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COMPARING THE UNIFORMITY INVARIANTS OF NULL SETS FOR DIFFERENT MEASURES

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Abstract. It is shown to be consistent with set theory that the uniformity invariant for Lebesgue measure is strictly greater than the corresponding invariant for Hausdorff r-dimensional measure where $0 < r < 1$.

Contents

1. Introduction

The uniformity invariant for Lebesgue measure is defined to be the least cardinal of a non-measurable set of reals, or, equivalently, the least cardinal of a set of reals which is not Lebesgue null. This has been studied intensively for the past 30 years and much of what is known can be found in [?] and other standard sources. Among the well known results about this cardinal invariant of the continuum is that it can equally well be defined using Lebesgue measure on $\mathbb{R}^n$ without changing the value of the cardinal. Indeed, equivalent definitions will result by using any Borel probability measure on any Polish space. However, the question of the values of uniformity invariants for other, non-σ-finite Borel measures is not so easily answered. This paper will deal with the most familiar class of such measures, the Hausdorff measures for fractional dimension. Observe that by the previous remarks, the least cardinal of any non-measurable subset of any σ-finite set will be the same as the uniformity invariant for Lebesgue measure. In other words, this paper will be concerned with the uniformity invariant of the ideal of σ-finite sets with respect to a Hausdorff measure.

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It will be shown that given any real number r in the interval $(0,1)$ it is consistent with set theory that every set of reals of size $\aleph_1$ is Lebesgue measurable yet there is a set of reals of size $\aleph_1$ which is not a null set with respect to r-dimensional Hausdorff measure. This answers Question FQ from D. Fremlin's list of open questions [1]. However, the motivation was an attempt to resolve the following question posed by P. Komjáth.

Question 1.1. Suppose that every set of size $\aleph_1$ has Lebesgue measure zero. Does it follow that the union of any set of $\aleph_1$ lines in the plane has Lebesgue measure zero?

It is worth noting that this is really a geometric question since [7] provides a negative answer to the version of the problem in which lines are replaced by their topological and measure theoretic equivalents. To see the relationship between Question 1.1 and the topic of this paper consider that it is easy to find countably many unit squares in the plane such that each line passes through either the top and bottom or the left and right sides of at least one of these squares. It is therefore possible to focus attention on all lines which pass through the top and bottom of the unit square. For any such line L there is a pair (a,b) such that both the points $(a,0)$ and $(1,b)$ belong to L. If the mapping which sends a line L to this pair (a,b) is denoted by β then it is easy to see that β is continuous and that if $S \subseteq [0,1]^2$ is a square of side ϵ then the union of $\beta^{-1}S$ has measure ϵ while S itself has measure ϵ^2. In other words, the Lebesgue measure of the union of the lines belonging to $\beta^{-1}X$ is no larger than the 1-dimensional Hausdorff measure of X for any $X \subseteq [0,1]^2$. In other words, a negative answer to Question 1.1 would imply that there is $X \subseteq [0,1]^2$ of size $\aleph_1$ which is not null with respect to linear Hausdorff measure even though every set of reals of size $\aleph_1$ is null. The consistency of this will be a consequence of Corollary 6.2.

The proof will rely partially on arguments from [5] and [6] in which a single stage of a forcing iteration that would achieve the desired model was described. The material in §3 and §4 is a reorganized and simplified version of §4, §5 and §6 of [6] which has been suitably modified for the current context. The approach taken here differs from the earlier attempt in that the forcing used is finite branching rather than infinite branching as in [6] and this allows the use of product forcing along lines similar to those in [2]. The new ingredient needed is described in §5. The arguments presented in §6 use ideas explained in greater detail in [4], however familiarity with that paper is not be required in order to follow the reasoning presented here.

The logical requirements guiding the organization of the paper may not have resulted in optimal organization for the purposes of comprehension. Some readers may prefer to start with §6 after having read §2, Definition 3.2 and Definition 4.2.

2. Notation

If $X \subseteq \mathbb{R}$ and $r \in (0,1)$ then the infimum of all $\sum_{i=0}^{\infty}(b_i - a_i)^r$ where $\{(a_i,b_i)\}_{i=0}^{\infty}$ is a cover of X by intervals of length less than ϵ is often denoted by $\mathcal{H}^r_\epsilon(X)$. The r-dimensional Hausdorff capacity of X is denoted by $\mathcal{H}^r_\infty(X)$ and is defined to be the infimum of all $\sum_{i=0}^{\infty}(b_i - a_i)^r$ where $\{(a_i,b_i)\}_{i=0}^{\infty}$ is a cover of X by arbitrary intervals. The r-dimensional Hausdorff measure of a set X is denoted by $\mathcal{H}^r(X)$ and, when defined, is equal to its outer measure $\lim_{\epsilon\to 0}\mathcal{H}^r_\epsilon(X)$. Since it can be shown[1] that $\mathcal{H}^r_\infty(X) = 0$ if and only if $\mathcal{H}^r(X) = 0$, in order to establish the main result it suffices to show that it is consistent with set theory that every set of reals of size $\aleph_1$ is a Lebesgue null set yet there is a set of reals of size $\aleph_1$ which is not a null set with respect to r-dimensional Hausdorff capacity. Actually, it will be more convenient to work with measures and capacities on the Cantor set and to replace intervals by dyadic intervals. This amounts to dealing with net measures as described in [3]. Since r-dimensional net measures have the same null sets as r-dimensional Hausdorff measures this will not be of significance to the results of this paper.

[1]See Lemma 4.6 in [3] for example.

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Notation 2.1. For the rest of this paper let r be a fixed real number such that $0 < r < 1$.

Notation 2.2. Let $\mathbb{C}$ denote $2^{\mathbb{N}}$ with the usual product topology. Let $\mathbb{Q}$ be the tree $\{f \restriction n : f \in \mathbb{C}$ and $n \in \mathbb{N}\}$. For $a \in \mathbb{Q}$ let $[a] = \{x \in \mathbb{C} : a \subseteq x\}$ and for $A \subseteq \mathbb{Q}$ let $[A] = \bigcup_{a\in A}[a]$. Let

$$\mathbb{C}^\infty_\epsilon = \left\{A \subseteq \mathbb{Q} : \sum_{a\in A} 2^{-|a|r} \leq \epsilon\right\}$$

and let $\mathbb{C}_\epsilon = \{A \in \mathbb{C}^\infty_\epsilon : |A| < \aleph_0\}$. For consistency of notation, let $[\mathbb{Q}]^{<\aleph_0}$ be denoted by $\mathbb{C}_\infty$. Let $\mathbb{C}^1_\epsilon = \{A \subseteq \mathbb{Q} : \sum_{t\in A} 2^{-|t|} \leq \epsilon\}$. For $X \subseteq \mathbb{C}$ define $\lambda^r_\infty(X) = \inf\{\epsilon : (\exists A \in \mathbb{C}^\infty_\epsilon)\ X \subseteq [A]\}$. The usual product measure on $\mathbb{C}$ will be denoted by λ.

Notation 2.3. Some notation concerning trees will be established. By a sequence will always be meant a function $f : n \to X$ where $n \in \mathbb{N}$ and X is some set. Sequences will occasionally be denoted as n-tuples $(x_1, x_2, \ldots, x_n)$ and, in particular, singleton sequences will be denoted by (x). If t and s are sequences the concatenation of s followed by t will be denoted by (s, t) . This is consistent with considering X^k to be a set of sequences because if $x \in X^k$ and $y \in Y^m$ then $(x, y) \in X^k \times Y^m$. If T is a tree then T is a set of sequences closed under restriction to initial segments. If $t \in T$ then $\mathcal{S}_T(t) = \{x : (t, (x)) \in T\}$. Furthermore, $T\langle t\rangle = \{s \in T : t \subseteq s \text{ or } s \subseteq t\}$. If $m \in \mathbb{N}$ then $T[m] = \{t \in T : |t| = m\}$ and $T[< m] = \{t \in T : |t| < m\}$ and $T[\leq m] = \{t \in T : |t| \leq m\}$.

Notation 2.4. For any $X \subseteq \mathbb{C}^m$ and $z \in \mathbb{C}^k$ the set $\{x \in \mathbb{C}^{m-k} : (z, x) \in X\}$ will be denoted by $X(z, \cdot)$. Similarly, if $F : \mathbb{C}^m \to X$ is a function and $z \in \mathbb{C}^k$ then $F(z, \cdot)$ will represent the function $F(z, \cdot) : \mathbb{C}^{m-k} \to X$ defined by $F(z, \cdot)(x) = F(z, x)$.

3. Continuous mappings between Lebesgue and Hausdorff measures

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The goal of this section is to exploit the difference between Lebesgue measure and r-dimensional Hausdorff measure. It will be shown that for any continuous function from the reals to the reals there are arbitrary small sets in the sense of Lebesgue measure whose pre-image is as large as desired in the sense of r-dimensional Hausdorff measure. For reasons which will reveal themselves in §4 it will also be necessary to consider similar results for products of the reals.

Definition 3.1. If $\delta > 0$ is a real number and $X \subseteq \mathbb{C}$ define $\square_\delta(X)$ by

$$\square_\delta(X) = \inf\{\lambda^r_\infty(X \setminus Z) : Z \subseteq \mathbb{C} \text{ and } \lambda(Z) < \delta\}.$$

Lemma 3.1. *If $1 > \gamma > 0$ and $\epsilon > 0$ then there is $\eta > 0$ such that for all but finitely many $m \in \mathbb{N}$, any measurable $E \subseteq \mathbb{C}$ and any measurable $D \subseteq E$*

$$\square_\eta(D) \geq \square_\gamma(E)$$

provided that $\lambda(D \cap [s]) \geq \lambda(E \cap [s])\epsilon$ for each $s \in \mathbb{Q}[m]$.

Proof. Let $\eta > 0$ be sufficiently small that

$$\frac{\gamma}{2-\gamma} > \frac{4\eta}{\epsilon\gamma}$$

and then let $m \in \mathbb{N}$ be so large that the inequality

$$\frac{2^{m(1-r)}\epsilon^r\gamma^r}{4^r}\left(\frac{\gamma}{2-\gamma} - \frac{4\eta}{\epsilon\gamma}\right) > \square_\gamma(E)$$

is satisfied.

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Suppose that $\lambda(Z) < \eta$ and $A \subseteq \mathbb{Q}$ are such that $D \subseteq Z \cup [A]$ and

$$\sum_{a\in A} 2^{-|a|r} < \square_\gamma(E).$$

Let $A^* = \mathbb{Q}[\leq m] \cap A$ and let $E^* = E \setminus [A^*]$. Let $B_0 = \{b \in \mathbb{Q}[m] \ : \ [b] \cap [A^*] = \emptyset\}$ and note that $E^* \subseteq [B_0]$ and that if $b \in B_0$ then $E \cap [b] = E^* \cap [b]$. Now let

$$B_1 = \left\{ b \in B_0 \ : \ \lambda(E \cap [b]) > \frac{1}{2}\frac{\lambda(E^*)}{2^m} \right\}.$$

Notice that $\lambda(E^*) \geq \gamma$ because $\sum_{a\in A^*} 2^{-|a|r} < \square_\gamma(E)$ and $E \subseteq E^* \cup [A^*]$. It follows from a Fubini argument that

$$|B_1| \geq \frac{2^m\lambda(E^*)}{2-\lambda(E^*)} \geq \frac{2^m\gamma}{2-\gamma}. \tag{3.1}$$

Next, let

$$B_2 = \left\{ b \in B_1 \ : \ \lambda(Z \cap [b]) > \frac{\epsilon\lambda(E \cap [b])}{2} \right\}$$

and in order to see that

$$|B_2| \leq \frac{2^{m+2}\eta}{\epsilon\gamma} \tag{3.2}$$

assume the opposite. Then the following sequence of inequalities

$$\eta > \lambda(Z) \geq \sum_{b\in B_2} \lambda(Z \cap [b]) \geq \frac{\epsilon}{2}\sum_{b\in B_2} \lambda(E \cap [b]) \geq \epsilon \sum_{b\in B_2} \frac{\lambda(E^*)}{2^{m+2}} \geq |B_2|\frac{\epsilon\gamma}{2^{m+2}} \geq \eta$$

yields a contradiction. (The fourth inequality uses that $B_2 \subseteq B_1$.)

Let $B_3 = B_1 \setminus B_2$ and observe that it follows from Inequalities 3.1 and 3.2 that

$$B_3 \geq \frac{2^m\gamma}{2-\gamma} - \frac{2^{m+2}\eta}{\epsilon\gamma} = 2^m\left(\frac{\gamma}{2-\gamma} - \frac{4\eta}{\epsilon\gamma}\right). \tag{3.3}$$

It follows that

$$\sum_{a\in A} 2^{-|a|r} \geq \sum_{b\in B_3}\left(\sum_{a\in A\setminus A^*, a\supseteq b} 2^{-|a|r}\right) \geq \sum_{b\in B_3}(\lambda(D\cap[b]\setminus Z))^r \geq \sum_{b\in B_3}(\lambda(D\cap[b]) - \lambda(Z\cap[b]))^r.$$

Since $B_3 \cap B_2 = \emptyset$ it follows, using the hypothesis on D, that the last term dominates

$$\sum_{b\in B_3}(\lambda(E\cap[b])\epsilon - (\epsilon/2)\lambda(E\cap[b]))^r \geq \frac{\epsilon^r}{2^r}\sum_{b\in B_3}(\lambda(E\cap[b]))^r \geq \frac{\epsilon^r}{2^r}\sum_{b\in B_3}\left(\frac{\lambda(E^*)}{2^{m+1}}\right)^r \geq$$

$$\frac{\epsilon^r}{4^r}|B_3|\left(\frac{\lambda(E^*)}{2^m}\right)^r \geq \frac{2^{m(1-r)}\epsilon^r\gamma^r}{4^r}\left(\frac{\gamma}{2-\gamma} - \frac{4\eta}{\epsilon\gamma}\right)$$

and hence $\sum_{a\in A} 2^{-|a|r} > \square_\gamma(E)$ which is impossible. □

If $X \subseteq \mathbb{C}$ then $F : X \to \mathbb{C}$ will be said to have small fibres if and only if $\lambda(F^{-1}\{x\}) = 0$ for each $x \in \mathbb{C}$. The proof of Theorem 3.1 and the lemmas preceding it will rely on decomposing an arbitrary continuous function into a piece that has small fibres and a piece which has countable range.

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Lemma 3.2. *Let $0 < \mu < 1$ and suppose that $\{X_s\}_{s\in\mathbb{Q}}$ is an indexed family of mutually independent $\{0,1\}$-valued random variables, each with mean μ. Suppose that $C \subseteq \mathbb{C}$ is a measurable set and that $F_j : C \to \mathbb{C}$ is a measurable function with small fibres for $1 \le j \le n$. For any $\epsilon > 0$ for all but finitely many $m \in \mathbb{N}$ the probability that*

$$\lambda\left(\bigcap_{j=1}^{n}\left(\bigcup_{s\in\mathbb{Q}[m],X_s=1} F_j^{-1}[s]\right)\right) > \frac{\mu^n\lambda(C)}{2}$$

is greater than $1-\epsilon$.

Proof. This is Lemma 3.1 in [5] or Lemma 6.2 in [6] except that it is stated here for $\mathbb{C}$ rather than $[0,1]$. □

Lemma 3.3. *Suppose that*

- *$E \subseteq \mathbb{C}$ is a measurable set*
- *$\mathcal{F}$ is a finite family of measurable functions with small fibres from E to $\mathbb{C}$*
- *$\gamma > 0$, $\mu > 0$.*

Then there is $\eta > 0$ such that for any $\epsilon > 0$ and for any mutually independent, $\{0,1\}$-valued random variables $\{X_s\}_{s\in\mathbb{Q}}$ with mean μ and for all but finitely many $m \in \mathbb{N}$ the probability that the inequality

$$\square_\eta\left(\bigcap_{F\in\mathcal{F}} F^{-1}\left(\bigcup_{s\in\mathbb{Q}[m],X_s=1}[s]\right)\right) \ge \square_\gamma(E) \tag{3.4}$$

holds is greater than $1-\epsilon$.

Proof. Let $|\mathcal{F}| = n$. Use Lemma 3.1 to choose $\eta > 0$ and an integer k such that if $D \subseteq E$ is a measurable set such that for each $s \in \mathbb{Q}[k]$

$$\lambda(D \cap [s]) \ge \frac{\mu^n}{2}\lambda(E\cap[s])$$

then $\square_\eta(D) \ge \square_\gamma(E)$. Let $\epsilon > 0$. Now use Lemma 3.2 to conclude that for each $t \in \mathbb{Q}[k]$ for all but finitely many $m \in \mathbb{N}$ and any mutually independent, $\{0,1\}$-valued random variables $\{X_s\}_{s\in\mathbb{Q}[m]}$ with mean μ, the probability that

$$\lambda\left([t] \cap \bigcap_{F\in\mathcal{F}} F^{-1}\left(\bigcup_{s\in\mathbb{Q}[m],X_s=1}[s]\right)\right) \ge \frac{\mu^n}{2}\lambda(E\cap[t])$$

is greater than $1-\epsilon 2^{-k}$. Hence the probability that this holds for all $t \in \mathbb{Q}[k]$ is greater than $1-\epsilon$ and so the hypothesis on k guarantees that Inequality 3.4 holds with at least the same probability. □

Corollary 3.1. *Suppose that C is a measurable subset of $\mathbb{C}^{d+1}$ and $\mathcal{F}$ is a finite family of measurable functions from C to $\mathbb{C}$ such that $F(x,\cdot)$ has small fibres for each $x \in \mathbb{C}^d$ and $F \in \mathcal{F}$. If $\mu > 0$ and $\gamma > 0$ then there is $\eta > 0$ such that for all $\epsilon > 0$ there is some $a \in \mathbb{C}^1_\mu$ such that the Lebesgue measure of*

$$\left\{x \in \mathbb{C}^d : \square_\eta\left(\bigcap_{F\in\mathcal{F}} F(x,\cdot)^{-1}a\right) \ge \square_\gamma(C(x,\cdot))\right\}$$

is at least $1-\epsilon$.

Proof. This is a standard application of Fubini's Theorem using Lemma 3.3 and the Law of Large Numbers. □

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Lemma 3.4. *Let $E \subseteq \mathbb{C}$ be a measurable set and $\mathcal{F}$ a finite family of measurable functions from E to $\mathbb{C}$. Then for any $\gamma > 0$ and any $\mu > 0$ there is $\eta > 0$ and $a \in \mathbb{C}^1_\mu$ such that*

$$\Box_\eta\left(\bigcap_{F\in\mathcal{F}} F^{-1}a\right) \geq \Box_\gamma(E).$$

Proof. Let $|\mathcal{F}| = n$. For each $F \in \mathcal{F}$ let $Y_F = \cup\{F^{-1}\{y\} \ : \ \lambda(F^{-1}\{y\}) > 0\}$ and let $\bar{F}$ be defined by

$$\bar{F}(z) = \begin{cases} F(z) & \text{if } z \in E \setminus Y_F \\ z & \text{if } z \in Y_F. \end{cases}$$

Since each $\bar{F}$ is measurable and has small fibres it is possible to use Lemma 3.3 to conclude that there is $\eta > 0$ and an integer m and mutually independent, $\{0,1\}$-valued random variables $\{X_s\}_{s\in\mathbb{Q}[m]}$ with mean $\mu/3$ such that the probability that

$$\Box_{2\eta}\left(\bigcap_{F\in\mathcal{F}} \bar{F}^{-1}\left(\bigcup_{s\in\mathbb{Q}[m],X_s=1}[s]\right)\right) \geq \Box_\gamma(E)$$

is greater than 1/2.

Since the mean of each X_s is $\mu/3$ it is possible to choose m so large that the probability that

$$\lambda\left(\bigcup_{s\in\mathbb{Q}[m],X_s=1}[s]\right) < \frac{\mu}{2}$$

is also greater than 1/2. Hence there is $a' \in \mathbb{C}^1_{\mu/2}$ such that

$$\Box_{2\eta}\left(\bigcap_{F\in\mathcal{F}} \bar{F}^{-1}a'\right) \geq \Box_\gamma(E).$$

Now for each $F \in \mathcal{F}$ choose a finite set $A_F \subseteq \mathbb{C}$ such that $\lambda(Y_F \setminus F^{-1}A_F) < \eta/n$. Then let $a \in \mathbb{C}^1_\mu$ be such that $a' \cup \bigcup_{F\in\mathcal{F}} A_F \subseteq a$. It follows that

$$\Box_\eta\left(\bigcap_{F\in\mathcal{F}} F^{-1}a\right) \geq \Box_\eta\left(\bigcap_{F\in\mathcal{F}} \bar{F}^{-1}a \setminus \bigcup_{F\in\mathcal{F}}(Y_F \setminus F^{-1}A_F)\right) \geq \Box_\gamma(E)$$

because $F^{-1}a \supseteq \bar{F}^{-1}a \setminus (Y_F \setminus F^{-1}A_F)$ for each $F \in \mathcal{F}$. □

For the next definition recall Definition 3.1.

Definition 3.2. If $\Theta : \mathbb{N} \to \mathbb{R}^+$, $\Gamma : \mathbb{N} \to \mathbb{R}^+$ and $X \subseteq \mathbb{C}^d$ then the relation $\Box_{\Theta,\Gamma}(X)$ will be defined to hold by induction on d. If $d = 1$ then $\Box_{\Theta,\Gamma}(X)$ if and only if $\Box_{\Theta(0)}(X) \geq \Gamma(0)$ whereas, if $d > 1$, then $\Box_{\Theta,\Gamma}(X)$ holds if and only if $\Box_{\Theta(d-1)}(\{x \in \mathbb{C} : \Box_{\Theta,\Gamma}(X(x,\cdot))\}) \geq \Gamma(d-1)$. Define $\Theta^+(i) = \Theta(i+1)$ and $\Gamma^+(i) = \Gamma(i+1)$.

The next lemma establishes that the top-down and bottom-up definitions of the relation $\Box_{\Theta,\Gamma}$ are the same.

Lemma 3.5. *Let $d \geq 2$. For any $\Theta : \mathbb{N} \to \mathbb{R}^+$, $\Gamma : \mathbb{N} \to \mathbb{R}^+$ and $X \subseteq \mathbb{C}^{d+1}$ the relation $\Box_{\Theta,\Gamma}(X)$ holds if and only if $\Box_{\Theta^+,\Gamma^+}\left(\{z \in \mathbb{C}^d \ : \ \Box_{\Theta(0)}(X(z,\cdot)) \geq \Gamma(0)\}\right)$.*

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Proof. Proceed by induction on d and observe that the case $d = 2$ is immediate from Definition 3.2. Assuming the lemma established for d let $X \subseteq \mathbb{C}^{d+2}$. Then the following sequence of equivalences establishes the lemma:

$$\square_{\Theta,\Gamma}(X)$$

$$\square_{\Theta(d+1)}\left(\left\{z \in \mathbb{C}^1 \ : \ \square_{\Theta,\Gamma}(X(z,\cdot))\right\}\right) \geq \Gamma(d+1)$$

$$\square_{\Theta(d+1)}\left(\left\{z \in \mathbb{C}^1 \ : \ \square_{\Theta^+,\Gamma^+}\left(\{w \in \mathbb{C}^d \ : \ \square_{\Theta(0)}(X(z,\cdot)(w,\cdot)) \geq \Gamma(0)\}\right)\right\}\right) \geq \Gamma(d+1)$$

$$\square_{\Theta^+(d)}\left(\left\{z \in \mathbb{C}^1 \ : \ \square_{\Theta^+,\Gamma^+}\left(\{w \in \mathbb{C}^d \ : \ \square_{\Theta(0)}(X((z,w),\cdot)) \geq \Gamma(0)\}\right)\right\}\right) \geq \Gamma^+(d)$$

$$\square_{\Theta^+,\Gamma^+}\left(\left\{(z,w) \in \mathbb{C}^1 \times \mathbb{C}^d \ : \ \square_{\Theta(0)}(X((z,w),\cdot)) \geq \Gamma(0)\right\}\right)$$

$$\square_{\Theta^+,\Gamma^+}\left(\left\{z \in \mathbb{C}^{d+1} \ : \ \square_{\Theta(0)}(X(z,\cdot)) \geq \Gamma(0)\right\}\right)$$

□

Lemma 3.6. *If $\square_{\Theta,\Gamma}(X)$ holds and $X \subseteq \mathbb{C}^d$ and $\eta < \Theta(i)$ for each $i < d$ and $A \subseteq \mathbb{C}^d$ is such that $\lambda(A) < \eta^d$ then $\square_{\Theta-\eta,\Gamma}(X \setminus A)$ holds where $(\Theta - \eta)(i) = \Theta(i) - \eta$.*

Proof. Proceed by induction on d using Fubini's Theorem. □

Theorem 3.1. *Let $\Gamma : \mathbb{N} \to \mathbb{R}^+$ and $\Theta : \mathbb{N} \to \mathbb{R}^+$ be functions such that $\Theta(i) + \Gamma(i) < 1$ for each i. Suppose that C is a closed subset of $\mathbb{C}^d$ such that $\square_{\Theta,\Gamma}(C)$ holds and that $\mathcal{F}$ is a finite family of continuous functions from C to $\mathbb{C}$. If $\mu > 0$ then there is $a \in \mathbb{C}^1_\mu$ and $\eta : \mathbb{N} \to \mathbb{R}^+$ such that $\square_{\eta,\Gamma}\left(\bigcap_{F\in\mathcal{F}} F^{-1}a\right)$.*

Proof. Proceed by induction on d, noting that if $d = 1$ then this follows from Lemma 3.4 by setting $\gamma = \Theta(0)$ in that lemma. So assume that the lemma has been established for d and that C is a closed subset of $\mathbb{C}^{d+1}$, $\square_{\Theta,\Gamma}(C)$ holds and $\mathcal{F}$ is a finite family of continuous functions from C to $\mathbb{C}$ and that $\mu > 0$. For each $F \in \mathcal{F}$ let

$$Y_F = \{(x,y) \in \mathbb{C}^d \times \mathbb{C} \ : \ \lambda(\{z \in \mathbb{C} \ : \ F(x,y) = F(x,z)\}) > 0\}$$

and define $\bar{F} : C \to \mathbb{C}$ by

$$\bar{F}(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \notin Y_F \\ y & \text{otherwise.} \end{cases}$$

Observe that $\bar{F}(x,\cdot)$ has small fibres for each $x \in \mathbb{C}^d$. By Corollary 3.1 there is $\delta > 0$ such that for all $\epsilon > 0$ there is some $a \in \mathbb{C}^1_{\mu/2}$ such that the Lebesgue measure of

$$B(a) = \left\{x \in \mathbb{C}^d : \square_\delta\left(\bigcap_{F\in\mathcal{F}} \bar{F}(x,\cdot)^{-1}a\right) \geq \square_{\Theta(0)}(C(x,\cdot)) \geq \Gamma(0)\right\}$$

is at least $1 - \epsilon$.

Since each of the relations Y_F is Borel, it is possible to find a finite family of functions $\mathcal{G}$ from $\mathbb{C}^d$ to $\mathbb{C}$ such that

$$\sum_{F\in\mathcal{F}} \int_{x\in\mathbb{C}^d} \lambda\left(Y_F(x,\cdot) \setminus F(x,\cdot)^{-1}\{G(x)\}_{G\in\mathcal{G}}\right) dx < \frac{\delta^{d+1}}{2^{d+1}}.$$

It is then possible to find a closed set $D \subseteq \mathbb{C}^d$ such that

- $\lambda(D) > \Theta(i+1) + \Gamma(i+1)$ for each $i < d$ (using Hypothesis 3.9)
- $\lambda(D) > 1 - \delta^d/2^d$
- $\lambda\left(Y_F(x,\cdot) \setminus F(x,\cdot)^{-1}\{G(x)\}_{G\in\mathcal{G}}\right) < \delta/2$ for each $x \in D$
- each $G \in \mathcal{G}$ is continuous on D.

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Condition 3 guarantees that $\square_{\Theta^+,\Gamma^+}(D)$ holds. It is therefore possible to apply the induction hypothesis to Θ^+, Γ^+, $\mathcal{G}$ and D to get $a_0 \in \mathbb{C}^1_{\mu/2}$ and $\eta : \mathbb{N} \to \mathbb{R}^+$ such that

$$\square_{\eta,\Gamma^+}\left(D \cap \bigcap_{G\in\mathcal{G}} G^{-1}a_0\right).$$

Let $0 < \epsilon' < \eta(i)$ for $i < d$. Now choose $a_1 \in \mathbb{C}^1_{\mu/2}$ such that $\lambda(B(a_1)) > 1 - \epsilon'$. Then

$$\square_{\eta-\epsilon',\Gamma^+}\left(D \cap B(a_1) \cap \bigcap_{G\in\mathcal{G}} G^{-1}a_0\right)$$

holds by Lemma 3.6 and Condition 3 and note that by Condition 3

$$\square_{\delta/2}\left(\bigcap_{F\in\mathcal{F}} \bar{F}(x,\cdot)^{-1}a_1 \cap Y_F(x,\cdot) \setminus F(x,\cdot)^{-1}\{G(x)\}_{G\in\mathcal{G}}\right) \geq \Gamma(0)$$

for $x \in D \cap B(a_1)$. Let $\bar{\eta}$ be defined by

$$\bar{\eta}(i) = \begin{cases} \eta(i-1) - \epsilon' & \text{if } d \geq i > 0 \\ \eta(i-1) & \text{if } d < i \\ \delta/2 & \text{if } i = 0 \end{cases}$$

and define Z to be the set of all $(x, w) \in \mathbb{C}^{d+1}$ such that the following three conditions are satisfied:

$$x \in D \cap B(a_1) \cap \bigcap_{G\in\mathcal{G}} G^{-1}a_0 \tag{3.5}$$

$$(\forall F \in \mathcal{F})\ \bar{F}(x, w) \in a_1 \tag{3.6}$$

$$(\forall F \in \mathcal{F})\ F(x, w) \in Y_F(x,\cdot) \text{ and } F(x, w) \neq G(x) \tag{3.7}$$

It follows from Lemma 3.5 that $\square_{\bar{\eta},\Gamma}(Z)$. Hence it suffices to observe that if $(x, w) \in Z$ then $F(x, w) \in a_0 \cup a_1$. To see this note that if $F(x, w) = \bar{F}(x, w)$ this is immediate from 3.6. Otherwise $F(x, w) \in Y_F$ and hence it follows from 3.7 that there is some $G \in \mathcal{G}$ such that $F(x, w) = G(x)$. From 3.5 it can be concluded that $G(x) \in a_0$. □

Notation 3.1. For the rest of the paper, fix a pair of functions $\Gamma : \mathbb{N} \to \mathbb{R}^+$ and $\Theta : \mathbb{N} \to \mathbb{R}^+$ such that

$$\Theta(n) > \Theta(n+1) \text{ and } \lim_{n\to\infty} \Theta(n) = 0 \tag{3.8}$$

$$(\forall j)\ \Theta(j) + \Gamma(j) < 1 \tag{3.9}$$

$$(\forall j)\ \Theta(j) + \Gamma(j+1) < 1 \tag{3.10}$$

$$\text{the range of } \Gamma \text{ is a dense subset of } (0,1). \tag{3.11}$$

Definition 3.3. For any $d \in \mathbb{N}$ and for $X \subseteq \mathbb{C}^d$ define $\square(X) = \square_{\Theta,\Gamma}(X)$ and define $\square_*(X)$ to hold if and only if there is some $\eta : \mathbb{N} \to \mathbb{R}^+$ such that $\square_{\eta,\Gamma}(X)$ holds.

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4. A preliminary norm

This section will introduce three norms[2] on subsets of $\mathbb{C}_\infty$ — these will be denoted ρ, ν^∞ and ν. The norms of [4] typically enjoy some form of sub-additivity but this will not be the case for any of these three, at least not explicitly. Nevertheless, Lemma 4.2 can be considered a substitute for this. The only norm used in the definition of the partial order in §6 will be ν. The role of the norm ρ will be to establish a connection between ν^∞ and the results of §3. The norm ν^∞ is an intermediary between ρ and ν and, furthermore, it has the advantage of allowing the compactness argument of §5 to work.

Notation 4.1. For any Polish space X let $\mathcal{K}(X)$ denote the spaces of compact subsets of X with the Hausdorff metric and let $\mathcal{C}(X)$ denote the space of continuous $\mathbb{C}$-valued functions with the uniform metric.

Definition 4.1. Define ρ to be a function from $\mathcal{P}(\mathbb{C}_\infty)$ to $\mathbb{N}\cup\{\infty\}$ by first defining $\rho^*(\mathcal{X})$ to be

$$\left\{d > 1 \ : \ (\forall C \in \mathcal{K}(\mathbb{C}^d))(\forall F \in \mathcal{C}(C))(\exists x \in \mathcal{X})\square(C) \Rightarrow \square_*(F^{-1}x)\right\}$$

and let $\rho(\mathcal{X}) = \sup(\rho^*(\mathcal{X}))$. If $\rho^*(\mathcal{X}) = \emptyset$ but $\mathcal{X} \neq \emptyset$ then define $\rho(\mathcal{X}) = 0$.

The following is a direct Corollary of Theorem 3.1.

Corollary 4.1. *If $\mu > 0$ then $\rho(\mathbb{C}^1_\mu) = \infty$.*

Lemma 4.1. *If $d \geq 1$ then $\{C \subseteq \mathcal{K}(\mathbb{C}^d) \ : \ \square(C)\}$ is a Borel set.*

Proof. It suffices to show $\square(C)$ fails if and only if there is some finite subset $\mathcal{C} \subseteq \mathbb{Q}^d$ such that

$$C \subseteq [\mathcal{C}] = \bigcup_{(t_1,t_2,\ldots,t_d)\in\mathcal{C}} [t_1] \times [t_2] \times \ldots \times [t_d]$$

and $\square([\mathcal{C}])$ also fails. To prove this proceed by induction on d. The case $d = 1$ is easy. Assuming the result for d let $C \subseteq \mathbb{C}^{d+1}$ be compact and suppose that $\square(C)$ fails. Using Definition 3.2 there is $\mathcal{C}_1 \in \mathbb{C}_\gamma$ for some $\gamma < \Gamma(d)$ such that $\{z \in \mathbb{C} \ : \ \square_{\Theta,\Gamma}(C(x,\cdot))\} \subseteq [\mathcal{C}_1]$. For $x \in \mathbb{C} \setminus [\mathcal{C}_1]$ it is possible to use the induction hypothesis to find a finite $\mathcal{C}_x$ such that $C(x,\cdot) \subseteq [\mathcal{C}_x]$ and $\square_{\Theta,\Gamma}(\mathcal{C}_x)$ fails. . The compactness of C yields an integer k_x such that $C(y,\cdot) \subseteq [\mathcal{C}_x]$ for each $y \in \mathbb{C}$ such that $y \restriction k_x = x \restriction k_x$. Let U_x denote this neighbourhood and choose a finite $A \subseteq \mathbb{C}$ such that $[\mathcal{C}_1]$ and $\{U_a\}_{a\in A}$ cover $\mathbb{C}$. It is then easy to extract the desired finite set from the cover $\{[\mathcal{C}_1] \times \mathbb{C}^d\} \cup \{U_a \times [\mathcal{C}_a]\}_{a\in A}$. □

Corollary 4.2. *For any $d \geq 1$ the set $\{C \subseteq \mathcal{K}(\mathbb{C}^d) \ : \ \square_*(C)\}$ is Borel.*

Proof. According to Definition 3.1, $\square_*(C)$ holds if and only if there is some $\eta : \mathbb{N} \to \mathbb{R}^+$ such that $\square_{\eta,\Gamma}(C)$ holds. But notice that η can be assumed to be a constant function with rational value. □

Notation 4.2. The notation $\langle z, F\rangle$ will be introduced for any $z \in \mathbb{C}$ and any function from $F : W \to \mathcal{P}(\mathbb{Q})$ to denote $\{w \in W : z \notin [F(w)]\}$.

Corollary 4.3. *Suppose that $\mathcal{X} \subseteq \mathbb{C}_\infty$ and $B : \mathcal{X} \to \mathcal{P}(\mathbb{C}_\infty)$. Then $\{z \in \mathbb{C} \ : \ \rho(\langle z, B\rangle < j\}$ is analytic for any integer j.*

Proof. This is immediate from Definition 4.1, Lemma 4.1 and Corollary 4.2. □

Lemma 4.2. *Suppose that $\mathcal{X} \subseteq \mathbb{C}_\infty$ and $\rho(\mathcal{X}) = j \geq 2$. Then*

$$\square_{\Theta(j-2)}\left(\{z \in \mathbb{C} \ : \ \rho(\langle z, F\rangle) < j-1\}\right) < \Gamma(j-1)$$

for each function F from $\mathcal{X}$ to $\mathbb{C}^\infty_{\Gamma(j-1)/2}$.

[2]This is the terminology of [4]. The exact meaning of the term "norm" is not required for the arguments used here.

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Proof. Let $S = \{z \in \mathbb{C} \ : \ \rho(\langle z, F\rangle) < j-1\}$ and assume that the lemma fails. In other words, $\square_{\Theta(j-2)}(S) \geq \Gamma(j-1)$. Let

$$E \subseteq \mathbb{C} \times \mathcal{K}(\mathbb{C}^{j-1}) \times \mathcal{C}(\mathbb{C}^{j-1})$$

be the set of all triples (z, C, f) such that

$$\square(C) \tag{4.1}$$

$$(\forall x \in \langle z, F\rangle)\ \neg\square_*(C \cap f^{-1}x) \tag{4.2}$$

It follows from Definition 4.1 that S is contained in the domain of E. From Lemma 4.1 and Corollary 4.2 it follows that E is a Borel set. From Corollary 4.3 it follows that S is measurable and so it is possible to use the von Neumann Selection Theorem and Egeroff's Theorem to find a closed set S^* in the domain of E and a continuous function $T \subseteq E$ with domain S^* such that $\lambda(S \setminus S^*) < \Theta(j-2) - \Theta(j-1)$. Let $T(s) = (C_s, f_s)$ and observe that the continuity of T guarantees that $C^* = \bigcup_{s \in S^*}\{s\} \times C_s$ is a closed set. Moreover, $\square_{\Theta(j-1)}(S^*) \geq \square_{\Theta(j-2)}(S) \geq \Gamma(j-1)$. A calculation using Definition 3.2 reveals that $\square(C^*)$ holds. Let g be defined on C^* by $g(c_1, c_2, \ldots, c_j) = f_{c_1}(c_2, c_3, \ldots, c_j)$ recalling that $j \geq 2$.

Since $\rho(\mathcal{X}) \geq j$ it is possible to find $x \in \mathcal{X}$ such that $\square_*(g^{-1}x)$. Let

$$W = \{z \in S^* \ : \ \square_*(g^{-1}x(z, \cdot))\} = \{z \in S^* \ : \ \square_*(g(z, \cdot)^{-1}x)\}$$

and note that it follows that $\lambda^r_\infty(W) \geq \Gamma(j-1)$. Since $F(x) \in \mathbb{C}_{\Gamma(j-1)/2}$ it is possible to choose $w \in W \setminus F(x)$. Then $x \in \langle w, F\rangle$ and so, by Condition 4.2 in the definition of E, it follows that $\square_*(g(w, \cdot)^{-1}x)$ fails since $g(w, \cdot) = f_w$. This contradicts that $w \in W$. □

Definition 4.2. The norms ν and ν^∞ will be defined for the subsets of $\mathbb{C}_\infty$ by first using induction to define an associated sequence of sets:

$$\mathcal{N}_0 = \mathcal{N}_0^\infty = \{\mathcal{X} \subseteq \mathbb{C}_\infty \ : \ \mathcal{X} \neq \emptyset\}$$

$$\mathcal{N}_1 = \mathcal{N}_1^\infty = \{\mathcal{X} \subseteq \mathbb{C}_\infty \ : \ [\cup\mathcal{X}] = \mathbb{C}\}$$

$$\mathcal{N}_{j+1} = \{\mathcal{X} \subseteq \mathbb{C}_\infty \ : \ (\forall F : \mathcal{X} \to \mathbb{C}_{\Gamma(j)/2})\ \square_{\Theta(j-1)}(\{z \in \mathbb{C} : \langle z, F\rangle \notin \mathcal{N}_j\}) < \Gamma(j)\} \tag{4.3}$$

$$\mathcal{N}^\infty_{j+1} = \{\mathcal{X} \subseteq \mathbb{C}_\infty \ : \ (\forall F : \mathcal{X} \to \mathbb{C}_{\Gamma(j)/2})\ \square_{\Theta(j-1)}\left(\{z \in \mathbb{C} : \langle z, F\rangle \notin \mathcal{N}^\infty_j\}\right) < \Gamma(j)\}. \tag{4.4}$$

Then define $\nu(\mathcal{X})$ to be the supremum of all j such that $\mathcal{X} \in \mathcal{N}_j$ and $\nu^\infty(\mathcal{X})$ to be the supremum of all j such that $\mathcal{X} \in \mathcal{N}^\infty_j$.

It must be noted that $\mathcal{N}_j \supseteq \mathcal{N}_{j+1}$ and $\mathcal{N}^\infty_j \supseteq \mathcal{N}^\infty_{j+1}$ for each integer j and hence the supremum in Definition 4.2 is taken over an initial segment of the integers. This will be used implicitly in what follows.

Corollary 4.4. *If* $\mathcal{X} \subseteq \mathbb{C}_\infty$ *then* $\rho(\mathcal{X}) \leq \nu^\infty(\mathcal{X})$.

Proof. Proceed by induction on $j = \rho(\mathcal{X})$. The case $j = 0$ is trivial but the case $j = 1$ is less so. To see that $\rho(\mathcal{X}) = 1$ implies that $\mathcal{X}$ covers $\mathbb{C}$ let $z \in \mathbb{C}$. Note that $\square(\mathbb{C})$ holds by Condition 3.10 of Notation 3.1. Letting F be the function on $\mathbb{C}$ with constant value z it follows from the definition of $\rho(\mathcal{X}) = 1$ that there is some $x \in \mathcal{X}$ such that $\square_*(F^{-1}x)$ holds. In particular, $F^{-1}x \neq \emptyset$ and so if $w \in F^{-1}x$ then $z = F(w) \in x$.

Therefore it can be assumed that $2 \leq j+1 = \rho(\mathcal{X})$ and that the lemma holds for j. In order to show that $\nu^\infty(\mathcal{X}) \geq j+1$ let $F : \mathcal{X} \to \mathbb{C}^\infty_{\Gamma(j)/2}$. By Lemma 4.2 it must be that

$$\square_{\Theta(j-1)}(\{z \in \mathbb{C} \ : \ \rho(\langle z, F\rangle) < j\}) < \Gamma(j).$$

From the induction hypothesis it follows that if $\nu^\infty(\langle z, F\rangle) < j$ then $\rho(\langle z, F\rangle) < j$. Hence

$$\square_{\Theta(j-1)}(\{z \in \mathbb{C} \ : \ \nu^\infty(\langle z, F\rangle) < j\}) < \Gamma(j)$$

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and this establishes that $\nu^\infty(\mathcal{X}) \geq j+1$. $\square$

5. Finding finite sets with large norm

It will be shown that for any j there is a finite $\mathcal{X}$ such that $\nu^\infty(\mathcal{X}) \geq j$. This will establish that the $\mathfrak{C}(k)$ required in the definition of the partial order $\mathbb{P}$ in §6 actually do exist. Each of the next lemmas is a step towards this goal.

Lemma 5.1. *If $\mathcal{X} \subseteq \mathbb{C}_\infty$ then $\nu^\infty(\mathcal{X}) \leq \nu(\mathcal{X})$.*

Proof. This follows from the definitions by an argument using induction on j to show that if $\nu^\infty(\mathcal{X}) \geq j+1$ then $\nu(\mathcal{X}) \geq j+1$. $\square$

Lemma 5.2. *If $\{\mathcal{A}_n\}_{n=0}^\infty$ is an increasing sequence of finite subsets of $\mathbb{C}_\infty$ then*

$$\nu^\infty\left(\bigcup_{n=0}^\infty \mathcal{A}_n\right) = \lim_{n\to\infty} \nu^\infty(\mathcal{A}_n).$$

Proof. Proceed by induction on j to show that if $\nu^\infty(\mathcal{A}_n) < j$ for each n then $\nu^\infty\left(\bigcup_{n=0}^\infty \mathcal{A}_n\right) < j$. For $j=0$ this trivial and if $j=1$ this is simply a restatement of the compactness of $\mathbb{C}$. Therefore assume that $j \geq 1$, that the lemma is true for j, that $\nu^\infty(\mathcal{A}_n) \leq j$ for each n yet $\nu^\infty\left(\bigcup_{n=0}^\infty \mathcal{A}_n\right) > j$. Let $F_n : \mathcal{A}_n \to \mathbb{C}^\infty_{\Gamma(j)/2}$ witness that $\nu^\infty(\mathcal{A}_n) \not\geq j+1$. In other words, using Equality 4.4 of Definition 4.2, $\square_{\Theta(j-1)}(S_n) \geq \Gamma(j)$ where $S_n = \{z \in \mathbb{C} : \nu^\infty(\langle z, F_n\rangle) < j\}$.

Claim 1. If $\{\delta_n\}_{n=0}^\infty$ is a sequence of positive reals and and $\{A_n\}_{n=0}^\infty$ is a sequence of elements of $\mathbb{C}_1^\infty$ then there are two increasing sequences of integers $\{k_n\}_{n=0}^\infty$ and $\{m_n\}_{n=0}^\infty$ such that, letting

$$D_n = A_{m_n} \cap \mathbb{Q}[< k_n]$$

the following hold:

$$\text{if } i \geq n \text{ then } D_n = D_i \cap \mathbb{Q}[< k_n] \tag{5.1}$$

$$\lambda\left(\bigcup_{i=n}^\infty [A_{m_i} \setminus D_n]\right) < \delta_n. \tag{5.2}$$

Moreover, the increasing sequence $\{k_n\}_{n=0}^\infty$ can be chosen from any given infinite set K.

Proof. Let K be given and, using the fact that $0 < r < 1$, let $\{k_n\}_{n=0}^\infty \subseteq K$ be such that for each n

$$\sum_{i=n}^\infty 2^{-k_i(1-r)} \leq \delta_n/2$$

and then choose the sequence $\{m_n\}_{n=0}^\infty$ such that Conclusion 5.1 holds. Then Conclusion 5.2 follows from the following inequalities

$$\lambda\left(\bigcup_{i=n}^\infty [A_{m_i} \setminus D_n]\right) \leq \lambda\left(\bigcup_{i=n}^\infty [A_{m_i} \setminus D_i] \cup [D_{i+1} \setminus D_i]\right) \leq \sum_{i=n}^\infty \lambda([A_{m_i} \setminus D_i]) + \lambda([D_{i+1} \setminus D_i])$$

$$\leq \sum_{i=n}^\infty \left(\sum_{t \in A_{m_i} \setminus D_i} 2^{-|t|} + \sum_{t \in D_{i+1} \setminus D_i} 2^{-|t|}\right) \leq \sum_{i=n}^\infty \left(\sum_{t \in A_{m_i} \setminus D_i} 2^{-|t|r} 2^{-|t|(1-r)} + \sum_{t \in D_{i+1} \setminus D_i} 2^{-|t|r} 2^{-|t|(1-r)}\right)$$

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$$\leq \sum_{i=n}^{\infty}\left(2^{-k_i(1-r)}\left(\sum_{t\in A_{m_i}\setminus D_i}2^{-|t|r}+\sum_{t\in D_{i+1}\setminus D_i}2^{-|t|r}\right)\right)\leq \sum_{i=n}^{\infty}2^{-k_i(1-r)}(1+1)\leq \delta_n.$$

□

Before continuing, let $\{a_i\}_{i=0}^{\infty}$ enumerate $\bigcup_{n=0}^{\infty}\mathcal{A}_n$ and, without loss of generality, assume that $\mathcal{A}_n=\{a_\ell\}_{\ell=0}^{n}$. Using the claim and its final clause and a diagonalization argument, find two increasing sequences of integers $\{k_n\}_{n=0}^{\infty}$ and $\{m_n\}_{n=0}^{\infty}$ such that for each i and each $n\geq i$, letting $F_{n,i}=F_{m_n}(a_i)\cap\mathbb{Q}[<k_n]$, Conclusion 5.1 and Conclusion 5.2 of the claim hold for a summable sequence of δ. To be more precise, $F_{n+1,i}$ is an end extension of $F_{n,i}$ and

$$\lambda\left(\bigcup_{j=n}^{\infty}\left[F_{m_j}(a_i)\setminus F_{n,i}\right]\right)<2^{-i-n}$$

for any $n\geq i$. Define $F(a_i)=\bigcup_{n=i}^{\infty}F_{n,i}$ and note that $F(a_i)\in\mathbb{C}^{\infty}_{\Gamma(j)/2}$ since $F_{n,i}\in\mathbb{C}_{\Gamma(j)/2}$ for each n and the $F_{n,i}$ are increasing with respect to n.

Now let $S=\{z\in\mathbb{C}:\nu^{\infty}(\langle z,F\rangle)<j\}$. Because

$$\nu^{\infty}\left(\bigcup_{n=0}^{\infty}\mathcal{A}_n\right)\geq j+1$$

it follows that $\square_{\Theta(j-1)}(S)<\Gamma(j)$ and, hence, it is possible to choose Y such that $\lambda^r_{\infty}(S\setminus Y)<\Gamma(j)$ and $\lambda(Y)<\Theta(j-1)$. Choose M so large that letting

$$W=\bigcup_{i=0}^{\infty}\left(\bigcup_{n=\max(M,i)}^{\infty}\left[F_{m_n}(a_i)\setminus F_{\max(M,i),i}\right]\right)$$

it follows that $\lambda(W)<\Theta(j-1)-\lambda(Y)$.

Now define $F^*(a_i)$ by

$$F^*(a_i)=\begin{cases}F_{M,i} & \text{if } i\leq M\\ F_{i,i} & \text{if } i>M\end{cases}$$

and note that $F^*(a_i)\subseteq F(a_i)$. Now let $S^*_m=\{z\in\mathbb{C}:\nu^{\infty}(\langle z,F^*\restriction\mathcal{A}_m\rangle)<j\}$ and note that $S^*_m\supseteq S^*_{m+1}$ and they are all compact. Indeed,

$$S^*_m=\bigcup_{\mathcal{B}\subseteq\mathcal{A}_m,\nu^{\infty}(\mathcal{B})<j}\left(\bigcap_{a\in\mathcal{A}_m\setminus\mathcal{B}}[F^*(a)]\right)$$

and each $[F^*(a)]$ is compact.

Claim 2. If $z\notin W$ and $i\geq M$ then $\langle z,F^*\restriction\mathcal{A}_{m_i}\rangle\subseteq\langle z,F_{m_i}\rangle$.

Proof. Let $a_\ell\notin\langle z,F_{m_i}\rangle$ and assume that $\ell\leq m_i$ since otherwise it is immediate that $a_\ell\notin\langle z,F^*\restriction\mathcal{A}_{m_i}\rangle$. There are two cases to consider. First assume that $\ell\leq M$. Then $z\in[F_{m_i}(a_\ell)]$ but since $z\notin W$ and $i\geq M$ it follows that $z\notin[F_{m_i}(a_\ell)\setminus F_{M,\ell}]$. Hence $z\in[F_{M,\ell}]=[F^*(a_\ell)]$. In other words, $a_\ell\notin\langle z,F^*\restriction\mathcal{A}_{m_i}\rangle$. If $\ell>M$ a similar argument works. □

It follows from the claim that if $z\notin W$ and $i\geq M$ then $\nu^{\infty}(\langle z,F^*\restriction\mathcal{A}_{m_i}\rangle)\leq\nu^{\infty}(\langle z,F_{m_i}\rangle)$. Hence, if $i\geq M$ and $z\in S_{m_i}\setminus W$ then $\nu^{\infty}(\langle z,F^*\restriction\mathcal{A}_{m_i}\rangle)<j$. In other words, $S_{m_i}\setminus W\subseteq S^*_{m_i}\setminus W$. Therefore,

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$\square_{\Theta(j-1)-\lambda(W)}(S^*_{m_i} \setminus W) \geq \square_{\Theta(j-1)-\lambda(W)}(S_{m_i} \setminus W) \geq \Gamma(j)$. Since each of the $S^*_{m_i} \setminus W$ are compact it follows that

$$\square_{\Theta(j-1)-\lambda(W)}\left(\bigcap_{i=M}^{\infty} S^*_{m_i} \setminus W\right) \geq \Gamma(j).$$

Let $S^* = \bigcap_{i=M}^{\infty} S^*_i$. Since $\lambda(Y) < \Theta(j-1) - \lambda(W)$ it follows that $\lambda^r_\infty(S^* \setminus (W \cup Y)) \geq \Gamma(j)$.

Since $\lambda^r_\infty(S \setminus Y) < \Gamma(j)$ it is possible to select $z \in S^* \setminus (S \cup W \cup Y)$. Then $\nu^\infty(\langle z, F^* \restriction \mathcal{A}_{m_i}\rangle) < j$ for each $i \geq M$ and so the induction hypothesis guarantees that

$$\nu^\infty\left(\bigcup_{i=M}^{\infty} \langle z, F^* \restriction \mathcal{A}_{m_i}\rangle\right) < j.$$

But

$$\bigcup_{i=M}^{\infty} \langle z, F^* \restriction \mathcal{A}_{m_i}\rangle = \langle z, \bigcup_{i=M}^{\infty} F^* \restriction \mathcal{A}_{m_i}\rangle = \langle z, F^*\rangle$$

and so $\nu^\infty(\langle z, F^*\rangle) < j$. Since $F^*(a) \subseteq F(a)$ for each a it is immediate that $\langle z, F^*\rangle \supseteq \langle z, F\rangle$ and so $\nu^\infty(\langle z, F\rangle) < j$. This contradicts that $z \notin S$. □

Corollary 5.1. *For any $j \in \mathbb{N}$ and $\mu > 0$ there is a finite set $\mathcal{X} \subseteq \mathbb{C}^1_\mu$ such that $\nu(\mathcal{X}) \geq j$.*

Proof. Combine Corollary 4.1 and Corollary 4.4 to conclude that $\nu^\infty(\mathbb{C}^1_\mu) = \infty$. Then use Lemma 5.2 to find a finite $\mathcal{X} \subseteq \mathbb{C}^1_\mu$ such that $\nu^\infty(\mathcal{X}) \geq j$. Finally apply Lemma 5.1. □

6. The forcing partial order

Using Corollary 5.1 let $\mathfrak{C}(n)$ be a finite subset of $\mathbb{C}^1_{2^{-n}}$ such that $\nu(\mathfrak{C}(n)) \geq n$ for each $n \in \mathbb{N}$. Recalling the notation concerning trees in §2, let $\mathbb{P}$ consist of all trees T such that:

$$(\forall t \in T)(\forall i < |t|)\ t(i) \in \mathfrak{C}(i) \tag{6.1}$$

$$(\forall k \in \omega)(\forall t \in T)(\exists s \in T)\ t \subseteq s \text{ and } \nu(\mathcal{S}_T(s)) > k \tag{6.2}$$

and let this be ordered under inclusion. The methods of [4] can be used to establish that $\mathbb{P}$ is an ω^ω-bounding proper partial order.

Lemma 6.1. *Let $V \subseteq W$ be models of set theory and suppose that $G \subseteq \mathbb{P} \cap V$ is generic over W. Then $W[G] \models \lambda(W \cap [0,1]) = 0$.*

Proof. If $G \subseteq \mathbb{P} \cap V$ is generic over W then let $B_G \in \prod_{n=0}^{\infty} \mathfrak{C}(n)$ be the generic branch determined by G. Note that $\lambda([B_G(j)]) < 2^{-j}$ for each j and so $\lambda(\bigcup_{j=0}^{\infty}[B_G(j)]) < \infty$. Also, for every $T \in \mathbb{P}$ there is some $t \in T$ such that $V \models \nu(\mathcal{S}_T(t)) \geq 1$. Since $\nu(\mathcal{S}_T(t)) \geq 1$ is equivalent to $\cup\mathcal{S}_T(t) = \mathbb{C}$, and this is absolute, it follows from genericity that $\mathbb{C} \cap W \subseteq \bigcup_{j=m}^{\infty}[B_G(j)]$ for every m. □

Definition 6.1. Let $\mathbb{P}^\kappa$ be the countable support product of κ copies of $\mathbb{P}$.

Corollary 6.1. *If $\kappa \geq \aleph_1$ and $G \subseteq \mathbb{P}^\kappa$ is a filter generic over V then in $V[G]$ every set of reals of size less than κ has Lebesgue measure 0.*

In light of Corollary 6.1 the goal now is to establish that if $\mathbb{P}^\kappa$ is the countable support product of κ copies of $\mathbb{P}$ and $G \subseteq \mathbb{P}^\kappa$ is a filter generic over V then $\lambda^r_\infty(\mathbb{C} \cap V) = 1$ in $V[G]$.

Lemma 6.2. *Suppose that $p \in \mathbb{P}^\kappa$ and*

$$p \Vdash_{\mathbb{P}^\kappa} \text{“}C \in \mathbb{C}^\infty_\delta\text{”}$$

and $\delta < 1$. Then there are $A \subseteq \mathbb{C}$ and $B \subseteq \mathbb{C}$ such that $\lambda^r_\infty(A) + \lambda(B) < 1$ and $q \leq p$ such that $q \not\Vdash_{\mathbb{P}^\kappa}$ “$z \in [C]$” for each $z \in \mathbb{C} \setminus (A \cup B)$.

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Proof. Let $1 > \delta_1 > \delta$ and choose a monotonically increasing function $\delta_2 : \mathbb{N} \to \mathbb{R}$ such that $\delta_1 + \lim_{n\to\infty} \delta_2(n) < 1$. For later use, define $k(i) = \prod_{j=0}^{i} |\mathfrak{C}(j)|$. The proof is based on a standard fusion argument. Induction will be used to construct p_i, S_i, ϵ_i, a_i, b_i and L_i satisfying the following conditions:

(1) $p_i \in \mathbb{P}^\kappa$ and $p_0 \leq p$
(2) $p_{i+1} \leq p_i$
(3) $S_i : \kappa \to \mathbb{N}$ is 0 at all but finitely many ordinals — these will be denoted by $D(S_i)$
(4) $S_i(\sigma) = S_{i+1}(\sigma)$ for all but one $\sigma \in \kappa$
(5) $S_{i+1}(\sigma) \leq S_i(\sigma) + 1$ for all $\sigma \in \kappa$
(6) $L_i \in \mathbb{N}$ and $L_i < L_{i+1}$
(7) $p_{i+1}(\sigma)[L_i] = p_i(\sigma)[L_i]$ for each $\sigma \in D(S_i)$
(8) if $S_i(\sigma) > 0$ there is a maximal antichain $\mathcal{C}_{i,\sigma} \subseteq p_i(\sigma)[< L_i]$ such that $\nu(\mathcal{S}_{p_i(\sigma_j)}(s)) \geq S_i(\sigma)$ for each $s \in \mathcal{C}_{i,\sigma}$
(9) for each $\tau \in \prod_{\sigma\in D(S_i)} p_i(\sigma)[L_i]$ there is some finite $C_{i,\tau}$ and $k_{i,\tau} \in \mathbb{N}$ such that

$$p_i\langle\tau\rangle \Vdash_{\mathbb{P}^\kappa} \text{“}\check{C}_{i,\tau} = C \cap \mathbb{Q}[< k_{i,\tau}] \text{ and } C \setminus \check{C}_{i,\tau} \in \mathbb{C}^\infty_{\epsilon_i}\text{”}$$

where $p_i\langle\tau\rangle$ is defined by $p_i\langle\tau\rangle(\alpha) = p_i(\alpha)\langle\tau(\alpha)\rangle$
(10) $a_i \in \mathbb{C}_{\alpha_i}$ for some $\alpha_i < \delta_1$ and $a_i \subseteq a_{i+1}$
(11) $\lambda([b_i]) < \delta_2(i)$ and $b_i \subseteq b_{i+1}$
(12) for each $z \in \mathbb{C} \setminus [a_i \cup b_i]$ there is $T_{\sigma,z,i} \in \mathbb{P}$ for each $\sigma \in D(S_i)$ such that
   (a) $T_{\sigma,z,i} \subseteq p_i(\sigma)[\leq L_i]$
   (b) if $s \in T_{\sigma,z,i}[< L_i] \cap \mathcal{C}_{j,\sigma}$ then $\nu(\mathcal{S}_{T_{\sigma,z,i}}(s)) \geq S_j(\sigma)$
   (c) $z \notin [C_{i,\tau}]$ for any $\tau \in \prod_{\sigma\in D(S_i)} T_{\sigma,z,i}[L_i]$
   (d) $T_{\sigma,z,i}[L_i] = T_{\sigma,z,i+1}[L_i]$
(13) $2\epsilon_i k(L_i)^{2i} < \delta_1 - \alpha_i$
(14) $\lim_{n\to\infty} S_n(\sigma) = \infty$ for each element σ of the domain of some p_i

Assuming this can be done, let $A = \bigcup_{i=0}^\infty [a_i]$ and $B = \bigcup_{i=0}^\infty [b_i]$ and define

$$q(\sigma) = \bigcup_{i=0}^\infty p_i(\sigma)[L_i]$$

for each $\sigma \in \kappa$ which is in the domain of some p_i. It follows from Conditions 8 and 7 that $q \in \mathbb{P}^\kappa$. Then if $z \in \mathbb{C} \setminus [A \cup B]$ let

$$q_z(\sigma) = \bigcup_{j=0}^\infty T_{\sigma,z,j}$$

and note that it follows from Condition 12b that $q_z \in \mathbb{P}^\kappa$. From Conditions 9 and 12c it follows that $q_z \Vdash_{\mathbb{P}^\kappa}$ “$z \notin C$”.

To see that the induction can be carried out, let i be given and suppose that p_i, S_i, ϵ_i, a_i, b_i and L_i satisfying the induction hypothesis have been chosen. Choose $\bar{\sigma}$ according to some scheme which will satisfy Condition 14 and define

$$S_{i+1}(\sigma) = \begin{cases} S_i(\sigma) & \text{if } \sigma \neq \bar{\sigma} \\ S_i(\sigma) + 1 & \text{if } \sigma = \bar{\sigma}. \end{cases}$$

Let $\bar{p}_0 \leq p_i$ be such that $\bar{p}_0 = p_i$ if $S_i(\bar{\sigma}) > 0$ and, otherwise, $\bar{p}_0(\sigma) = p_i(\sigma)$ for $\sigma \neq \bar{\sigma}$ and $|\bar{p}_0(\bar{\sigma})[L_i]| = 1$. Let $m \geq S_{i+1}(\bar{\sigma})$ be such that

$$\frac{\Gamma(m)}{2} > \epsilon_i k(L_i)^i \tag{6.3}$$

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$$\Gamma(m)k(L_i) < \delta_1 - \alpha_i \tag{6.4}$$

$$\Theta(m-1) < \frac{\delta_2(i+1) - \delta_2(i)}{k(L_i)} \tag{6.5}$$

and for each $t \in \bar{p}_0(\bar{\sigma})[L_i]$ there is some $t^* \in \bar{p}_0(\bar{\sigma})$ such that $t \subseteq t^*$ and such that

$$\nu(\mathcal{S}_{\bar{p}_0(\bar{\sigma})}(t^*)) \geq m+1. \tag{6.6}$$

Let $L_{i+1} > L_i$ be so large that $|t^*| < L_{i+1}$ for each $t \in \bar{p}_0(\bar{\sigma})[L_i]$. Then let $\bar{p}_1 \leq \bar{p}_0$ be such that

- $\mathcal{C}_{i+1,\bar{\sigma}} = \{t^* \ : \ t \in \bar{p}_0(\bar{\sigma})[L_i]\}$ is a maximal antichain in $\bar{p}_1(\bar{\sigma})$
- if $t \in \bar{p}_1(\bar{\sigma})$ and $L_i \leq |t| \leq L_{i+1}$ then either
  - $|\mathcal{S}_{\bar{p}_1(\bar{\sigma})}(t)| = 1$ or
  - $\mathcal{S}_{\bar{p}_1(\bar{\sigma})}(t) = \mathcal{S}_{\bar{p}_0(\bar{\sigma})}(t)$
- $\bar{p}_1(\sigma) = \bar{p}_0(\sigma)$ for $\sigma \neq \bar{\sigma}$.

Let $\mathcal{C}_{i+1,\sigma} = \mathcal{C}_{i,\sigma}$ if $\sigma \neq \bar{\sigma}$. Let $\bar{p}_2 \leq \bar{p}_1$ be such that $\bar{p}_2(\sigma) = \bar{p}_1(\sigma)$ for $\sigma \notin D(S_i) \setminus \{\bar{\sigma}\}$ and if $\sigma \in D(S_i) \setminus \{\bar{\sigma}\}$ and $t \in \bar{p}_2(\sigma)$ and $L_i \leq |t| \leq L_{i+1}$ then $|\mathcal{S}_{\bar{p}_2(\sigma)}(t)| = 1$. Let $\epsilon_{i+1} > 0$ be so small that $2\epsilon_{i+1}k(L_{i+1})^{2(i+1)} < \delta_1 - \alpha_i - \epsilon_i k(L_i)$.

Choose $p_{i+1} \leq \bar{p}_2$ such that $p_{i+1}(\sigma)[L_{i+1}] = \bar{p}_2(\sigma)[L_{i+1}]$ for each $\sigma \in D(S_{i+1})$ — this implies that Condition 7 holds — and for each $\tau \in \prod_{\sigma \in D(S_{i+1})} p_{i+1}(\sigma)[L_{i+1}]$ there is some $C_{i+1,\tau}$ and $k_{i+1,\tau}$ such that

$$p_{i+1}\langle\tau\rangle \Vdash_{\mathbb{P}^\kappa} \text{“}\check{C}_{i+1,\tau} = C \cap \mathbb{Q}[k_{i+1,\tau}] \text{ and } C \setminus \check{C}_{i+1,\tau} \in \mathbb{C}^\infty_{\epsilon_{i+1}}\text{”}.$$

For each $t \in p_i(\bar{\sigma})[L_i]$ and $x \in \mathcal{S}_{p_i(\bar{\sigma})}(t^*)$ and $\tau \in \prod_{\sigma \in D(S_i)\setminus\{\bar{\sigma}\}} p_i(\sigma)[L_i]$ let $\rho(t,x,\tau)$ be the unique element of $\prod_{\sigma \in D(S_{i+1})} p_{i+1}(\sigma)[L_{i+1}]$ such that $\rho(t,x,\tau) \supseteq \tau(\sigma)$ for each $\sigma \in D(S_i) \setminus \{\bar{\sigma}\}$ and $\rho(t,x,\tau) \supseteq t^*$ and $\rho(t,x,\tau)(|t^*|) = x$. Let $\rho^*(t,x,\tau)$ be the unique element of $\prod_{\sigma \in D(S_i)} p_i(\sigma)[L_i]$ such that $\rho^*(t,x,\tau) = \tau$ if $S_i(\bar{\sigma}) = 0$ and $\rho^*(t,x,\tau) \supseteq \tau$ and $\rho^*(t,x,\tau)(\bar{\sigma}) = t$ if $S_i(\bar{\sigma}) > 0$.

Note that $C_{i+1,\rho(t,x,\tau)} \setminus C_{i,\rho^*(t,x,\tau)} \in \mathbb{C}_{\epsilon_i}$ by Condition 9. Define $F_t : \mathcal{S}_{p_{i+1}(\bar{\sigma})}(t^*) \to \mathbb{C}_{\epsilon_i k(L_i)^i}$ by

$$F_t(x) = \bigcup_{\tau \in \prod_{\sigma \in D(S_i)\setminus\{\bar{\sigma}\}} p_i(\sigma)[L_i]} C_{i+1,\rho(t,x,\tau)} \setminus C_{i,\rho^*(t,x,\tau)}$$

and observe that $F_t(x) \in \mathbb{C}_{\Gamma(m)/2}$ by 6.3 since

$$\left|\prod_{\sigma \in D(S_i)} p_i(\sigma)[L_i]\right| \leq k(L_i)^{|D(S_i)|} \leq k(L_i)^i.$$

By 6.6 it follows that

$$\square_{\Theta(m-1)}(\{z \in \mathbb{C} : \nu(\langle z, F_t\rangle) < m\}) < \Gamma(m)$$

for each $t \in p_i(\bar{\sigma})[L_i]$. Let b^t_{i+1} be such that $\lambda(b^t_{i+1}) < \Theta(m-1)$ and such that there is $a^t_{i+1} \in \mathbb{C}_{\Gamma(m)}$ such that

$$[a^t_{i+1}] \supseteq \{z \in \mathbb{C} : \nu(\langle z, F_t\rangle) < m\} \setminus [b^t_{i+1}]. \tag{6.7}$$

It follows from 6.5 and Induction Hypothesis 11 that if b_{i+1} is defined to be $b_i \cup \bigcup_{t \in p_i(\bar{\sigma})[L_i]} b^t_{i+1}$ then Condition 11 is satisfied by b_{i+1}. Similarly, if a_{i+1} is defined to be $a_i \cup \bigcup_{t \in p_i(\bar{\sigma})[L_i]} a^t_{i+1}$ then $a_{i+1} \in \mathbb{C}_{\alpha_{i+1}}$ where $\alpha_{i+1} = \alpha_i + \Gamma(m)k(L_i) < \delta_1$ by 6.4 and Induction Hypothesis 10.

In order to verify that Condition 12 holds let $z \in \mathbb{C} \setminus [a_{i+1} \cup b_{i+1}]$. Let $T_{z,\sigma,i+1} = T_{z,\sigma,i}$ if $\sigma \neq \bar{\sigma}$ and let $T_{z,\bar{\sigma},i+1}$ be the set of all $s \in p_{i+1}(\bar{\sigma})[\leq L_{i+1}]$ such that if $t^* \subseteq s$ then $s(|t^*|) \in \langle z, F_t\rangle$. In order to show that Condition 12b holds it suffices to show that

$$\nu(\mathcal{S}_{T_{\bar{\sigma},z,i+1}}(t^*)) \geq m > S_{i+1}(\bar{\sigma})$$

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for any $t \in p_i(\bar\sigma)[L_i]$. For any given t this follows from the fact that $z \notin [a^t_{i+1} \cup b^t_{i+1}]$ and 6.7.

In order to show that Condition 12c holds let $\tau \in \prod_{\sigma \in D(S_{i+1})} T_{\sigma,z,i}[L_{i+1}]$. Let $\bar\tau = \tau \restriction (D(S_i) \setminus \{\bar\sigma\})$. Let $t \in p_i(\bar\sigma)[L_i]$ be such that $t \subseteq \tau(\bar\sigma)$ and note that the definition of $T_{\bar\sigma,z,i}$ guarantees that $\tau(\bar\sigma)(|t^*|) \in \langle z, F_t\rangle$. In other words, $z \notin [F_t(\tau(\bar\sigma)(|t^*|))] \supseteq [C_{i+1,\rho(t,\tau(\bar\sigma)(|t^*|),\bar\tau)} \setminus C_{i,\rho^*(t,\tau(\bar\sigma)(|t^*|),\bar\tau)}]$. Moreover, since $\rho^*(t,\tau(\bar\sigma)(|t^*|),\bar\tau) \in \prod_{\sigma\in D(S_i)} p_i(\sigma)[L_i]$ it follows that $z \notin [C_{i,\rho^*(t,\tau(\bar\sigma)(|t^*|),\bar\tau)}]$ by the Induction Hypothesis 12c. Hence $z \notin [C_{i+1,\tau}]$. □

Corollary 6.2. *If $G \subseteq \mathbb{P}$ is generic over V then $V[G] \models \lambda^r_\infty(V \cap \mathbb{C}) = 1$.*

Proof. Suppose not and that

$$p \Vdash_{\mathbb{P}^\kappa} \text{“}C \in \mathbb{C}^\infty_\delta \text{ and } V \cap \mathbb{C} \subseteq [C]\text{”}$$

and $\delta < 1$. Using Lemma 6.2 find $A \subseteq \mathbb{C}$ and $B \subset \mathbb{C}$ such that $\lambda^r_\infty(A) + \lambda(B) < 1$ and $q \leq p$ such that $q \not\Vdash_{\mathbb{P}^\kappa}$ “$z \in [C]$” for each $z \in \mathbb{C} \setminus (A \cup B)$. Choose any $z \in \mathbb{C} \setminus (A \cup B)$ and $q' \leq q$ such that $q' \Vdash_{\mathbb{P}^\kappa}$ “$z \notin [C]$”. □

7. Remarks and open questions

One might expect that the methods developed here could be used to prove that for any two reals s and t such that $0 < t < s < 1$ it is consistent that sets of size $\aleph_1$ are null with respect to s-dimensional Hausdorff measure but that this is not so for t-dimensional Hausdorff measure. While this is true of most of the argument there are some slippery spots. For example. the use of Y_F in the proof of Lemma 3.4 assumes the σ-finiteness of Lebesgue measure. Lemma 3.2 might also pose some challenges to generalization. Let $\mathfrak{n}_s$ denote the least cardinal of a set which is not null with respect to s-dimensional Hausdorff measure Hence the following question remains open:

Question 7.1. Is it consistent that $0 < t < s < 1$ and $\mathfrak{n}_t < \mathfrak{n}_s$?

Question 7.2. Is it consistent that $0 < u < v < w < 1$ and $\mathfrak{n}_u < \mathfrak{n}_v < \mathfrak{n}_w$?

Question 7.3. How big can the cardinality of $\{\mathfrak{n}_s\}_{s\in(0,1)}$ be?

However, the main open problem in this area still remains Question 1.1. It would be interesting to know what the answer to this question is in the model described in §6.

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