

**FORCING AXIOM
FAILURE FOR ANY $\lambda > \aleph_1$
SH784**

SAHARON SHELAH

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Mathematics Department
New Brunswick, NJ USA

Annotated Content

§1 A forcing axiom for $\lambda > \aleph_1$ fails

[The forcing axiom is: if \mathbb{P} is a forcing notion preserving stationary subsets of any regular uncountable $\mu \leq \lambda$ and \mathcal{S}_i is dense open subset of \mathbb{P} for $i < \lambda$ then some directed $G \subseteq \mathbb{P}$ meets every \mathcal{S}_i .

We prove (in ZFC) that it fails for every regular $\lambda > \aleph_1$. In our counterexample the forcing notion \mathbb{P} adds no new sequence of ordinals of length $< \lambda$).

§2 There are $\{\aleph_1\}$ -semi-proper forcing notions

I would like to thank Alice Leonhardt for the beautiful typing.
Publ. 784 - Revised with proofreading for the Journal
Latest version - 04/Mar/2

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

§1 A FORCING AXIOM FOR $\lambda > \aleph_0$ FAIL

David Aspero asks on the possibility of, see Definition below, the forcing axiom $\text{FA}(\mathfrak{K}, \aleph_2)$ for the case $\mathfrak{K} =$ the class of forcing notions preserving stationarity of subsets of \aleph_1 and of \aleph_2 . We answer negatively for any regular $\lambda > \aleph_1$ (even demanding adding no new sequence of ordinals of length $< \lambda$), see 1.16 below.

1.1 Definition. 1) Let $\text{FA}(\mathfrak{K}, \lambda)$, the λ -forcing axiom for \mathfrak{K} mean that \mathfrak{K} is a family of forcing notions and for any $\mathbb{P} \in \mathfrak{K}$ and dense open sets $\mathcal{J}_i \subseteq \mathbb{P}$ for $i < \lambda$ there is a directed $G \subseteq \mathbb{P}$ meeting every \mathcal{J}_i .

2) If $\mathfrak{K} = \{\mathbb{P}\}$ we may write \mathbb{P} instead of \mathfrak{K} .

1.2 Definition. Let λ be regular uncountable. We define a forcing notion $\mathbb{P} = \mathbb{P}_\lambda^2$ as follows:

(A) if $p \in \mathbb{P}$ iff $p = (\alpha, \bar{S}, \bar{W}) = (\alpha^p, \bar{S}^p, \bar{C}^p)$ satisfying

(a) $\alpha < \lambda$

(b) $\bar{S}^p = \langle S_\beta : \beta \leq \alpha \rangle = \langle S_\beta^p : \beta \leq \alpha \rangle$

(c) $\bar{C}^p = \langle C_\beta : \beta \leq \alpha \rangle = \langle C_\beta^p : \beta \leq \alpha \rangle$

such that

(d) S_β is a stationary subset of λ consisting of limit ordinals

(e) C_β is a closed subset of β

(f) if $\beta \leq \alpha$ is a limit ordinal then C_β is a closed unbounded subset of β

(g) if $\gamma \in C_\beta$ then $C_\gamma = \gamma \cap C_\beta$

(h) $C_\beta \cap S_\beta = \emptyset$

(i) for every $\beta \leq \alpha$ and $\gamma \in C_\beta$ we have $S_\gamma = S_\beta$

(B) order: natural

$p \leq q$ iff $\alpha^p \leq \alpha^q$, $\bar{S}^p = \bar{S}^q \upharpoonright (\alpha^p + 1)$ and $\bar{C}^p = \bar{C}^q \upharpoonright (\alpha^p + 1)$.

1.3 Observation: 1) \mathbb{P}_λ^2 is a (non empty) forcing notion of cardinality 2^λ .

2) $\mathcal{J}_i = \{p \in \mathbb{P}_\lambda^2 : \alpha^p \geq i\}$ is dense open for any $i < \lambda$.

Proof. 1) Obvious.

2) Given $p \in \mathbb{P}_\lambda^2$ if $\alpha^p \geq i$ we are done. So assume $\alpha^p < i$ and for $\gamma \in (\alpha^p, i]$ let S_γ^q be S^* for any stationary subset S^* of $\{\delta < \lambda : \delta > i \text{ a limit ordinal}\}$ which does not belong to $\{S_\beta^p : \beta \leq \alpha^p\}$ and let $C_\gamma^q = \{j : \alpha^p < j < \gamma\}$ and

$$q = (i, \bar{S}^p \wedge \langle S_\gamma^q : \gamma \in (\alpha^p, i] \rangle, \bar{C}^p \wedge \langle C_\gamma^q : \gamma \in (\alpha^p, i] \rangle).$$

It is easy to check that $p \leq q \in \mathbb{P}_\lambda^2$ and $q \in \mathcal{J}_i$. □_{1.3}

1.4 Claim. *Let $\lambda = \text{cf}(\lambda)$ be regular uncountable and $\mathbb{P} = \mathbb{P}_\lambda^2$. For any stationary $S \subseteq \lambda$ and \mathbb{P}_λ^2 -name f of a function from $\gamma^* \leq \lambda$ to the ordinals or just to \mathbf{V} and $p \in \mathbb{P}$ there are q, δ such that:*

- ⊠(i) $p \leq q \in \mathbb{P}$
- (ii) $\alpha^q = \delta + 1$
- (iii) $\delta \in S$ if $\gamma^* = \lambda$
- (iv) q forces a value to $f \upharpoonright (\delta \cap \gamma^*)$
- (v) if $\beta < \delta \cap \gamma^*$ and $\Vdash_{\mathbb{P}} \text{“Rang}(f) \subseteq \lambda\text{”}$ then $q \Vdash_{\mathbb{P}} \text{“}f(\beta) < \delta\text{”}$.

Proof. Without loss of generality S is a set of limit ordinals. We prove this by induction on γ^* , so without loss of generality $\gamma^* = |\gamma^*|$ and without loss of generality $\gamma^* < \lambda \Rightarrow \gamma^* = \text{cf}(\gamma^*)$, but if $\gamma^* < \lambda$ the set S is immaterial so without loss of generality

$$\otimes \gamma^* < \lambda \ \& \ \delta \in S \Rightarrow \text{cf}(\delta) \geq \gamma^*.$$

Let χ be large enough (e.g. $\chi = (\beth_3(\lambda))^+$), $\langle \cdot, \cdot \rangle_\chi^*$ is a well ordering of $\mathcal{H}(\chi)$ and choose $\bar{N} = \langle N_i : i < \lambda \rangle$ such that

- ⊙(a) $N_i \prec (\mathcal{H}(\chi), \in, \langle \cdot, \cdot \rangle_\chi^*)$ is increasing continuous
- (b) λ, p, f, S belongs to N_i hence $\mathbb{P} \in N_i$
- (c) $\|N_i\| < \lambda$
- (d) $N_i \cap \lambda \in \lambda$
- (e) $\langle N_j : j \leq i \rangle$ belong to N_{i+1} ; hence $i \subseteq N_i$ so $\lambda \subseteq \cup\{N_i : i < \lambda\}$.

Let $\delta_i = N_i \cap \lambda$, and let $i(*) = \text{Min}\{i : i < \lambda \text{ is a limit ordinal and } \delta_i \in S\}$, it is well defined as $\langle \delta_i : i < \lambda \rangle$ is strictly increasing continuous hence $\{\delta_i : i < \lambda\}$ is a club of λ ; so by \otimes we know that $\gamma^* < \lambda \Rightarrow \text{cf}(i(*)) = \text{cf}(\delta_{i(*)}) \geq \gamma^*$. Let α_i^* be δ_i for $i \leq i(*)$ a limit ordinal and be $\delta_i + 1$ for $i < i(*)$ a non limit ordinal. Now by induction on $i \leq i(*)$ choose p_i^- and if $i < i(*)$ also p_i and prove on them the following:

- (*) (i) $p_i, p_i^- \in \mathbb{P} \cap N_{i+1}$
- (ii) p_i is increasing

- (iii) $\alpha^{p_i} > \alpha_i^*$ (and $\delta_{i+1} > \alpha^{p_i}$ follows from $p \in \mathbb{P} \cap N_{i+1}$)
- (iv) $S_{\alpha_i^*}^{p_i} = S$ and $C_{\alpha_i^*}^{p_i} = \{\alpha_j^* : j < i\}$
- (v) p_i^- is the $<_{\chi}^*$ -first q satisfying:
 - $q \in \mathbb{P}$
 - $j < i \Rightarrow p_j \leq q$
 - $\alpha^q > \delta_i$
 - $S_{\alpha_i^*}^q = S$ and
 - $C_{\alpha_i^*}^q = \{\alpha_j^* : j < i\}$
- (vii) p_i is the $<_{\chi}^*$ -first q such that:
 - $q \in \mathbb{P}$
 - $p_i^- \leq q$
 - q forces a value to $f(i)$ if $\gamma^* < \lambda$
 - q forces a value to $f \upharpoonright \delta_i$ if $\gamma^* = \lambda$.

There is no problem to carry the definition, recalling the inductive hypothesis on γ^* and noting that $\langle p_j^-, p_j \rangle : j < i \rangle \in N_{i+1}$ by the “ $<_{\chi}^*$ -first” being used to make our choices as $\langle N_j : j \leq i \rangle \in N_{i+1}$ hence $\langle \delta_j : j \leq i \rangle \in N_{i+1}$ and also $\langle \alpha_j^* : j \leq i \rangle \in N_{i+1}$ (and $p, f \in N_0 \prec N_{i+1}$).

Now $p_{i(*)}^-$ is as required. $\square_{1.4}$

1.5 Conclusion: Let $\lambda = \text{cf}(\lambda) > \aleph_0$. Forcing with \mathbb{P}_{λ}^2 add no bounded subset of λ and preserve stationarity of subsets of λ (and add no new sequences of ordinals of length $< \lambda$).

Proof. Obvious from 1.4.

1.6 Claim. Let $\lambda = \text{cf}(\lambda) > \aleph_0$. If $\text{FA}(\mathbb{P}_{\lambda}^2)$, (the forcing axiom for the forcing notion $\mathbb{P}_{\lambda}^2, \lambda$ dense sets) holds, then there is a witness (\bar{S}, \bar{C}) to λ where

1.7 Definition. 1) For λ regular uncountable, we say that (\bar{S}, \bar{C}) is a witness to λ or (\bar{S}, \bar{C}) is a λ -witness if:

- (a) $\bar{S} = \langle S_{\beta} : \beta < \lambda \rangle$
- (b) $\bar{C} = \langle C_{\beta} : \beta < \lambda \rangle$
- (c) for every $\alpha < \lambda$, $(\alpha, \bar{S} \upharpoonright (\alpha + 1), \bar{C} \upharpoonright (\alpha + 1)) \in \mathbb{P}_{\lambda}^2$.

2) For (\bar{S}, \bar{C}) a witness for λ , let $F = F_{(\bar{S}, \bar{C})}$ be the function $F : \lambda \rightarrow \lambda$ defined by

$$F(\alpha) = \text{Min}\{\beta : S_\alpha = S_\beta\}.$$

3) For $\beta < \lambda$ let $W_{(\bar{S}, \bar{C})}^\beta = \{\alpha < \lambda : F_{(\bar{S}, \bar{C})}(\alpha) = \beta\}$.

Proof of 1.6. Let $\mathcal{J}_i = \{p \in \mathbb{P}_\lambda^2 : \alpha^p \geq i\}$, by 1.3(2) this is a dense open subset of \mathbb{P}_λ^2 , hence by the assumption there is a directed $G \subseteq \mathbb{P}_\lambda^2$ such that $i < \lambda \Rightarrow \mathcal{J}_i \cap G \neq \emptyset$. Define

$S_\alpha = S_\alpha^p, C_\alpha = C_\alpha^p$ for every $p \in G$ such that $\alpha^p \geq \alpha$.

Now check. □_{1.6}

1.8 Observation: Let (\bar{S}, \bar{C}) be a witness for λ and $F = F_{(\bar{S}, \bar{C})}$.

- 1) If $\alpha < \lambda$ then $F(\alpha) \leq \alpha$.
- 2) If $\alpha < \lambda$ is limit then $F(\alpha) < \alpha$.
- 3) If $\alpha < \lambda$ then $\alpha \in W_{(\bar{S}, \bar{C})}^{F(\alpha)}$.
- 4) If $\alpha < \lambda$ and $i = F(\alpha)$ and $\beta \in C_\alpha$ then $\beta \notin S_\alpha = S_i$.

Proof. Easy (for part (4) remember that each S_α is a set of limit ordinals $< \lambda$ and that for limit $\alpha \leq \alpha^p, p \in \mathbb{P}_\lambda^2$ we have $\alpha = \sup(C_\alpha)$ and $\alpha \in S_\beta \Rightarrow C_\alpha \cap S_\beta = \emptyset$).

1.9 Claim. Assume (\bar{S}, \bar{C}) is a λ -witness and $S^* \subseteq \lambda$ satisfies $\delta \in S^* \Rightarrow \text{cf}(\delta) \geq \theta > \aleph_0$ and $F_{(\bar{S}, \bar{C})} \upharpoonright S^*$ is constant and S^* is stationary. Then there is a club E^* of λ such that: $(\bar{S}, \bar{C}, S^*, E^*)$ is a strong (λ, θ) -witness, where

1.10 Definition. 1) We say that $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ is a strong λ -witness if

- (a) (\bar{S}, \bar{C}) is a λ -witness
- (b) $S^* \subseteq \lambda$ is a set of limit ordinals and is a stationary subset of λ
- (c) E^* is a club of λ
- (d) for every club E of λ , for stationarily many $\delta \in S^*$ we have

$$\delta = \sup\{\alpha \in C_\delta : \alpha < \text{Suc}_{C_\delta}^1(\alpha, E^*) \in E\}$$

where

- (*) (i) $\text{Suc}_{C_\delta}^0(\alpha) = \text{Min}(C_\delta \setminus (\alpha + 1))$,
- (ii) $\text{Suc}_{C_\delta}^1(\alpha, E^*) = \sup(E^* \cap \text{Suc}_{C_\delta}^0(\alpha))$.

2) We say $(\bar{S}, \bar{C}, S^*, E^*)$ is a strong (λ, θ) -witness if in addition

$$(e) \delta \in S^* \Rightarrow \text{cf}(\delta) \geq \theta.$$

3) For $(\bar{S}, \bar{C}, S^*, E^*)$ a strong λ -witness we let $\bar{C}' = \langle C'_\delta : \delta \in S^* \cap \text{acc}(E^*) \rangle$, $C'_\delta = C_\delta \cup \{\text{Suc}_{C_\delta}^1(\alpha, E^*) : \alpha \in C_\delta\}$; if $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ we write $\bar{C}' = \bar{C}'_{\mathbf{p}}$ and $\bar{S}_{\mathbf{p}} = \bar{S}, \bar{C}_{\mathbf{p}} = \bar{C}, S_{\mathbf{p}}^* = S^*, E_{\mathbf{p}}^* = E^*$. We call $(\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ an expanded strong λ -witness (or (λ, θ) -witness).

1.11 Observation. In Definition 1.10(3) for $\delta \in S^* \cap \text{acc}(E^*)$ we have:

- ⊗ C'_δ is a club of δ , $\text{Min}(C'_\delta) \geq \sup(E^* \cap \text{Min}(C_\delta))$ and if $\gamma_1 < \gamma_2$ are successive members of C_δ then $C'_\delta \cap (\gamma_1, \gamma_2)$ has at most one member (which necessarily is $\sup(E^* \cap \gamma_2)$) hence $\text{acc}(C'_\delta) = \text{acc}(C_\delta)$ and $\alpha \in C_\delta \wedge \alpha < \text{Suc}_{C_\delta}^1(\alpha) \Rightarrow \alpha \notin C'_\delta \setminus C_\delta$ and $\text{acc}(C_\delta) = \text{acc}(C'_\delta)$.

Proof of 1.9. As in [Sh:g, III], but let us elaborate, so assume toward contradiction that for no club E^* of λ is $(\bar{S}, \bar{C}, S^*, E^*)$ a strong (λ, θ) -witness. We choose by induction on n sets E_n^*, E_n, A_n such that:

- (a) E_n^*, E_n are clubs of λ
 (b) $E_0^* = \lambda$
 (c) E_n is a club of λ such that the following set is not stationary (in λ)

$$A_n = \{\delta \in S^* : \delta \in \text{acc}(E_n^*) \text{ and} \\ \delta = \sup\{\alpha \in C_\delta : \alpha < \text{Suc}_{C_\delta}^1(\alpha, E_n^*) \in E_n\}\}$$

- (d) E_{n+1}^* is a club of λ included in $\text{acc}(E_n^* \cap E_n)$ and disjoint to A_n .

For $n = 0$, E_n^* is defined by clause (b).

If E_n^* is defined, choose E_n as in clause (c), possible by our assumption toward contradiction, also $A_n \subseteq S^*$ is defined and not stationary. So obviously E_{n+1}^* as required in clause (d) exists.

So $E^* =: \cap\{E_n^* : n < \omega\}$ is a club of λ and let $\alpha(*)$ be the constant value of $F_{(\bar{S}, \bar{C})} \upharpoonright S^*$, exists by an assumption of the claim. Recall that $S_{\alpha(*)}$ is a stationary subset of λ , so clearly $E^{**} =: \{\delta \in E^* : \delta = \sup(\delta \cap E^* \cap S_{\alpha(*)})\}$ is a club of λ . As S^* is a stationary subset of λ , we can choose $\delta^* \in S^* \cap E^{**}$. For each $n < \omega$ we

have $\delta^* \in S^* \cap E^{**} \subseteq E^{**} \subseteq E^* \subseteq E_{n+1}^*$ hence $\delta^* \notin A_n$ hence $\beta_n^* = \sup\{\beta \in C_{\delta^*} : \beta < \text{Suc}_{C_{\delta^*}}^1(\beta, E_n^*) \in E_n\}$ is $< \delta^*$ but $\in C_{\delta^*}$. But $\delta^* \in S^*$ so $\text{cf}(\delta^*) \geq \theta > \aleph_0$, hence $\beta^* = \sup\{\beta_n^*, \text{Min}(C_{\delta^*}) : n < \omega\}$ is $< \delta^*$ but $\geq \text{Min}(C_{\delta^*})$ and it belongs to C_{δ^*} . As $\delta^* \in E^{**}$, we know that $\delta^* = \sup(\delta^* \cap E^* \cap S_{\alpha(*)})$ hence there is $\gamma^* \in E^* \cap S_{\alpha(*)} \cap (\text{Suc}_{C_{\delta^*}}^0(\beta^*), \delta^*)$. But $\delta^* \in S^* \subseteq S_{\alpha(*)}$ recalling by the choice of $\alpha(*)$ above $F_{(\bar{S}, \bar{C})}(\delta^*) = \alpha(*)$ hence by Claim 1.8(4), i.e., Definition 1.2(1), clause (A)(h) and Definition 1.7(1) we have $C_{\delta^*} \cap S_{\alpha(*)} = \emptyset$ hence $\gamma^* \notin C_{\delta^*}$. But $\delta^* > \gamma^* > \beta^* \geq \text{Min}(C_{\delta^*})$ and C_{δ^*} is a closed subset of δ^* hence $\zeta^* = \max(C_{\delta^*} \cap \gamma^*)$ is well defined and so, recalling $\beta^* \in C_{\delta^*}$ we have

$$(\forall n < \omega)(\beta_n^* \leq \beta^* < \text{Suc}_{C_{\delta^*}}^0(\beta^*) \leq \zeta^* \in C_{\delta^*}).$$

Let $\xi^* = \text{Suc}_{C_{\delta^*}}^0(\zeta^*)$ so clearly $\gamma^* \in (\zeta^*, \xi^*)$. Now for every n we have $\sup(\xi^* \cap E_n^*) \in [\gamma^*, \xi^*]$ as $\gamma^* \in E^* \cap S_{\alpha(*)} \subseteq E^* \subseteq E_n^*$.

So recalling $\zeta^* < \gamma^*$ clearly $\zeta^* < \sup(\xi^* \cap E_n^*)$; if also $\sup(\xi^* \cap E_n^*) \in E_n$ then recalling $\xi^* = \text{Suc}_{C_{\delta^*}}^0(\zeta^*)$, $\text{Suc}_{C_{\delta^*}}^1(\zeta^*, E_n^*) \equiv \sup(\xi^* \cap E_n^*)$ we have $\zeta^* \leq \beta_n^*$ (see its choice and see the choice of β_n^* above), but this contradicts $\zeta^* \geq \text{Suc}_{C_{\delta^*}}^0(\beta^*) > \beta^* \geq \beta_n^*$ and the definition of A_n (see clause (c) of (*)), contradiction. So necessarily $\sup(\xi^* \cap E_n^*)$ does not belong to E_n hence does not belong to E_{n+1}^* , hence $\sup(\xi^* \cap E_n^*) > \sup(\xi^* \cap E_{n+1}^*)$.

So $\langle \sup(\xi^* \cap E_n^*) : n < \omega \rangle$ is a strictly decreasing sequence of ordinals, contradiction. $\square_{1.9}$

1.12 Definition. Assume

- (*)₁ $(\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ is an expanded strong λ -witness so $\bar{C}' = \langle C'_\delta : \delta \in S^* \rangle$, $C'_\delta = C_\delta \cup \{\text{Suc}_{C_\delta}^1(\alpha, E^*) : \alpha \in C_\delta\}$ or just
- (*)₂ $S^* \subseteq \lambda$ is a stationary set of limit ordinals, $\bar{C}' = \langle C'_\delta : \delta \in S^* \rangle$, C'_δ is a club of δ and E^* is a club of λ .

We define a forcing notion $\mathbb{P} = \mathbb{P}_{\bar{C}'}$

- (A) $c \in \mathbb{P}$ iff
 - (a) c is a closed bounded subset of λ
 - (b) if $\delta \in S^* \cap c$ then $\{\alpha \in C'_\delta : \text{Suc}_{C'_\delta}^0(\alpha) \in c\}$ is bounded in δ

Let $\alpha^c = \sup(c)$.

- (B) order: $c_1 \leq c_2$ iff c_1 is an initial segment of c_2 .

1.13 Claim. Let $\mathbb{P} = \mathbb{P}_{\bar{C}}$, be as in Definition 1.12.

- 1) \mathbb{P} is a (non empty) forcing notion.
- 2) For $i < \lambda$ the set $\mathcal{J}_i = \{c \in \mathbb{P} : i < \text{sup}(c)\}$ is dense open.

Proof. 1) Trivial.

2) If $c \in \mathbb{P}$, $i < \lambda$ and $c \notin \mathcal{J}_i$ then let $c_2 = c \cup \{i+1\}$, clearly $(c_2 \setminus c) \cap S^* = \emptyset$ as S^* is a set of limit ordinals hence $c_2 \in \mathbb{P}$ and obviously $c \leq c_2 \in \mathcal{J}_i$. $\square_{1.13}$

1.14 Claim. Assume $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*, \bar{C}')$ is an expanded strong λ -witness.

Forcing with $\mathbb{P} = \mathbb{P}_{\bar{C}}$, add no new bounded subsets of λ , no new sequence of ordinals of length $< \lambda$ and preserve stationarity of subsets of λ .

Proof. Assume $p \in \mathbb{P}$, $\gamma^* \leq \lambda$ and f is a \mathbb{P} -name of a function from γ^* to the ordinals or just to \mathbf{V} and $S \subseteq \lambda$ is stationary and we shall prove that there are q, δ satisfying (the parallel of) \boxtimes of 1.4, i.e.,

- $\boxtimes(i)$ $p \leq q \in \mathbb{P}$
- (ii) $\alpha^q = \delta + 1$
- (iii) $\delta \in S$ if $\gamma^* = \lambda$
- (iv) q forces a value to $f \upharpoonright (\delta \cap \gamma^*)$
- (v) if $\beta < \delta \cap \gamma^*$ and $\Vdash "p : \gamma^* \rightarrow \lambda"$ then $q \Vdash_{\mathbb{P}} "f(\beta) < \delta"$.

This is clearly enough for all the desired consequences. We prove this by induction on γ^* , so without loss of generality $\gamma^* = |\gamma^*|$ and without loss of generality $\gamma^* < \lambda \Rightarrow \gamma^* = \text{cf}(\gamma^*)$, but if $\gamma^* < \lambda$ then S is immaterial so without loss of generality $\gamma^* < \lambda$ & $\delta \in S \Rightarrow \text{cf}(\delta) \geq \gamma^*$. Also we can shrink S as long as it is a stationary subset of λ and recall that $F_{(\bar{S}, \bar{C})}$ is regressive on limit ordinals (see Observation 1.8(2)) so without loss of generality $F_{(\bar{S}, \bar{C})} \upharpoonright S$ is constantly say $\alpha(*)$.

Let χ be large enough and choose $\bar{N} = \langle N_i : i < \lambda \rangle$ such that

- $\odot(a)$ $N_i \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is increasing continuous
- (b) λ, p, f, S belongs to N_i hence $\mathbb{P} \in N_i$
- (c) $\|N_i\| < \lambda$
- (d) $N_i \cap \lambda \in \lambda$
- (e) $\langle N_j : j \leq i \rangle$ belong to N_{i+1} (hence $i \subseteq N_i$, so $\lambda \subseteq \cup\{N_i : i < \lambda\}$)
- (f) $N_{i+1} \cap \lambda \in S_{\alpha(*)}$ and $N_0 \cap \lambda \in S_{\alpha(*)}$.

Let $\delta_i = N_i \cap \lambda$ and $i(*) = \text{Min}\{i : i < \lambda \text{ is a limit ordinal and } \delta_i \in S\}$, it is well defined as $\langle \delta_i : i < \lambda \rangle$ is (strictly increasing continuous) hence $\{\delta_i : i < \lambda\}$ is a club of λ , hence $\gamma^* < \lambda \rightarrow \text{cf}(i(*)) = \text{cf}(\delta_{i(*)}) \geq \gamma^*$.

Let¹ $W =: \{i \leq i(*) : i > 0 \text{ and if } i < i(*) \text{ and } j < i \text{ then } C'_{\delta_{i(*)}} \cap \delta_{i+1} \not\subseteq \delta_{j+1}\}$. Clearly $W \cap i(*)$ is a closed subset of $i(*)$ and as $\delta_{i(*)} = \sup(C_{\delta_{i(*)}})$, also $W \cap i(*)$ is unbounded in $i(*)$. Also as by 1.11 we have $(\alpha \in \text{acc } C'_{\delta_{i(*)}}) \Rightarrow \alpha \in C_{\delta_{i(*)}} \Rightarrow C'_\alpha = C'_{\delta_{i(*)}} \cap \alpha$ clearly

(*) if $i \in W$ then $\langle N_j : j \in W \cap (i+1) \rangle \in N_{i+1}$.

Also note that

(**) if $i < i(*)$ is nonlimit, then $\delta_i > \sup(C_{\delta_{i(*)}} \cap \delta_i)$ hence $\delta_i > \sup(C'_{\delta_{i(*)}} \cap \delta_i)$. [Why? By 1.8(4) as $\delta_{i(*)} \in S \subseteq S_{\alpha(*)}$ recalling the choice of $\alpha(*)$ clearly $C_{\delta_{i(*)}} \cap S_{\alpha(*)} = \emptyset$ but by clause $\odot(f)$ we have $\delta_i \in S_{\alpha(*)}$ so $\delta_i \notin C_{\delta_{i(*)}}$. But $C_{\delta_{i(*)}}$ is a closed subset of $\delta_{i(*)}$ hence $\delta_i > \sup(C_{\delta_{i(*)}} \cap \delta_i)$, and $C'_{\delta_{i(*)}} \cap \delta_i \setminus \sup(C_{\delta_{i(*)}} \cap \delta_i)$ has at most two members (see 1.11) so $C'_{\delta_{i(*)}} \cap \delta_i$ is a bounded subset of δ_i so we are done.]

Now by induction on $i \in W$ we choose p_i, p_i^- and prove on them the following:

- (*) (i) $p_i, p_i^- \in \mathbb{P} \cap N_{i+1}$
- (ii) p_i is increasing (in \mathbb{P})
- (iii) $\max(p_i) > \delta_i$
(of course $\delta_{i+1} > \max(p_i)$ as $p_i \in \mathbb{P} \cap N_{i+1}$)
- (iv) $p_i^- = p \cup \{\sup(\delta_i \cup (C'_{\delta_{i(*)}} \cap \delta_{i+1})) + 1\}$ if $i = \text{Min}(W)$
- (v) if $0 < i = \sup(W \cap i)$ and $\gamma_i = \max(C'_{\delta_{i(*)}} \cap \delta_{i+1})$ so $\delta_i \leq \gamma_i < \delta_{i+1}$ then $p_i^- = \cup\{p_j : j \in W \cap i\} \cup \{\delta_i, \gamma_i + 1\}$
- (vi) if $j < i$ are in W then $p_j \leq p_i^- \leq p_i$
- (vii) $i \in W, i < i(*)$ and $j < i$ satisfies $j = \text{Max}(W \cap i)$ and $\gamma_i = \max(\{\delta_i\} \cup (C'_{\delta_{i(*)}} \cap \delta_{i+1}))$ so $\delta_i \leq \gamma_i < \delta_{i+1}$ then $p_i^- = p_j \cup \{\gamma_i + 1\}$
- (viii) p_i is the $<^*_\chi$ -first $q \in \mathbb{P}$ satisfying
 - (α) $p_i^- \leq q \in \mathbb{P}$
 - (β) if $\gamma^* < \lambda$ then q forces a value to $f(\text{otp}(\{j < i : j \in W \text{ and } \text{otp}(j \cap W) \text{ is a successor ordinal}\}))$
 - (γ) if $\gamma^* = \lambda$ then q forces a value to $f \upharpoonright \delta_i$

¹if $\text{cf}(\delta_{i(*)}) > \aleph_0$ then $W = \{i < i(*) : \delta_i \in C_{\delta_{i(*)}}\} \cup \{\delta_{i(*)}\}$ is O.K.

(ix) $p_i^- \setminus \bigcup_{j < i} p_j$ and $p_i \setminus p_i^-$ are disjoint to $C'_{\delta_{i(*)}} \setminus \text{acc}(C'_{\delta_{i(*)}})$, which include the set $\{\text{Suc}_{C'_{\delta_{i(*)}}}^1(\alpha, E^*) : \alpha \in C_{\delta_{i(*)}}$ and $\alpha < \text{Suc}_{C'_{\delta_{i(*)}}}^1(\alpha)\}$.

Note that clause (ix) follows from the rest; we now carry the induction.

Case 1: $i = \text{Min}(W)$.

Choose p_i^- just to fulfill clauses (iv), note that $\delta_i \leq \gamma_i < \delta_{i+1}$ as $i \in W \cap i(*)$ and then choose p_i to fulfill clause (viii).

Case 2: $i = \text{Min}(W \setminus (j+1))$ and $j \in W$.

Choose p_i^- by clauses (vii) and then p_i by clause (viii).

Case 3: $0 < i = \text{sup}(W \cap i)$.

A major point is $\langle p_j : j < i \rangle \in N_{i+1}$, this holds as $\langle p_j^-, p_j, j \in i \cap W \rangle$ is definable from $\bar{N} \upharpoonright \delta_i, \underline{f}, p, C'_{\delta_{i(*)}} \cap N_{i+1}$ all of which belong to N_{i+1} and $N_{i+1} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$.

Let p_i^- be defined by clause (v), note that $\delta_i \leq \gamma_i < \delta_{i+1}$ as $i \in W$ and $p_i^- \in \mathbb{P}$ as:

(α) $(\forall j < i)[p_j \in \mathbb{P}]$ and

(β) $\delta_i = \text{sup}(\cup\{\delta_j : j < i \text{ and } j \in W\})$.

[Why? As $\delta_i < \max(p_j) < \delta_{i+1}$ by clause (iii)] and

(γ) $\alpha \in p_i^- \cap S^* \Rightarrow \text{sup}(p_i^- \cap C'_\alpha \setminus \text{acc}(C'_\alpha)) < \alpha$.

[Why? If $\alpha < \delta_i$ then for some $j \in i \cap W$ we have $\alpha < \delta_j$ so p_j is an initial segment of p_i^- hence $\text{sup}(p_i^- \cap C'_\alpha) = \text{sup}(p_j \cap C'_\alpha) < \alpha$. If $\alpha = \delta_i$ we can assume $\alpha \in S^*$ but clearly $\alpha = \delta_i \in C'_{\delta_{i(*)}}$ by the definition of W and the assumption of case 3; so by (\bar{S}, \bar{C}) being a λ -witness, $C'_{\delta_i} = C'_{\delta_{i(*)}} \cap \delta_i$ so by clause (ix) the demand (in (γ)) hold.]

So easily p_i^- is as required. If $i < i(*)$ we can choose p_i by clause (viii) using the induction hypothesis if $\gamma^* = \lambda$. So we have carried the definition and $p_{i(*)}^-$ is as required. $\square_{1.14}$

1.15 Conclusion: 1) If $\mathbf{p} = (\bar{S}, \bar{C}, S^*, E^*)$ is a strong λ -witness and $\bar{C}' = \bar{C}'_{\mathbf{p}}$ and $\mathbb{P} = \mathbb{P}_{\bar{C}'}$, then $\text{FA}(\mathbb{P}, \lambda)$ fails.

2) In part (1), $\mathbb{P}_{\bar{C}'}$ is a forcing of cardinality $\leq 2^{<\lambda}$, add no new sequence of ordinals of length $< \lambda$ and preserve stationarity of subsets of any $\theta = \text{cf}(\theta) \in [\aleph_1, \lambda]$.

Proof. 1) Recall that by Claim 1.13(2), \mathcal{J}_i is a dense open subset of \mathbb{P} . Now if $G \subseteq \mathbb{P}_{\bar{C}'}$ is directed not disjoint to \mathcal{J}_i for $i < \lambda$, let $E = \cup\{p : p \in G\}$. By the definition of $\mathbb{P}_{\bar{C}'}$ and \mathcal{J}_i clearly E is an unbounded subset of λ and by the definition of $\mathbb{P}_{\bar{C}'}$ and G being directed, $p \in G \Rightarrow E \cap (\max(p) + 1) = p$ and (p is closed) hence E is a closed unbounded subset of λ . So E contradicts the definition of “ $(\bar{S}, \bar{C}, \bar{S}^*, \bar{E}^*, \bar{C}')$ being a strong λ -witness”.

2) Follows from 1.14 and direct checking. □_{1.15}

1.16 Conclusion: Let λ be regular $> \aleph_1$.

Then there is a forcing notion \mathbb{P} such that:

- (α) \mathbb{P} of cardinality $\leq 2^\lambda$
- (β) forcing with \mathbb{P} add no new sequences of ordinals of length $< \lambda$
- (γ) forcing with \mathbb{P} preserve stationarity of subsets of λ (and by clause (β) also of any $\theta = \text{cf}(\theta) \in [\aleph_1, \lambda)$)
- (δ) FA(\mathbb{P}, λ) fail.

Proof. We try \mathbb{P}_λ^2 , it satisfies clause (α), (β), (γ) (see 1.3(1), 1.5, 1.6). If it satisfies also clause (δ) we are done otherwise by Claim 1.6 there is a λ -witness (\bar{S}, \bar{C}) . Let $S^* \subseteq \{\delta < \lambda : \text{cf}(\delta) > \aleph_0\}$ be stationary, so by 1.9 for some club E^* of λ , the quadruple $\mathbf{p} = (\bar{C}, \bar{S}, S^*, E^*)$ is a strong λ -witness (see Definition 1.10), and let $\bar{C}' = \bar{C}'_{\mathbf{p}}$.

Now the forcing notion $\mathbb{P} = \mathbb{P}_{\bar{C}'}$ (see Definition 1.12) satisfies clauses (α), (β), (γ) by claims 1.15(2) and also clause (δ) by claim 1.15(1). So we are done. □_{1.16}

§2 THERE ARE $\{\aleph_1\}$ -SEMI-PROPER NOT PROPER FORCING NOTION²

By [Sh:f, XII,§2], it was shown that if “remnant of large cardinal properties holds” (e.g. $\neg 0^\#$) then every quite semi-proper forcing is proper, more fully UReg-semi-properness implies properness. This leaves the problem

- (*) is the statement (for every forcing notion \mathbb{P} , “ \mathbb{P} is proper” follows from \mathbb{P} is “semi-proper, i.e., $\{\aleph_1\}$ -semi proper”) consistent or is the negation provable in ZFC.

David Asparó raises the question and we answer affirmatively: there are such forcing notions. So the iteration theorem for semi proper forcing notions in [Sh:f, X] is not covered by the one on proper forcing notions even if $0^\#$ does not exist.

2.1 Claim. *There is a forcing notion \mathbb{P} of cardinality 2^{\aleph_2} which is not proper but is $\{\aleph_1\}$ -semi proper. This follows from 2.2 using $\kappa = \aleph_2$.*

2.2 Claim. *Assume $\kappa = \text{cf}(\kappa) > \aleph_1, \lambda = 2^\kappa$. Then there is \mathbb{P} such that*

- (a) \mathbb{P} is a forcing notion of cardinality 2^κ
- (b) if $\chi > \lambda, p \in \mathbb{P} \in N \prec (\mathcal{H}(\chi), \in), N$ countable, then there is $q \in \mathbb{P}$ above p such that $q \Vdash “N \cap \kappa \triangleleft N[G_{\mathbb{P}}] \cap \kappa”$ (\triangleleft means initial segment); this gives \mathbb{P} is $\{\aleph_1\}$ -semi proper and more
- (c) there is a stationary $\mathcal{S} \subseteq [\lambda]^{\aleph_0}$ such that $\Vdash_{\mathbb{P}} “\mathcal{S}$ is not stationary”
- (d) \mathbb{P} is not proper.

Proof. We give many details.

Stage A: Preliminaries.

Let $M^* = (\lambda, F_{n,m})_{n,m < \omega}$, with $F_{n,m}$ an $(n+1)$ -place function be such that for every $n < \omega$ and n -place function f from κ to κ there is $m < \omega$ such that $(\forall i_1, \dots, i_n < \kappa)(\exists \alpha < \kappa)[f(i_1, \dots, i_n) = F_{n,m}(\alpha, i_1, \dots, i_n)]$.

Let S_1, S_2 be disjoint stationary subsets of κ of cofinality \aleph_0 (i.e. $\delta \in S_1 \cup S_2 \Rightarrow \text{cf}(\delta) = \aleph_0$).

Let

²done 2001/8/8

$$\mathcal{S} = \left\{ a \in [\lambda]^{\aleph_0} : \text{for some } b \in [\lambda]^{\aleph_0} \text{ we have} \right.$$

$$\begin{aligned} (\alpha) \quad & a \subseteq b \text{ are closed under } F_{n,m} \text{ for } n, m < \omega, \\ (\beta) \quad & \sup(a \cap \kappa) \in S_1, \sup(b \cap \kappa) \in S_2 \\ (\gamma) \quad & (a \cap \kappa) \triangleleft (b \cap \kappa) \text{ (}\triangleleft \text{ is being an initial segment)} \end{aligned} \left. \right\}$$

$\mathbb{P} = \mathbb{P}_{\mathcal{S}} = \{ \bar{a} : \bar{a} = \langle a_i : i \leq \alpha \rangle \text{ is an increasing continuous sequence}$
of members of $[\lambda]^{\aleph_0} \setminus \mathcal{S}$ of length $\alpha < \omega_1 \}$

Clearly clause (a) of 2.2 holds.

Stage B: \mathcal{S} is a stationary subset of $[\lambda]^{\aleph_0}$.

Why? Let N^* be a model with universe λ and countable vocabulary, it is enough to find $a \in \mathcal{S}$ such that $N^* \upharpoonright a \prec N$. Without loss of generality N^* has Skolem functions and N^* expands M^* . Choose for $\alpha < \kappa$, $N_\alpha \prec N^*$, $\|N_\alpha\| < \kappa$, $\beta < \alpha \Rightarrow N_\beta \subseteq N_\alpha$, $\alpha \subseteq N_\alpha$, N_α increasing continuous.

So $C = \{ \delta < \kappa : \delta \text{ a limit ordinal and } N_\delta \cap \kappa = \delta \}$ is a club of κ . Choose $\delta_1 < \delta_2$ from C such that $\delta_1 \in S_1$, $\delta_2 \in S_2$. Choose a countable $c_1 \subseteq \delta_1$ unbounded in δ_1 , and a countable $c_2 \subseteq \delta_2$ unbounded in δ_2 .

Choose a countable $M \prec N_{\delta_2}$ such that $M \cap N_{\delta_1} \prec N_{\delta_1}$ and $c_1 \cup c_2 \subseteq \delta$. Let $a = M \cap N_{\delta_1}$, $b = M \cap N_{\delta_2}$. As N^* expands M^* , clearly a, b are closed under the functions of M^* . Also $c_1 \subseteq M \cap \delta_1 = M \cap (N_{\delta_1} \cap \kappa) = a \cap \kappa \subseteq N_{\delta_1} \cap \kappa = \delta_1$ hence $\delta_1 = \sup(c_1) \leq \sup(a \cap \kappa) \leq \delta_1$ so $\sup(a \cap \kappa) = \delta_1$. Similarly $\sup(b \cap \kappa) = \delta_2$. Lastly, obviously $a \cap \kappa \triangleleft b \cap \kappa$ so b witnesses $a \in \mathcal{S}$, as required.

Stage C: $\Vdash_{\mathbb{P}} \text{“}\mathcal{S} \text{ is not stationary”}$.

Why? Define $\mathcal{a}_\alpha^* = \{ a_\alpha : \bar{a} \in \mathcal{G}_{\mathbb{P}}, \ell g(\bar{a}) > \alpha \}$. Clearly

(*)₀ $\mathbb{P} \neq \emptyset$.

[Why? Trivial.]

(*)₁ for $\alpha < \omega_1$, $\mathcal{S}_\alpha^1 = \{ \bar{a} \in \mathbb{P} : \ell g(\bar{a}) > \alpha \}$ is a dense open subset of \mathbb{P} .

[Why? If $\langle a_i : i \leq j \rangle \in \mathbb{P}$, $j < \gamma < \omega_1$ we let $a_i =: a_j$ for $i \in (j, \gamma]$ and then $\langle a_i : i \leq j \rangle \leq_{\mathbb{P}} \langle a_i : i \leq \gamma \rangle$.]

Also

(*)₂ for $\beta < \lambda$, $\mathcal{I}_\beta^2 = \{\bar{a} \in \mathbb{P} : \beta \in a_\alpha \text{ for some } \alpha < \text{lg}(\bar{a})\}$ is a dense open subset of \mathbb{P} .

[Why? Given $\bar{a} = \langle a_i : i \leq j \rangle$. Choose $\delta \in S_2$ such that $\delta > \sup(\kappa \cap (a_j \cup \{\beta\}))$ let $c \subseteq \delta$ be countable unbounded in δ and let $a_{j+1} = a_j \cup \{\beta\} \cup c$; so trivially $\sup(a_{j+1} \cap \kappa) = \delta \in S_2$ hence $a_{j+1} \notin \mathcal{S}$. Now let $\bar{a}^+ = \langle a_i : i \leq j+1 \rangle$. Now check.]

So

(*)₃ $\Vdash_{\mathbb{P}}$ “ $\langle a_i : i < \omega_1 \rangle$ is an increasing continuous sequence of members of $([\lambda]^{\aleph_0})^{\mathbf{V}} \setminus \mathcal{S}$ whose union is λ ”
hence

(*)₄ $\Vdash_{\mathbb{P}}$ “ $\langle a_i : i < \omega_1 \rangle$ witness \mathcal{S} is not stationary (subset) of $[\lambda]^{\aleph_0}$ ”.

So we have finished Stage C.

Stage D: Clauses (c),(d) of 2.2 holds.

Why? By Stage B and Stage C.

Stage E: Clause (b) of 2.2 holds.

So let $\chi > \lambda$, N a countable elementary submodel of $(\mathcal{H}(\chi), \in, <_\chi^*)$ to which \mathbb{P} and $p \in \mathbb{P}$ belong hence $M^*, \kappa, \lambda, S \in N$ (they are definable from \mathbb{P} or demand it). In the next stage we prove

☐ there is a countable $M \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ such that $N \prec M$, $(N \cap \kappa) \trianglelefteq (M \cap \kappa)$ and $M \cap \lambda \notin \mathcal{S}$.

Let $\langle \mathcal{I}_n : n < \omega \rangle$ list the dense open subsets of \mathbb{P} which belong to M . Choose by induction on n , $p_n \in N \cap \mathbb{P} : p_0 = p, p_n \leq_{\mathbb{P}} p_{n+1} \in \mathcal{I}_n$. So let $p_n = \langle a_i : i \leq \gamma_n \rangle$, by (*)₁ of Stage C the sequence $\langle \gamma_n : n < \omega \rangle$ is not eventually constant. Define q by: $q = \langle a_i : i \leq \gamma \rangle$ where $\gamma = \cup \{\gamma_n : n < \omega\}$ and $a_\gamma = M \cap \lambda$. Trivially $a_i \subseteq M \cap \lambda$ and by (*)₂ of Stage C clearly $a_\gamma = \cup \{a_i : i < \gamma\}$ hence $\langle a_i : i \leq \gamma \rangle$ is increasing continuous and $i \leq \gamma \Rightarrow a_i \in [\lambda]^{\leq \aleph_0}$ and $i < \gamma \Rightarrow a_i \in [\lambda]^{\aleph_0} \setminus \mathcal{S}$. So the only non trivial point is $a_\gamma \notin S$ which holds by ☐.

Clearly $p \leq q$ and q is (M, \mathbb{P}) -generic hence $q \Vdash$ “ $N[G] \subseteq M[G]$ and $N \cap \kappa \subseteq (N[G] \cap \kappa) \subseteq M[G] \cap \kappa = M \cap \kappa$ ” so as $(N \cap \kappa) \triangleleft (M \cap \kappa)$ necessarily $(N[G] \cap \kappa) \triangleleft (M[G] \cap \kappa)$ as required.

Stage F: Proving ☐.

If $N \cap \lambda \notin \mathcal{S}$ let $M = N$ and we are done so assume $M \cap \lambda \in \mathcal{S}$. Let $a = N \cap \lambda \in [\lambda]^{\aleph_0}$ and let $b \in [\lambda]^{\aleph_0}$ witness $a = N \cap \lambda \in \mathcal{S}$ [the rest should by

now be clear but we elaborate]. Let M be the Skolem Hull in $(\mathcal{H}(\chi), \in, <^*_\chi)$ of $N \cup (b \cap \kappa)$ (exists as $<^*_\chi$ is a well ordering of $\mathcal{H}(\chi)$ so $(\mathcal{H}(\chi), \in, <^*_\chi)$ has (definable) Skolem functions).

If $\gamma \in M \cap \kappa$ then we can find a definable function f of $(\mathcal{H}(\chi), \in, <^*)$ and $x \in N$ (recall that in N we can use m -tuple for every m) and $\alpha_1 \dots \alpha_n \in b \cap \kappa$ such that $\gamma = f(x, \alpha_1, \dots, \alpha_n)$. Fixing x, f the mapping $(\alpha_1, \dots, \alpha_n) \mapsto f(x, \alpha_1, \dots, \alpha_n)$ is an n -place function from κ to κ definable in N hence belong to N and $M^* \in N$ hence for some $\beta \in N \cap \lambda$ and $m < \omega$ we have $(\forall \alpha_1, \dots, \alpha_n < \kappa)[f(x, \alpha_1, \dots, \alpha_n) = F_{n,m}(\beta, \alpha_1, \dots, \alpha_n)]$.

But $\alpha_1, \dots, \alpha_n \in b \cap \kappa \subseteq b$ and $\beta \in N \cap \lambda \subseteq b \cap \lambda = b$ and as b being in \mathcal{S} is closed under $F_{n,m}$ clearly $\gamma = f(x, \alpha_1, \dots, \alpha_n) = F_{n,m}(\beta, \alpha_1, \dots, \alpha_n) \in b$ but $\gamma \in \kappa$ so $\gamma \in b \cap \kappa$. So $M \cap \kappa \subseteq b$ but of course $b \cap \kappa \subseteq M \cap \kappa$ so $b \cap \kappa = M \cap \kappa$. So $a \cap \kappa = (N \cap \lambda) \cap \kappa = N \cap \kappa$; but $a \cap \kappa \triangleleft b \cap \kappa$ by the choice of b so $N \cap \kappa = a \cap \kappa \triangleleft b \cap \kappa = M \cap \kappa$.

Lastly, $\sup(M \cap \kappa) = \sup(b \cap \kappa) \in S_2$ hence $M \cap \kappa \notin S$. So M is as required in \boxtimes and we are done. $\square_{2.2}$

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