# POSITIVE PARTITION RELATIONS FOR $P_\kappa(\lambda)$

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**Abstract.** Let $\kappa$ a regular uncountable cardinal and $\lambda$ a cardinal $> \kappa$, and suppose $\lambda^{<\kappa}$ is less than the covering number for category $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. Then (a) $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda}, \omega+1)^2$, (b) $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} [I^+_{\kappa,\lambda}]^2_{\kappa^+}$ if $\kappa$ is a limit cardinal, and (c) $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda})^2$ if $\kappa$ is weakly compact.

## 0. Introduction

Let $\kappa$ be a weakly compact cardinal. Then $\kappa \longrightarrow (\kappa)^2$ and more generally for any cardinal $\lambda \geq \kappa$, $\{P_\kappa(\lambda)\} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda})^2$ ([M4]), which means that for any $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$, there is $A \subseteq P_\kappa(\lambda)$ such that $A$ does not belong to $I_{\kappa,\lambda}$ (the ideal of noncofinal subsets of $P_\kappa(\lambda)$) and $F$ is constant on

$$\{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}.$$

Now if $J$ is the ideal of noncofinal subsets of $\kappa$, then $J^+ \longrightarrow (J^+)^2$ since $(A, <)$ is isomorphic to $(\kappa, <)$ for any $A \in J^+$. So it is natural to ask whether $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda})^2$ for every $\lambda > \kappa$. It turns out that the answer is negative. This is not surprising since it is well-known that some members of $I^+_{\kappa,\lambda}$ may be quite different from $P_\kappa(\lambda)$. To give an example , if the GCH holds and $\lambda$ is the successor of a cardinal of cofinality $< \kappa$, then $\overline{cof}(I_{\kappa,\lambda} \mid A) < \overline{cof}(I_{\kappa,\lambda})$ for some $A \in I^+_{\kappa,\lambda}$ ([MPéS2 ]). We prove that $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda})^2$ if and only if $\lambda^{<\kappa}$ is less than $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ (a generalization of the covering number for category $\mathbf{cov}(\mathbf{M})$).

Let $\kappa$ be an arbitrary regular uncountable cardinal. Dushnik and Miller [DMi] established that $\kappa \longrightarrow (\kappa, \omega)^2$. This was improved to $\kappa \longrightarrow (\kappa, \omega+1)^2$ by Erdös and Rado [ER]. The Erdös-Rado result generalizes ([M3]) : for every cardinal $\lambda \geq \kappa, \{P_\kappa(\lambda)\} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda}, \omega+1)^2$ (i.e. for any $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$, there is either $A \in I^+_{\kappa,\lambda}$ such that $F$ is identically 0 on

$$\{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\},$$

or $a_0, a_1, \ldots, a_\omega$ in $P_\kappa(\lambda)$ such that $a_0 \subset a_1 \subset \ldots \subset a_\omega, \cup(a_0 \cap \kappa) < \cup(a_1 \cap \kappa) < \ldots < \cup(a_\omega \cap \kappa)$ and $F$ is identically 1 on $\{(\cup(a_n \cap \kappa), a_q) : n < q \leq \omega\}$). Here we show that $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} (I^+_{\kappa,\lambda}, \omega+1)^2$ if $\lambda^{<\kappa}$ is less than $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. In the other direction we prove that $I^+_{\kappa,\lambda} \overset{\kappa}{\not\longrightarrow} (I^+_{\kappa,\lambda}, 3)^2$ if $\lambda$ is greater than or equal to $\mathfrak{d}_\kappa$ (or even $\overline{\mathfrak{d}}_\kappa$).

It is a result of [M5] that $\{P_\kappa(\lambda)\} \overset{\kappa}{\not\longrightarrow} [I^+_{\kappa,\lambda}]^2_\lambda$ for any $\lambda > \kappa$ if $\kappa$ is a successor cardinal such that $\kappa \not\longrightarrow [\kappa]^2_\kappa$. In contrast to this, we show that $I^+_{\kappa,\lambda} \overset{\kappa}{\longrightarrow} [I^+_{\kappa,\lambda}]^2_{\kappa^+}$ if $\kappa$ is a limit cardinal and $\lambda$ a cardinal $> \kappa$ with $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. It is also shown that $I^+_{\kappa,\lambda} \overset{\kappa}{\not\longrightarrow} [I^+_{\kappa,\lambda}]^2_\lambda$ if $\lambda \geq \overline{\mathfrak{d}}_\kappa$.

**Throughout the remainder of this paper $\kappa$ will denote a regular uncountable cardinal and $\lambda$ a cardinal $> \kappa$.**

The paper is organized as follows. Section 1 reviews a number of standard definitions concerning ideals on $\kappa$ and $P_\kappa(\lambda)$. Sections 2-7 give results about combinatorics on $\kappa$ that are needed for our study of $P_\kappa(\lambda)$. Sections 2 and 3 review some facts concerning, respectively, the dominating number $\mathfrak{d}_\kappa$ and the

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covering number for category $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. Section 4 deals with the problem of determining the value of the unequality number $\mathfrak{U}_\kappa$ in the case where $\kappa$ is a successor cardinal. In Section 5 we show that if $2^{<\kappa} = \kappa$ and $\mathfrak{U}_\kappa < \kappa^{+\omega}$, then $\mathfrak{U}_\kappa = \mathbf{non}_\kappa$(weakly selective). Sections 6 and 7 review some material concerning, respectively, the unbalanced partition relation $J^+ \longrightarrow (J^+, \rho)^2$ and the square bracket partition relation $J^+ \longrightarrow [J^+]^2_\rho$.

Sections 8-15 are concerned with combinatorial properties of ideals on $P_\kappa(\lambda)$. Section 8 gives two characterizations of $\mathfrak{d}^\kappa_{\kappa,\lambda}$ : one as the least cofinality of any $\kappa$-complete fine ideal on $P_\kappa(\lambda)$ that is not a weak $\pi$-point, and the other as the least cofinality of any $\kappa$-complete fine ideal on $P_\kappa(\lambda)$ that admits a maximal almost disjoint family of size $\kappa$. In Section 9 we show that any $\kappa$-complete fine ideal on $P_\kappa(\lambda)$ with cofinality $< \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ is a weak $\chi$-point. Conversely if $\kappa$ is inaccessible and $I_{\kappa,\lambda}$ is a weak $\chi$-point, then $cof(I_{\kappa,\lambda}) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. Sections 10-13 deal with unbalanced partition relations. Given an infinite cardinal $\theta \leq \kappa$ such that $\kappa \rightarrow (\kappa, \theta)^2$, we show that (a) $u(\kappa, \lambda) \cdot \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta)^2)$ is the least cofinality of any $\kappa$-complete fine ideal $H$ on $P_\kappa(\lambda)$ such that $H^+ \overset{\kappa}{\underset{\kappa}{\not\longrightarrow}} (H^+; \theta)^2$, (b) If $H$ is a $\kappa$-complete fine ideal on $P_\kappa(\lambda)$ with $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ (respectively, $cof(H) <$ $\mathbf{non}_\kappa$(weakly selective)), then $H^+ \longrightarrow (H^+, \theta)^2$ (respectively, $H^+ \overset{\kappa}{\underset{\kappa}{\longrightarrow}} (H^+, \theta)^2$), and (c) Conversely, if $\theta = \kappa$ and $I^+_{\kappa,\lambda} \overset{\kappa}{\underset{\kappa}{\longrightarrow}} (H^+, \theta)^2$, then $cof(I_{\kappa,\lambda}) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. The last two sections are concerned with square bracket partition relations. We show that if $\kappa$ is a limit cardinal, then $H^+ \overset{\kappa}{\longrightarrow} [H^+]^2_{\kappa^+}$ (respectively, $H^+ \overset{\kappa}{\underset{\kappa}{\longrightarrow}} [H^+]^2_\kappa$) for every ideal $H$ on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ (respectively, $cof(H) <$ $\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\kappa)$). In the other direction, $\lambda \geq \overline{\mathfrak{d}}_\kappa$ implies that $I^+_{\kappa,\lambda} \overset{\kappa}{\not\longrightarrow} [I^+_{\kappa,\lambda}]^2_\lambda$ (and $I^+_{\kappa,\lambda} \overset{\kappa}{\underset{\kappa}{\not\longrightarrow}} [I^+_{\kappa,\lambda}]^2_\kappa$ if $\kappa$ is a limit cardinal such that $2^{<\kappa} = \kappa$).

## 1. Ideals

In this section we review some standard definitions and a few basic facts concerning ideals on $\kappa$ and $P_\kappa(\lambda)$.

Given a cardinal $\mu$ and a set $A$, let $P_\mu(A) = \{a \subseteq A : \mid a \mid < \mu\}$.

Given an infinite set $S$, an *ideal on* $S$ is a collection $K$ of subsets of $S$ such that (i) $\{s\} \in K$ for every $s \in S$, (ii) $P(A) \subseteq K$ for every $A \in K$, (iii) $A \cup B \in K$ whenever $A, B \in K$, and (iv) $S \notin K$.

Given an ideal $K$ on $S$, let $K^+ = P(S) - K$ and $K \mid A = \{B \subseteq S : B \cap A \in K\}$ for $A \in K^+$. $sat(K)$ is the least cardinal $\tau$ with the property that for every $Y \subseteq K^+$ with $\mid Y \mid = \tau$, there exist $A, B \in Y$ such that $A \neq B$ and $A \cap B \in K^+$.

$cof(K)$ is the least cardinality of any $X \subseteq K$ such that $K = \bigcup_{A \in X} P(A)$. $K$ is $\kappa$-*complete* if $\bigcup X \in K$ for every $X \in P_\kappa(K)$. Assuming that $K$ is $\kappa$-complete and $\bigcup Y \in K^+$ for some $Y \subseteq K$ with $\mid Y \mid = \kappa$, $\overline{cof}(K)$ is the least cardinality of any $X \subseteq K$ such that $K = \bigcup\{P(\cup x) : x \in P_\kappa(X)\}$.

**We adopt the convention that the phrase "ideal on $\kappa$" means "$\kappa$-complete ideal on $\kappa$".**

Note that the smallest ideal on $\kappa$ is $P_\kappa(\kappa)$.

Given two sets $A$ and $B$ and $f \in {}^A B$, $f$ is *regressive* if $f(a) \in a$ for all $a \in A$.

An ideal $J$ on $\kappa$ is *normal* if given $A \in J^+$ and a regressive $f \in {}^A\kappa$, there is $B \in J^+ \cap P(A)$ such that $f$ is constant on $B$.

$NS_\kappa$ denotes the nonstationary ideal on $\kappa$.

$\kappa$ is *inaccessible* if $2^\mu < \kappa$ for every cardinal $\mu < \kappa$.

Let $[A]^2 = \{(\alpha, \beta) \in A \times A : \alpha < \beta\}$ for any $A \subseteq \kappa$. Given an ordinal $\alpha \geq 2, \kappa \longrightarrow (\kappa, \alpha)^2$ means that for every $f : [\kappa]^2 \longrightarrow 2$, there is $A \subseteq \kappa$ such that either $A$ has order type $\kappa$ and $f$ is identically 0 on $[A]^2$, or $A$ has order type $\alpha$ and $f$ is identically 1 on $[A]^2$. The negation of this and other partition relations is indicated by crossing the arrow. $\kappa \longrightarrow (\kappa)^2$ means that $\kappa \longrightarrow (\kappa, \kappa)^2$.

$\kappa$ is *weakly compact* if $\kappa \longrightarrow (\kappa)^2$.

If $\kappa$ is weakly compact, then it is inaccessible (see e.g. Proposition 4.4 in [Ka]).

An ideal $J$ on $\kappa$ is a *weak P-point* if given $A \in J^+$ and $f \in {}^A\kappa$ with $\{f^{-1}(\{\alpha\}) : \alpha < \kappa\} \subseteq J$, there is $B \in J^+ \cap P(A)$ such that $f$ is $< \kappa$-to-one on $B$. $J$ is a *local Q-point* if given $g \in {}^\kappa\kappa$, there is $B \in J^+$ such that $g(\alpha) < \beta$ for any $(\alpha, \beta) \in [B]^2$. $J$ is a *weak Q-point* if $J \mid A$ is a local $Q$-point for every $A \in J^+$.

It is well-known (see [M1] for a proof) that an ideal $J$ on $\kappa$ is a weak $Q$-point if and only if given $A \in J^+$ and a $< \kappa$-to-one $f : A \longrightarrow \kappa$, there is $B \in J^+ \cap P(A)$ such that $f$ is one-to-one on $B$.

An ideal $J$ on $\kappa$ is *weakly selective* if it is both a weak $P$-point and a weak $Q$-point.

Given a cardinal $\rho$ with $2 \leq \rho \leq \kappa$ and an ideal $J$ on $\kappa, J^+ \longrightarrow [J^+]^2_\rho$ means that for every $A \in J^+$ and every $f : [A]^2 \longrightarrow \rho$, there is $B \in J^+ \cap P(A)$ such that $f''[B]^2 \neq \rho$. $\kappa \longrightarrow [\kappa]^2_\rho$ means that $(P_\kappa(\kappa))^+ \longrightarrow [(P_\kappa(\kappa))^+]^2_\rho$.

Note that $\kappa \longrightarrow [\kappa]^2_2$ if and only if $\kappa \longrightarrow (\kappa)^2$.

Let $P$ be a property such that at least one ideal on $\kappa$ does not satisfy $P$. Then $\mathbf{non}_\kappa(P)$ (respectively, $\overline{\mathbf{non}}_\kappa(P)$) denotes the least cardinal $\tau$ for which one can find an ideal $J$ on $\kappa$ such that $cof(J) = \tau$ (respectively, $\overline{cof}(J) = \tau$) and $J$ does not satisy $P$.

Notice that $\lambda^{<\kappa} < \overline{\mathbf{non}}_\kappa(P)$ if and only if $\lambda^{<\kappa} < \mathbf{non}_\kappa(P)$.

$I_{\kappa,\lambda}$ denotes the set of all $A \subseteq P_\kappa(\lambda)$ such that $A \cap \{b \in P_\kappa(\lambda) : a \subseteq b\} = \phi$ for some $a \in P_\kappa(\lambda)$. An ideal $H$ on $P_\kappa(\lambda)$ is *fine* if $I_{\kappa,\lambda} \subseteq H$.

**We adopt the convention that the phrase "ideal on $P_\kappa(\lambda)$" means " $\kappa$-complete fine ideal on $P_\kappa(\lambda)$".**

Note that $I_{\kappa,\lambda}$ is the smallest ideal on $P_\kappa(\lambda)$.

$u(\kappa, \lambda)$ denotes the least cardinality of any $A \in I^+_{\kappa,\lambda}$.

The following facts are well-known (see e.g. [MPéS1]) : (1) $u(\kappa, \lambda) \geq \lambda$ ; (2) $\lambda^{<\kappa} = 2^{<\kappa} \cdot u(\kappa, \lambda)$ ; (3) $u(\kappa, \lambda) = cof(I_{\kappa,\lambda} \mid A)$ for every $A \in I^+_{\kappa,\lambda}$ ; (4) $u(\kappa, \kappa^{+n}) = \kappa^{+n}$ whenever $0 < n < \omega$.

$\mathcal{K}(\kappa, \lambda)$ denotes the set of all cardinals $\sigma \geq \lambda$ with the property that there is $T \subseteq P_\kappa(\lambda)$ such that $|T| = \sigma$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$.

It is simple to see that $\sigma \leq u(\kappa, \lambda)$ for every $\sigma \in \mathcal{K}(\kappa, \lambda)$. Notice that $\lambda \in \mathcal{K}(\kappa, \lambda)$. More generally, if $\tau$ is an infinite cardinal $\leq \kappa$ such that $|P_\tau(\nu)| < \kappa$ for every infinite cardinal $\nu < \kappa$, then $\lambda^{<\tau} \in \mathcal{K}(\kappa, \lambda)$. It follows that $\lambda^{<\kappa} \in \mathcal{K}(\kappa, \lambda)$ if $\kappa$ is inaccessible. It can be shown (see Remark 11.4 in [To 2] and Theorem 4.1 in [CFMag]) that $\lambda^+ \in \mathcal{K}(\kappa, \lambda)$ if $\square^*_\kappa$ holds and $cf(\lambda) < \kappa$.

An ideal $H$ on $P_\kappa(\lambda)$ is *$\kappa$-normal* if given $A \in H^+$ and a regressive $f \in {}^A\kappa$, there is $B \in H^+ \cap P(A)$ such that $f$ is constant on $B$. The smallest $\kappa$-normal ideal on $P_\kappa(\lambda)$ is denoted by $NS^\kappa_{\kappa,\lambda}$.

## 2. Domination

In this section we recall some characterizations of the dominating number $\mathfrak{d}_\kappa$.

**Definition.** $\mathfrak{d}_\kappa$ is the least cardinality of any $X \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $f \in X$ such that $g(\alpha) < f(\alpha)$ for all $\alpha < \kappa$.
$\overline{\mathfrak{d}}_\kappa$ is the least cardinality of any $X \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $x \in P_\kappa(X)$ such that $g(\alpha) < \bigcup_{f \in x} f(\alpha)$ for all $\alpha < \kappa$.

**PROPOSITION 2.1.**

- **(i)** ([L1]) $\mathfrak{d}_\kappa = cof(NS_\kappa)$.
- **(ii)** ([MRoS]) $\overline{\mathfrak{d}}_\kappa = \overline{cof}(NS_\kappa)$.

**Definition.** Given an ideal $J$ on $\kappa$, $\mathcal{M}_J^{\geq\kappa}$ is the set of all $Q \subseteq J^+$ such that (i) $|Q| \geq \kappa$, (ii) $A \cap B \in J$ for all $A, B \in Q$ with $A \neq B$, and (iii) for every $C \in J^+$, there is $A \in Q$ with $A \cap C \in J^+$.
$\mathfrak{a}_J$ is the least cardinality of any member of $\mathcal{M}_J^{\geq\kappa}$ if $\mathcal{M}_J^{\geq\kappa} \neq \phi$, and $(2^\kappa)^+$ otherwise.

**THEOREM 2.2.** ([Laf], [MP2]) $\mathfrak{d}_\kappa = \mathbf{non}_\kappa(\mathfrak{a}_J > \kappa) = \mathbf{non}_\kappa(\text{weak } P\text{-point})$.

**PROPOSITION 2.3.** $\overline{\mathfrak{d}}\kappa \geq \overline{\mathbf{non}}_\kappa(\mathfrak{a}_J > \kappa) \geq \overline{\mathbf{non}}_\kappa(\text{weak } P\text{-point})$.

**Proof.** The first inequality follows from Proposition 2.1 (ii) since $\mathfrak{a}_{NS_\kappa} = \kappa$ ([MP2]). To prove the second inequality, argue as for Lemma 8.5 below. □

**QUESTION.** Is it consistent that $\overline{\mathfrak{d}}_\kappa > \overline{\mathbf{non}}_\kappa(\text{weak } P\text{-point})$ ?

## 3. Covering for category

Throughout this section $\nu$ will denote a fixed regular infinite cardinal.

We will review some basic facts concerning the covering number $\mathbf{cov}(\mathbf{M}_{\nu,\nu})$.

**Definition.** Suppose $\rho$ is a cardinal $\geq \nu$.
Let $Fn(\rho, 2, \nu) = \cup\{{}^a2 : a \in P_\nu(\rho)\}$. $Fn(\rho, 2, \nu)$ is ordered by : $p \leq q$ if and only if $q \subseteq p$.
${}^\rho 2$ is endowed with the topology obtained by taking as basic open sets $\phi$ and $O_s^\rho$ for $s \in Fn(\rho, 2, \nu)$, where $O_s^\rho = \{f \in {}^\rho 2 : s \subseteq f\}$.
$\mathbf{M}_{\nu,\rho}$ is the set of all $W \subseteq {}^\rho 2$ such that $W \cap (\cap X) = \phi$ for some collection $X$ of dense open subsets of ${}^\rho 2$ with $0 < |X| \leq \nu$.
$\mathbf{cov}(\mathbf{M}_{\nu,\rho})$ is the least cardinality of any $Y \subseteq \mathbf{M}_{\nu,\rho}$ such that ${}^\rho 2 = \cup Y$.

**PROPOSITION 3.1.**

- **(i)** ([L2],[Mil2]) $\mathbf{cov}(\mathbf{M}_{\nu,\rho}) \geq \nu^+$ *for every cardinal* $\rho \geq \nu$.
- **(ii)** ([L2],[Mil2]) *Suppose that* $\rho$ *and* $\mu$ *are two cardinals such that* $\nu \leq \mu \leq \rho$. *Then* $\mathbf{cov}(\mathbf{M}_{\nu,\mu}) \geq \mathbf{cov}(\mathbf{M}_{\nu,\rho})$.
- **(iii)** ([L2]) *Suppose* $2^{<\nu} > \nu$. *Then* $\mathbf{cov}(\mathbf{M}_{\nu,\nu}) = \nu^+$.

**PROPOSITION 3.2.** *Suppose that* $\rho$ *is a cardinal* $> \nu$ *and* $V \models 2^{<\nu} = \nu$. *Then setting* $P = Fn(\rho, 2, \nu)$ :

**(i)** ([L2],[Mil2]) $V^P \models \mathbf{cov}(\mathbf{M}_{\nu,\rho}) \geq \rho$.

**(ii)** ([L2],[Mil2]) *If* $cf(\rho) \leq \nu$, *then* $V^P \models \mathbf{cov}(\mathbf{M}_{\nu,\nu}) > \rho$.

**(iii)** *Let* $\mu$ *be any regular cardinal* $> \nu$. *Then* $(\mathfrak{d}_\mu)^{V^P} = (\mathfrak{d}_\mu)^V$ *and* $(\overline{\mathfrak{d}}_\mu)^{V^P} \leq (\overline{\mathfrak{d}}_\mu)^V$.

**Proof.** (iii) : The conclusion easily follows from the following observation : Suppose that $\sigma$ is a cardinal $> 0$ and $F \in V^P$ is a function from $\sigma \times \mu$ to $\mu$. Then by Lemma VII.6.8 of [K], there is $H : \sigma \times \mu \longrightarrow P_{\nu^+}(\mu)$ such that $H \in V$ and $F(\alpha, \beta) \in H(\alpha, \beta)$ (so $F(\alpha, \beta) \leq \cup H(\alpha, \beta)$) for every $(\alpha, \beta) \in \sigma \times \mu$. □

**Remark.** It is not known whether it is consistent that $cf(\mathbf{cov}(\mathbf{M}_{\nu,\nu}) \leq \nu$.

## 4. Unequality

Our main concern in this section is with the problem of evaluating the unequality number $\mathfrak{U}_\kappa$ when $\kappa$ is a successor cardinal.

**Definition.** $\mathfrak{U}_\kappa$ (respectively, $\mathfrak{U}'_\kappa$) is the least cardinality of any $F \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $f \in F$ such that $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} = \phi$ (respectively $|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa$).

The following is readily checked.

**PROPOSITION 4.1.** $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{U}_\kappa \leq \mathfrak{d}_\kappa$.

**Remark.** It is shown in [MRoS] that if $V \models$ GCH, then there is a $\kappa$-complete $\kappa^+$−cc forcing notion $P$ such that

$$V^P \models \text{“}\overline{\mathfrak{d}}_\kappa = \kappa^{+\omega} \text{ and } \mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) = 2^\kappa = \kappa^{+(\omega+1)}\text{”}.$$

For models where $\mathfrak{d}_\kappa > \kappa^+$ see also [CS].

**PROPOSITION 4.2.** $\mathfrak{U}_\kappa = \mathfrak{U}'_\kappa$.

**Proof.** Fix $F \subseteq {}^\kappa\kappa$ with the property that for every $g \in {}^\kappa\kappa$, there is $f \in F$ such that

$$|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| < \kappa.$$

For $f \in F$ and $\gamma, \delta < \kappa$, define $f_{\gamma,\delta} \in {}^\kappa\kappa$ by : $f_{\gamma,\delta}(\alpha) = f(\alpha)$ if $\alpha \geq \gamma$, and $f_{\gamma,\delta}(\alpha) = \delta$ otherwise. Then for every $g \in {}^\kappa\kappa$, there are $f \in F$ and $\gamma, \delta < \kappa$ such that $\{\alpha \in \kappa : f_{\gamma,\delta}(\alpha) = g(\alpha)\} = \phi$. □

The following is due to Landver [L2].

**PROPOSITION 4.3.** $cf(\mathfrak{U}_\kappa) > \kappa$.

**Proof.** Suppose otherwise. Set $\nu = cf(\mathfrak{U}_\kappa)$ and fix $F \subseteq {}^\kappa\kappa$ so that $|F| = \mathfrak{U}_\kappa$ and for every $g \in {}^\kappa\kappa$, there exists $f \in F$ with $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} = \phi$. Let $< F_\beta : \beta < \nu >$ be such that (a) $|F_\beta| < \mathfrak{U}_\kappa$ for any $\beta$, and (b) $\bigcup_{\beta<\nu} F_\beta = F$. Select $A_\beta \subseteq \kappa$ for $\beta < \nu$ so that (i) $|A_\beta| = \kappa$ for every $\beta < \nu$, (ii) $A_\beta \cap A_\gamma = \phi$ whenever $\gamma < \beta < \nu$, and (iii) $\bigcup_{\beta<\nu} A_\beta = \kappa$. For each $\beta < \nu$, there is $g_\beta : A_\beta \longrightarrow \kappa$ such that

$$\{\alpha \in A_\beta : (f \restriction A_\beta)(\alpha) = g_\beta(\alpha)\} \neq \phi$$

for every $f \in F_\beta$. Set $g = \bigcup_{\beta<\nu} g_\beta$. Then clearly, $\{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \neq \phi$ for all $f \in F$. This is a contradiction. □

We now turn our attention to the task of computing $\mathfrak{U}_\kappa$. We begin with the case when $\kappa$ is a successor cardinal.

**THEOREM 4.4.** *Suppose $\kappa$ is the successor of a regular infinite cardinal $\nu$. Then*

$$\mathfrak{U}_\kappa \geq \min(\mathfrak{d}_\kappa, \mathbf{cov}(\mathbf{M}_{\nu,\kappa})).$$

**Proof.** Fix $F \subseteq {}^\kappa\kappa$ with $0 < |F| < \min(\mathfrak{d}_\kappa, \mathbf{cov}(\mathbf{M}_{\nu,\kappa}))$. Pick $k : \kappa \longrightarrow \kappa - \nu$ so that

$$|\{\alpha < \kappa : k(\alpha) > f(\alpha)\}| = \kappa$$

for every $f \in F$. Select a bijection $j : \kappa \times \nu \longrightarrow \kappa$ and a bijection $i_\alpha : k(\alpha) \longrightarrow \nu$ for each $\alpha < \kappa$. Given $A \subseteq \kappa$ and $t \in {}^A 2$, define a partial function $\overline{t}$ from $\kappa$ to $\kappa$ by stipulating that $\overline{t}(\alpha) = \gamma$ if and only if (a) $\gamma < k(\alpha)$, (b) $\{j(\alpha, \eta) : \eta < i_\alpha(\gamma)\} \subseteq t^{-1}(\{0\})$, and (c) $j(\alpha, i_\alpha(\gamma)) \in t^{-1}(\{1\})$. For $f \in F$, let $D_f$ be the set of all $s \in Fn(\kappa, 2, \nu)$ such that there is $\alpha \in dom(\overline{s})$ with $k(\alpha) > f(\alpha)$ and $\overline{s}(\alpha) = f(\alpha)$. Clearly, each $D_f$ is a dense subset of $Fn(\kappa, 2, \nu)$, so we can find $g \in {}^\kappa 2$ with the property that for every $f \in F$, there is $a \in P_\nu(\kappa)$ with $g \restriction a \in D_f$. Then

$$\{\alpha \in dom(\overline{g}) : \overline{g}(\alpha) = f(\alpha)\} \neq \phi$$

for every $f \in F$. $\square$

**THEOREM 4.5.** *Suppose $\kappa$ is a successor cardinal. Then* $\mathfrak{U}_\kappa \geq \overline{\mathfrak{d}}_\kappa$.

**Proof.** Fix $F \subseteq {}^\kappa\kappa$ with $0 < |F| < \overline{\mathfrak{d}}_\kappa$. Set $\kappa = \nu^+$. Pick $k : \kappa \longrightarrow \kappa - \nu$ so that

$$|\{\alpha < \kappa : f(\alpha) < k(\alpha)\}| = \kappa$$

for every $f \in F$. For $\alpha < \kappa$, select a bijection $\pi_\alpha : k(\alpha) \longrightarrow \nu$. Given $f \in F$, there exists $i_f \in \nu$ such that the set

$$A_f = \{\alpha < \kappa : f(\alpha) < k(\alpha) \text{ and } \pi_\alpha(f(\alpha)) = i_f\}$$

has size $\kappa$. Define $g_f \in {}^\kappa\kappa$ by

$$g_f(\beta) = \text{ least } \alpha \in A_f \text{ such that } \alpha \geq \beta.$$

It is shown in [MRoS] that $\overline{\mathfrak{d}}_\kappa$ is the least cardinality of any $X \subseteq {}^\kappa\kappa$ with the property that for every $h \in {}^\kappa\kappa$, there is $x \in P_\kappa(X)$ such that the set $\{\beta < \kappa : h(\beta) \geq \bigcup_{f \in x} f(\beta)\}$ is nonstationary in $\kappa$. Hence there is $h \in {}^\kappa\kappa$ such that the set

$$B_x = \{\beta < \kappa : h(\beta) \geq \bigcup_{f \in x} g_f(\beta)\}$$

is stationary in $\kappa$ for every $x \in P_\kappa(F)$.

Define $J \subseteq P(\kappa)$ by : $D \in J$ if and only if there is $x \in P_\kappa(F)$ such that $D \cap B_x \in NS_\kappa$. Then $J$ is an ideal on $\kappa$. Since $sat(J) > \nu$ by a result of Ulam (see [Ka], 16.3), there exist pairwise disjoint $D_i \in J^+$ for $i < \nu$ with $\bigcup_{i<\nu} D_i = \kappa$.

Let $C$ be the set of all infinite limit ordinals $\delta < \kappa$ such that $h(\xi) < \delta$ for every $\xi < \delta$. Then $C$ is a closed unbounded subset of $\kappa$. Define $t \in {}^\kappa\kappa$ so that for every $\eta < \kappa, t(\eta) < k(\eta)$ and $c_\eta \in D_{\pi_\eta(t(\eta))}$, where $c_\eta = \cup(C \cap \eta)$.

Now fix $f \in F$. Pick $\zeta \in D_{i_f} \cap C \cap B_{\{f\}}$ and set $\eta = g_f(\zeta)$. Notice that $\zeta \leq \eta$ by the definition of $g_f$. Also, $\eta \leq h(\zeta)$ since $\zeta \in B_{\{f\}}$. Hence $c_\eta = \zeta$ by the definition of $C$ and the fact that $\zeta \in C$. It now follows from the definition of $t$ and the fact that $\zeta \in D_{i_f}$ that $\pi_\eta(t(\eta)) = i_f$. On the other hand, $\eta \in A_f$ since $\eta = g_f(\zeta)$, so $f(\eta) < k(\eta)$ and $\pi_\eta(f(\eta)) = i_f$. Thus $t(\eta) = f(\eta)$. □

**Remark.** It follows from Proposition 4.1 and Theorem 4.5 that $\mathfrak{U}_\kappa = \mathfrak{d}_\kappa$ if $\kappa$ is a successor cardinal and $\mathfrak{d}_\kappa < \kappa^{+\omega}$.

**THEOREM 4.6.** *Suppose that* $\kappa$ *is a successor cardinal and* $2^{<\kappa} = \kappa$. *Then* $\mathfrak{U}_\kappa = \mathfrak{d}_\kappa$.

**Proof.** By Proposition 4.1 it suffices to prove that $\mathfrak{U}_\kappa \geq \mathfrak{d}_\kappa$. Set $\kappa = \nu^+$ and select a one-to-one

$$j : \bigcup_{\alpha<\kappa} {}^{[\alpha,\alpha+\nu)}\kappa \longrightarrow \kappa.$$

Now fix $F \subseteq {}^\kappa\kappa$ with $0 < |F| < \mathfrak{d}_\kappa$. Select $g \in {}^\kappa\kappa$ so that for every $f \in F$, there is $\beta_f < \kappa$ with

$$j\big(f \restriction [\beta_f, \beta_f + \nu)\big) < g(\beta_f).$$

Let $C$ be the set of all $\gamma < \kappa$ such that $\beta + \nu < \gamma$ and $g(\beta) < \gamma$ for every $\beta < \gamma$. Then $C$ is a closed unbounded subset of $\kappa$. Let $< \gamma_\delta : \delta < \kappa >$ be the increasing enumeration of $C$. For $\delta < \kappa$, set

$$W_\delta = \Big\{ t \in \bigcup_{\gamma_\delta \leq \alpha < \gamma_{\delta+1}} {}^{[\alpha,\alpha+\nu)}\kappa : j(t) < \gamma_{\delta+1} \Big\}.$$

Then define $k_\delta \in {}^{[\gamma_\delta,\gamma_{\delta+1})}\kappa$ so that for every $t \in W_\delta$, there is $\zeta \in dom(t)$ with $k_\delta(\zeta) = t(\zeta)$. Set $k = \bigcup_{\delta<\kappa} k_\delta$.

Given $f \in F$, let $\delta_f < \kappa$ be such that $\gamma_{\delta_f} \leq \beta_f < \gamma_{\delta_f+1}$. Then $f \restriction [\beta_f, \beta_f + \nu) \in W_{\delta_f}$. Hence $k(\zeta) = f(\zeta)$ for some $\zeta \in [\beta_f, \beta_f + \nu)$. □

**QUESTION.** Is it consistent that $\kappa$ is a successor cardinal and $\mathfrak{U}_\kappa < \mathfrak{d}_\kappa$ ?

**QUESTION.** Is it consistent that $\kappa$ is a successor cardinal such that $2^{<\kappa} = \kappa$ and $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) < \mathfrak{U}_\kappa$ ?

Let us now consider the case when $\kappa$ is a limit cardinal. By a result of Bartoszyński [B] and Miller [Mil1], $\mathfrak{U}_\omega = \mathbf{cov}(\mathbf{M}_{\omega,\omega})$. Landver [L2] was able to show that this fact generalizes to uncountable inaccessible cardinals :

**THEOREM 4.7.** *If* $\kappa$ *is an inaccessible cardinal, then* $\mathfrak{U}_\kappa = \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$.

**QUESTION.** Is it consistent that $\kappa$ is a limit cardinal and $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) < \mathfrak{U}_\kappa$ ?

## 5. Weak selectivity

The following is due to Baumgartner, Taylor and Wagon [BauTW].

**PROPOSITION 5.1.** *If* $\kappa$ *is a successor cardinal, then every ideal on* $\kappa$ *is a weak* $Q$*-point.*

By Proposition 5.1 and Theorem 2.2 $\mathbf{non}_\kappa$(weakly selective) $= \mathfrak{d}_\kappa$ if $\kappa$ is a successor cardinal. The remainder of the section is primarily concerned with the value of $\mathbf{non}_\kappa$(weakly selective) in the case when $\kappa$ is a limit cardinal.

**Remark.** It is easy to see that $\kappa^+ \leq \mathbf{non}_\kappa$(weak $Q$-point) if $\kappa$ is a limit cardinal.

**Definition.** An ideal $J$ on $\kappa$ is a weak semi-$Q$-point if given $A \in J^+$ and a $<\kappa$-to-one function $f$ from $A$ to $\kappa$, there is $C \in J^+ \cap P(A)$ such that $|C \cap f^{-1}(\{\alpha\})| \leq |\alpha|$ for every $\alpha \in \kappa$.
$J$ is weakly semiselective if $J$ is both a weak semi-$Q$-point and a weak $P$-point.
$J$ is weakly rapid if given $A \in J^+$ and $f \in {}^\kappa\kappa$, there is $C \in J^+ \cap P(A)$ such that o.t.$(C \cap f(\alpha)) \leq \alpha + 1$ for every $\alpha \in \kappa$.

**Remark.** It is simple to see that every weak $Q$-point ideal on $\kappa$ is weakly rapid, and every weakly rapid ideal on $\kappa$ is a weak semi-$Q$-point.

Every weak semi-$Q$-point ideal on $\omega$ is weakly rapid ([MP1]). We will show that this does not generalize.

**Definition.** An ideal $J$ on $\kappa$ is a semi-$Q$-point if given a $<\kappa$-to-one function $f$ from $\kappa$ to $\kappa$, there is $B \in J$ such that $|f^{-1}(\{\alpha\}) - B| \leq |\alpha|$ for every $\alpha \in \kappa$.

**PROPOSITION 5.2.** *Suppose $\kappa$ is a limit cardinal. Then there exists a semi-$Q$-point ideal on $\kappa$ that is not weakly rapid.*

**Proof.** Let $Y$ be the set of all infinite cardinals $< \kappa$. Select $h \in {}^Y\kappa$ so that (a) $h(\mu)$ is a regular infinite cardinal $\leq \mu$ for every $\mu \in Y$, and (b) $\{\mu \in Y : h(\mu) \geq \theta\}$ is stationary in $\kappa$ for every $\theta \in Y$. For $A \subseteq \kappa$ and $\theta \in Y$, let $T^A_\theta$ be the set of all $\mu \in Y$ such that $h(\mu) \geq \theta$ and $|A \cap [\mu, \mu + h(\mu))| = h(\mu)$. Now let $J_h$ be the set of all $A \subseteq \kappa$ such that $T^A_\theta$ is a nonstationary subset of $\kappa$ for some $\theta \in Y$. It is simple to check that $J_h$ is an ideal on $\kappa$.

Let us remark in passing that if $\kappa$ is weakly Mahlo and $h$ is defined by : $h(\mu) = \omega$ if $\mu$ is singular, and $h(\mu) = \mu$ otherwise, then a subset $A$ of $\kappa$ lies in $J_h$ if and only if the set of all $\mu \in Y$ such that $\mu$ is regular and $|A \cap [\mu, \mu + \mu)| = \mu$ is nonstationary in $\kappa$.

Let us show that $J_h$ is a semi-$Q$-point. Thus fix a $<\kappa$-to-one function $f : \kappa \longrightarrow \kappa$. Then

$$C = \Big\{\mu \in Y : \mu = \bigcup_{\alpha < \mu} f^{-1}(\{\alpha\})\Big\}$$

is a closed unbounded subset of $\kappa$. Set $Q = \bigcup_{\mu \in C} [\mu, \mu + h(\mu))$. It is immediate that $\kappa - Q \in J_h$. Now fix $\alpha \in \kappa$ such that $Q \cap f^{-1}(\{\alpha\}) \neq \phi$. Pick $\nu \in C$ so that

$$[\nu, \nu + h(\nu)) \cap f^{-1}(\{\alpha\}) \neq \phi.$$

Clearly, $\alpha \geq \nu$ and $\nu \cap f^{-1}(\{\alpha\}) = \phi$. Let $\rho$ be the least element of $C$ that is $> \nu$. Then $\alpha < \rho$ and $f^{-1}(\{\alpha\}) \subseteq \rho$. Thus

$$Q \cap f^{-1}(\{\alpha\}) \subseteq [\nu, \nu + h(\nu))$$

and consequently

$$|Q \cap f^{-1}(\{\alpha\})| \leq h(\nu) \leq \nu \leq |\alpha| .$$

It remains to show that $J$ is not weakly rapid. Fix $D \in J_h^+$. Then

$$S = \{\mu \in T^D_\omega : |T^D_\omega \cap \mu| = \mu\}$$

is a stationary subset of $\kappa$. Given $\mu \in S$, $|D \cap \mu| = \mu$ since

$$D \cap [\rho, \rho + h(\rho)) \subset D \cap \mu$$

for every $\rho \in \mu \cap T^D_\omega$, and hence

$$\text{o.t.}(D \cap (\mu + h(\mu)) > \mu + 1.$$ □

**THEOREM 5.3.** *Suppose $\kappa$ is a limit cardinal. Then* $\mathfrak{U}_\kappa \leq$ $\mathbf{non}_\kappa$(weak semi-$Q$-point).

**Proof.** Let $J$ be an ideal on $\kappa$ with $cof(J) < \mathfrak{U}_\kappa$. Let us show that $J$ is a weak semi-$Q$-point. Thus fix $A \in J^+$ and a $<\kappa$-to-one function $f : A \longrightarrow \kappa$. Select $B_\beta \in J$ for $\beta < cof(J)$ so that $J = \bigcup_{\beta < cof(J)} P(B)$.
For $\beta < cof(J)$, define $g_\beta \in {}^\kappa\kappa$ by :

$$g_\beta(\alpha) = \text{ least element of } (\bigcup_{\gamma > \alpha} f^{-1}(\{\gamma\})) - B_\beta.$$

There is $h \in {}^\kappa\kappa$ such that $\{\alpha \in \kappa : g_\beta(\alpha) = h(\alpha)\} \neq \phi$ for every $\beta < cof(J)$. Define $C \subseteq ran(h)$ by : $h(\alpha) \in C$ just in case $h(\alpha) \in \bigcup_{\gamma > \alpha} f^{-1}(\{\gamma\})$. Then clearly $C \in J^+ \cap P(A)$. Moreover, $C \cap f^{-1}(\{\alpha\}) \subseteq \{h(\gamma) : \gamma < \alpha\}$ for every $\alpha < \kappa$. □

**THEOREM 5.4.** *Suppose that $\kappa$ is a limit cardinal and $2^{<\kappa} = \kappa$. Then*

$$\overline{\mathbf{non}}_\kappa(\text{weakly semiselective}) \leq \mathfrak{U}_\kappa \leq \mathbf{non}_\kappa(\text{weak } Q\text{-point}).$$

**Proof.** The proof of the first inequality is an easy modification of that of Lemma 6.1 in [MP1] (which should be corrected by substituting "$e \in [\omega]^{<\omega}$ such that $B \subseteq \bigcup_{j \in e} \omega^{E_j} \cup \bigcup_{f \in z} B_f$" for "$e \in [\bigcup_{j \in \omega} \omega^{E_j}]^{<\omega}$ such that $B \subseteq e \cup \bigcup_{f \in z} B_f$"). The second inequality is proved as Proposition 5.3 in [MP1]. □

**Remark.** Suppose that $\kappa$ is a limit cardinal, $2^{<\kappa} = \kappa$ and $\mathbf{non}_\kappa(\text{weakly semiselective}) < \kappa^{+\omega}$. Then by Proposition 4.1 and Theorems 2.2 and 5.4,

$$\mathfrak{U}_\kappa = \mathbf{non}_\kappa(\text{weakly selective}) = \mathbf{non}_\kappa(\text{weakly semiselective}).$$

**Remark.** It is consistent (see [MP1]) that $\mathfrak{U}_\omega < \mathbf{non}_\omega(\text{weak } Q\text{-point})$, and that $\mathbf{non}_\omega(\text{weak } Q\text{-point}) < \mathbf{non}_\omega(\text{weak semi-}Q\text{-point})$. We do not know whether these results can be generalized.

**QUESTION.** Is it consistent that $\kappa$ is a limit cardinal, $2^{<\kappa} > \kappa$ and $\kappa^+ < \mathbf{non}_\kappa(\text{weak } Q\text{-point})$ ?

**QUESTION.** By a result of [MP1], $cf(\mathbf{non}_\omega(\text{weak } Q\text{-point})) > \omega$. Does this generalize ?

## 6. $\mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta)^2)$

In this section we use standard material to discuss the value of $\mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta)^2)$ for a cardinal $\theta \in [3, \kappa]$.

**THEOREM 6.1.**

(i) $\mathfrak{d}_\kappa \geq \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, 3)^2)$.

(ii) $\overline{\mathfrak{d}}_\kappa \geq \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow (J^+, 3)^2)$.

(iii) $\overline{\mathbf{non}}_\kappa(\text{weak } P\text{-point}) \geq \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow (J^+, \omega)^2)$.

**Proof.** (i) and (ii) : By a straightforward generalization of Lemma 4.4 in [M2], there exists an ideal $J$ on $\kappa$ such that $\overline{cof}(J) \leq \overline{\mathfrak{d}}_\kappa, cof(J) \leq \mathfrak{d}_\kappa$ and $J^+ \not\longrightarrow (J^+, 3)^2$.
(iii) : Baumgartner, Taylor and Wagon [BauTW] established that if $J$ is an ideal on $\kappa$ such that $J^+ \longrightarrow (J^+, \omega)^2$, then $J$ is a weak $P$-point. □

**Definition.** Given an ideal $J$ on $\kappa$, $A \in J^+$ and $F : \kappa \times \kappa \longrightarrow 2$, $(J, A, F)$ is 0-good if there is $D \in J^+ \cap P(A)$ such that $\{\beta \in D : F(\alpha, \beta) = 1\} \in J$ for every $\alpha \in D$.

The following is readily checked.

**LEMMA 6.2.** *Suppose that $J$ is weakly selective and $(J, A, F)$ is 0-good, where $J$ is an ideal on $\kappa$, $A \in J^+$ and $F : \kappa \times \kappa \longrightarrow 2$. Then there is $B \in J^+ \cap P(A)$ such that $F$ is constantly 0 on $[B]^2$.*

**LEMMA 6.3.** *Suppose that $(J, A, F)$ is not 0-good, where $J$ is an ideal on $\kappa$, $A \in J^+$ and $F : \kappa \times \kappa \longrightarrow 2$. Then :*

- **(i)** *There is $B \subseteq A$ such that* $\text{o.t.}(B) = \omega + 1$ *and $F$ is identically 1 on $[B]^2$.*
- **(ii)** *Suppose that $\mathfrak{a}_J > \kappa$ and $\theta$ is an uncountable cardinal $< \kappa$ such that $\kappa \longrightarrow (\kappa, \theta)^2$. Then there is $C \subseteq A$ such that* $\text{o.t.}(C) = \theta + 1$ *and $F$ is identically 1 on $[C]^2$.*

**Proof.** The proof is similar to that of Lemma 10.4 below. □

**THEOREM 6.4.**

- **(i)** $\overline{\mathbf{non}}_\kappa(J^+ \longrightarrow (J^+, \omega + 1)^2) \geq \overline{\mathbf{non}}_\kappa(\text{weakly selective})$.
- **(ii)** *Suppose that $\theta$ is an infinite cardinal $< \kappa$ such that $\kappa \longrightarrow (\kappa, \theta)^2$. Then*

$$\mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta + 1)^2) \geq \mathbf{non}_\kappa(\text{weakly selective}).$$

**Proof.** (i) : Baumgartner, Taylor and Wagon [BauTW] showed that $J^+ \longrightarrow (J^+, \omega + 1)^2$ for every weakly selective ideal $J$ on $\kappa$.

(ii) : By Lemmas 6.2 and 6.3. □

**Remark.** Suppose that $\kappa$ is a successor cardinal and $\theta$ is cardinal $\geq 2$ such that $\kappa \longrightarrow (\kappa, \theta)^2$. Then by Theorems 6.1 (i), 6.4 (ii) and 2.2 and Proposition 5.1, $\mathfrak{d}_\kappa = \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta + 1)^2)$.

**Remark.** It is consistent (see [M2]) that $\mathfrak{d} > \mathbf{non}_\omega(J^+ \longrightarrow (J^+, 3)^2)$. We do not know whether this can be generalized.

**THEOREM 6.5.** *Suppose $\kappa$ is a weakly compact cardinal. Then :*

- **(i)** $\overline{\mathbf{non}}_\kappa(\text{weak } Q\text{-point}) \geq \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow (J^+, \kappa)^2)$.
- **(ii)** $\mathbf{non}_\kappa(J^+ \longrightarrow (J^+)^2) = \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \kappa)^2) = \mathbf{non}_\kappa(\text{weakly selective})$.

**Proof.** The result follows from Theorems 2.2 and 6.1 (i) and the following two well-known facts : (1) Every ideal $J$ on $\kappa$ such that $J^+ \longrightarrow (J^+, \kappa)^2$ is a weak $Q$-point ; (2) If $\kappa$ is weakly compact, then $J^+ \longrightarrow (J^+)^2$ for every weakly selective ideal $J$ on $\kappa$ such that $\mathfrak{a}_J > \kappa$. □

## 7. $\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\rho)$

In this section we consider the cardinal $\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\rho)$, where $3 \leq \rho \leq \kappa$, about which little is known. We begin with the case where $\rho = 3$. The following is due to Blass [Bl].

**LEMMA 7.1.** *Suppose $J$ is an ideal on $\kappa$ such that $J^+ \longrightarrow [J^+]^2_3$. Then $J$ is a weak $P$-point.*

**Proof.** Fix $A \in J^+$ and $f \in {}^A\kappa$ with $\{f^{-1}(\{\gamma\}) : \gamma \in \kappa\} \subseteq J$. Define $g : [A]^2 \longrightarrow 3$ by stipulating that $g(\alpha, \beta) = 0$ if and only if $f(\alpha) < f(\beta)$, and $g(\alpha, \beta) = 1$ if and only if $f(\alpha) = f(\beta)$. There are $B \in J^+ \cap P(A)$ and $i < 3$ such that $i \notin g''[B]^2$. It is simple to see that $i \neq 0$, so $f$ is $< \kappa$-to-one on $B$. □

The following is proved by adapting an argument of Baumgartner and Taylor [BauT].

**LEMMA 7.2.** *Suppose $J$ is an ideal on $\kappa$ such that $J^+ \longrightarrow [J^+]^2_3$, and $(J, A, F)$ is 0-good, where $A \in J^+$ and $F : \kappa \times \kappa \longrightarrow 2$. Then either there exists $C \in J^+ \cap P(A)$ such that $F$ is constantly 0 on $[C]^2$, or for every $\delta < \kappa$, there exists $Q \subseteq A$ such that $\text{o.t.}(Q) = \delta$ and $F$ is constantly 1 on $[Q]^2$.*

**Proof.** Select $B \in J^+ \cap P(A)$ so that $\{\beta \in B : F(\alpha, \beta) = 1\} \in J$ for every $\alpha \in B$. By Lemma 7.1, there exists $S \in J^+ \cap P(B)$ so that $|\{\beta \in S : F(\alpha, \beta) = 1\}| < \kappa$ for every $\alpha \in S$. Define $\delta_\xi$ for $\xi < \kappa$ by :

(i) $\delta_0 = \cap S$;

(ii) $\delta_{\xi+1}$ = the least $\zeta < \kappa$ with the property that $\zeta > \beta$ for every $\beta \in S$ such that $F(\alpha, \beta) = 1$ for some $\alpha \in S \cap \delta_\zeta$ ;

(iii) $\delta_\xi = \bigcup_{\zeta < \xi} \delta_\zeta$ if $\xi$ is a limit ordinal $> 0$.

Let $X$ be the set of all limit ordinals $< \kappa$. For $\eta \in X$, $n \in \omega$ and $j < 2$, set

$$d^j_{\eta,n} = S \cap [\delta_{\eta+2n+j}, \delta_{\eta+2n+j+1}).$$

For $j < 2$, let

$$D^j = \cup\{d^j_{\eta,n} : \eta \in X \text{ and } n \in \omega\}.$$

Select $k < 2$ so that $D^k \in J^+$. Notice that $F(\alpha, \beta) = 0$ if $(\alpha, \beta) \in [D^k]^2$ and $\{\alpha, \beta\} \not\subseteq d^k_{\eta,n}$ for all $\eta \in X$ and $n \in \omega$.

Define $h : [D^k]^2 \longrightarrow 3$ by stipulating that $h(\alpha, \beta) = 0$ if and only if $\{\alpha, \beta\} \not\subseteq d^k_{\eta,n}$ for all $\eta \in X$ and $n \in \omega$, and $h(\alpha, \beta) = 1$ if and only if $F(\alpha, \beta) = 1$. There are $W \in J^+ \cap P(D^k)$ and $i < 3$ so that $i \notin h''[W]^2$. Clearly, $i \neq 0$. If $i = 1$, $F$ is identically 0 on $[W]^2$. Now assume $i = 2$. Let $Z$ be the set of all $(\eta, n) \in X \times \omega$ such that $W \cap d^k_{\eta,n} \neq \phi$. Suppose that there is $\gamma < \kappa$ such that $\text{o.t.}(W \cap d^k_{\eta,n}) \leq \gamma$ for every $(\eta, n) \in Z$. Then there exists $C \in J^+ \cap P(W)$ such that $|C \cap d_{\eta,n}| = 1$ for any $(\eta, n) \in Z$. Clearly, $F$ takes the constant value 0 on $[T]^2$. □

**PROPOSITION 7.3.** *Suppose $\theta \in (2, \kappa)$ is a cardinal such that $\kappa \longrightarrow (\kappa, \theta)^2$. Then*

$$\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_3) \leq \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \theta + 1)^2).$$

**Proof.** By Theorem 2.2 and Lemmas 6.3, 7.1 and 7.2. □

Let us now consider the partition relation $J^+ \longrightarrow [J^+]^2_\kappa$. We begin with the following observation.

**PROPOSITION 7.4.** *Suppose* $\kappa$ *is inaccessible. Then there is an ideal* $J$ *on* $\kappa$ *such that* (a) $J^+ \not\longrightarrow [J^+]^2_\kappa$, (b) $J$ *is not a weak semi-Q*-point, (c) $\mathfrak{a}_J > \kappa$, *and* (d) $J^+ \longrightarrow (J^+, \alpha)^2$ for every $\alpha < \kappa$.

**Proof.** Let $< \rho_\alpha : \alpha < \kappa >$ be the increasing enumeration of all strong limit infinite cardinals $< \kappa$. let $Z$ be the set of all regular infinite cardinals $< \kappa$. For $\mu \in Z$, set $\nu_\mu = (\rho_\mu)^{++}$. Then $\nu_\mu \not\longrightarrow [\nu_\mu]^2_{\nu_\mu}$ by a result of Todorcevic [To1]. On the other hand, by a result of Erdös and Rado (see [EHMáR], Corollary 17.5), $\nu_\mu \longrightarrow (\nu_\mu, \tau)^2$ for every infinite cardinal $\tau < \mu$. Pick pairwise disjoint $A_\mu$ for $\mu \in Z$ so that $|A_\mu| = \nu_\mu$ for any $\mu \in Z$, and $\bigcup_{\mu \in Z} A_\mu = \kappa$. Let $J$ be the set of all $B \subseteq \kappa$ such that

$$|\{\mu \in Z : |B \cap A_\mu| = \nu_\mu\}| < \kappa.$$

It is simple to see that $J$ is an ideal on $\kappa$.

For $\mu \in Z$, pick $g_\mu : [A_\mu]^2 \longrightarrow \nu_\mu$ so that $g''_\mu [B]^2 = \nu_\mu$ for every $B \subseteq A_\mu$ with $|B| = \nu_\mu$. Let $G : [\kappa]^2 \longrightarrow \kappa$ be such that $\bigcup_{\mu \in Z} g_\mu \subseteq G$. Then clearly $G''[C]^2 = \kappa$ for any $C \in J^+$.

Define $f \in {}^\kappa\kappa$ by stipulating that $f^{-1}(\{\mu\}) = A_\mu$ for every $\mu \in Z$. Clearly, there is no $S \in J^+$ so that $|S \cap f^{-1}(\{\alpha\}| \leq |\alpha|$ for all $\alpha < \kappa$. Hence $J$ is not a weak semi-$Q$-point.

Let us next show that $\mathfrak{a}_J > \kappa$. Thus suppose that $B_\alpha \in J^+$ for $\alpha < \kappa$, and $B_\alpha \cap B_\beta \in J$ whenever $\beta < \alpha < \kappa$. Select a strictly increasing function $k : \kappa \longrightarrow Z$ so that

$$|(B_\alpha - (\bigcup_{\beta < \alpha} B_\beta)) \cap A_{k(\alpha)}| = \nu_{k(\alpha)}$$

for any $\alpha < \kappa$. Set

$$T = \bigcup_{\alpha < \kappa} ((B_\alpha - (\bigcup_{\beta < \alpha} B_\beta)) \cap A_{k(\alpha)}).$$

Then $T \in J^+$ and moreover $|T \cap B_\alpha| < \kappa$ for every $\alpha < \kappa$.

It remains to prove (d). Thus fix $A \in J^+$ and $F : \kappa \times \kappa \longrightarrow 2$. Suppose that there is $\eta < \kappa$ such that for every $Q \subseteq A$ with o.t.$(Q) = \eta$, $F$ is not constantly 1 on $[Q]^2$. Since by Theorem 17.1 of [EHMáR] $\kappa \longrightarrow (\kappa, \alpha)^2$ for every $\alpha < \kappa$, it follows from Lemma 6.3 that $(J, A, F)$ is 0-good. Select $D \in J^+ \cap P(A)$ so that $\{\beta \in D : F(\alpha, \beta) = 1\} \in J$ for every $\alpha \in D$. Define $D_\gamma$ for $\gamma < \kappa$ and a strictly increasing function $h : \kappa \longrightarrow Z$ so that

(0) $$D_\gamma = D - (\bigcup_{\delta < \gamma} \bigcup_{\alpha \in D_\delta \cap A_{h(\delta)}} \{\beta \in D : F(\alpha, \beta) = 1\}) ;$$

(1) $$|D_\gamma \cap A_{h(\gamma)}| = \nu_{h(\gamma)}.$$

For $\gamma \in (|\eta|^+, \kappa)$, select $X_\gamma \subseteq D_\gamma \cap A_{h(\gamma)}$ so that $|X_\gamma| = \nu_{h(\gamma)}$ and $F$ is constantly 0 on $[X_\gamma]^2$. Set $Y = \bigcup_{|\eta|^+ < \gamma < \kappa} X_\gamma$. Then clearly $Y \in J^+ \cap P(A)$. Moreover, $F$ takes the constant value 0 on $[Y]^2$. □

**Remark.** $J^+ \longrightarrow (J^+, \kappa)^2$ does not necessarily imply that $J^+ \longrightarrow [J^+]^2_\kappa$. This follows from the following two facts : (0) If $\kappa$ is weakly compact, then there exists a normal ideal $J$ on $\kappa$ such that $J^+ \longrightarrow (J^+, \kappa)^2$ ([Bau1], [Bau2]) ; (1) Assuming $V = L$, $\kappa$ is completely ineffable if and only if there is a normal ideal $J$ on $\kappa$ such that $J^+ \longrightarrow [J^+]^2_\kappa$ ([M4]).

Recall that for $S \subseteq \kappa, \diamondsuit^*_\kappa(S)$ means that there are $s_\alpha \in P_{|\alpha|^+}(\alpha)$ for $\alpha \in S$ such that for every $A \subseteq \kappa$, there exists a closed unbounded subset $C$ of $\kappa$ with the property that $A \cap \alpha \in s_\alpha$ for every $\alpha \in C \cap S$.

**PROPOSITION 7.5.-** *Suppose that* $\diamondsuit^*_\kappa(S)$ *holds for some stationary subset* $S$ *of* $\kappa$. *Then* $\mathfrak{d}_\kappa \geq$ $\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\kappa)$ *and* $\overline{\mathfrak{d}}_\kappa \geq \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow [J^+]^2_\kappa)$.

**Proof.** By a result of [M4], the hypothesis implies that $NS^+_\kappa \not\longrightarrow [NS^+_\kappa]^2_\kappa$. □

**Remark.** It is shown in [S] that if (a) $\kappa$ is a successor cardinal $\geq \omega_2$ with $2^{<\kappa} = \kappa$, and (b) setting $\kappa = \nu^+$, $\mu^\tau \leq \nu$ for every infinite cardinal $\mu < \nu$, where $\tau = \aleph_1$ if $cf(\nu) = \omega$ and $\tau = \aleph_0$ otherwise, then there is a stationary subset $S$ of $\kappa$ such that $\diamondsuit^*_\kappa(S)$ holds.

**Remark.** We do not know whether it is consistent that the conclusion of Proposition 7.5 fails. Results of Section 15 (below) imply that

$$\mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\kappa) \leq (\overline{\mathfrak{d}}_\kappa)^{<\kappa}$$

if $\kappa$ is a limit cardinal such that $2^{<\kappa} = \kappa$.

## 8. $\mathfrak{d}^\kappa_{\kappa,\lambda}$

We now start our study of combinatorial properties of ideals on $P_\kappa(\lambda)$. The aim of this section is to present a two-cardinal version of Theorem 2.2.

**Definition.** $\mathfrak{d}^\kappa_{\kappa,\lambda}$ is the least cardinality of any $F \subseteq {}^\kappa(P_\kappa(\lambda))$ with the property that for every $g \in {}^\kappa(P_\kappa(\lambda))$, there is $f \in F$ such that $g(\alpha) \subseteq f(\alpha)$ for all $\alpha \in \kappa$.

**Remark.** It is shown in [MPéS1] that $\mathfrak{d}^\kappa_{\kappa,\lambda} = \mathfrak{d}_\kappa \cdot u(\kappa^+, \lambda)$.

**Definition.** Given an ideal $H$ on $P_\kappa(\lambda)$, $\mathcal{M}^{\geq\kappa}_H$ is the set of all $Q \subseteq H^+$ such that (i) $|Q| \geq \kappa$, (ii) $A \cap B \in H$ for all $A, B \in Q$ with $A \neq B$, and (iii) for every $C \in H^+$, there is $A \in Q$ with $A \cap C \in H^+$. $\mathfrak{a}_H$ is the least cardinality of any member of $\mathcal{M}^{\geq\kappa}_H$ if $\mathcal{M}^{\geq\kappa}_H \neq \phi$, and $2^{(\lambda^{<\kappa})^+}$ otherwise.

The following is proved as Proposition 11.2 of [MP2].

**PROPOSITION 8.1.** *Given a $\kappa$-normal ideal $H$ on $P_\kappa(\lambda)$, the following are equivalent :*

- **(i)** $\mathfrak{a}_H = \kappa$.
- **(ii)** $sat(H) > \kappa$.

**COROLLARY 8.2.** *Let $A \in (NS^\kappa_{\kappa,\lambda})^+$ and set $H = NS^\kappa_{\kappa,\lambda} \mid A$. Then $\mathfrak{a}_H = \kappa$.*

**Proof.** The result follows from Proposition 8.1 since $sat(H) > \kappa$ by a result of Abe [A]. □

The following is proved as Proposition 11.1 (ii) of [MP2].

**PROPOSITION 8.3.** *Given an ideal $H$ on $P_\kappa(\lambda)$, the following are equivalent :*

- **(i)** $\mathfrak{a}_H = \kappa$.
- **(ii)** *There exist $A_\alpha \in H^+$ for $\alpha < \kappa$ such that (a) $A_\alpha \subseteq A_\beta$ whenever $\beta < \alpha < \kappa$, and (b) for every $C \in H^+$, there is $\alpha < \kappa$ such that $C - A_\alpha \in H^+$.*

**Definition.** An ideal $H$ on $P_\kappa(\lambda)$ is a weak $\pi$-point if given $f \in {}^\kappa H$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $B \cap f(\alpha) \in I_{\kappa,\lambda}$ for every $\alpha \in \kappa$.

**THEOREM 8.4.** *Let $H$ be an ideal on $P_\kappa(\lambda)$ such that $cof(H) < \mathfrak{d}^\kappa_{\kappa,\lambda}$. Then $\mathfrak{a}_H > \kappa$ and $H$ is a weak $\pi$-point.*

**Proof.** Let $A_\alpha \in H^+$ for $\alpha < \kappa$ be such that $A_\alpha \subseteq A_\beta$ for all $\beta < \alpha$. Select $X \subseteq H$ so that $|X| = cof(H)$ and $H = \bigcup_{B \in X} P(B)$. For $B \in X$, define $f_B \in {}^\kappa(P_\kappa(\lambda))$ so that $f_B(\alpha) \in A_\alpha - B$. There is $g \in {}^\kappa(P_\kappa(\lambda))$ such that $\{\alpha < \kappa : g(\alpha) \not\subseteq f_B(\alpha)\} \neq \phi$ for every $B \in X$. Set $C = \bigcup_{\alpha\in\kappa} \{a \in A_\alpha : g(\alpha) \not\subseteq a\}$. Then $C \in H^+$, and moreover $C - A_\alpha \in I_{\kappa,\lambda}$ for any $\alpha < \kappa$.

**Definition.** $\overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda}$ is the least cardinality of any $X \subseteq {}^\kappa(P_\kappa(\lambda))$ with the property that for every $g \in {}^\kappa(P_\kappa(\lambda))$, there is $x \in P_\kappa(X)$ such that $g(\alpha) \subseteq \bigcup_{f\in x} f(\alpha)$ for every $\alpha < \kappa$.

**Remark.** It is shown in [MRoS] that $\overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda} = \overline{\mathfrak{d}}_\kappa \cdot \mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$, where $\mathrm{cov}(\lambda, \kappa^+, \kappa^+, \kappa)$ denotes the least cardinality of any $X \subseteq P_{\kappa^+}(\lambda)$ such that for every $b \in P_{\kappa^+}(\lambda)$, there is $x \in P_\kappa(X)$ with $b \subseteq \cup x$.

**Remark.** It is immediate that $I_{\kappa,\lambda}$ is a weak $\pi$-point. On the other hand $\mathfrak{a}_{I_{\kappa,\lambda}} > \kappa$ does not necessarily hold. In fact if $cf(\lambda) \neq \kappa$ and $\overline{\mathfrak{d}}^{\kappa}_{\kappa,\sigma} \leq \lambda$ for every cardinal $\sigma \in [\kappa, \lambda)$, then $\mathfrak{a}_{I_{\kappa,\lambda}} = \kappa$ ([M6]).

**LEMMA 8.5.** *Suppose that $H$ is an ideal on $P_\kappa(\lambda)$ with $\mathfrak{a}_H = \kappa$. Then there is an ideal $K$ on $P_\kappa(\lambda)$ such that (a) $K$ is not a weak $\pi$-point, (b) $cof(K) \leq cof(H)$, and (c) $\overline{cof}(K) \leq \overline{cof}(H)$.*

**Proof.** Select $A_\alpha \in H^+$ for $\alpha < \kappa$ so that ($\alpha$) $A_\alpha \subseteq A_\beta$ whenever $\beta < \alpha < \kappa$, and ($\beta$) for any $C \in H^+$, there is $\alpha < \kappa$ with $C - A_\alpha \in H^+$. Let $K$ be the set of all $B \subseteq P_\kappa(\lambda)$ such that $B \cap A_\alpha \in H$ for some $\alpha < \kappa$. It is simple to check that $K$ is as desired. □

**THEOREM 8.6.**

**(i)** *There is an ideal $H$ on $P_\kappa(\lambda)$ such that (a) $\mathfrak{a}_H = \kappa$, (b) $cof(H) = \mathfrak{d}^{\kappa}_{\kappa,\lambda}$, and* (c) *$\overline{cof}(H) \leq \overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda}$.*

**(ii)** *There is an ideal $K$ on $P_\kappa(\lambda)$ such that (a) $K$ is not a weak $\pi$-point, (b) $cof(K) = \mathfrak{d}^{\kappa}_{\kappa,\lambda}$, and (c) $\overline{cof}(K) \leq \overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda}$.*

**Proof.** (i) : Set $H = NS^{\kappa}_{\kappa,\lambda}$. Then $\mathfrak{a}_H = \kappa$ by Corollary 8.2. Moreover, $cof(H) = \mathfrak{d}^{\kappa}_{\kappa,\lambda}$ ([MPéS1]) and $\overline{cof}(H) = \overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda}$ ([MRoS]).

(ii) : By (i), Lemma 8.5 and Theorem 8.4. □

**Remark.** Theorem 8.6 is not optimal, even under GCH. In fact, suppose that the GCH holds, $\lambda = \sigma^+$, where $\sigma$ is a cardinal of cofinality $< \kappa$, and $\kappa$ is not the successor of a cardinal of cofinality $\leq cf(\sigma)$. Then $\overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda} = \lambda$ ([MRoS]). Moreover, there is $A \in (NS^{\kappa}_{\kappa,\lambda})^+$ such that $\overline{cof}(NS^{\kappa}_{\kappa,\lambda} \mid A) = \sigma$ ([MPéS2]). Hence there is by Corollary 8.2 an ideal $H$ on $P_\kappa(\lambda)$ (namely $H = NS^{\kappa}_{\kappa,\lambda} \mid A$) such that $\overline{cof}(H) < \overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda}$ and $\mathfrak{a}_H = \kappa$, and by Lemma 8.5 an ideal $K$ on $P_\kappa(\lambda)$ such that $\overline{cof}(K) < \overline{\mathfrak{d}}^{\kappa}_{\kappa,\lambda}$ and $K$ is not a weak $\pi$-point.

## 9. Weak $\chi$-pointness

**Definition.** An ideal $H$ on $P_\kappa(\lambda)$ is a weak $\chi$-point if given $A \in H^+$ and $g \in {}^\kappa(P_\kappa(\lambda))$, there is $B \in H^+ \cap P(A)$ such that $g(\cup(a \cap \kappa) \subseteq b$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

Our primary concern in this section is with the problem of determining when $I_{\kappa,\lambda}$ is a weak $\chi$-point. We will first give a sufficient condition and then prove that this condition is necessary if $\kappa$ is inaccessible.

The following is proved as Lemma 2.1 in [M2].

**THEOREM 9.1.** *Let $H$ be an ideal on $P_\kappa(\lambda)$ such that $cof(H) <$ $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. Then $H$ is a weak $\chi$-point.*

**QUESTION.** Is it consistent that $2^{<\kappa} > \kappa$ and $I_{\kappa,\kappa^+}$ is a weak $\chi$-point ?

**THEOREM 9.2.** *Suppose that for all* $A \in I^+_{\kappa,\lambda}$ *with* $A \subseteq \{a : \cup(a \cap \kappa) \in a\}$, *there is* $B \in I^+_{\kappa,\lambda} \cap P(A)$ *such that* $\cup(a \cap \kappa) \in b$ *for all* $a, b \in B$ *with* $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. *Then* $\sigma < \overline{\mathbf{non}}_\kappa$(weakly selective) *for every* $\sigma \in \mathcal{K}(\kappa, \lambda)$.

**Proof.** Suppose that $T \subseteq P_\kappa(\lambda - \kappa)$ is such that $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$, and $J$ is an ideal on $\kappa$ with $\overline{cof}(J) \leq |T|$. Select $D_d \in J$ for $d \in T$ so that for every $W \in J$, there is $u \in P_\kappa(T) - \{\phi\}$ with $W \subseteq \bigcup_{d \in u} D_d$. Now fix $G_\alpha \in J$ for $\alpha < \kappa$. Define $A \subseteq P_\kappa(\lambda)$ by stipulating that $a \in A$ if and only if there is $\delta < \kappa$ such that (a) $\delta = \max(a \cap \kappa)$, (b) $\delta \notin \bigcup_{d \in T \cap P(a)} D_d$, and (c) $\delta \notin G_\alpha$ for every $\alpha \in a \cap \delta$.

Let us show that $A \in I^+_{\kappa,\lambda}$. Given $c \in P_\kappa(\lambda)$, pick $\delta < \kappa$ so that $\delta \notin \bigcup_{d \in T \cap P(c)} D_d$ and for every $\alpha \in c \cap \kappa, \delta > \alpha$ and $\delta \notin G_\alpha$. Set $e = c \cup \{\delta\}$. Then $e \in A$.

By our assumption there is $B \in I^+_{\kappa,\lambda} \cap P(A)$ such that $\cup(a \cap \kappa) \in b$ for all $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Set $C = \{\cup(a \cap \kappa) : a \in B\}$. Then $C \in J^+$. Moreover, $\xi \notin G_\zeta$ for all $\zeta, \xi \in C$ with $\zeta < \xi$. □

We mention the following partial converse to Theorem 9.2.

**PROPOSITION 9.3.** *Suppose that* $2^{<\kappa} = \kappa$ *and* $H$ *is an ideal on* $P_\kappa(\lambda)$ *such that* $cof(H) <$ $\mathbf{non}_\kappa$(weakly selective). *Then for all* $f \in {}^\kappa\kappa$ *and* $A \in H^+$, *there is* $B \in H^+ \cap P(A)$ *such that* $f(\cup(a \cap \kappa)) \subseteq b$ *for all* $a, b \in B$ *with* $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

**Proof.** Fix $f \in {}^\kappa\kappa$ and $A \in H^+$. For $D \subseteq P_\kappa(\kappa)$, set $Z_D = \{a \in P_\kappa(\lambda) : a \cap \kappa \in D\}$. It is simple to see that (a) $Z_{P_\kappa(\kappa)} = P_\kappa(\lambda)$, (b) $Z_{\cup \mathfrak{D}} Z_D$ for $\mathfrak{D} \subseteq P(P_\kappa(\kappa))$, (c) $Z_D \in I_{\kappa,\lambda}$ for every $D \subseteq P_\kappa(\kappa)$ with $|D| = 1$, and (d) $Z_{D'} \subseteq Z_D$ for all $D, D' \subseteq P_\kappa(\kappa)$ such that $D' \subseteq D$. Hence

$$K = \{D \subseteq P_\kappa(\kappa) : Z_D \in H \mid A\}$$

is a $\kappa$-complete ideal on $P_\kappa(\kappa)$. For $C \subseteq P_\kappa(\lambda)$, let $W_C$ be the set of all $d \in P_\kappa(\kappa)$ such that

$$\{a \in P_\kappa(\lambda) : a \cap \kappa = d\} \subseteq C.$$

If $C \in H \mid A$, then $W_C \in K$ since $Z_{W_C} \subseteq C$. Moreover, if $D \subseteq P_\kappa(\kappa)$ and $Z_D \subseteq C \subseteq P_\kappa(\lambda)$, then $D \subseteq W_C$. Hence

$$cof(K) \leq cof(H \mid A) \leq cof(H).$$

For $d \in P_\kappa(\kappa)$, let $S_d$ be the set of all $e \in P_\kappa(\kappa)$ such that $f(\cup d) \not\subseteq e$ or $\cup e \leq \cup d$. Then $S_d \in K$ since

$$\{a \in Z_{S_d} : f(\cup d) \cup \{(\cup d) + 1\} \subseteq a\} = \phi.$$

Select a bijection $\ell : P_\kappa(\kappa) \longrightarrow \kappa$. Since $cof(K) < \mathbf{non}_\kappa$ (weakly selective), there is $D \in K^+$ such that $e \notin S_d$ for all $d, e \in D$ such that $\ell(d) < \ell(e)$. Set

$$B = A \cap Z_D = \{a \in A : a \cap \kappa \in D\}.$$

Then $B \in H^+$. Now fix $a, b \in B$ with $\cup(a \cap \kappa) < \cup(b \cap \kappa)$. Then clearly $\ell(a \cap \kappa) \neq \ell(b \cap \kappa)$. In fact $\ell(a \cap \kappa) < \ell(b \cap \kappa)$ (since otherwise $a \cap \kappa \notin S_{b \cap \kappa}$ and therefore $\cup(a \cap \kappa) > \cup(b \cap \kappa)$). Hence $b \cap \kappa \notin S_{a \cap \kappa}$, so $f(\cup(a \cap \kappa)) \subseteq b \cap \kappa$. □

**Definition.** For $A \subseteq P_\kappa(\lambda)$, let

$$[A]^2_\kappa = \{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } \cup(a \cap \kappa) < \cup(b \cap \kappa)\}$$

**Remark.**

$$[P_\kappa(\lambda)]^2_\kappa = \{(\alpha, b) \in \kappa \times P_\kappa(\lambda) : \alpha < \cup(b \cap \kappa)\}.$$

**Definition.** For $a, b \in P_\kappa(\lambda)$, let $a \prec b$ just in case $a \subseteq b$ and $\cup(a \cap \kappa) < \cup(b \cap \kappa)$.

**Definition.** For $A \subseteq P_\kappa(\lambda)$, let

$$[A]^2_\prec = \{(\cup(a \cap \kappa), b) : a, b \in A \text{ and } a \prec b\}.$$

**Remark.** $[P_\kappa(\lambda)]^2_\prec = [P_\kappa(\lambda)]^2_\kappa$.

**THEOREM 9.4.** *Suppose that* $\kappa$ *is inaccessible and* $H$ *is an ideal on* $P_\kappa(\lambda)$ *such that* $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$, *and let* $A \in H^+$. *Then there is* $C \in H^+ \cap P(A)$ *such that* $[C]^2_\kappa = [C]^2_\prec$.

**Proof.** For $\alpha < \kappa$, set $A_\alpha = \{a \in A : \cup(a \cap \kappa) = \alpha\}$. By induction on $\alpha < \kappa$, we define $c_k \in \{\phi\} \cup A_\alpha$ for $k \in {}^\alpha 2$ as follows. Given $k \in {}^\alpha 2$, set

$$e_k = \bigcup \{c_{k \restriction \beta} : \beta \in k^{-1}(\{1\})\}$$

and

$$Z_k = \{a \in A_\alpha : e_k \subseteq a\}.$$

If $Z_k \neq \phi$, let $c_k$ be an arbitrary member of $Z_k$. Otherwise let $c_k = \phi$.

Set $\nu = cof(H)$ and pick $B_\xi \in H$ for $\xi < \nu$ so that $H = \bigcup_{\xi < \nu} P(B_\xi)$. Let $\xi < \nu$. For $\alpha < \kappa$, let $D^\alpha_\xi$ be the set of all $s \in {}^{(\alpha+1)}2$ such that (i) $s(\alpha) = 1$, and (ii) there is $a \in A_\alpha - B_\xi$ with the property that

$$(\forall \beta \in \alpha \cap s^{-1}(\{1\}))(\forall k \in {}^\beta 2) \quad c_k \subseteq a.$$

Then let $D_\xi = \bigcup_{\alpha < \kappa} D^\alpha_\xi$ and $U_\xi = \bigcup_{s \in D_\xi} O^\kappa_s$. Let us prove that the open set $U_\xi$ is dense. Thus let $\gamma < \kappa$ and $p \in {}^\gamma 2$. Pick $a \in (\bigcup_{\gamma \leq \delta < \kappa} A_\delta) - B_\xi$ so that

$$(\forall \beta \in p^{-1}(\{1\}))(\forall k \in {}^\beta 2) \quad c_k \subseteq a.$$

Set $\alpha = \cup(a \cap \kappa)$ and define $s \in {}^{(\alpha+1)}2$ by : $s \restriction \gamma = p$, $s(\delta) = 0$ if $\gamma \leq \delta < \alpha$, and $s(\alpha) = 1$. It is immediate that $s \in D^\alpha_\xi$.

Select $f \in \bigcap_{\xi < \nu} U_\xi$. For each $\xi < \nu$, there is $s_\xi \in D_\xi$ such that $s_\xi \subset f$. Let $\alpha_\xi < \kappa$ be such that $s_\xi \in D^{\alpha_\xi}_\xi$. Set $T = \{\alpha_\xi : \xi < \nu\}$ and define $g \in {}^\kappa 2$ so that $g^{-1}(\{1\}) = T$. For $\xi < \nu$, set

$$d_\xi = \bigcup \{c_{g \restriction \beta} : \beta \in T \cap \alpha_\xi\}$$

and

$$C_\xi = \{b \in A_{\alpha_\xi} : d_\xi \subseteq b\}.$$

Finally, let $C = \bigcup_{\xi < \nu} C_\xi$.

Let us verify that $C$ is as desired. It is clear that $C \subseteq A$. Let $\xi < \nu$. There is $a_\xi \in A_{\alpha_\xi} - B_\xi$ such that

$$(\forall \beta \in \alpha_\xi \cap s_\xi^{-1}(\{1\}))(\forall k \in {}^\beta 2) \quad c_k \subseteq a_\xi.$$

Put $k_\xi = g \restriction \alpha_\xi$. Then $a_\xi \in Z_{k_\xi}$ since $s_\xi(\beta) = f(\beta) = 1$ for every $\beta \in T \cap \alpha_\xi$. It follows that $c_{k_\xi} \in Z_{k_\xi}$. It is immediate that $Z_{k_\xi} = C_\xi$. Thus we have shown that (a) $C_\xi - B_\xi \neq \phi$ for every $\xi < \nu$, and (b) $c_{g \restriction \alpha_\xi} \in C_\xi$ for every $\xi < \nu$. It follows from (a) that $C \in H^+$, and from (b) that $[C]^2_\kappa = [C]^2_\prec$ since given $\xi, \zeta < \nu$ with $\alpha_\xi < \alpha_\zeta$, we have $c_{g \restriction \alpha_\xi} \subseteq b$ for every $b \in C_\zeta$. □

**QUESTION.** Is the assumption that $\kappa$ is inaccessible necessary in the statement of Theorem 9.4 ?

**Remark.** Suppose $\kappa$ is inaccessible. Then by Theorems 9.1, 9.2, 9.4, 5.4 and 4.7, $I_{\kappa,\lambda}$ is a weak $\chi$-point if and only if $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$ if and only if $\{C : [C]^2_\kappa = [C]^2_\prec\}$ is dense in $(I^+_{\kappa,\lambda}, \subseteq)$.

## 10. $H^+ \overset{\kappa}{\longrightarrow} (H^+, \alpha)^2$

**Definition.** Let $H$ be an ideal on $P_\kappa(\lambda)$ and $\alpha$ an ordinal. $H^+ \overset{\kappa}{\longrightarrow} (H^+, \alpha)^2$ means that given $F : [P_\kappa(\lambda)]^2_\kappa \longrightarrow 2$ and $A \in H^+$, there is $B \subseteq A$ such that either $B \in H^+$ and $F$ is identically 0 on $[B]^2_\kappa$ or $(B, \prec)$ has order type $\alpha$ and $F$ is identically 1 on $[B]^2_\kappa$.

In this section we show that $H^+ \overset{\kappa}{\longrightarrow} (H^+, \omega+1)^2$ for every ideal $H$ on $P_\kappa(\lambda)$ with $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$.

**Definition.** Suppose that $H$ is an ideal on $P_\kappa(\lambda), A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$. Then $(H, A, F)$ is 0-good if there is $D \in H^+ \cap P(A)$ such that $\{b \in D : F(\cup(a \cap \kappa), b) = 1\} \in H$ for any $a \in D$.

The following is straightforward.

**LEMMA 10.1.** *Suppose that $(H, A, F)$ is 0-good, where $H$ is an ideal on $P_\kappa(\lambda)$ which is both a weak $\pi$-point and a weak $\chi$-point, $A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$. Then $F$ is identically 0 on $[C]^2_\kappa$ for some $C \in H^+ \cap P(A)$.*

**Definition.** Given an ideal $H$ on $P_\kappa(\lambda)$ and $B \in H^+$, let $M^{\mathrm{d}}_{H,B}$ be the set of all $Q \subseteq H^+ \cap P(B)$ such that (i) any two distinct members of $Q$ are disjoint, and (ii) for every $A \in H^+ \cap P(B)$, there is $C \in Q$ with $A \cap C \in H^+$.

**LEMMA 10.2.** *Suppose that $(H, A, F)$ is not 0-good, where $H$ is an ideal on $P_\kappa(\lambda)$, $A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$, and let $B \in H^+ \cap P(A)$. Then there exist $Q_B \in M^{\mathrm{d}}_{H,B}$ and $\varphi_B : Q_B \longrightarrow B$ such that (i) $\varphi_B(D) \prec b$ and $F(\cup(\varphi_B(D) \cap \kappa), b) = 1$ whenever $b \in D \in Q_B$, and (ii) $\cup(\varphi_B(D) \cap \kappa) \neq \cup(\varphi_B(D') \cap \kappa)$ for any two distinct members $D$ and $D'$ of $Q_B$.*

**Proof.** Set $T = \{\cup(a \cap \kappa) : a \in B\}$ and define $\psi : T \times (H^+ \cap P(B)) \to P(B)$ by $\psi(\alpha, C) = \{b \in C : F(\alpha, b) = 1\}$. Now using induction, define $\eta \leq \kappa$ and $\alpha_\delta \in T$ and $B_\delta \in H^+ \cap P(B)$ for $\delta < \eta$ so that :

(0) If $\delta < \eta$, $B - (\bigcup_{\xi<\delta} B_\xi) \in H^+$,

$$\alpha_\delta = \text{ least } \alpha \in T \text{ such that } \psi(\alpha, B - (\bigcup_{\xi<\delta} B_\xi)) \in H^+$$

and $B_\delta = \psi(\alpha_\delta, B - (\bigcup_{\xi<\delta} B_\xi))$.

(1) If $\eta < \kappa$, $B - (\bigcup_{\xi<\eta} B_\xi) \in H$.

Notice that if $\gamma < \delta < \eta$, then

$$\psi(\alpha_\delta, B - (\bigcup_{\xi<\delta} B_\xi)) \subseteq \psi(\alpha_\delta, B - (\bigcup_{\zeta<\delta} B_\zeta))$$

and consequently $\alpha_\gamma \leq \alpha_\delta$. In fact $\alpha_\gamma < \alpha_\delta$ as $\psi(\alpha_\gamma, B - (\bigcup_{\xi<\delta} B_\xi)) = \phi$ (since $(B - \bigcup_{\xi<\delta} B_\xi)) \cap B_\gamma = \phi$ and $B_\gamma = \{b \in B - (\bigcup_{\zeta<\gamma} B_\zeta) : F(\alpha_\gamma, b) = 1\}$).

We claim that $\{B_\delta : \delta < \eta\} \in M^{\mathrm{d}}_{H,B}$. Suppose otherwise. Then there exists $E \in H^+ \cap P(B)$ such that $E \cap B_\xi \in H$ for every $\xi < \eta$. Since

$$E - (\bigcup_{\xi<\delta} B_\xi) \in H^+ \cap P(B - (\bigcup_{\xi<\delta} B_\xi))$$

for every $\delta < \kappa$, we must have $\eta = \kappa$. Set

$$\beta = \text{ least } \alpha \in T \text{ such that } \psi(\alpha, E) \in H^+.$$

Then for each $\delta < \kappa$,

$$\psi(\beta, E) - (\bigcup_{\xi<\delta} B_\xi) \in H^+ \cap P(\psi(B, B - (\bigcup_{\xi<\delta} B_\xi)))$$

and therefore $\beta \geq \alpha_\delta$, which is a contradiction.

For each $\delta < \eta$, pick $s_\delta \in B$ so that $\cup(s_\delta \cap \kappa) = \alpha_\delta$, and put

$$S_\delta = \{b \in B_\delta : s_\delta \cup (\alpha_\delta + 2) \subseteq b\}.$$

Finally, set $Q_B = \{S_\delta : \delta < \eta\}$ and define $\varphi_B : Q_B \longrightarrow B$ by $\varphi_B(S_\delta) = s_\delta$. □

**LEMMA 10.3.** *Suppose that $H$ is an ideal on $P_\kappa(\lambda)$ and $A \in H^+$. Suppose further that $C \in H^+ \cap P(A)$ and $Q_\alpha \in M^{\mathrm{d}}_{H,A}$ for $\alpha < \beta$, where $\beta$ is a limit ordinal with $0 < \beta < \kappa$. Then*

$$\{a \in C : (\forall h \in \prod_{\alpha<\beta} Q_\alpha) \quad a \notin \bigcap_{\alpha<\beta} h(\alpha)\} \in H.$$

**Proof.** It suffices to observe that for each $a \in \bigcap_{\alpha<\beta} (C \cap (\cup Q_\alpha))$, there is $h \in \prod_{\alpha<\beta} Q_\alpha$ such that $a \in \bigcap_{\alpha<\beta} h(\alpha)$. □

**LEMMA 10.4.** *Suppose that $(H, A, F)$ is not 0-good, where $H$ is an ideal on $P_\kappa(\lambda)$, $A \in H^+$ and $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$. Then :*

- **(i)** *There is $C \subseteq A$ such that $(C, \prec)$ has order type $\omega + 1$ and $F$ is identically 1 on $[C]^2_\kappa$.*
- **(ii)** *Suppose that $\mathfrak{a}_H > \kappa$ and $\theta$ is uncountable cardinal $< \kappa$ such that $\kappa \longrightarrow (\kappa, \theta)^2$. Then there is $C \subseteq A$ such that $(C, \prec)$ has order type $\theta + 1$ and $F$ is identically 1 on $[C]^2_\kappa$.*

**Proof.** We prove (ii) and leave the proof of (i) to the reader. By Corollary 19.7 in [EHMáR], we have that $\mu^\tau < \kappa$ whenever $\mu$ and $\tau$ are cardinals such that $\theta \leq \mu < \kappa$ and $0 < \tau < \theta$. Using this and Lemmas 10.2 and 10.3, define $R_\beta, Q_\beta \in \{W \in M^{\mathrm{d}}_{H,A} : |W| < \kappa\}$ and $\varphi_\beta : Q_\beta \longrightarrow A$ for $\beta < \theta$ by :

(0) $R_0 = \{A\}$ ;

(1) $Q_\beta = \bigcup_{B \in R_\beta} Q_B$ ;

(2) $R_{\beta+1} = Q_\beta$ ;

(3) $R_\beta = H^+ \cap \{\bigcap_{\alpha<\beta} h(\alpha) : h \in \prod_{\alpha<\beta} Q_\alpha\}$ if $\beta$ is a limit ordinal $> 0$ ;

(4) $\quad \varphi_\beta = \bigcup_{B\in R_\beta} \varphi_B$.

Select $b \in \bigcap_{\beta<\theta} (\cup Q_\beta)$. There must be $k \in \prod_{\beta<\theta} Q_\beta$ such that $b \in \bigcap_{\beta<\theta} k(\beta)$. Then

$$C = \{\varphi_\beta(k(\beta)) : \beta < \theta\} \cup \{b\}$$

is as desired. $\square$

**THEOREM 10.5.** *Suppose $\theta$ is an infinite cardinal $< \kappa$ such that $\kappa \longrightarrow (\kappa, \theta)^2$. Then $H^+ \overset{\kappa}{\longrightarrow} (H^+, \theta+1)^2$ for every ideal $H$ on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$.*

**Proof.** Let $H$ be an ideal on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. Then $H$ is a weak $\chi$-point by Theorem 9.1. Moreover, $H$ is a weak $\pi$-point and $\mathfrak{a}_H > \kappa$ by Theorem 8.4 since $\mathbf{cov}(\mathbf{M}_{\kappa,\kappa}) \leq \mathfrak{d}^\kappa_{\kappa,\lambda}$ by Proposition 4.1. Hence, $H^+ \overset{\kappa}{\longrightarrow} (H^+, \theta+1)^2$ by Lemmas 10.1 and 10.4. $\square$

## 11. $H^+ \underset{\kappa}{\overset{\kappa}{\longrightarrow}} (H^+, \alpha)^2$

**Definition.** For $A \subseteq P_\kappa(\lambda)$, let

$$[A]^2_{\kappa,\kappa} = \{(\cup(a\cap\kappa), \cup(b\cap\kappa)) : a, b \in A \text{ and } \cup(a\cap\kappa) < \cup(b\cap\kappa)\}$$

**Remark.** $[P_\kappa(\lambda)]^2_{\kappa,\kappa} = [\kappa]^2$.

**Definition.** Let $H$ be an ideal on $P_\kappa(\lambda)$ and $\alpha$ an ordinal. $H^+ \underset{\kappa}{\overset{\kappa}{\longrightarrow}} (H^+, \alpha)^2$ means that given $F : [P_\kappa(\lambda)]^2_{\kappa,\kappa} \longrightarrow 2$ and $A \in H^+$, there is $B \subseteq A$ such that either $B \in H^+$ and $F$ is identically 0 on $[B]^2_{\kappa,\kappa}$, or $(B, \prec)$ has order type $\alpha$ and $F$ is identically 1 on $[B]^2_{\kappa,\kappa}$.

We will show that $H^+ \underset{\kappa}{\overset{\kappa}{\longrightarrow}} (H^+, \omega+1)^2$ for every ideal $H$ on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{non}_\kappa$(weakly selective).

**Definition.** For an ideal $H$ on $P_\kappa(\lambda)$, $J_H = \{B \subseteq \kappa : U_B \in H\}$, where

$$U_B = \{a \in P_\kappa(\lambda) : \cup(a\cap\kappa) \in B\}.$$

**LEMMA 11.1.** *Let $H$ be an ideal on $P_\kappa(\lambda)$. Then $J_H$ is an ideal on $\kappa$. Moreover, $cof(J_H) \leq cof(H)$.*

**Proof.** It is simple to see that (a) $U_\kappa = P_\kappa(\lambda)$, (b) $U_{\cup\mathfrak{B}} \subseteq \bigcup_{B\in\mathfrak{B}} U_B$ for $\mathfrak{B} \subseteq P(\kappa)$, (c) $U_C \subseteq U_B$ if $C \subseteq B \subseteq K$, and (d) $U_B \in I_{\kappa,\lambda}$ for every $B \subseteq \kappa$ with $|B| = 1$. The first assertion immediately follows.

For $C \subseteq P_\kappa(\lambda)$, let $Y_C$ be the set of all $\delta \in \kappa$ such that

$$\{a \in P_\kappa(\lambda) : \cup(a\cap\kappa) = \delta\} \subseteq C.$$

If $C \in H$, then $Y_C \in J_H$ since $U_{Y_C} \subseteq C$. Moreover if $B \subseteq \kappa$ and $U_B \subseteq C \subseteq P_\kappa(\lambda)$, then $B \subseteq Y_C$. Hence $cof(J_H) \leq cof(H)$. $\square$

**Remark.** Let $H$ be an ideal on $P_\kappa(\lambda)$. Then

$$\{\cup(a\cap\kappa) : a \in A\} \in (J_{H|A})^+$$

for every $A \in H^+$.

The following is readily checked.

**LEMMA 11.2.** *Given an ideal $H$ on $P_\kappa(\lambda)$, the following are equivalent :*

**(i)** *$J_H$ is a local $Q$-point.*

**(ii)** *For every $g \in {}^\kappa\kappa$, there is $B \in H^+$ such that $g(\cup(a\cap\kappa)) < \cup(b\cap\kappa)$ for all $a, b \in B$ with $\cup(a\cap\kappa) < \cup(b\cap\kappa)$.*

Suppose $\kappa$ is a limit cardinal. If $\kappa^+ < \mathbf{non}_\kappa$(weak $Q$-point), then by Lemma 11.1 $J_{I_{\kappa,\kappa^+}|A}$ is a local $Q$-point for every $A \in I^+_{\kappa,\kappa^+}$. The following shows that this implication can be reversed.

**PROPOSITION 11.3.** *Suppose that $\kappa$ is a limit cardinal and $J_{I_{\kappa,\lambda}|A}$ is a local $Q$-point for every $A \in I^+_{\kappa,\lambda}$. Then $\sigma < \overline{\mathbf{non}}_\kappa$(weak $Q$-point) for every $\sigma \in \mathcal{K}(\kappa,\lambda)$.*

**Proof.** Suppose that $J$ is an ideal on $\kappa$ and $T \subseteq P_\kappa(\lambda - \kappa)$ is such that $\overline{cof}(J) \leq |T|$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$. Select $B_d \in J$ for $d \in T$ so that for every $D \in J$, there is $u \in P_\kappa(T) - \{\phi\}$ with $D \subseteq \bigcup_{d\in u} B_d$. Let $A$ be the set of all $a \in P_\kappa(\lambda)$ such that $\cup(a\cap\kappa) \notin B_d$ for every $d \in T \cap P(a-\kappa)$. It is simple to see that $A \in I^+_{\kappa,\lambda}$. Now fix $g \in {}^\kappa\kappa$. By Lemma 11.2, there is $C \in (I_{\kappa,\lambda} \mid A)^+$ such that $g(\cup(a\cap\kappa)) < \cup(b\cap\kappa)$ for all $a, b \in C$ with $\cup(a\cap\kappa) < \cup(b\cap\kappa)$. Set

$$D = \{\cup(a\cap\kappa) : a \in C \cap A\}.$$

Then $D \in J^+$. Moreover $g(\alpha) < \beta$ for all $\alpha, \beta \in D$ with $\alpha < \beta$. Hence $J$ is a local $Q$-point. □

**THEOREM 11.4.** *Suppose that $\theta$ is an infinite cardinal $< \kappa$ such that $\kappa \longrightarrow (\kappa,\theta)^2$, and $H$ is an ideal on $P_\kappa(\lambda)$ with $cof(H) < \mathbf{non}_\kappa$(weakly selective). Then $H^+ \xrightarrow[\kappa]{\kappa} (H^+, \theta+1)^2$.*

**Proof.** Fix $G : \kappa \times \kappa \longrightarrow 2$ and $A \in H^+$. Define $F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$ by $F(\alpha, b) = G(\alpha, \cup(b\cap\kappa))$.

First suppose $(H, A, F)$ is 0-good. Pick $D \in H^+ \cap P(A)$ so that

$$\{b \in D : F(\cup(a\cap\kappa), b) = 1\} \in H$$

for any $a \in D$. Set $B_\alpha = \{\delta < \kappa : G(\alpha,\delta) = 1\}$ for $\alpha < \kappa$. Then $B_{\cup(a\cap\kappa)} \in J_{H|D}$ for every $a \in D$ since

$$D \cap U_{B_{\cup(a\cap\kappa)}} = \{b \in D : G(\cup(a\cap\kappa), \cup(b\cap\kappa)) = 1\} = \{b \in D : F(\cup(a\cap\kappa), b) = 1\}.$$

By Lemma 11.1 $cof(J_{H|D}) < \mathbf{non}_\kappa$(weak $P$-point) so there is $G \in (J_{H|D})^+$ such that $|G \cap B_{\cup(a\cap\kappa)}| < \kappa$ for every $a \in D$. Notice that $D \cap U_G \in H^+$. Select $g \in {}^\kappa\kappa$ so that $\cup(b\cap\kappa) \notin B_{\cup(a\cap\kappa)}$ for all $a, b \in D \cap U_G$ such that $g(\cup(a\cap\kappa)) < \cup(b\cap\kappa)$. By Lemma 11.1

$$cof(J_{H|(D\cap U_G)}) < \mathbf{non}_\kappa(\text{weak } Q\text{-point})$$

and hence by Lemma 11.2 there is $R \in (H \mid (D \cap U_G))^+$ such that $g(\cup(a\cap\kappa)) < \cup(b\cap\kappa)$ for all $a, b \in R$ with $\cup(a\cap\kappa) < \cup(b\cap\kappa)$. Then $R \cap D \cap U_G \in H^+ \cap P(A)$ and moreover $F$ is identically 0 on $[R \cap D \cap U_G]^2_{\kappa,\kappa}$.

Finally, suppose $(H, A, F)$ is not 0-good. Since $\mathfrak{a}_H > \kappa$ by Theorems 2.2 and 8.4, there is by Lemma 10.4 $C \subseteq A$ such that $(C, \prec)$ has order type $\theta + 1$ and $F$ is identically 1 on $[C]^2_\kappa$. It is immediate that $G$ is constantly 1 on $[C]^2_{\kappa,\kappa}$. $\square$

**Remark.** Suppose $\kappa$ is a successor cardinal. Then by Theorem 11.4 $\kappa^+ < \mathfrak{d}_\kappa$ implies that $I^+_{\kappa,\kappa^+} \xrightarrow[\kappa]{\kappa} (I^+_{\kappa,\kappa^+}, \theta + 1)^2$ for every cardinal $\theta \geq 2$ such that $\kappa \longrightarrow (\kappa, \theta)^2$. Conversely, it will be shown in the next section that $I^+_{\kappa,\kappa^+} \xrightarrow[\kappa]{\kappa} (I^+_{\kappa,\kappa^+}, 3)^2$ implies that $\kappa^+ < \mathfrak{d}_\kappa$.

## 12. $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$

**Definition.** Given an ideal $H$ on $P_\kappa(\lambda)$ and an ordinal $\alpha, H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$ means that for all $F : [P_\kappa(\lambda)]^2_{\kappa,\kappa} \longrightarrow 2$ and $A \in H^+$, there is $B \subseteq A$ such that either $B \in H^+$ and $F$ is identically 0 on $[B]^2_{\kappa,\kappa}$, or $\{\cup(a \cap \kappa) : a \in B\}$ has order type $\alpha$ and $F$ is identically 1 on $[B]^2_{\kappa,\kappa}$.

**Remark.** $H^+ \xrightarrow{\kappa} (H^+, \alpha)^2 \Rightarrow H^+ \xrightarrow[\kappa]{\kappa} (H^+, \alpha)^2 \Rightarrow H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2 \Rightarrow \kappa \longrightarrow (\kappa, \alpha)^2$.

We will prove that $I^+_{\kappa,\kappa} \xrightarrow[\kappa]{\kappa} (I^+_{\kappa,\kappa^+}; \alpha)^2$ if and only if $\kappa^+ < \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \alpha)^2)$.

**THEOREM 12.1.** *Suppose that* $3 \leq \alpha \leq \kappa$ *and* $H$ *is an ideal on* $P_\kappa(\lambda)$ *such that* $cof(H) < \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \alpha)^2)$. *Then* $H^+ \xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$.

**Proof.** By Lemma 11.1, $(J_{H|A})^+ \longrightarrow ((J_{H|A})^+, \alpha)^2$ for every $A \in H^+$. The desired conclusion easily follows. $\square$

**THEOREM 12.2.** *Suppose that* $3 \leq \alpha \leq \kappa$ *and* $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{\kappa} (I^+_{\kappa,\lambda}; \alpha)^2$. *Then* $\sigma < \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow (J^+, \alpha)^2)$ *for every* $\sigma \in \mathcal{K}(\kappa, \lambda)$.

**Proof.** The proof is an easy modification of that of Proposition 11.3. $\square$

**Remark.** Suppose that $\kappa$ is inaccessible and $3 \leq \alpha \leq \kappa$. Then by Theorems 12.1 and 12.2, $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{\kappa} (I^+_{\kappa,\lambda}; \alpha)^2$ if and only if $\lambda^{<\kappa} < \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \alpha)^2)$.

Let us finally observe that for $3 \leq \alpha \leq \kappa$, there always exists an ideal $H$ on $P_\kappa(\lambda)$ of the least possible cofinality such that $H^+ \not\xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$ :

**PROPOSITION 12.3.** *Given* $3 \leq \alpha \leq \kappa$, *there is an ideal* $H$ *on* $P_\kappa(\lambda)$ *such that* (a) $H^+ \not\xrightarrow[\kappa]{\kappa} (H^+; \alpha)^2$, (b) $cof(H) = u(\kappa, \lambda) \cdot \mathbf{non}_\kappa(J^+ \longrightarrow (J^+, \alpha)^2)$, *and* (c) $\overline{cof}(H) \leq \lambda \cdot \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow (J^+, \alpha)^2)$.

**Proof.** Argue as for Lemma 5.1 of [M2]. $\square$

## 13. $H^+ \xrightarrow{\kappa} (H^+)^2$

**Definition.** Given an ideal $H$ on $P_\kappa(\lambda)$, $H^+ \xrightarrow{\kappa} (H^+)^2$ (respectively, $H^+ \xrightarrow[\kappa]{\kappa} (H^+)^2$) means that for all $F : [P_\kappa(\lambda)]^2_\kappa \longrightarrow 2$ (respectively, $F : [P_\kappa(\lambda)]^2_{\kappa,\kappa} \longrightarrow 2$) and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $F$ is constant on $[B]^2_\kappa$ (respectively, $[B]^2_{\kappa,\kappa}$).

**THEOREM 13.1.** *Suppose $\kappa$ is weakly compact. Then $H^+ \xrightarrow{\kappa} (H^+)^2$ for every ideal $H$ on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$.*

**Proof.** Suppose that $H$ is an ideal on $P_\kappa(\lambda)$ with $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa}), F : \kappa \times P_\kappa(\lambda) \longrightarrow 2$ and $A \in H^+$. Then $cof(H) < \mathfrak{d}^\kappa_{\kappa,\lambda}$ by Proposition 4.1 and therefore by a result of [M5] there are $B \in H^+ \cap P(A)$ and $i < 2$ such that

$$\{b \in B : F(\cup(a \cap \kappa), b) \neq i\} \in I_{\kappa,\lambda}$$

for every $a \in B$. Since $H$ is a weak $\chi$-point by Theorem 9.1, there is $C \in H^+ \cap P(B)$ such that $F$ takes the constant value $i$ on $[C]^2_\kappa$. □

**Remark.** It follows from Theorem 6.5 (ii) and Theorem 15.1 (below) that if $\kappa$ is weakly compact, then $H^+ \xrightarrow[\kappa]{\kappa} (H^+)^2$ for every ideal $H$ on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{non}_\kappa$(weakly selective).

**COROLLARY 13.2.** *The following are equivalent :*

- **(i)** $\kappa$ is weakly compact and $\lambda^{<\kappa} < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$.
- **(ii)** $I^+_{\kappa,\lambda} \xrightarrow{\kappa} (I^+_{\kappa,\lambda})^2$.
- **(iii)** $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{\kappa} (I^+_{\kappa,\lambda}; \kappa)^2$.

**Proof.** (i) $\to$ (ii) : By Theorem 13.1.
(ii) $\to$ (iii) : Trivial.
(iii) $\to$ (i) : By Theorems 12.2, 6.5 (i), 6.1 (iii), 5.4 and 4.7. □

## 14. $H^+ \xrightarrow{\kappa} [H^+]^2_\rho$

**Definition.** Given a cardinal $\rho$ with $2 \le \rho \le \lambda^{<\kappa}$ and an ideal $H$ on $P_\kappa(\lambda), H^+ \xrightarrow{\kappa} [H^+]^2_\rho$ means that for all $F : [P_\kappa(\lambda)]^2_\kappa \longrightarrow \rho$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $F''[B]^2_\kappa \neq \rho$.

**THEOREM 14.1.** *Suppose that $\kappa$ is a limit cardinal and $H$ is an ideal on $P_\kappa(\lambda)$ such that $cof(H) < \mathbf{cov}(\mathbf{M}_{\kappa,\kappa})$. Then $H^+ \xrightarrow{\kappa} [H^+]^2_{\kappa^+}$.*

**Proof.** Fix $F : \kappa \times P_\kappa(\lambda) \longrightarrow \kappa^+$ and $A \in H^+$. Since $cof(H) < \mathfrak{d}^\kappa_{\kappa,\lambda}$ by Proposition 4.1, there are $B \in H^+ \cap P(A)$ and $\xi \in \kappa^+$ such that $\{b \in B : F(\cup(a \cap \kappa), b) = \xi\} \in I_{\kappa,\lambda}$ for every $a \in B$ ([M5]). Now $H$ is a weak $\chi$-point by Theorem 9.1 and so $\xi \notin F''[C]^2_\kappa$ for some $C \in H^+ \cap P(B)$. □

Let us now show that $I^+_{\kappa,\lambda} \overset{\kappa}{\not\longrightarrow} [I^+_{\kappa,\lambda}]^2_\lambda$ if $\lambda \ge \overline{\mathfrak{d}}_\kappa$. We will need some definitions.

**Definition.** Given $f \in \prod_{\alpha \in \kappa} (\kappa - \alpha)$, we define $\widetilde{f} \in {}^\kappa\kappa$ by stipulating that

(i) $\widetilde{f}(0)=0$ ;

(ii) $\widetilde{f}(\xi+1)=f(\widetilde{f}(\xi))+1$ ;

(iii) $\widetilde{f}(\xi)=\bigcup_{\zeta<\xi}\widetilde{f}(\zeta)$ if $\xi$ is a limit ordinal $>0$.

**Remark.** $\widetilde{f}$ is a strictly increasing function.

**Remark.** If $g\in{}^{\kappa}\kappa$ is a strictly increasing function such that $g(\alpha)\leq f(\alpha)$ for all $\alpha<\kappa$, then $g(\widetilde{f}(\xi))\in[\widetilde{f}(\xi),\widetilde{f}(\xi+1))$ for every $\xi<\kappa$.

**Definition.** Given $f\in\prod_{\alpha\in\kappa}(\kappa-\alpha)$ and a cardinal $\tau\in(0,\kappa)$, we define $c_{f,\tau}:\widetilde{f}(\tau)\longrightarrow\tau$ by stipulating that $c_{f,\tau}$ takes the constant value $\xi$ on $[\widetilde{f}(\xi),\widetilde{f}(\xi+1))$.

**Definition.** Suppose that $T\subseteq P_\kappa(\lambda-\kappa)$ is such that (a) $|T|\geq\overline{\mathfrak{d}}_\kappa$, and (b) $|T\cap P(a)|<\kappa$ for every $a\in P_\kappa(\lambda)$.
Let $\psi_T:T\longrightarrow{}^{\kappa}\kappa$ be such that given $g\in{}^{\kappa}\kappa$, there is $u\in P_\kappa(T)-\{\phi\}$ such that

$$g(\alpha)\leq\bigcup_{d\in u}(\psi_T(d))(\alpha)$$

for all $\alpha<\kappa$.
For $e\in P_\kappa(\lambda-\kappa)$, let $\tau_{T,e}=|T\cap P(e)|$ and select a bijection $k_{T,e}:\tau_{T,e}\longrightarrow T\cap P(e)$.
Also, define $f_{T,e}\in{}^{\kappa}\kappa$ by

$$f_{T,e}(\alpha)=\max(\alpha,\bigcup_{d\in T\cap P(e)}(\psi_T(d))(\alpha)).$$

Finally, let $A_T$ be the set of all $a\in P_\kappa(\lambda)$ such that (i) $T\cap P(a-\kappa)\neq\phi$, and (ii) $\cup(a\cap\kappa)\geq\widetilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})$.

**Remark.** $A_T\in I^+_{\kappa,\lambda}$.

**THEOREM 14.2.** *Suppose that* $\rho\in\mathcal{K}(\kappa,\lambda)$ *and* $\rho\geq\overline{\mathfrak{d}}_\kappa$. *Then* $I^+_{\kappa,\lambda}\overset{\kappa}{\not\longrightarrow}[I^+_{\kappa,\lambda}]^2_\rho$.

**Proof.** Select $T\subseteq P_\kappa(\lambda-\kappa)$ so that $|T|=\rho$ and $|T\cap P(a)|<\kappa$ for every $a\in P_\kappa(\lambda)$. We define a partial function $F$ from $\kappa\times A_T$ to $T$ by stipulating that

$$F(\beta,a)=k_{T,a-\kappa}(c_{f_{T,a-\kappa},\tau_{T,a-\kappa}}(\beta))$$

if $a\in A_T$ and $\beta<\widetilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})$.

Now fix $B\in I^+_{\kappa,\lambda}\cap P(A_T)$ and $x\in T$. Let $g\in{}^{\kappa}\kappa$ be the increasing enumeration of the elements of the set $\{\cup(b\cap\kappa):b\in B\}$. Select $u\in P_\kappa(T)-\{\phi\}$ so that $g(\alpha)\leq\bigcup_{d\in u}(\psi_T(d))(\alpha)$ for all $\alpha<\kappa$. Now pick $a\in B$ so that $x\cup(\cup u)\subseteq a$. Notice that $g(\alpha)\leq f_{T,a-\kappa}(\alpha)$ for every $\alpha\in\kappa$. Let $\xi\in\tau_{T,a-\kappa}$ be such that $k_{T,a-\kappa}(\xi)=x$. Then

$$\widetilde{f}_{T,a-\kappa}(\xi)\leq g(\widetilde{f}_{T,a-\kappa}(\xi))<\widetilde{f}_{T,a-\kappa}(\xi+1)\leq\widetilde{f}_{T,a-\kappa}(\tau_{T,a-\kappa})\leq\cup(a\cap\kappa).$$

Moreover,

$$F(g(\widetilde{f}_{T,a-\kappa}(\xi)),a)=k_{T,a-\kappa}(\xi)=x.$$

since

$$c_{f_{T,a-\kappa},\tau_{T,a-\kappa}}(g(\widetilde{f}_{T,a-\kappa}(\xi)))=\xi$$

□

## 15. $H^+ \xrightarrow[\kappa]{\kappa} [H^+]^2_\rho$

**Definition.** Given a cardinal $\rho \in [2, \kappa]$ and an ideal $H$ on $P_\kappa(\lambda)$, $H^+ \xrightarrow[\kappa]{\kappa} [H^+]^2_\rho$ means that for all $F : [P_\kappa(\lambda)]^2_{\kappa,\kappa} \longrightarrow \rho$ and $A \in H^+$, there is $B \in H^+ \cap P(A)$ such that $F''[B]^2_{\kappa,\kappa} \neq \rho$.

**Remark.** $\kappa \not\longrightarrow [\kappa]^2_\rho \Rightarrow H^+ \not\xrightarrow[\kappa]{\kappa} [H^+]^2_\rho \Rightarrow H^+ \not\xrightarrow{\kappa} [H^+]^2_\rho$.

The following result shows that $I^+_{\kappa,\kappa^+} \xrightarrow[\kappa]{\kappa} [I^+_{\kappa,\kappa^+}]^2_\rho$ if and only if $\kappa^+ < \mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\rho)$.

**THEOREM 15.1.** *Let* $\rho$ *be a cardinal with* $2 \leq \rho \leq \kappa$. *Then :*

**(i)** $H^+ \xrightarrow[\kappa]{\kappa} [H^+]^2_\rho$ *for every ideal* $H$ *on* $P_\kappa(\lambda)$ *such that* $cof(H) < \mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\rho)$.

**(ii)** *If* $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{\kappa} [I^+_{\kappa,\lambda}]^2_\rho$, *then* $\sigma < \overline{\mathbf{non}}_\kappa(J^+ \longrightarrow [J^+]^2_\rho)$ *for every* $\sigma \in \mathcal{K}(\kappa, \lambda)$.

**Proof.** (i) : Use Lemma 11.1.
(ii) : Argue as for Proposition 11.3. □

**Remark.** Thus assuming $\kappa$ is inaccessible, $I^+_{\kappa,\lambda} \xrightarrow[\kappa]{\kappa} [I^+_{\kappa,\lambda}]^2_\rho$ if and only if $\lambda^{<\kappa} < \mathbf{non}_\kappa(J^+ \longrightarrow [J^+]^2_\rho)$.

Finally, we show that if $\lambda \geq \overline{\mathfrak{d}}_\kappa$ and $\kappa$ is a limit cardinal such that $2^{<\kappa} = \kappa$, then $I^+_{\kappa,\lambda} \not\xrightarrow[\kappa]{\kappa} [I^+_{\kappa,\lambda}]^2_\kappa$.

**THEOREM 15.2.** *Suppose that (a)* $\kappa$ *is a limit cardinal such that* $2^{<\kappa} = \kappa$, *and (b) either* $\lambda > \overline{\mathfrak{d}}_\kappa$, *or* $\overline{\mathfrak{d}}_\kappa \in \mathcal{K}(\kappa, \lambda)$. *Then* $I^+_{\kappa,\lambda} \not\xrightarrow[\kappa]{\kappa} [I^+_{\kappa,\lambda}]^2_\kappa$.

**Proof.** Select $T \subseteq P_\kappa(\lambda - \kappa)$ so that $|T| = \lambda \cdot \overline{\mathfrak{d}}_\kappa$ and $|T \cap P(a)| < \kappa$ for every $a \in P_\kappa(\lambda)$. Also, select $\chi : \kappa \longrightarrow \bigcup_{\gamma<\kappa} {}^\gamma\kappa$ so that $|\chi^{-1}(\{z\})| = \kappa$ for every $z \in \bigcup_{\gamma<\kappa} {}^\gamma\kappa$. Now let $A$ be the set of all $a \in A_T$ such that

$$\chi(\cup(a \cap \kappa)) = c_{f_{T,a-\kappa}, \tau_{T,a-\kappa}}.$$

Notice that $A \in I^+_{\kappa,\lambda}$. We define a partial function $F$ from $\kappa \times \kappa$ to $\kappa$ by stipulating that $F(\delta, \eta) = (\chi(\eta))(\delta)$ if $\eta \in \kappa$ and $\delta \in \mathrm{dom}(\chi(\eta))$.

Now fix $B \in I^+_{\kappa,\lambda} \cap P(A)$ and $\xi \in \kappa$. Let $g \in {}^\kappa\kappa$ be the increasing enumeration of the elements of the set $\{\cup(b \cap \kappa) : b \in B\}$. Select $u \in P_\kappa(T) - \{\phi\}$ so that $g(\alpha) \leq \bigcup_{d \in u} (\psi_T(d))(\alpha)$ for all $\alpha < \kappa$. Pick $a \in B$ so that $\cup u \subseteq a$ and $|T \cap P(a)| > \xi$. Then

$$g(\widetilde{f}_{T,a-\kappa}(\xi)) < \cup(a \cap \kappa)$$

and

$$\xi = c_{f_{T,a-\kappa}, \tau_{T,a-\kappa}}(g(\widetilde{f}_{T,a-\kappa}(\xi))) = (\chi(\cup(a \cap \kappa))(g(\widetilde{f}_{T,a-\kappa}(\xi))) = F(g(\widetilde{f}_{T,a-\kappa}(\xi)), \cup(a \cap \kappa)).$$

□

**Remark.** Theorems 14.2, 15.1 and 15.2 (as well as e.g. Theorems 9.2, 9.4, 12.1 and 12.2, Propositions 9.3 and 11.3 and Corollary 13.2) are also true for $\kappa = \omega$. This gives (a) $\mathfrak{d} \geq \mathbf{non}_\omega(J^+ \longrightarrow [J^+]^2_\omega)$, and (b) if $\lambda \geq \mathfrak{d}$, then $I^+_{\omega,\lambda} \not\xrightarrow{\omega} [I^+_{\omega,\lambda}]^2_\lambda$ and $I^+_{\omega,\lambda} \not\xrightarrow[\omega]{\omega} [I^+_{\omega,\lambda}]^2_\omega$.

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