangle\) is a \((\lambda, \kappa, \chi^+)\)-Md-parameter with the \((\Upsilon, \theta)\)-Md-property. Let \(\text{ID}_{\lambda, \kappa, \chi, \Upsilon, \theta}(\bar{C}) = \{A \subseteq \lambda : \bar{C} \upharpoonright (S \cap A)\) does not have the \((\Upsilon, \theta)\)-Md-property\}. If \(\Upsilon = \theta\) we may omit it.
1.23 Claim. Let $\bar{C}, \lambda, \kappa, \chi, \Upsilon, \theta$ be as above in Definition 1.22 and $J = \text{ID}_{\lambda, \kappa, \chi, \Upsilon, \theta}(\bar{C})$.

1) $J$ is an ideal on $\lambda$ such that $\lambda \setminus S \in J, \lambda \notin J$ if $\Upsilon \geq \aleph_0$.
2) $J$ is $\lambda$-complete and even normal if $\Upsilon \geq \lambda$ (if $\aleph_0 \leq \Upsilon < \lambda$ it is still $\Upsilon^+$-complete).
3) If $S \subseteq \lambda$ and $S \neq \emptyset \mod J$, then for any $F^i = \langle F^i_\delta : \delta \in S \rangle, F^i_\delta : (\text{Md})_\theta \rightarrow \theta$, $\lambda$ can be partitioned to $\theta$ subsets each having a $F^i$-Md-sequence.

Proof. 1) Let $\text{cd}: \Upsilon \times \Upsilon \rightarrow \Upsilon$ be one-to-one onto and $\text{cd}_\ell : \Upsilon \rightarrow \Upsilon$ be such that $\text{cd}_\ell(\text{cd}(\langle \alpha_0, \alpha_\ell \rangle)) = \alpha_\ell$. Clearly $J$ is a subfamily of $\mathcal{P}(\lambda)$ and it is closed under subsets, so we need just to prove that it is closed under union of two. So suppose $A = A_1 \cup A_2, A_\ell \in J$ for $\ell = 1, 2$. As $J$ is closed under subsets, without loss of generality $A \cap A_2 = \emptyset$. Let the sequence $\bar{F}_\ell = \langle F^\ell_\delta : \delta \in S \cap A_\ell \rangle$ exemplify $A_\ell \in J$. Now we define $\bar{F} = \langle F_\delta : \delta \in S \cap A \rangle$ as follows:

- if $\delta \in S \cap A$ and $f \in (C_\delta) Y$ then:
  $\delta \in A_\ell \Rightarrow F_\delta(f) = F^\ell_\delta(\text{cd}_\ell \circ f)$, i.e., $F_\delta(f) = F^\ell_\delta(f^\ell)$ where $\alpha \in C_\delta \Rightarrow f^\ell(\alpha) = \text{cd}_\ell(f(\alpha))$.

2) Straight as the sequence $\langle F_\delta : \delta \in S \rangle$ is not required to be nicely definable.
3) Guessing $\theta \times \theta$ is the same as guessing $\theta$. $\square_{1.23}$

Well, are there good $(\lambda, \kappa, \kappa^+)$-parameters? Some version of this is crucial for having interesting consequences of 1.10, but we shall have to quote (on $I[\lambda]$ see Definition 0.5; see on it [Sh 420, §1]).

1.24 Claim. 1) If $S$ is a stationary subset of a regular cardinal $\lambda$ and $S \in I[\lambda]$, see Definition 0.5, and $(\forall \delta \in S)[\text{cf}(\delta) = \kappa]$, then

(a) for some club $E$ of $\lambda$, there is a good $^+(S \cap E, \kappa, \kappa^+)$-Md-parameter. (Of course, we can replace the $\kappa^+$ by any $\chi \leq \lambda$) (recall that we do not required the $C_\delta$’s to be a closed subset of $\delta$)

(b) for some club $E$ of $\lambda$ there is a weakly good $^+(S \cap E, \kappa, \kappa^+)$-Md-parameter consisting of clubs (this is of interest when $\kappa > \aleph_0$)

(c) for some club $E$ of $\lambda$ there is a good $(S \cap E, \kappa, \kappa^+)$-Md-parameter $\bar{C} = \langle C_\delta : \delta \in S \cap E \rangle$ such that

(i) $\bar{C}$ is weakly good $^+$, i.e., $\alpha \in \text{nacc}(C_{\delta_1}) \cap \text{nacc}(C_{\delta_2}) \Rightarrow C_{\delta_1} \cap \alpha = C_{\delta_2} \cap \alpha$

\[\text{This means: the parameter } \bar{C} = \langle C_\delta : \delta \in S \cap E \rangle, C_\delta \text{ a club of } \delta\]
(ii) \( C \) is good, i.e., for each \( \alpha < \lambda \) we have \( \lambda > |\{ \alpha \cap C_\delta : \delta \in S, \alpha \in C_\delta \}| \)

(iii) \( \tilde{C} \) is \( \Theta_\lambda \)-closed that is each \( C_\delta \) is \( \Theta_\lambda \)-closed which means \( \alpha = \sup(C_\delta \cap \alpha) \) & \( \text{cf}(\alpha) \in \Theta_\lambda \Rightarrow \alpha \in C_\delta \) where \( \Theta_\lambda = \{ \sigma : \sigma = \text{cf}(\sigma) \text{ and } \alpha < \lambda \Rightarrow |\alpha|^{<\sigma_\forall} < \lambda \}. \)

2) If \( \kappa = \text{cf}(\kappa), \kappa^+ < \lambda = \text{cf}(\lambda), \) then there is a stationary \( S \in I[\lambda] \) with \( (\forall \delta \in S) |\text{cf}(\delta)|^{<\sigma_\forall} < \lambda \).

3) In part (1), moreover, we can find a good \( (S \cap E, \kappa, \kappa^+) \)-Md-parameter and a \( \lambda \)-complete filter \( D \) on \( \lambda \) such that: for every club \( E' \) of \( \lambda \), the set \( \{ \delta \in S \cap E : C_\delta \subseteq E' \cap E \} \) belongs to \( D \).

Proof. 1) By the definition of \( I[\lambda] \) see Definition 0.4 and 1.5 above (see more in [Sh 420, §1]) and for clause (\( \gamma \)) the definition of \( |\alpha|^{<\sigma_\forall} \).

2) By [Sh 420, §1].

3) By [Sh:g, III]. \( \square_{1.24} \)

1.25 Definition. Let \( \tilde{C} \) be a \( (\lambda, \kappa, \chi) \)-Md-parameter

(1) For a family \( \mathcal{F} \) of \( \tilde{C} \)-colourings and \( \mathcal{P} \subseteq \lambda^2 \), let \( \text{id}_{\tilde{C}, \mathcal{F}, \mathcal{P}} \) be

\[
\left\{ W \subseteq \lambda : \text{for some } F \in \mathcal{F} \text{ for every } \tilde{c} \in \mathcal{P} \right.
\begin{align*}
&\text{for some } \eta \in \lambda^{\lambda} \text{ the set } \{ \delta \in W \cap S : F(\eta \restriction C_\delta) = c_\delta \} \\
&\text{is a non stationary subset of } \lambda \} \right.
\]

(2) If \( \mathcal{F} \) is the family of all \( \tilde{C} \)-colouring we may omit it. If we write \( \text{Def} \) instead of \( \mathcal{F} \) this mean as in [Sh 576, §1] (not used so the reader can ignore this).

We can strengthen 1.5(2) as follows.

1.26 Definition. We say the \( \lambda \)-colouring \( F \) is \( (S, \kappa, \chi) \)-Md-good if:

(a) \( S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \} \) is stationary

(b) we can find \( E \) and \( \langle C_\delta : \delta \in S \cap E \rangle \) such that

(\( \alpha \)) \( E \) is a club of \( \lambda \)
\( \beta \) \( C_\delta \) is an unbounded subset of \( \delta, |C_\delta| < \chi \)

\( \gamma \) if \( \rho, \rho' \in \delta, \delta \in S \cap E \), and \( \rho' \upharpoonright C_\delta = \rho \upharpoonright C_\delta \) then \( F(\rho') = F(\rho) \)

\( \delta \) for every \( \alpha < \lambda \) we have \( \lambda > |\{ C_\delta \cap \alpha : \delta \in S \cap E \text{ and } \alpha \in C_\delta \}| \)

\( \epsilon \) \( \delta \in S \Rightarrow |\delta|^{< \text{cf}(\delta)} < \lambda \) or just \( \delta \in S \Rightarrow \lambda > |\{ C : C \subseteq \delta \text{ is unbounded in } \delta \text{ and for every } \alpha \in C \text{ for some } \gamma \in S \text{ we have } C \cap \alpha = C_{\gamma} \cap \alpha \text{ and } \alpha \in C_{\gamma} \}| \).

We can sum up (relying again on some quotations).

1.27 Conclusion. If \( \lambda = \text{cf}(2^\mu) \) and we let \( \Theta =: \{ \theta : \theta = \text{cf}(\theta) \text{ and } (\forall \alpha < 2^\mu)(|\alpha|^{< \text{th} \theta} < 2^\mu) \} \) then

\( a \) \( \Theta \) is “large” e.g. contains every large enough \( \theta \in \text{Reg} \cap \beth_\omega \) if \( \beth_\omega < \lambda \)

\( b \) if \( \theta \in \Theta \chi \land \theta^+ < \lambda \) then there is a stationary \( S \in I[\lambda] \) such that \( \delta \in S \Rightarrow \text{cf}(\delta) = \theta \)

\( c \) if \( \theta, S \) are as in clause \( b \) above then for some club \( E \) of \( \lambda \)

\( i \) there is a weakly good \( (\lambda, \kappa, \kappa^+)-\text{Md-parameter} \bar{C} = \langle C_\delta : \delta \in S \cap E \rangle \)

\( ii \) there is a good \( (\lambda, \kappa, \kappa^+)-\text{Md-parameter} \langle C_\delta : \delta \in E \cap S \rangle \)

\( iii \) there is a good\(^+ \) \( (\lambda, \kappa, \kappa^+)-\text{Md-parameter} \bar{C} = \langle C_\delta : \delta \in E \cap S \rangle \)

\( d \) for \( \theta, S, \bar{C} \) as in clause \( b \), \( c \) above and \( \Upsilon \leq 2^\mu \), if \( F \) is a \( (\bar{C}, \Upsilon, \theta) \)-colouring and \( D = D_\lambda \) (or \( D \) is a \( \mu^+ \)-complete filter on \( \lambda \) extending \( D_\lambda \)), and lastly \( S \in D^+ \) then there is a \( F \)-middle diamond sequence for \( C \) and there is an \( D - F \)-Md-sequence; see Definition 1.5(3),(3A)

\( e \) if \( \theta, S, \bar{C} \) as in clauses \( b \), \( c \); \( \Upsilon \leq 2^\mu \), \( \text{then } \bar{C} \) has the \( (\Upsilon, \theta) \)-property

\( f \) for \( \theta, S \) as in clause \( b \) above \( \Upsilon \leq 2^\mu \), if \( \theta > \aleph_0 \) \( 3^\Upsilon = \langle C'_{\delta} : \delta \in S \rangle, C_{\delta} \) a stationary subset of \( \delta \) for each \( \delta \) then we can find \( C''_{\delta} = \langle C''_{\delta} : \delta \in E \cap S \rangle \) such that \( C''_{\delta} \) is a club of \( \delta \) for each \( \delta \) and \( \langle C'_{\delta} \cap C''_{\delta} : \delta \in S \rangle \) has the \( (\Upsilon, \theta) \)-Md-property (really \( C'' \) does not depend on \( C' \))

\( g \) in clause \( f \) if \( \theta = \theta^+, \sigma \geq \kappa, 2^\lambda \leq \theta \) we can conclude also that \( \langle C'_{\delta} \cap C''_{\delta} : \delta \in S \rangle \) has the \( (\chi, \sigma) \)-MD\( _2 \)-property.

Proof. Clause \( a \) holds by [Sh 460], clause \( b \), \( c \) holds by 1.24, clause \( d \) by the major claim 1.10 and clause \( e \) is a rephrasing of clause \( d \), and lastly for clause
(f) use above case (iii) of clause (c) and use the obvious monotonicity (and $C'_\delta \cap C''_\delta$ is unbounded in $\delta$ being stationary).

\[ \square_{1.27} \]

Note some open questions.

1.28 Question: 1) Assume $\lambda$ is a weakly inaccessible which is not strong limit (i.e. a limit regular uncountable cardinal but not strong limit) and $\langle 2^\theta : \theta < \lambda \rangle$ is not eventually constant.

Does $\lambda$ have the weak diamond? Other consequences? In this case $\text{cf}(2^{<\lambda}) = \lambda$ as $\langle 2^\theta : \theta < \lambda \rangle$ witness it hence $2^\lambda > 2^{<\lambda} = \Sigma \{2^\theta : \theta < \lambda\}$. If $\lambda = 2^{<\lambda}$ then $\lambda = \lambda^{<\lambda}$ and such $\lambda$’s are investigated in [Sh 755] (for weak diamond on many stationary sets).

2) Consider generalizing [Sh:f, AP, §3] to our present context. Also we may consider $\langle c_\eta : \eta \in \lambda^+ \times \lambda^{+(n-3)} \times \ldots \times \lambda \rangle$ where $c_\eta \in \theta$.

1.29 Claim. Assume

$\otimes_{\mu, \theta}(i) \quad \theta \leq \mu$

$\langle 2^\theta : \theta < \lambda \rangle$ witness it hence $2^\lambda > 2^{<\lambda} = \Sigma \{2^\theta : \theta < \lambda\}$. If $\lambda = 2^{<\lambda}$ then $\lambda = \lambda^{<\lambda}$ and such $\lambda$’s are investigated in [Sh 755] (for weak diamond on many stationary sets).

2) Even if $2^\mu$ is singular still in Claim 1.10 we can replace clause (f) in the assumption by $\otimes_{\mu, \theta}$.

Remark. On relatives of $\otimes_{\mu, \theta}$ see $\epsilon_\kappa$, see [MPSH 713].

Proof. 1) Clearly $\mu^\theta$ has cardinality $2^\mu$, so let $\langle \nu_\alpha : \alpha < 2^\mu \rangle$ be a list without repetitions of $\mu^\theta$. For $\alpha \leq 2^\mu$, let $\Gamma_\alpha = \{\nu_\beta : \beta < \alpha\}$ and we choose $\tilde{f}^\alpha$ by induction on $\alpha \leq 2^\mu$ such that

\begin{enumerate}
  \item $\tilde{f}^\alpha = \langle f^\alpha_\varepsilon : \varepsilon < \mu \rangle$
  \item $f^\alpha_\varepsilon$ is a function from $\Gamma_\alpha$ to $\theta$
  \item $f^\alpha_\varepsilon$ is increasing continuous in $\alpha$, i.e., $\beta < \alpha \Rightarrow f^\beta_\varepsilon = f^\alpha_\varepsilon \upharpoonright \text{Dom}(f^\beta_\varepsilon)$
  \item for every $\gamma < \beta < \alpha$ the set $\{\varepsilon < \mu : f^\alpha_\varepsilon(\nu_\beta) = \nu_\gamma(\varepsilon)\}$ is not empty.
\end{enumerate}
If we succeed, then we let $f_\varepsilon = f_\varepsilon^{2^\mu}$ that is $f_\varepsilon = \bigcup \{ f_\alpha^\varepsilon : \alpha < 2^\mu \}$ for $\varepsilon < \mu$ and $\langle f_\varepsilon : \varepsilon < \mu \rangle$ is clearly as required.

For clause (d) it is enough to consider the case $\alpha = \beta + 1$, as the non trivial case in the induction is $\alpha = \beta + 1$. We can choose $\langle f_\varepsilon^{\beta+1}(\nu_\beta) : \varepsilon < \mu \rangle \in ^\mu \theta$ such that it holds by the assumption $(\ast)$.

2) In the proof of 1.10, now $\rho \in ^\mu \theta \Rightarrow |\text{Sol}_\rho| < 2^\mu$ does not suffice. However, before choosing $\langle f_\varepsilon : \varepsilon < \mu \rangle$ we are given $\mathcal{F}$ and we can choose $cd, \langle cd_\varepsilon : \varepsilon < \mu \rangle$ and define $T, \langle \eta_\alpha : \alpha < 2^\mu \rangle, T_{<\alpha}$ and $\langle \mathcal{P}_\delta : \delta \in S \rangle$. Note that $\alpha < \lambda \Rightarrow |h(\alpha)|^{<\kappa^{<\mu}} < 2^\mu$ hence $\alpha < \lambda \Rightarrow |\bigcup \{ \mathcal{P}_\delta : \delta \in S \cap \alpha \}| < 2^\mu$. We can also choose $\langle \nu_{\eta_\varepsilon} : \eta \in \mathcal{P}_\delta, \varepsilon < \mu \rangle$ and define $\rho_\eta \in ^\mu \theta$ for $\eta \in \mathcal{P}_\delta, \delta \in S$.

Now we can phrase our demands on $\langle f_\varepsilon : \varepsilon < \mu \rangle$, which is in fact weaker than the full generality of Sep($\mu, \theta$); it is

$$\square \ ^\mu \theta \neq \{ \nu \in ^\mu \theta : \text{for some } \eta \in \mathcal{P}_\delta \text{ we have } (\forall \varepsilon < \mu)(f_\varepsilon(\nu) \neq \rho_\eta(\varepsilon)) \}.$$  

For this it is enough that for the list $\langle \nu_\alpha : \alpha < 2^\mu \rangle$ chosen in the proof of part (1), for each $\delta \in S$, for some $\gamma < 2^\mu$ we have $\{ \rho_\eta : \eta \in \mathcal{P}_\delta \} \subseteq \{ \nu_\alpha : \alpha < \gamma \}$ which is easy to obtain.  

$\square_{1.29}$

1.30 Observation. If $\mu \geq \aleph_0, \theta = 2$, then $\odot_{\mu, \theta}$ holds.

Proof. If $\mathcal{F} \subseteq ^\mu \theta, |\mathcal{F}| < 2^\mu$, let $\nu \in ^\mu \theta \setminus \mathcal{F}$ and let $\rho = \langle 1 - \nu(\varepsilon) : \varepsilon < \mu \rangle$.  

$\square_{1.30}$
§2 Upgrading to larger cardinals

In §1 we get the $\theta$-middle diamond property for regular cardinals $\lambda$ of the form $\text{cf}(2^\mu)$ on “most” $S \subseteq S^\lambda_\kappa$ for “many” $\kappa = \text{cf}(\kappa)$ (and $\theta$). We would like to prove Theorem 0.1. For this we first give an upgrading, which is a stepping up claim (for $\text{MD}_\ell$) given as 2.1 below. Then we prove that it applies to $\lambda_2$ (for quite a few $\lambda_1$), when $\lambda_2$ is a successor of regular, then when $\lambda_2$ is successor of singular and lastly when $\lambda_2$ is weakly inaccessible (not strong limit). We can upgrade each of the variants from §1, but we shall concentrate on the one from 1.5(5A) because of Claim 1.20.

The main result of this section is claim 2.9.

**Question:** What occurs to $\lambda_2$ strongly inaccessible, does it have to have some Md-property?

Possibly combining §1 and §2 we may eliminate the $\lambda_2 > 2^{\lambda_1^{+\chi} + \theta}$ in 2.1, but does not seem urgent now.

### 2.1 Upgrading Claim.

1) Assume

(a) $\lambda_1 < \lambda_2$ are regular cardinals such that $\lambda_2 > 2^{\lambda_1^{+\chi} + \theta}$ (this last condition can be waived for “nice” colourings, see Definition 2.2 below)

(b) for some stationary $S_1 \subseteq S^{\lambda_1}_\kappa$, some sequence $\bar{C} = \langle C^1_\delta : \delta \in S_1 \rangle$ is a $(\lambda_1, \kappa, \chi)$-Md-parameter which has a $(D^{\lambda_1}_\kappa, \Upsilon, \theta)$-Md-property

(c) $\lambda_2 > \lambda_1^{+\chi}, \lambda_2$ is a successor of regular.

Then for some stationary $S_2 \subseteq S^{\lambda_2}_\kappa$, there is $\bar{C} = \langle C^2_\delta : \delta \in S_2 \rangle$, a good\(^5\) $(\lambda_2, \kappa, \chi)$-Md-parameters which has the $(D^{\lambda_2}_\kappa, \Upsilon, \theta)$-Md-property.

2) We can in part (1) replace clause (c) by any of the following clauses

(c)’ there is a $\leq \lambda_1$ square on some stationary $S \subseteq S^{\lambda_2}_{\lambda_1}$, (see Definition 0.4(1A); this holds if $\lambda_2 = \mu^+, \mu^{\lambda_1} = \mu$)

(c)’’ $\lambda_2$ has a $(\lambda_1, S_1)$-square (where $S_1$ is from clause (b)) which means:

$$\prod^{\lambda_2}_{\lambda_1, S_1} \lambda_1 < \lambda_2 \text{ are regular, } S_1 \subseteq \lambda_1 \text{ is stationary and there are } S_2 \subseteq S^{\lambda_2}_{\lambda_1} \text{ stationary and } C^2 = \langle C^2_\delta : \delta \in S_2 \rangle \text{ satisfying } C^2_\delta \subseteq \delta = \sup(C^2_\delta), C^2_\delta \text{ is closed, } \text{otp}(C^2_\delta) = \lambda_1 \text{ and if } \alpha \in C^2_{\delta_1} \cap C^2_{\delta_2}, \{\delta_1, \delta_2\} \subseteq S_2 \text{ and } \alpha = \sup(C^2_{\delta_1} \cap \alpha) \text{ and } \text{otp}(\alpha \cap C^2_{\delta_2}) \in S_1, \text{ then } \alpha \cap C^2_{\delta_2} = \alpha \cap C^2_{\delta_1}. $$

---

\(^5\)the goodness is a bonus
\( (c)^\prime \) \( \lambda_2 \) has a \((\lambda_1, C_1)\)-square \( \bar{C}_2 \) (where \( \bar{C}_1 \) is from clause (b)) which means:

\[
\square_{\lambda_1, C_1, C_2} (\lambda) \quad \bar{C}_2 = \langle C^2_\delta : \delta \in S_2 \rangle
\]

\( (\beta) \) \( S_2 \subseteq S_{\lambda_2^\ell} \) is stationary

\( (\gamma) \) \( C^2_\delta \) is a closed unbounded subset of \( \delta \) of order type \( \lambda_1 \)

\( (\delta) \) if \( \alpha \in C^2_{\delta_1} \cap C^2_{\delta_2} \) (so \( \{\delta_1, \delta_2\} \subseteq S_2 \)) and \( \text{otp} (\alpha \cap C^2_{\delta_1}) \in S_1 \), and \( \alpha = \sup(C^2_{\delta_1} \cap \alpha) \) then \( \text{otp} (\alpha \cap C^2_{\delta_2}) = \text{otp}(\alpha \cap C^2_{\delta_1}) \), call it \( \gamma \) and \( \{\beta \in C^2_{\delta_1} : \text{otp}(\beta \cap C^2_{\delta_1}) \in C^1_{\gamma}\} \)

\[ = \{\beta \in C^2_{\delta_2} : \text{otp}(\beta \cap C^2_{\delta_2}) \in C^1_{\gamma}\}. \]

3) Assume \( \ell \in \{0, 2\} \)

\( (a) \) \( \lambda_1 < \lambda_2 \) are regular uncountable

\( (b) \) some \((\lambda_1, \kappa, \chi)\)-Md-parameter has the \( \theta\)-MD\( \ell \)-property

\( (c) \) at least one of the variants of clause (\( c \)) above holds.

Then some \((\lambda_2, \kappa, \chi)\)-Md-parameter has the \( \theta\)-MD\( \ell \)-property.

2.2 Definition. We say that a \((\Upsilon, \theta)\)-colouring \( F = \langle F_\delta : \delta \in S \rangle \) for a \( \lambda\)-Md-parameter \( \bar{C} = \langle C_\delta : \delta \in S \rangle \) is nice when:

\[ \oplus \text{ if } \delta_1, \delta_2 \in S \text{ and } \text{otp}(C_{\delta_1}) = \text{otp}(C_{\delta_2}) \text{ and } \eta_1 \in \langle C_{\delta_1} \rangle \Upsilon, \eta_2 \in \langle C_{\delta_2} \rangle \Upsilon \text{ and } \alpha_1 \in C_{\delta_1} \land \alpha_2 \in C_{\delta_2} \land \text{otp}(C_{\delta_1} \cap \alpha_1) = \text{otp}(C_{\delta_2} \cap \alpha_2) \Rightarrow \eta_1(\alpha_1) = \eta_2(\alpha_2), \]

then \( F_{\delta_1}(\eta_1) = F_{\delta_2}(\eta_2). \)

Proof. 1) If clause (\( c \)) holds then clause (\( c \))' of part (2) holds by [Sh 351, §4], so it is enough to prove part (2).

2) Trivially clause (\( c \))' implies clause (\( c \))'' and clause (\( c \))'' implies clause (\( c \))''' so assume (\( c \))''; and let \( C^2 = \langle C^2_\delta : \delta \in S_2 \rangle \) be as there.

Let \( \bar{h} = \langle h_\delta : \delta \in S_2 \rangle \) be such that \( h_\delta \) is an increasing continuous function from \( \lambda_1 \) onto \( C^2_\delta \) recalling case (\( \gamma \)), i.e., \( C^2_{\delta_1} \) is a closed unbounded subset of \( \delta \) of order type \( \lambda_1 \). Let \( S = \{h_\delta(i) : \delta \in S_2 \text{ and } i \in S_1\} \) and if \( \alpha \in S, \alpha = h_\delta(i), \delta \in S_2 \) and \( i \in S_1 \) then we let \( C^\alpha_\delta = \{h_\delta(j) : j \in C^1_i\} \); and let \( \bar{C}^* = \langle C^\alpha_\delta : \alpha \in S \rangle \). Why is \( C^\alpha_\delta \) (hence \( \bar{C}^* \)) well defined? We shall check that if \( \alpha \in S, \alpha = h_\delta(i_\ell), \delta \in S_2, i_\ell \in S \) for \( \ell = 1, 2 \) then \( \{h_{\delta_1}(j) : j \in C^1_{i_1}\} = \{h_{\delta_2}(j) : j \in C^1_{i_2}\} \); this holds by clause (\( \delta \)) of the assumption (\( c \)). We should check the conditions for “\( \bar{C}^* \) is a good \((\lambda_2, \kappa, \chi)\)-Md-parameter with the \((D_{\lambda_2}, \Upsilon, \theta)\)-Md-property.
The main point is: assume that \( F \) is a \((\bar{C}^\ast, \Upsilon, \theta)\)-colouring, find a \( \bar{c} \) as required.

For each \( \delta \in S_2 \) we define a \((\bar{C}^1, \Upsilon, \theta)\)-colouring \( F^\delta \) by \( F^\delta(\eta) = F(\eta \circ h_\delta^{-1}) \) that is for \( i \in S_1 \) and \( \eta \in C^1_\delta \Upsilon \) we define \( F^\delta(\eta) = F(\nu) \) where \( \nu \in C^\ast_{\deltaS(i)} \Upsilon \), where \( \nu \) is defined by \( \forall \beta \in C^\ast_{\deltaS(i)} \) we let \( \nu(\beta) = \eta(h_\delta^{-1}(\beta)) \). As \( \lambda_2 > 2^{\lambda_1 + |\Upsilon|^\varsigma + \theta} \) because

\[
\theta^\Sigma(|\Upsilon|^{C^\ast_{\deltaS}}; \delta \in S_1) \leq 2^{\theta \lambda_1 \Sigma(|\Upsilon|^\varsigma; \alpha \leq \chi)} = 2^{\lambda_1 + |\Upsilon|^\varsigma + \theta},
\]

for some stationary \( S^\ast_2 \subseteq S_2 \) we have \( \delta \in S^\ast_2 \Rightarrow F^\delta = F^\ast \), so \( F^\ast \) is a \((\bar{C}^1, \Upsilon, \theta)\)-colouring, hence it has a good \((D_{\lambda_1}, \Upsilon, \theta)\)-Md-sequence \( \bar{c}^1 = \{c^1_i : i \in S_1\} \).

We define \( \bar{c}^\ast = \{c^\ast_\alpha : \alpha \in S\} \) by \( \delta \in S_2 \land i \in S_1 \Rightarrow c^\ast_{h_\delta(i)} = c^1_i \). By clause (\( \delta \)) of (\( c \))'' it is well defined. Now for any \( \eta \in (\lambda_2) \Upsilon \) and club \( E_2 \) of \( \lambda_2 \), we can find \( \delta \in S_2 \cap \text{acc}(E_2) \), hence \( E_1 = \{\alpha < \lambda_1 : h_\delta(\alpha) \in E_2\} \) is a club of \( \lambda_1 \) and let \( \eta_1 \in (\lambda_1) \Upsilon \) be \( \eta_1(\alpha) = \eta(h_\delta(\alpha)) \), so by the choice of \( \bar{c}^1 \) for some \( \delta_1 \in S \cap E_1, c^1_{\delta_1} = F^\ast(\eta_1 \upharpoonright C^1_{\delta_1}) \) and let \( \delta_2 = h_\delta(\delta_1) \), so \( \delta_2 \in E_2 \) and \( c^\ast_{\delta_2} = c^1_i = F^\ast(\eta_1 \upharpoonright C^1_{\delta_1}) = F^\delta(\eta_1 \upharpoonright C^1_{\delta_2}) = F(\eta \upharpoonright C^1_{\delta_2}) \), so we are done.

3) Similar only easier (or see the proof of 2.10). \( \square_{2.1} \)

### 2.3 Claim

We can strengthen the conclusion of 2.1(1) and 2.1(2) to: we can find a sequence \( \langle S_{2,i} : i < \lambda_2 \rangle \) of pairwise disjoint stationary substates of \( \lambda_2 \) and for each \( i < \lambda_2 \) a sequence \( \langle C^1_\delta : \delta \in S_{2,i} \rangle \) which is a \((\lambda_2, \kappa, \chi)\)-Md-parameter having the \((D_{\lambda_2}, \Upsilon, \theta)\)-Md-property.

**Proof.** By the proof of 2.1, without loss of generality clause (\( c \))'' of 2.1(2) and clauses (\( a \)), (\( b \)) of 2.1(1) hold. For some \( \zeta < \lambda_1 \) for unboundedly many \( i < \lambda_2 \) the set \( S_{2,i} : = \{\delta \in S_2 : i \text{ is the } \zeta \text{-th member of } C^2_{\delta}\} \) is stationary. \( \square_{2.3} \)

### 2.4 Conclusion

1) If \( \mu \) is a strong limit, \( \lambda \) is a successor cardinal, \( \lambda = \text{cf}(\lambda) > 2^\mu \), moreover \( \lambda > 2^\mu \) (superfluous for nice colourings, e.g., for \( \text{MD}_\ell, \ell \in \{0, 2\} \)), then for every large enough regular \( \kappa < \mu \), some good\( ^+ (\lambda, \kappa, \kappa^+)\)-Md-parameter \( C \) has the \((2^\mu, \theta)\)-Md-property for every \( \theta < \mu \).

2) We can add: \( C \) has the \( \theta\text{-MD}_\ell\)-property for \( \ell \in \{0, 2\} \).

3) We can find a sequence \( \langle S_i : i < \lambda \rangle \) of pairwise disjoint stationary substates of \( S^\lambda \) such that for each \( i < \lambda \) there is \( \bar{C}^i = \{C^i_\delta : \delta \in S\} \) as in parts (1) and (2).

**Proof.** 1) Let \( \kappa_0 < \mu \) be large enough such that \( |\theta \in \{\lambda, 2^\mu\} \land \alpha < \theta \land \kappa_0 \leq \kappa = \text{cf}(\kappa) < \mu| \Rightarrow |\alpha|^{[\kappa]} < \theta \), (see Definition 1.3, exists by [Sh 460]).

By 1.10, for any regular \( \kappa \in [\kappa_0, \mu) \) and \( \theta < \mu \) we have the result for \( \text{cf}(2^\kappa) \) in place of \( \lambda \). If \( \lambda \) is a successor of regular we can apply 2.1(1), i.e., for clause (\( c \))

\(^6\)only place this is used, if \( F \) is nicely defined this is not necessary
there is a square as required by [Sh 351, §4]. In the case that \( \lambda \) is a successor of a singular cardinal, by 2.7 below, clause \((c)''\) of 2.1(2) holds for \( \lambda_2 = \lambda, \lambda_1 = \text{cf}(2^\mu) \) so we can apply 2.1.

2) Follows by 1.20.

3) In the proof of parts (1) + (2) above, instead of 2.1 we use 2.3. \( \square_{2.4} \)

Below we generalize Engelking Karlovic theorem [EK] (which says that if \( \mu = \mu^\kappa, \lambda = 2^\mu \) then there are \( f_\varepsilon : \lambda \to \mu \) for \( \varepsilon < \mu \) such that every partial function from \( \lambda \) to \( \mu \) with domain of cardinality \( \leq \kappa \), is extended by \( f_\varepsilon \) for some \( \varepsilon < \mu \)); we do more than is necessary for the middle diamond.

2.5 Claim. 1) Assume

\((a)\) \( \mu = \text{cov}(\mu, \theta^+, \theta^+, \kappa^+) \)
\((b)\) \( 2^\theta \leq \mu \)
\((c)\) \( \mu < \lambda \leq 2^\mu \)

Then there is \( \bar{F} \) such that:

\((\alpha)\) \( \bar{F} = \langle F_\varepsilon : \varepsilon < \mu \rangle \)
\((\beta)\) each \( F_\varepsilon \) is a function from \( \lambda \) to \( \mu \)
\((\gamma)\) if \( u \in [\lambda]^{\leq \theta} \) then we can find \( \bar{E} \) such that
\((i)\) \( \bar{E} = \langle E_i : i < \kappa \rangle \) is a sequence of equivalence relations on \( u \)
\((ii)\) \( \alpha \in u \Rightarrow \{\alpha \} = \cap \{\alpha/E_i : i < \kappa \} \)
\((iii)\) if \( f : u \to \mu \) respects some \( E_i \) (i.e. \( \alpha E_i \beta \Rightarrow f(\alpha) = f(\beta) \))
then we can find a partition \( \langle u_\xi : \xi < \kappa \rangle \) of \( u \) such that \( \xi < \kappa \Rightarrow f \upharpoonright u_\xi \subseteq F_\varepsilon \) for some \( \varepsilon < \mu \).

2) In part (1) we can strengthen clause \((\gamma)\) to \((\gamma)\) below if \((d)\ell \) for some \( \ell \in \{1, \ldots, 5\} \) below holds

\((\gamma)\) if \( u \in [\lambda]^{\leq \theta} \) and \( f : u \to \mu, \) then we can find \( \bar{u} = \langle u_i : i < \kappa \rangle \) such that
\((i)\) \( u = \cup \{u_i : i < \kappa \} \) and
\((ii)\) for every \( i < \kappa \) for some \( \varepsilon < \mu \) we have \( f \upharpoonright u_i \subseteq F_\varepsilon \)

\((d)\) there is a sequence \( \bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \) of pairwise distinct functions from \( \mu \) to \( \mu \) such that: if \( u \in [\lambda]^{\leq \theta} \) then we can find sequences \( \bar{u} = \langle u_i : i < \kappa \rangle \) and \( \langle B_i : i < \kappa \rangle \) such that \( u = \cup \{u_i : i < \kappa \}, B_i \in [\mu]^{<\kappa_0} \) for \( i < \kappa \) and for each \( i < \kappa \) the sequence \( \langle f_\alpha \upharpoonright B_i : \alpha \in u_i \rangle \) is with no repetitions.
(d)₂ there is a sequence \( \vec{f} = \langle f_\alpha : \alpha < \lambda \rangle \) of functions from \( \kappa \) to \( \mu \) and an ideal \( J \) on \( \kappa \) such that \( \vec{f} \) is \( \theta^+ \)-free modulo \( J \), i.e., \((\theta^+, J)\)-free (which means that for \( u \in [\lambda]^{\leq \theta} \), we can find a sequence \( \vec{a} = \langle a_\alpha : \alpha \in u \rangle \) of subsets of \( \kappa \) such that \( a_\alpha = \kappa \) mod \( J \) and \( i \in a_\alpha \cap a_\beta \) & \( \alpha \neq \beta \Rightarrow f_\alpha(i) \neq f_\beta(i) \))

(d)₃ like (d)₂ but \( a_\alpha \in J^+ \) that is \( a_\alpha \notin J \)

(d)₄ \((\forall \alpha < \mu)(|\alpha|^\theta < \mu)\) and \( \lambda < \text{pp}_J(\mu) \) for some ideal \( J \) on \( \kappa \)

(d)₅ \( \mu = \mu^{\aleph_0} < \lambda < \mu^\kappa \).

3) If we weaken the conclusion of part (2), replacing \((\gamma)\) by \((\gamma)''_{\theta,\kappa} \) for every club \( E \) of \( \lambda \) for some closed \( u \subseteq E \) of order type \( \theta \) there is \( \langle u_i : i < \kappa \rangle \) such that \( u_i \subseteq u \) and \( \zeta \in u \) & \( \text{cf}(\zeta) > \kappa \Rightarrow \zeta \in \cup\{u_i : i < \kappa\} \) and \((\forall i < \kappa)(\exists \varepsilon < \mu)(f_i \mid u_i \subseteq F_\varepsilon)\)

then we can weaken the assumption in a parallel way, e.g.

(d)⁻₁,θ,κ there is \( \vec{f} = \langle f_\alpha : \alpha < \lambda \rangle \), \( f_\alpha \in \mu^\kappa \) such that for any club \( E \) of \( \lambda \) we can find a closed \( e \subseteq E \) of order type \( \theta + 1 \), and \( \vec{u} = \langle u_i : i < \kappa \rangle \), \( \vec{B} = \langle B_i : i < \kappa \rangle \) such that \( B_i \in [\mu_1]^< \aleph_0, u_i \subseteq \theta \) and \( \zeta \in u \) & \( \text{cf}(\zeta) \geq \kappa \Rightarrow \zeta \in \cup\{u_i : i < \kappa\} \) and for each \( i \), \( \langle f_\alpha \mid B_i : \alpha \in u_i \rangle \) is with no repetitions (and in (d)₃ it becomes easy).

Remark. We can replace \( \lambda \) (as a domain) by \([\lambda]^2\), etc.

**Proof.** 1) Let \( \mathcal{A} \) be a family of \( \leq \mu \) subsets of \( \mu \) each of cardinality \( \leq \theta \) such that any \( A \in [\mu]^{\leq \theta} \) is included in the union of \( \leq \kappa \) many of them; it exists by assumption (a). By assumption (b) without loss of generality \( A \in \mathcal{A} \) & \( B \subseteq A \Rightarrow B \in \mathcal{A} \) so we replace “included” by “equal” and also without loss of generality \( A_1 \in \mathcal{A} \) & \( A_2 \in \mathcal{A} \Rightarrow A_1 \cup A_2 \in \mathcal{A} \). Let \( \text{pr} \) be a pairing function on \( \mu \).

As \( \mu < \lambda \leq 2^\mu \) there is a sequence \( \langle f_\zeta^* : \zeta < \lambda \rangle \) of pairwise distinct functions from \( \mu \) to \( \{0, 1\} \).

Let

\[
\mathcal{Y} = \{(A, B, \vec{g}, \vec{\beta}) : B \in \mathcal{A}, \vec{g} = \langle g_i : i < \theta \rangle, g_i \in B^2 \text{ and } i < j < \theta \Rightarrow g_i \neq g_j, \text{ and } \vec{\beta} = \langle \beta_i : i < \theta \rangle \text{ satisfies } \{\beta_i : i < \theta\} \subseteq A \in \mathcal{A}\}.
\]
Clearly \( |\mathcal{Y}| \leq \mu \times 2^{\theta} = \mu \). For each \( x = (A, \bar{g}, \bar{\beta}) \in \mathcal{Y} \) we define a function \( F_x : \lambda \rightarrow \mu \) as follows:

\( F_x(\zeta) = \beta_i \) if \( g_i \subseteq f^*_\zeta \) and is 0 if there is no such \( i \) (clearly we cannot have \( i_0 \neq i_1 < \theta \) \& \( g_{i_0} \subseteq f^*_\zeta \) \& \( g_{i_1} \subseteq f^*_\zeta \)).

Let \( \bar{F} = \langle F_x : \varepsilon < \mu \rangle \) list \( \{ F_x : x \in \mathcal{Y} \} \), clearly \( \bar{F} \) satisfied demand \((\alpha) + (\beta)\). As for demand \((\gamma)\), let \( u \in [\lambda]^{\leq \theta} \) be given. For any \( \zeta_1 \neq \zeta_2 \in u \) let \( \alpha = \alpha(\zeta_1, \zeta_2) < \mu \) be minimal such that \( f^*_{\zeta_1}(\alpha) \neq f^*_{\zeta_2}(\alpha) \), exists as \( f^*_{\zeta_2} \neq f^*_{\zeta_1} \in \mu^2 \) and let \( B = \{ \alpha(\zeta_1, \zeta_2) : \zeta_1 < \zeta_2 \text{ are from } u \} \). By the choice of \( \mathcal{A} \) we can find pairwise disjoint \( B_\zeta \in \mathcal{A} \) for \( \zeta < \kappa \) such that \( B = \bigcup \{ B_\zeta : \zeta < \kappa \} \). Lastly, we define \( \tilde{E} = \langle E_\zeta : \zeta < \kappa \rangle \) by \( \bigoplus \) for \( \zeta_1, \zeta_2 \in u, \zeta_1 \in \zeta_2 \) iff \( f^*_i \uparrow B_\zeta = f^*_i \uparrow B_\zeta \).

Assume that \( f : u \rightarrow \mu \) respects \( E_\zeta \), then we can find non empty \( A_\zeta \in \mathcal{A} \) for \( \zeta < \kappa \) such that \( \text{Rang}(f) = \bigcup A_\zeta \). Now for any pair \( (\zeta_1, \zeta_2) \in \kappa \times \kappa \), let \( u_{\zeta_1} = \text{Min}\{ \zeta \in u : f(\zeta) \in A_\zeta \} \), let \( \langle \zeta_i : i < \theta \rangle \) list \( u_{\zeta_2} \), let \( x_{\zeta_1, \zeta_2} \) be \( (A_{\zeta_2}, B_{\zeta_1} \cup B_{\zeta_2}, \langle f^*_i \uparrow B_\zeta : i < \theta \rangle, \langle f(\zeta) : i < \theta \rangle) \). Now check.

2) Naturally the proof splits to five cases, but first let \( \mathcal{A} \) be as in the proof of part (1), \( \text{pr}(..., ... \rangle \) be a pairing function on \( \mu \), let \( \langle f^*_i : \zeta < \lambda \rangle \) be a sequence of functions from \( \mu \) to \( \mu \) (chosen separately in each case). Let \( \mathcal{X} = \{ x : x = (A, \bar{g}, \bar{\beta}) \text{ satisfies} \)

\begin{align*}
(i) & \quad A \in \mathcal{A} \\
(ii) & \quad \bar{g} = \langle g_i : i < \theta \rangle, \\
(iii) & \quad \text{each } g_i \text{ is a partial function from } A \text{ to } A \\
(iv) & \quad \bar{\beta} = \langle \beta_i : i \leq \theta \rangle \text{ and } i \leq \theta \Rightarrow \beta_i \in A \}.
\end{align*}

Clearly \( |\mathcal{X}| \leq \mu \) and for each \( x = (A, \bar{g}, \bar{\beta}) \in \mathcal{X} \) we define \( F_x : \lambda \rightarrow \mu \) by

\[ F_x(\zeta) = \beta_i \iff i = i_x(\zeta) = \text{Min}\{ \varepsilon \leq \theta : \text{if } \varepsilon < \theta \text{ then } g_\varepsilon \subseteq f^*_\zeta \}. \]

So let \( u \in [\lambda]^{\leq \theta} \) and \( f : u \rightarrow \mu \) be given and let \( \langle v_i : i < \kappa \rangle \) be such that \( u = \bigcup \{ v_i : i < \kappa \} \) and \( \langle A_i : i < \kappa \rangle \) be such that \( i < \kappa \Rightarrow A_i \in \mathcal{A} \) and \( \text{Rang}(f^{v_i}) \subseteq A_i \).

**Case 1:** Clause \((d)_1\) holds.

We choose \( f^*_\zeta = f_\zeta \) for every \( \zeta < \lambda \) where \( f_\zeta \) is from clause \((d)_1\). Let \( \langle (u_i, B_i) : i < \kappa \rangle \) be as guaranteed to exist for \( u \) in clause \((d)_1\). For each \( i, j < \kappa \) such that \( u_i \cap v_j \neq \emptyset \) and let \( \langle \zeta^{i,j} : \varepsilon < \theta \rangle \) list \( u_i \cap v_j \) (possibly with repetition) and let \( x_{i,j} = (A^{i,j}_i, g^{i,j}_\varepsilon : \varepsilon < \theta), \langle \beta^{i,j}_\varepsilon : \varepsilon < \theta \rangle \) where
\((a)\) \(A^*_{i,j} = B_i \cup A_j\)
\((b)\) \(g^{i,j}_\varepsilon = f^*_{\zeta^{i,j}_\varepsilon} \upharpoonright B_i\)
\((c)\) \(\beta^{i,j}_\varepsilon = f(\zeta^{i,j}_\varepsilon)\).

We can check that
\[(a)\] \(x_{i,j} \in \mathcal{X}^*\) and
\[(b)\] \(\bigcup \{u_i \cap v_j : i, j < \kappa\} = u\)
\[(c)\] \(F_{x_{i,j}} \supseteq f \upharpoonright (u_i \cap v_j)\).

[Why? Let \(\zeta(*) \in u_i \cap v_j\) so \(\zeta(*) = \zeta^{i,j}_{\varepsilon(*)}\) for some \(\varepsilon(*) < \theta\) so \(g^{i,j}_{\varepsilon(*)} = f^*_{\zeta^{i,j}_{\varepsilon(*)}} \upharpoonright B_i\) by clause (b) above hence \(\varepsilon(*) \in \{\varepsilon \subseteq \theta : \text{if } \varepsilon < \theta \text{ then } g_{\varepsilon} \subseteq f^*_{\zeta^{i,j}_{\varepsilon(*)}}\}. But \(\langle f^*_{\zeta^{i,j}_{\varepsilon(*)}} \upharpoonright B_i : \zeta \in u_i \rangle\) is a sequence without repetitions, so

\[\varepsilon < \varepsilon(*) \Rightarrow \zeta^{i,j}_{\varepsilon} \subseteq u_i \cap \zeta \Rightarrow g^{i,j}_{\varepsilon} = f^*_{\zeta^{i,j}_{\varepsilon}} \upharpoonright B_i\]
\[\neq f^*_{\zeta^{i,j}_{\varepsilon(*)}} = g^{i,j}_{\varepsilon(*)}\]

(the middle inequality holds by the choice of \(\langle (u_i, B_i) : i < \kappa\rangle\) above, i.e., by clause \((d)_1\)).

So \(x_{i,j}^*(\zeta(*)) = \varepsilon(*)\) so \(F_{x_{i,j}}^*(\zeta(*)) = \beta^{i,j}_{\varepsilon(*)} = f(\zeta^{i,j}_{\varepsilon(*)})\), the last equality holds by clause (c). So we are done.

**Case 2**: Clause \((d)_2\) holds.

Follows from Case 3.

**Case 3**: \((d)_3\) holds.

Let \(f'_\zeta\) be any function from \(\mu\) to \(\mu\) extending \(f_\zeta\). It suffices to show that \(\langle f'_\zeta : \zeta < \lambda\rangle\) satisfies clause \((d)_1\). So let \(u \in [\lambda]^{<\theta}\), so by \((d)_3\) we can find a sequence \(\bar{a} = \langle a_\zeta : \zeta \in u \rangle\) of subsets of \(\kappa\) such that \(a_\alpha \notin J\) and

\[\zeta_1 \neq \zeta_2 \in u \& i \in a_{\zeta_1} \cap a_{\zeta_2} \Rightarrow f_\alpha(i) \neq f_\beta(i)\]

Let \(B_i = \{i\} \in [\mu]^{<\theta}\) and \(u_i = \{\zeta \in u : i \in a_\zeta\}\), clearly they are as required.

**Case 4**: Clause \((d)_4\) holds.

This implies Clause \((d)_3\) by [Sh:g, II,1.3,p.46+1.4(3),p.50 (or see [Sh 506]).

**Case 5**: \((d)_5\).
Let $\mu_1 = \text{Min}\{\partial : \partial^\kappa > \lambda\}$, so clearly $\mu_1 \leq \mu$. As $\kappa \leq \theta, 2^\theta \leq \mu$ we have $2^\kappa < \mu_1$, hence $\kappa_1 = \text{cf}(\mu_1)$ is $\leq \kappa$ and $(\forall \alpha < \mu_1)(|\alpha|^\kappa < \mu_1)$. By (d) we have $\kappa_1 > \aleph_0$. So by [Sh:g, VIII], pp$p_{\kappa^2}(\mu_1) \geq (\mu_1)^\kappa = \mu^\kappa > \lambda$ so Clause (d) holds.

\[\square_{2.5}\]

2.6 Claim. 1) Assume

(a) $\mu = \text{cov}(\mu, \theta^+, \theta^+, \kappa^+)$ so $\mu > \theta \geq \kappa$

(b) $2^\theta \leq \mu$

(c) $\mu < \lambda = \text{cf}(\lambda) \leq \mu^\kappa$

(d) $\theta = \text{cf}(\theta), S \subseteq S^\lambda_{\theta}$ is stationary and $S \in I[\lambda]$

(e) $T \subseteq \theta$ and $[\alpha < \lambda \& \delta \in T \Rightarrow |\alpha|^{<\text{cf}(\delta)} > \mu < \lambda]$.

Then there is $\bar{F}$ such that

(\alpha) $\bar{F} = \langle F_\varepsilon : \varepsilon < \mu \rangle$

(\beta) $F_\varepsilon$ is a function from $\lambda$ to $\mu$

(\gamma) for a club of $\delta \in S$, there is an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ with limit $\delta$ such that:

(*) for every function $f$ from $\{\alpha_\varepsilon : \varepsilon \in T\}$ to $\mu$ we can find a sequence $\langle u_i : i < \kappa \rangle$ of subsets of $T$ with union $T$ such that $(\forall i < \kappa)(\exists \varepsilon < \mu)(f \upharpoonright \{\alpha_j : j \in u_i\} \subseteq F_\varepsilon)$.

Proof. 1) Let $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ and $E_0$ as in 0.5(2) witness $S \in I[\lambda]$. Let $M = \langle M_\alpha : \alpha < \lambda \rangle$ be an increasing continuous sequence of elementary submodels of $(\mathcal{H}(\lambda), \in, <^*_\lambda)$, each of cardinality $< \lambda$ satisfying $M \upharpoonright (\alpha + 1) \in M_{\alpha+1}, \|M_0\| = \mu, M_{1+\alpha} \cap \lambda \in \lambda$ and $\{\lambda, \theta, T, S, \bar{a}, E_0\} \cup (\mu + 1) \subseteq M_0$. Let $E = \{\delta < \lambda : M_\delta \cap \lambda = \delta\}$, clearly a club of $\lambda$ and $E \subseteq E_0$ as $E_0 \in M_0$.

Now we choose by induction on $\alpha < \lambda$ a function $f_\alpha$ from $\kappa$ to $\mu$ such that:

(*1) if $b \in [\alpha]^{\leq \theta} \& b \in M_{\alpha+1}$ then
\[
\text{Rang}(f_\alpha) \not\subseteq \{\text{Rang}(f_\beta) : \beta \in b\}
\]

(*2) $f_\alpha$ is the $<^*_\lambda$-first object satisfying (*1).

This is clearly possible as $\kappa \leq 2^\theta \leq \mu < \lambda \leq \mu^\kappa$.

For every $i < \kappa$ and $u \in M_0 \cap [\mu]^{\leq \theta}$ and $g : u \rightarrow \mu$ from $M_0$, let $F_{g,i}$ be the unique $F_{g,i} : \lambda \rightarrow \mu$ and it satisfies: if $f_\alpha(i) \in \text{Dom}(g)$ then $F_{g,i}(\alpha) = g(f_\alpha(i))$. [Why $F_{g,i} \in M_0$? As $\mu + 1 \subseteq M_0$ and $u, g \in M_0$.]
Let \( \langle g_\varepsilon, i_\varepsilon \rangle : \varepsilon < \mu \) list the set of all such pairs \((g,i)\) and let \( F_\varepsilon = F_{g_\varepsilon, i_\varepsilon} \). We shall show that \( \langle F_\varepsilon : \varepsilon < \mu \rangle \) is as required; so clauses \((\alpha) + (\beta)\) hold trivially.

Now if \( \delta \in \text{acc}(E) \cap S \subseteq S \cap E_0 \) so \( a_\delta \subseteq \delta = \sup(a_\delta) \), \( \text{otp}(a_\delta) = \text{cf}(\delta) = \theta \) and let \( \langle \alpha_\varepsilon : \varepsilon < \theta \rangle \) list the closure in \( \delta \) of \( a_\delta \) in the increasing order. So for some club \( C \) of \( \theta \) consisting of limit ordinals we have \( \varepsilon \in C \Rightarrow \alpha_\varepsilon \in E \). Let \( \Theta = \{ \text{cf}(\delta) : \delta \in T \} \), now suppose \( \varepsilon \in C \) is a limit ordinal with cofinality \( \Theta \). So for every \( \zeta < \varepsilon, \alpha_{\zeta+1} \in a_{\alpha_\varepsilon} \) so, as \( a \in M_0 < M_{\alpha_\varepsilon} \) & \( \alpha_{\zeta+1} \in M_{\alpha_\varepsilon} \) clearly \( a_{\alpha_{\zeta+1}} \in M_{\alpha_\varepsilon} \) but (see Definition 0.5(2)) \( \text{cf}(a_{\alpha_{\zeta+1}}) = \{ \alpha_\varepsilon : \varepsilon \leq \zeta \} \) hence \( \{ \alpha_\varepsilon : \varepsilon \leq \zeta \} \in M_{\alpha_\varepsilon} \). Now the tree \( \{v : \nu \in M_{\alpha_\varepsilon} \text{ is a sequence of ordinals of length } < \delta \} \subseteq \) is a tree which belongs to \( M_{\alpha_\varepsilon+1} \), has \( \varepsilon \) levels and \( |M_{\alpha_\varepsilon}| < \lambda \) nodes so by assumption \((e)\) it has \( < \lambda \) \( \varepsilon \)-branches. As this set (of \( \varepsilon \)-branches) belongs to \( M_{\alpha_\varepsilon+1} \), every \( \varepsilon \)-branch of this tree belongs to \( M_{\alpha_\varepsilon+1} \) but \( \langle \alpha_{\zeta} : \zeta < \varepsilon \rangle \) is (essentially) such a branch; so together \( \varepsilon \in C \wedge \text{cf}(\varepsilon) \in \Theta \Rightarrow \langle \alpha_{\zeta} : \zeta < \varepsilon \rangle \in M_{\alpha_\varepsilon+1} \).

Let

\[
v_i = \{ \varepsilon \in C : \text{cf}(\varepsilon) \in \Theta \text{ and } f_{\alpha_\varepsilon}(i) \notin \cup \{ \text{Rang}(f_\beta) : \beta \in \{ \alpha_{\zeta} : \zeta < \varepsilon \} \} \}.
\]

So \( \langle v_i : i < \kappa \rangle \) is a sequence of subsets of \( C \) with union \( \supseteq T \). For each \( i < \kappa \) by assumption \((a)\) there is \( \langle v_{i,j} : j < \kappa \rangle \) be a sequence of subsets of \( v_i \) with union \( v_i \) such that for every \( j < \kappa \{ f_{\alpha_\varepsilon}(i) : \varepsilon \in v_{i,j} \} \subseteq v^+_{i,j} \in M_0 \).

Let \( h \) be a function from \( \{ \alpha_\varepsilon : \varepsilon \in T \} \) to \( \mu \). So by assumption \((a)\) we can find a sequence \( \langle w_j : j < \kappa \rangle \) of subsets of \( \mu \) each of cardinality \( \leq \theta \) such that \( \text{Rang}(h) \subseteq \bigcup w_j \) and \( \{ w_j : j < \kappa \} \subseteq M_0 \). Let \( u_{i,j_1,j_2} = \{ \varepsilon : \varepsilon \in v_i, f_{\alpha_\varepsilon}(i) \in v^+_{i,j_1} \} \) and

\[
h(\alpha_\varepsilon) \in w_{j_2} \}.
\]

Hence for \( i, j_1, j_2 < \kappa \) there is \( g = g_{i,j_1,j_2} : \{ f_{\alpha_\varepsilon}(i) : \varepsilon \in u_{i,j_1,j_2} \} \to \mu \) satisfying \( \varepsilon \in u_{i,j_1,j_2} \Rightarrow g(f_{\alpha_\varepsilon}(i)) = h(\alpha_\varepsilon) \), see the choice of \( v_i, v_{i,j_1}, w_{j_1}, u_{i,j_1,j_2} \). Now as \( \varepsilon \subseteq \theta, \theta^2 \leq \mu, \mu + 1 \subseteq M_0 \) and \( v^+_{i,j_1}, w_{j_2} \in M_0 \) are subsets of \( \mu \) of cardinality \( \leq \theta \) clearly the set of partial functions from \( v^+_{i,j_1} \) to \( w_{j_2} \) belongs to \( M_0 \) and has cardinality \( \leq |u_{i,j_1}| \cdot |w_{j_2}| \leq \theta^\theta = 2^\theta \leq \mu \) hence is included in \( M_0 \). So \( g_{i,j} \) belongs to \( M_0 \) for every \( i, j < \kappa \). Also easily \( F_{g_{i,j},i} \) extends \( h \upharpoonright \{ \{ \alpha_\varepsilon : \varepsilon \in u_{i,j} \} \} \). So \( \{ \alpha_\varepsilon : \varepsilon \in u_{i,j} \} \) and \( \langle \alpha_\varepsilon : \varepsilon \in C \rangle \) are as required (modulo some renaming). But \( \text{Dom}(g_{i,j}) = \{ f_{\alpha_\varepsilon}(i) : \varepsilon \in u_{i,j} \} \subseteq w_j \).

\( \square 2.6 \)

Compare the following with [Sh 237e].

2.7 Claim. 1) Assume

\((a)\) \( \lambda = \mu^+ \) and \( \kappa = \text{cf}(\mu) < \mu \) and \( \aleph_0 < \theta = \text{cf}(\theta) < \mu \)
(b) (i)  \( \Theta = \{ \sigma : \aleph_0 \leq \sigma = \text{cf}(\sigma) < \mu \text{ and } \mu^{<\sigma^+} = \mu \} \)

(ii)  \( T_\theta = \{ \delta < \theta : \text{cf}(\delta) \in \Theta \} \),

(iii)  \( T_\theta^+ = T_\theta \cup \{ 0 \} \cup \{ \zeta + 1 : \zeta < \theta \} \),

(iv)  \( T_\theta^- = T_\theta^+ \setminus T_\theta \)

(c)  \( P \subseteq \{ \bar{A} : \bar{A} \text{ a sequence of length } \leq \mu \text{ of subsets of } \theta \text{ each included in } T_\theta^+ \text{ and including } T_\theta^- \} \)

(d)  \( \mathcal{P} \subseteq [\mu]^{<\theta} \) has cardinality \( \mu \)

(e)  if \( \alpha_\varepsilon < \mu \) for \( \varepsilon < \theta \) and \( \langle w_\varepsilon^i : i < \kappa \rangle \) a sequence of subsets of \( T_\theta^+ \) and including \( T_\theta^- \) then for some \( \bar{A} \in P \) we have:

(i)  for some \( \zeta < \ell g(\bar{A}) \) and \( j < \kappa \), \( A_\zeta \subseteq w_j^* \)

(ii)  \( \zeta < \ell g(\bar{A}) \) \& \( j < \kappa \) \& \( A_\zeta \subseteq w_j^* \Rightarrow \{ \alpha_\varepsilon : \varepsilon \in A_\zeta \} \in \mathcal{P} \).

Then we can find \( (\bar{C}_i, A_i^*, S_i) \) for \( i < \mu \) such that

(\( \alpha \))  \( \{ 0 \} \cup \{ \zeta + 1 : \zeta < \theta \} \subseteq A_i^* \subseteq \theta \)

(\( \beta \))  \( \bar{C}_i = \langle C_\delta^i : \delta \in S_i \rangle ; S_i \subseteq \lambda \) and \( \text{otp}(\alpha \cap C_\delta^i) \in A_i^* \& C_\delta^i \subseteq \delta \& \alpha \in C_\delta^i \Rightarrow \alpha \in S_i \& C_\alpha^i = \alpha \cap C_\delta^i \)

(\( \gamma \))  \( S_i^* = \{ \delta \in S_i : \text{cf}(\delta) = \theta \} \) is stationary

(\( \delta \))  for every club \( E \) of \( \lambda \) there are \( \delta^* \in E \) and a sequence \( \langle w_i^* : i < \kappa \rangle \) of subsets of \( T_\theta^+ \) including \( T_\theta^- \) with union \( T_\theta^+ \) such that for each \( j < \kappa \) there is \( \bar{A} \in P \) satisfying: \( ( \forall \zeta < \ell g(\bar{A})) (A_\zeta \subseteq w_j^* \rightarrow (\exists i < \mu)(\delta^* \in S_i^* \& A_i^* = A_\zeta) \)

(\( \varepsilon \))  if \( B \in I[\lambda] \) is stationary and \( B \subseteq S_0^\lambda \) then we can demand \( B \setminus \{ S_i^* : i < \mu \} \) is not stationary and in clause (\( \delta \)) any \( \delta \in B \) satisfies the requirement on \( \delta^* \).

2) Assume, in addition \( \alpha < \mu \Rightarrow |\alpha|^{\theta} < \mu, T_\theta' = \{ \delta < \theta : \text{cf}(\delta) \neq \kappa \} \) and we let \( P = \{ \bar{A} : \bar{A} = \langle A_i : i < \kappa \rangle \) is a sequence of sets with union \( T_\theta^+ \} \). Then \( T_\theta' = T_\theta^+ \) and for some \( \mathcal{P} \) all the assumptions hence the conclusion of part (1) holds.

3) Assume \( \mu_* \) is strong limit \( < \mu, \kappa = \text{cf}(\mu) < \mu, \lambda = \mu^+ \), then for some \( \theta_0 < \mu_* \), for any regular \( \theta \in (\theta_0, \mu_* \) the set \( T_\theta \) include \( \{ \delta < \theta : \text{cf}(\delta) \geq \theta_1 \) or \( \delta \) nonlimit \) and for \( P = \{ \bar{A} : \bar{A} \text{ is a sequence of } < \theta_0 \text{ subsets of } \theta \text{ with union } T_\theta \text{ such that } \) \( \{ A_\zeta : \zeta < \ell g(\bar{A}) \} \) is closed under finite unions \).

Then for some \( \mathcal{P} \) the assumptions hence the conclusion of part (1) holds.

4) In (1) (and (2), (3), if \( \bar{A} = \langle A_\varepsilon : \varepsilon < \varepsilon(*) \rangle \in P \) we can replace it by \( \bar{A}' = \langle A_\varepsilon' : \varepsilon < \varepsilon(*) \rangle, A_0' = A_0 \cup \{ \delta < \theta : \text{cf}(\delta) > \aleph_0 \text{ and } A_\varepsilon \cap \delta \text{ is stationary in } \delta \} \) and still obtain an element of \( P \).
2.8 Remark. 1) If \( \lambda \) is (weakly) inaccessible we have weaker results: letting \( \Theta = \{ \sigma : \sigma = \text{cf}(\sigma) \) and \( \alpha < \lambda \Rightarrow |\alpha|^{|\sigma|^{<\theta}} < \lambda \}, \) there is \( \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle, \mathcal{P}_\alpha \subseteq [\alpha]^{\theta}, |\mathcal{P}_\alpha| < \lambda \) such that for regular \( \theta < \lambda \) for every stationarily many \( \delta \in S^\lambda_{\theta} \) for some increasing continuous sequence \( \langle \alpha_\varepsilon : \varepsilon < \alpha \rangle \) with limit \( \delta, [\varepsilon < \theta \& \text{cf}(\varepsilon) \in \Theta_\lambda \Rightarrow \{ \alpha_\zeta : \zeta < \varepsilon \} \in \mathcal{P}_\alpha \} \) (see [Sh 420, §1]).

2) Can we restrict ourselves to: \( \langle w_j^* : j < \kappa \rangle \) is increasing continuous in part (1) and similarly in (2).

It seems very reasonable (if \( \lambda = \mu^+, \kappa = \text{cf}(\mu) < \mu \) that we can find \( \langle \lambda_i : i < \kappa \rangle, \) an increasing sequence of regular cardinals \( \kappa \) with limit \( \mu \) such that

\[
\lambda^+ = \text{tcf}(\prod_{i<\kappa} \lambda_i, \sigma \subseteq \mathcal{P}_\alpha) \text{ and } j < \kappa \Rightarrow \lambda_j > \max \text{pcf}(\lambda_i : i < j). \]

Why? This is trivial if \( \kappa = \aleph_0 \) and if \( \kappa > \aleph_0 \), it follows by the weak hypothesis (see [Sh 410]), whose negation is not known to be consistent and which has large consistency strength.

3) If there is \( \langle \lambda_i : i < \kappa \rangle \) as above then in part (1) of 2.7 we can strengthen clauses \( (\delta), (\varepsilon) \) in the conclusion. By demanding that “\( (w_j^* : j < \kappa) \)” is increasing continuous.

Proof. There is \( B \) as in clause \( (\varepsilon) \), that is \( B \in I[\lambda] \) and \( B \subseteq S^\lambda_{\theta} \) is a stationary subset of \( \lambda \), see Claim 1.24(2). So let \( B \) be such a set; clearly omitting from \( B \) a nonstationary subset of \( \lambda \) is O.K. as if \( B' \subseteq B, B \setminus B' \) is non-stationary and the demands in clause \( (\varepsilon) \) for \( B' \) are exemplified by the sequence \( \langle \bar{C}_i, A_i, S_i^\kappa : i < \mu \rangle \) then the same sequence exemplifies it for \( B \).

We can find \( \bar{a} = \langle a_\alpha : \alpha < \lambda \rangle \) witnessing \( B \in I[\lambda] \) see Definition 0.5(2), so (possibly omitting from \( B \) a nonstationary set) we have: \( \alpha \in B \Rightarrow a_\alpha \subseteq \alpha = \sup(a_\alpha) \) & \( \theta = \text{otp}(a_\alpha) \) and \( \beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha \) and \( [\alpha \in \lambda \setminus B \Rightarrow a_\alpha \subseteq \alpha \) & \( \text{otp}(a_\alpha) < \theta]. \)

Let \( \chi = \lambda^+ \) and \( \bar{M} = \langle M_\alpha : \alpha < \lambda \rangle \) be an increasing continuous sequence of elementary submodels of \( \mathcal{M}(\chi), \varepsilon, <_{\kappa}^\chi \) to each of which \( \{ \bar{a}, \mu, P, \mathcal{P} \} \) belongs, and satisfying \( \bar{M} \upharpoonright (\alpha + 1) \in M_{\alpha+1}, \mu \subseteq M_\alpha \) and \( \|M_\alpha\| < \lambda. \)

As \( \lambda = \mu^+ \) clearly \( \alpha < \lambda \Rightarrow \|M_\alpha\| = \mu. \) For any \( \alpha < \lambda \) let \( \langle b_{\alpha, \zeta} : \zeta < \mu \rangle \) list \( M_{\alpha+1} \cap [\alpha]^{<\theta}. \)

Let \( \langle \lambda_i : i < \kappa \rangle \) be an increasing sequence of regular cardinals with limit \( \mu, \lambda_i > \theta \) with \( \lambda = \text{tcf}(\prod_{i<\kappa} \lambda_i, <_{\kappa}^{\text{ind}}) \) and let \( \bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \) be an increasing sequence in
\[
\prod_{i < \kappa} \lambda_i, \langle \kappa, \mu \rangle \text{ cofinal in it and obeying } \overline{a} \text{ (i.e. } i < \kappa \text{ & } \beta \in a_\alpha \Rightarrow f_\beta(i) < f_\alpha(i) \text{) and } \\
\zeta < \lambda_i \wedge i < j < \kappa \wedge \beta \in b_{\alpha, \zeta} \Rightarrow f_\beta(j) < f_\alpha(j) \text{ (there is such } \langle \lambda_i : i < \kappa \rangle \text{ by [Sh:g, II]); there is such } f \text{ by } [Sh: \alpha, I]. \text{ For notational help later without loss of generality each } f_\alpha(i) \text{ is divisible by } \theta. \\
\text{Now for any } x = (h_1, h_2, A, i) = (h_1^x, h_2^x, A^x, i^x) \text{ where } h_1 \text{ is a partial function from } \mu \text{ to } \theta, h_2 \text{ a function from } A \text{ to } \mu \text{ where } T^-_\theta \subseteq A \subseteq T^+_\theta \text{ and } i < \kappa, \text{ we define} \\
S_x = \left\{ \alpha < \lambda \colon \text{we have } cf(\alpha) \neq \theta, f_\alpha(i) \in \text{ Dom}(h_1) \text{ and } \zeta = h_1(f_\alpha(i)) \in A \text{ satisfies} \right. \\
\left. \\
i \quad b_{\alpha, h_2(\zeta)} \text{ is a subset of } \alpha \text{ of order type } \zeta, \\
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\]
and for } \alpha \in S_x \text{ let } C^x_\alpha = b_{\alpha, h_2(\zeta)}(f_\alpha(i)). 

For } x \text{ as above let } S^*_x = \{ \delta < \lambda : cf(\delta) = \theta \text{ and there is an increasing continuous sequence } \langle \alpha_\varepsilon : \varepsilon < \theta \rangle \text{ with limit } \delta \text{ such that } \varepsilon \in A \Rightarrow \alpha_\varepsilon \in S_x \text{ & } C^x_{\alpha_\varepsilon} = \{ \alpha_\zeta : \zeta < \theta \} \}. \text{ Now for any such } x, \text{ for } \delta \in S^*_x \text{ choose } \langle \alpha_\varepsilon : \varepsilon < \theta \rangle \text{ witnessing it and let } C^x_\delta = \bigcup \{ C^x_{\alpha_\varepsilon} : \varepsilon \in A \}; \text{ note that } S_x \cap S^*_x = \emptyset \text{ because } [\alpha \in S_x \Rightarrow cf(\alpha) \neq \theta], [\alpha \in S^*_x \Rightarrow cf(\alpha) = \theta]. 

Let } \langle x_\varepsilon : \varepsilon < \varepsilon^* \leq \mu \rangle \text{ list the set of } x \in M_\theta \text{ which are as above and } S^*_x \text{ is stationary, and for each } \varepsilon < \varepsilon^*, \text{ let } S_\varepsilon = S_{x_\varepsilon} \cup S^*_x, C_\varepsilon = \langle C^x_{\alpha_\varepsilon} : \alpha \in S_\varepsilon \rangle, A_\varepsilon = A^{x_\varepsilon}. \text{ We have to check the five clauses of 2.7(1):} 

\text{Clause (} \alpha \text{): } \text{See the demand on } x \text{ above.} 

\text{Clause (} \beta \text{): } \text{Holds by clause (iii) in the definition of } \langle \alpha_\varepsilon : \varepsilon < \varepsilon^* \rangle \text{ and of } C^x_\alpha \text{ for } \alpha \in S_x \text{ and similarly for } \alpha \in S^*_x. 

\text{Clause (} \gamma \text{): } \text{Holds by the choice of } \langle x_\varepsilon : \varepsilon < \varepsilon^* \rangle. 

\text{Clause (} \delta \text{) + (} \varepsilon \text{): } \text{Let } \delta \in B, \text{ and we shall show that } \delta \in \bigcup \{ S^*_x : i < \mu \}, \text{ this clearly suffices for } (\varepsilon) \text{ hence for } (\delta). \text{ Let } \langle \alpha_\zeta : \zeta \leq \theta \rangle \text{ list the closure of } a_\delta \text{ in the increasing order by the choice of } \overline{a}, a_\delta \text{ is a set of non-limit ordinals hence } cf(\alpha_\zeta) = \theta \Leftrightarrow \zeta = \theta \text{ and } \alpha_\theta = \delta. 

Now for every } \zeta \in T^+_\theta, \text{ for some } \xi(\zeta) < \mu \text{ we have } \{ \alpha_\xi : \xi < \zeta \} = b_{\alpha, \xi(\zeta)} \text{ (} \xi(\zeta) \text{ exists as } \mu^{cf(\zeta)} > \varepsilon^* = \mu \text{ by the definition of } T^+_\theta \) \text{ and let } j(\zeta) = \text{ Min}\{ j < \kappa : f_\alpha(j) > f_{\alpha_\varepsilon}(i) \text{ for every } \varepsilon < j \}. 

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For each $j < \kappa$ let $w^*_j = \{\zeta : \zeta \in T^{-}_\theta \text{ or } \zeta \in T_{\theta} \& \ j(\zeta) = j\}$ and apply clause (e) of the assumptions to the sequence $\langle \alpha'_\zeta : \zeta < \theta \rangle$ where $\alpha'_\zeta = \text{cd}(j(\zeta), f_{\alpha}(j, \zeta), \xi(\alpha'))$, $\text{cd}$ a coding function on $\mu$ so (by (e)) we can find $\bar{A} \in \mathcal{P}$ satisfying clauses (i) and (ii) of assumption (e). So by (i) of (e) there are $i(*) < \ell g(\bar{A})$ and $j(*) < \kappa$ such that $A_{i(*)} \subseteq w^*_{j(*)}$.

For any pair $(i, j)$ such that $i < \ell g(\bar{A}), j < \kappa$ and $A_i \subseteq w^*_j$ we shall define $x = x_{i,j}$. By (ii) of (e) we have $\{\alpha'_\epsilon : \epsilon \in A_i\} \in \mathcal{P}$.

We now define $x = (h_1^\epsilon, h_2^\epsilon, A^x, i^x)$ as follows: we let $i^x = j, A^x = A_i, \text{Dom}(h_1^\epsilon) = \{f_{\alpha}(j) : \epsilon \in A^x\}, h_1^\epsilon(f_{\alpha}(j)) = \epsilon, \text{Dom}(h_2^\epsilon) = A^x, h_2^\epsilon(\epsilon) = \xi(\alpha')$. Easily $x = x_{i,j} \in M_0$ (as $\mathcal{P} \in M_0, |\mathcal{P}| \leq \mu, \mu + 1 \subseteq M_0$) and $\alpha' \in S^x_{\epsilon}$, so we are done.

2) Let $\mathcal{P} = \{A : A \text{ is a bounded subset of } \mu \text{ of cardinality } \leq \theta\}$. Clearly $|\mathcal{P}| \leq \Sigma\{|A|^\theta : \alpha < \mu\} \leq \Sigma\{|A| : \alpha < \mu\} = \mu$, so clause (d) holds.

Now if $\sigma = \text{cf}(\sigma) < \theta$ and $\sigma \neq \kappa$ then $\mu^{<\sigma>^\mu} = \mu$. [Why? For completeness if $T$ is a tree with set of nodes $\subseteq \mu$ and $\sigma$ levels, then every $\sigma$-branch of $T$ includes a $<T$-unbounded subset which is bounded as a subset of $\mu$ hence belongs to $\mathcal{P}$, and clearly the branch is determined by any $<T$-unbounded subset hence the number of $\sigma$-branches of $T$ is $\leq |\mathcal{P}| \leq \mu$.] So $T' \theta = T_\theta$ and the non-obvious clause to check is (e). So assume $\alpha_\epsilon < \mu$ for $\epsilon < \theta$ and $\langle w^*_i : i < \kappa\rangle$ is a sequence of subsets of $T^+_\theta$.

Let $\mu = \Sigma\{\mu_i : i < \kappa\}, \mu_i < \mu$ and let $\bar{A} = \langle A_i : i < \kappa\rangle$ list the family $\{\{\alpha_\epsilon : \epsilon \in w^*_i\}$ and $\alpha_\epsilon < \mu \mid j\} : i, j < \kappa\}$.

3), 4) Left to the reader. \qed

2.9 Claim. 1) If $\lambda$ is regular and not strongly inaccessible and $\mu < \lambda$ is a strong limit singular of uncountable cofinality, then for every large enough regular $\kappa < \mu$, for every $\Upsilon, \theta < \mu$, some $(\lambda, \kappa, \kappa^+)$-parameter $C$ has the $(\Upsilon, \theta)$-Md-property (see Definition 1.19(4)); in fact, $C$ is a sequence of clubs.

2) Moreover, we can find such $C^\alpha$ for $\alpha < \lambda$ satisfying: $\langle \text{Dom}(C^\alpha) : \alpha < \lambda\rangle$ are pairwise disjoint.

3) Moreover for every large enough regular $\kappa < \mu$ for every $\chi, \sigma < \mu$ some $(\lambda, \kappa, \kappa^+)$-parameter has the $(\chi, \sigma)$-Md$^\ell$-property for $\ell = 0, 1, 2$.

Proof. 1) Let $\lambda_1 < \lambda$ be such that $\lambda \leq 2^{\lambda_1}$ and let $\kappa_0 < \mu$ be large enough such that

$(*)_1(i)$ if $\lambda$ is not a successor of regular, then $\alpha < \lambda \Rightarrow \text{cov}(|\alpha|, \mu, \mu, \kappa_0) < \lambda$

$(ii)$ if $\lambda$ is inaccessible, then $\text{cov}(\lambda_1, \mu, \mu, \kappa_0) = \lambda_1$.

(Exist by [Sh 460].)

Note that $\lambda_1$ is needed only if $\lambda$ is (weakly) inaccessible.
Let $\mu_* \in (\kappa_0, \mu)$ be strong limit; this is possible as $\mu$ is a strong limit of uncountable cofinality. Let $\kappa_1 = \kappa_1[\mu_*] < \mu_*$ be $> \kappa_0$ and such that $\alpha < 2^{\mu_*} \Rightarrow \text{cov}([\alpha], \mu, \mu, \kappa_1) < 2^{\mu_*}$; (such $\kappa_1$ exists by [Sh 460]).

Let $\kappa \in \text{Reg} \cap \mu \setminus \kappa_1$ and we shall prove the result for $\kappa$ and every $\Upsilon, \theta < \mu_*$, so without loss of generality $\Upsilon = \theta$. By Fodor’s Lemma on $\mu$ this is enough.

Let $\lambda_* = \text{cf}(2^{\mu_*})$. By 1.24(1) for every stationary $S \subseteq \{ \delta < \lambda_* : \text{cf}(\delta) = \kappa \}$ from $\mathcal{I}[\lambda_*]$ (and there are such, see 1.24(2)) there are $\tilde{C}^S = \langle C^S_\delta : \delta \in S \rangle$, $D$ such that $\tilde{C}^S$ is a weakly good $^+ (\lambda_*, \kappa, \kappa^+)$-Md-parameter, $D$ is a $\lambda_*$-complete filter on $\lambda_*$ containing the clubs of $\lambda_*$ and by 1.10 + 1.11 $S \in D^+$, (e.g., $D = \emptyset_\lambda$), the pair $(\tilde{C}^S, D)$ has the $\theta$-Md-property.

For the case $\lambda_*$ is (weakly) inaccessible we add “each $C_\delta$ is a club of $\delta$”. This is permissible as we can use clause $(\beta)$ of 1.24(1). Fix such $S_1 \subseteq S^\lambda_*$ and $\tilde{C} = \langle C_\delta : \delta \in S_1 \rangle = \tilde{C}^{S_1}$.

Note that we actually have

\[ \exists(a) \text{ if } S' \subseteq S_1 \text{ is stationary, } C'_\delta \subseteq C_\delta \subseteq \delta = \sup(C'_\delta) \text{ for } \delta \in S' \text{ then } \tilde{C}' = \langle C'_\delta : \delta \in S' \rangle \text{ is a weakly good } {}^+ (\lambda_*, \kappa, \kappa^+)\text{-Md-parameter and } \tilde{C}' \text{ has the } (D, \theta)\text{-Md-property} \]

\[ (b) \text{ if } \lambda_* = \bigcup \{ A_i : i < i(*) < \kappa \} \text{ then for some } i, j < i(*), \tilde{C}' \text{ is a good } (\lambda_*, \kappa, \kappa^+)\text{-Md-parameter and } \tilde{C}' \text{ has the } (D, \theta)\text{-Md-property where } S' =: \{ \delta \in S_1 : \delta \in A_j \text{ and } \delta = \sup(\text{nacc}(C_\delta) \cap A_i) \} \text{ and for } \delta \in S' \text{ we let } C'_\delta =: \text{nacc}(C_\delta) \cap A_j. \]

[Why? Let $S'_{i,j} = \{ \delta \in S_1 : \delta \in A_j \text{ and } \delta = \sup(\text{nacc}(C_\delta) \cap A_i) \}$. Clearly $\langle S_{i,j} : i, j < i(*) \rangle$ is a sequence of subsets of $S_1$ with union $S_1$. By Claim 1.23 for some $i, j < i(*)$ the sequence $\tilde{C} \upharpoonright S_{i,j}$ has the $(D, \theta)$-Md-property. By monotonicity, i.e., Claim 1.14(1), the sequence $\langle \text{nacc}(C_\delta) \cap A_i : \delta \in S_{i,j} \rangle$ has the $(D, \theta)$-Md-property.]

Now

Case 1: $\lambda_*$ is the successor of regulars.

We apply 2.1(1) with $(\lambda, \lambda_*)$ here standing for $(\lambda_2, \lambda_1)$ there; there is suitable partial square (in ZFC by [Sh 351, §4]). Note that we do not “lose” any $\kappa < \mu$ because of “the lifting to” $\lambda$.

Case 2: $\lambda_*$ is the successor of a singular.

The proof is similar only now we use 2.7(3) to get the appropriate partial square using $\exists$ to derive the $\tilde{C}'$ used as input for it (recalling that each sequence $\tilde{A} \in \mathcal{P}$ is closed under the union of two!)

Case 3: $\lambda_*$ is (weakly) inaccessible.
Really the proof in this case covers the other cases (but \( \lambda_1 \) in this case contributes to increasing \( \kappa_0 \)). This time we use 2.6 + 2.10 which is proved below.

Let \( S_2 \) be a stationary subset of \( S^\lambda_{\kappa^+} \) from \( I[\lambda] \), it exists, see 1.24(2). Let \( \langle F_\varepsilon : \varepsilon < \lambda_1 \rangle \) be as in 2.6 with \( (\lambda, \lambda_1, \lambda_*, \kappa) \) here standing for \( (\lambda, \mu, \theta, \kappa) \) there. Let \( g^* \) be a function from \( \lambda_1 \) onto \( \mathcal{M}_{\leq \kappa}(\lambda_*). \) Possibly replacing \( S_2 \) by its intersection with some club of \( \lambda \) there is a good \( \lambda_* \)-Md-parameter \( \bar{C}^2 = \langle C^2_\delta : \delta \in S_2 \rangle \). Let \( \bar{e} = \langle e_\alpha : \alpha \in S^\lambda_{\kappa^+} \rangle, e_\alpha \) a club of \( \alpha \) of order type \( \kappa \). Clearly

\[
(\ast) \text{ for some } \varepsilon < \lambda_1 \text{ for any club } E \text{ of } \lambda \text{ for some increasing continuous sequence } \langle \alpha_i : i \leq \lambda_* \rangle \text{ of member of } E \text{ the set } B_\varepsilon(\langle \alpha_i : i \leq \lambda_* \rangle) = \{ i \in S_1 : g^*(F_\varepsilon(\alpha_i)) = \{ (i, \zeta, \text{otp}(\alpha_\zeta \cap C^2_\alpha), \text{otp}(\alpha_\zeta \cap e_{\alpha_i})) : \zeta < i \text{ and } \alpha_\zeta \in e_{\alpha_i} \} \}
\]

[Why? For every club \( E \) we can choose \( \alpha_i \in E (i \leq \lambda_*) \) increasing continuous so for some \( \varepsilon \) the set \( B_\varepsilon(\langle \alpha_i : i \leq \lambda_* \rangle) \) is a stationary subset of \( \lambda_* \). As the number of possible \( \varepsilon \) is \( \lambda_1 = \text{cf}(\lambda) \), there is \( \varepsilon \) as above.]

For this \( \varepsilon \) let

(i) \( S_3^\varepsilon = \{ \alpha < \lambda : \text{cf}(\alpha) = \kappa \text{ and } g^*(F_\varepsilon(\alpha)) \text{ has the right form} \} \)

(ii) \( \delta \in S_3^\varepsilon \) let \( C^3_\delta = \{ \beta \in \varepsilon_\delta : (i, \zeta, \text{otp}(\beta \cap e_\delta)) \in g^*(F_\varepsilon(\delta)) \} \)

(iii) let \( F' : \lambda \to \lambda_* \) be defined by \( F'(\alpha) = \zeta \) if: 

\[
\zeta = \zeta_1 \text{ for some } (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in g^*(F_\varepsilon(\alpha)).
\]

We apply 2.10.

2),3) Similarly using Claim 2.3 when using Claim 2.4.

We now state another version of the upgrading of middle diamonds.

**2.10 Claim.** 1) Assume

(a) \( \lambda_1 \prec \lambda_2 \) are regular cardinals

(b) \( S_1 \subseteq S^\lambda_{\kappa^+} \) is stationary, \( \bar{C}^1 = \langle C^1_\delta : \delta \in S_1 \rangle \) is a \( (\lambda_1, \kappa, \chi) \)-Md-parameter which has the \( (D_{\lambda_1}, \Upsilon, \theta) \)-Md\(_{\mathfrak{f}}\)-property, \( D_{\lambda_1} \) is a filter on \( \lambda_1 \) to which \( S_1 \) belongs

(c) \( S_2 \subseteq S^\lambda_{\kappa^2} \) is stationary and \( \bar{C}^2 = \langle C^2_\delta : \delta \in S_2 \rangle \) is a \( (\lambda_2, \kappa, \chi) \)-Md-parameter

(d) \( F : \lambda_2 \to \lambda_1 \) satisfies: for every club \( E \) of \( \lambda_2 \) there is an increasing continuous function \( h \) from \( \lambda_1 \) into \( \lambda_2 \) with \( \sup(\text{Rang}(h)) \in \text{acc}(E) \) such that 

\[
\{ \delta \in S_1 : h(\delta) \in S_2 \text{ and } C^2_{h(\delta)} \subseteq \{ h(\alpha) : \alpha \in C^1_\delta \} \text{ and } F(h(\delta)) = \delta \}
\]

belongs to \( D_{\lambda_1} \)

(e) \( 2^{2^{\lambda_1}} < \lambda_2 \).
Then $\mathcal{C}^2$ has the $(\Upsilon, \theta)$-Md$_{\ell}$-property.

2) Assume that in part (1) we omit assumption (e) and replace the Md-property by MD$_{\ell}$-property where $\ell \in \{0, 2\}$. 

Then $\mathcal{C}^2$ has the $(\Upsilon, \theta)$-MD$_{\ell}$-property.

3) In part (1) if we omit assumption (e) and strengthen clause (d) by demanding 
\[ \{ \delta \in S_1 : h(\delta) \in S_2 \text{ and } C_{h(\delta)}^2 = \{ h(\alpha) : \alpha \in C_{h(\delta)}^1 \} \text{ and } F(h(\delta)) = \delta \} \] belongs to $D_{\lambda_1}$, 
then we can still conclude that: for every nice $(\mathcal{C}^2, \Upsilon, \theta)$-colouring $\mathcal{F}$ (see below) there is an $\mathcal{F}$-middle diamond sequence.

Remark. 1) Is clause (d) of the assumption reasonable? Later in [Sh:F611, 4.5] we prove that it holds under reasonable conditions.

2) This + 2.11 are formalized elsewhere.

Proof of 2.10. Similar to 2.1.

First, assume $\ell = 2$ (so we are in part (2)) and let $\tau$ be a relational vocabulary of cardinality $< \Upsilon$ and by clause (b) of the assumption there is a sequence $M^1 = \langle M_\delta^1 : \delta \in S_1 \rangle$, $M_\delta^1$ a $\tau$-model with universe $C_{\delta}^1$ such that

\[ \circ \bigcirc \text{ if } M \text{ is a } \tau \text{-model with universe } \lambda_1, \text{ then the set of } \delta \in S_1 \text{ for which } M_\delta^1 = M \upharpoonright C_{\delta}^1 \text{ is } \not= \emptyset \text{ mod } D_{\lambda_1}. \]

Now for every $\delta \in S_2$ if $F(\delta) \in S_1$ and $\{ F(\alpha) : \alpha \in C_{\delta}^2 \} \subseteq C_{F(\delta)}^1$ is an unbounded subset of $f(\delta)$, then let $M_\delta^2$ be the unique $\tau$-model with universe $C_{\delta}^2$ such that $F \upharpoonright C_{\delta}^2$ is an isomorphism from $M_\delta^2$ onto $M_{F(\delta)}^1 \upharpoonright \{ F(\alpha) : \alpha \in C_{\delta}^2 \}$. If $\delta \in S_1$ does not satisfy the requirement above let $M_\delta^2$ be any $\tau$-model with universe $C_{\delta}^2$.

Now it is easy to check that $\bar{M}^2 = \langle M_\delta^2 : \delta \in S_2 \rangle$ is as required.

Second, assume $\ell = 1$ (and we are in part (1)) and let $\mathcal{F}^2$ be a $(\mathcal{C}^2, \Upsilon, \theta)$-colouring.

For each $(\mathcal{C}^1, \Upsilon, \theta)$-colouring $\mathcal{F}^1$ we ask:

Question on $\mathcal{F}^1$. Do we have, for every club $E$ of $\lambda_2$ an increasing continuous function $h : \lambda_1 \rightarrow \lambda_2$ such that

(i) $\sup(\text{Rang}(h)) \in E$

(ii) the following set is a stationary subset of $\lambda_1$
\[ \{ \delta \in S_1 : h(\delta) \in S_2 \text{ and } C_{h(\delta)}^2 \subseteq \{ h(\delta) : \alpha \in C_{\delta}^1 \} \text{ and } F(h(\delta)) = \delta \text{ and } \forall \eta_1 \in (C_{\delta}^1) \Upsilon, \forall \eta_2 \in (C_{h(\delta)}^2) \Upsilon, (\forall \alpha \in C_{\delta}^1) (\eta_1(\alpha) = \eta_2(h(\alpha))) \Rightarrow \mathcal{F}^1(\eta_1) = \mathcal{F}^2(\eta_2) \}. \]

If for some $\mathcal{F}^1$ the answer is yes, we proceed as in the proof for $\ell = 1$. Otherwise, for each such $\mathcal{F}^1$ there is a club $E[\mathcal{F}^1]$ exemplifying the negative answer. Let
$E = \bigcap \{ E[F^1] : F^1 \text{ is a } (C^1, \Upsilon, \theta)\text{-colouring} \}$. As there are $\leq 2^{2^{\lambda_1}}$ such functions $F^1$ clearly $E$ is a club of $\lambda_2$ and using (d) of the assumption, there is $h : \lambda_1 \to \lambda_2$ as there. Now we can define $F^1$ by $h$, $F^2$ naturally and get a contradiction.

The case $\ell = 0$ and part (3) is left to the reader. \(\square\)

Trying to apply 2.10, clause (d) of the assumption looks the most serious. By the following if each $C^1_\delta$ is a stationary subset of $\delta$ (so $\kappa > \aleph_0$), it helps.

2.11 Claim. In 2.10, if $2^{\lambda_1} < \lambda_2$, and we replace (d) by (d)$^-$ + (f) below, then we can conclude that for some $\bar{C} = \langle C^3_\delta : \delta \in S \rangle$ has the $\theta$-MD$^\ell$-property where $C^3_\delta \in D_\delta$.

\((f)\) $\tilde{D} = \langle D'_\delta : \delta \in S_2 \rangle$, $D'_\delta$ is a filter on $C^2_\delta$ containing the co-bounded subsets of $C^2_\delta$ (or just $D_\delta \subseteq \mathcal{P}(C^2_\delta)\setminus\{ A \subseteq C^2_\delta : \sup(A) < \delta \}$

\((d)^-\) $F : \lambda_2 \to \lambda_1$ satisfies: for every club $E$ of $\lambda$ for some increasing continuous function $h$ from $\lambda_1$ to $\lambda_2$ with $\sup(\text{Rang}(h)) \in \text{acc}(E)$ the set $\{ \delta \in S_1 : h(\delta) \in S_2 \text{ and } \{ h(\alpha) : \alpha \in C^1_\delta \} \cap C^2_\delta \in D'_\delta \}$ belong to $D_{\lambda_1}$.

Proof. Let $\mathcal{E} = \{ \bar{e} : \bar{e} = \langle e_\delta : \delta \in S_1 \rangle, e_\delta \text{ is a subset of } C^2_\delta \}.$

So $|\mathcal{E}| \leq 2^{\lambda_1} < \lambda$ and for each $\bar{e} \in E$ we define $\bar{C}^{e_\delta} = \langle C^{e_\delta}_\delta : \delta \in S^e_2 \rangle$ by $S^e_\delta = \{ \delta \in S_2 : C^{e_\delta}_\delta \in D_\delta \}$ where $C^{e_\delta}_\delta = \{ \alpha \in C^2_\delta : \text{otp}(C^2_\delta \cap \alpha) \in e_{F(\delta)} \}$. If for some $\bar{e} \in \mathcal{E}, C^{\bar{e}}$ is as required. \(\square\)
§0 Introduction
Theorem 0.1: Many cases of MD-diamond.
Theorem 0.2: With division to $\lambda$.
Conclusion 0.3: Connection to the principle from [Sh 667].
Definition 0.4: Having (partial) squares.

§1 Middle Diamond: sufficient conditions
Definition 1.1: Definition of $S$ being $F - \theta$-small and the normal filter.
Claim 1.2: The filter is normal and $\Upsilon$ does not matter.
Definition 1.3: Recalling pcf definitions.
Remark 1.4:
Definition 1.5: The main notions are defined:
- $\bar{C}$ is $\lambda$-Md-parameter, $(\lambda, \kappa, \chi)$-Md-parameter,
- $F$ is a $(\bar{C}, \mu, \theta)$-coloring,
- $\bar{C}$ is a $(D, F)$-middle diamond sequence,
- $\bar{C}$ is good and $\bar{C}$ has the $(D, \mu, \theta)$-Md-property or the $(D, \mu, \theta)$-Md-property, also (good$^{+}$)-Md-property and Md$^{\ell}$.

Definition 1.6:
Claim 1.8: Easy implications.
Major Claim 1.10: Sufficient conditions for $\bar{C}$ having the $(D, \theta)$-Md-properties
$(D$ a filter on $\lambda, \lambda = \text{cf}(2^\mu))$.
Claim 1.11: E.g., $\mu = \mu^\theta$ implies Sep$(\mu, \theta)$, one of the assumptions of 1.10.
Remark 1.12: On variants.
Definition 1.9: Sep$(\chi, \mu, \theta, \theta_1, \Upsilon)$, Sep$(\mu, \theta)$.
Claim 1.13: Monotonicity properties of Sep.
Claim 1.14: On some implications.
Definition 1.15: $\lambda$-MD-parameter.
Claim 1.16: Existence of good MD-parameters.
Claim 1.17: Like 1.10 for MD-parameters (1.14 - 1.17 are not central).
Claim 1.20: Getting, e.g., $(D, \Upsilon, \theta)$-MD-property, i.e., $(M_\delta : \delta \in S), |M_\delta| = C_\delta$
which guess (as in Theorem 0.1) for having $\bar{C}$ with the $(D, \theta)$-property; also md-properties MD-properties are defined.
Definition 1.22: $\theta$-MD$^{\ell}$-property and the ideal $\text{ID}_{\lambda, \kappa, \chi, \Upsilon, \theta}(\bar{C})$
Claim 1.23: Non saturation ideal above (needed in 1.20)
Claim 1.24: Quotations on the existence of good Md-parameters (by quoting).
Definition 1.25: A related ideal.
Definition 1.26: $F$ is $(S, \kappa, \chi)$-good (1.25, 1.26 not central).
Conclusion 1.27: Phrasing some conclusions.
Question 1.28: $\lambda$ weakly inaccessible, $\langle 2^\theta : \theta < \lambda \rangle$ is eventually constant.

§2 Upgrading to larger cardinals
Claim 2.1: For suitable $\lambda_2 > 2^{\lambda_1}$, we construct $(\lambda_2, \kappa, \chi)$-Md-parameters $\bar{C}$ which has the $(D_\lambda, \Upsilon, \theta)$-property from $(\lambda_1, \kappa, \chi)$-Md-parameters which has the $(D_\lambda, \Upsilon, \theta)$-parameters.
Claim 2.3: Getting $\bar{C}_i = \langle C_{\delta_i} : \delta \in S_i \rangle$ sets $S_i$ pairwise disjoint for $i < \lambda$ in 2.1
Conclusion 2.4: If $\mu$ is strong limit, $\lambda = \text{cf}(\lambda) > 2^{2^{\mu}}$ is successor of singulars then for every large enough $\kappa < \mu$, some good $(\lambda, \kappa, \kappa^+)$-Md-parameter has the $(D_\lambda, 2^{\mu}, \theta)$-Md-property for every $\theta < \mu$.
Claim 2.5: A generalization of Engelking Karlowic. [no pf]
Claim 2.7: Dealing with successor of singulars. [$\exists$ pf]
Claim 2.9: We advance the picture in 2.4 dealing with weakly inaccessible (i.e. regular limit) not strong limit cardinals; at a minor price - $\mu$ is of uncountable cofinality.
Claim 2.10: This is a stepping-up lemma used in 2.9 above.
Claim 2.11: Sufficient conditions for the “lifting” used in 2.10.
REFERENCES.


