

# ON THE COGENERATION OF COTORSION PAIRS

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ABSTRACT. Let  $R$  be a Dedekind domain. In [6], Enochs' solution of the Flat Cover Conjecture was extended as follows: (\*) If  $\mathfrak{C}$  is a cotorsion pair generated by a class of cotorsion modules, then  $\mathfrak{C}$  is cogenerated by a set. We show that (\*) is the best result provable in ZFC in case  $R$  has a countable spectrum: the Uniformization Principle  $UP^+$  implies that  $\mathfrak{C}$  is not cogenerated by a set whenever  $\mathfrak{C}$  is a cotorsion pair generated by a set which contains a non-cotorsion module.

## 1. INTRODUCTION

For any ring  $R$ , if  $\mathcal{S}$  is a class of (right)  $R$ -modules, we define

$${}^{\perp}\mathcal{S} = \{A : \text{Ext}_R^1(A, M) = 0 \text{ for all } M \in \mathcal{S}\}$$

and

$$\mathcal{S}^{\perp} = \{A : \text{Ext}_R^1(M, A) = 0 \text{ for all } M \in \mathcal{S}\}$$

If  $\mathcal{S}$  is a set (not a proper class), then  ${}^{\perp}\mathcal{S} = {}^{\perp}\{K\}$  where  $K$  is the direct product of the elements of  $\mathcal{S}$ , and  $\mathcal{S}^{\perp} = \{B\}^{\perp}$  where  $B$  is the direct sum of the elements of  $\mathcal{S}$ . (Henceforth, in an abuse of notation, we will write  ${}^{\perp}K$  instead of  ${}^{\perp}\{K\}$ , and  $B^{\perp}$  instead of  $\{B\}^{\perp}$ .)

A *cotorsion pair* (originally called a *cotorsion theory*) is a pair  $\mathfrak{C} = (\mathcal{F}, \mathcal{C})$  such that  $\mathcal{F} = {}^{\perp}\mathcal{C}$  and  $\mathcal{C} = \mathcal{F}^{\perp}$ .  $\mathfrak{C}$  is said to be *generated* (resp., *cogenerated*) by  $\mathcal{S}$  when  $\mathcal{F} = {}^{\perp}\mathcal{S}$  (resp.,  $\mathcal{C} = \mathcal{S}^{\perp}$ ).

A motivating example (for  $R$  a Dedekind domain) is the pair  $(\mathcal{F}, \mathcal{C})$  where  $\mathcal{F}$  is the class of torsion-free modules and  $\mathcal{C} = \mathcal{F}^{\perp}$ ; the members of  $\mathcal{C}$  are called cotorsion modules. Equivalently,  $K$  is cotorsion if and only if  $\text{Ext}_R^1(Q, K) = 0$ , where  $Q$  is the quotient field of  $R$  (cf. [8, §XIII.8]). Pure-injective modules are cotorsion, and torsion-free cotorsion modules are pure-injective.

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Cotorsion theories were first studied by Salce [10]; their study was given new impetus by the work of Göbel-Shelah [9]. (See, for example, [2, Chap. XVI] for an introduction to these concepts.)

In this paper we are interested in the question of when a cotorsion pair  $(\mathcal{F}, \mathcal{C})$  is cogenerated by a set, or, equivalently, when there is a single module  $B \in \mathcal{F}$  such that  $\mathcal{C} = B^\perp$ . One reason this question is of interest is that, by a result in [5], if  $(\mathcal{F}, \mathcal{C})$  is cogenerated by a set, then it is *complete*, that is, for every module  $M$ , there is an epimorphism  $\psi : N \rightarrow M$  such that  $N \in \mathcal{F}$  and  $\ker(\psi) \in \mathcal{C}$ ; in particular,  $\mathcal{F}$ -precovers exist for all  $R$ -modules. It is these ideas and results that are involved in the proof of the Flat Cover Conjecture by Enochs [1]; see the introduction to [6] for the historical sequence of events. (See also [7] and/or [14] for a comprehensive study of (pre)covers and their uses.)

The following is proved in [6]:

**Theorem 1.1.** *For any ring  $R$ , if  $\mathfrak{C} = (\mathcal{F}, \mathcal{C})$  is a cotorsion pair which is generated by a class of pure-injective modules, then  $\mathfrak{C}$  is cogenerated by a set. Moreover, if  $R$  is a Dedekind domain, the same conclusion holds when  $\mathfrak{C}$  is generated by a class of cotorsion modules, or, equivalently, when every element of  $\mathcal{C}$  is cotorsion.*

Note that  $(\mathcal{F}, \mathcal{C})$  is generated by a class of cotorsion modules if and only if  $Q \in \mathcal{F}$ , in which case every member of  $\mathcal{C}$  is cotorsion.

The case when  $\mathcal{C}$  contains non-cotorsion modules is more complicated, and the results depend on the extension of ZFC we work in. In [6] it is proved that it is consistent with ZFC that the conclusion of Theorem 1.3 holds for even more cotorsion pairs:

**Theorem 1.2.** *Gödel's Axiom of Constructibility ( $V = L$ ) implies that  $\mathfrak{C}$  is cogenerated by a set whenever  $\mathfrak{C}$  is a cotorsion pair generated by a set and  $R$  is a right hereditary ring.*

The main result of this paper is that Theorem 1.1 is the best that can be proved in ZFC (even in ZFC + GCH) for cotorsion pairs which are generated by a set — at least for certain rings, including  $\mathbb{Z}$ :

**Theorem 1.3.** *It is consistent with ZFC + GCH that if  $R$  is a Dedekind domain with a countable spectrum and  $\mathfrak{C} = (\mathcal{F}, \mathcal{C})$  is a cotorsion pair generated by a set which contains a non-cotorsion module, then  $\mathfrak{C}$  is not cogenerated by a set.*

The assumption that  $\mathfrak{C}$  is generated by a set is essential in 1.3: for example, by a classical result of Kaplansky, the cotorsion pair  $(\mathcal{P}_0, \text{Mod-}R)$  is cogenerated by a set (of countably generated modules), for any ring  $R$ . (Here,  $\mathcal{P}_0$  denotes the class of all projective modules.)

Putting together Theorems 1.1 and 1.3, we have:

**Corollary 1.4.** *Let  $R$  be a Dedekind domain with a countable spectrum, and let  $K$  be an  $R$ -module. It is provable in ZFC + GCH that there is a module  $B$  such that  $({}^\perp K)^\perp = B^\perp$  if and only if  $K$  is cotorsion.*

PROOF. If  $K$  is cotorsion, it is proved in [6] that  $B$  exists. (This is provable in ZFC alone.) The other direction follows immediately from Theorem 1.3 for the cotorsion pair  $({}^\perp K, ({}^\perp K)^\perp)$ .  $\square$

In [4] this result was proved for *countable torsion-free*  $\mathbb{Z}$ -modules  $K$ . It was also proved there that the cotorsion pair  $({}^\perp \mathbb{Z}, ({}^\perp \mathbb{Z})^\perp)$  is not complete.

Theorem 1.3 is proved in the next two sections. In the first one we prove in ZFC some preliminary results. In the following section we invoke the additional set-theoretic hypothesis  $\text{UP}^+$ .

## 2. RESULTS IN ZFC

We will make use of the following result from [5]. (See also [2, XVI.1.2 and XVI.1.3].)

**Theorem 2.1.** *Let  $B$  be an  $R$ -module and let  $\kappa$  be a cardinal  $> |R| + |B|$ . Let  $\mu$  be a cardinal  $> \kappa$  such that  $\mu^\kappa = \kappa$ . Then there is a module  $A \in B^\perp$  such that  $A = \bigcup_{\nu < \mu} A_\nu$  (continuous),  $A_0 = 0$  (or any given module of size  $< \kappa$ ), and such that for all  $\nu < \mu$ ,  $A_{\nu+1}/A_\nu$  is isomorphic to  $B$ .*

*Moreover, if, for some  $R$ -module  $K$ ,  $B \in {}^\perp K$ , then  $A/A_\nu \in {}^\perp K$  for all  $\nu < \mu$ .*  $\square$

The continuity condition on the  $A_\nu$  means that for every limit ordinal  $\sigma < \mu$ ,  $A_\sigma = \bigcup_{\nu < \sigma} A_\nu$ .

From now on,  $R$  will denote a Dedekind domain and  $Q$  will denote its quotient field. Moreover, we assume that  $Q$  is countably generated as an  $R$ -module, or, equivalently, that  $R$  has a countable spectrum.

The conditions on  $A$  in Theorem 2.1 motivate the hypotheses in the following lemmas. Recall that a module  $M$  is *reduced* if  $\text{Hom}_R(Q, M) = 0$ .

**Lemma 2.2.** *Let  $B$  be a torsion-free reduced module. Let  $\mu$  be a limit ordinal and suppose  $M = \bigcup_{\nu < \mu} M_\nu$  (continuous), where  $M_0 = 0$ , and for all  $\nu < \mu$ ,  $M_{\nu+1}/M_\nu$  is isomorphic to  $B$ . Then  $M$  is torsion-free and reduced.*

PROOF. It is clear that  $M$  is torsion-free. Suppose that there is a non-zero homomorphism, hence an embedding,  $\theta : Q \rightarrow M$ . Let  $\tau$  be minimal such that  $M_\tau$  contains a non-zero element,  $\theta(y)$ , of the range of  $\theta$ . Then  $\tau$  is not a limit ordinal; say  $\tau = \nu + 1$ , and  $\theta$  induces a non-zero map, hence an embedding, of  $Q$  into  $M/M_\nu$ . Since  $M/M_{\nu+1}$  has no torsion, this map embeds  $Q$  into  $M_{\nu+1}/M_\nu$ , which is a contradiction, since  $M_{\nu+1}/M_\nu \cong B$ .  $\square$

**Definition 2.3.** By hypothesis on  $R$  we can fix a countable set  $\{\rho_j : j \in \omega\}$  of non-units of  $R$  such that  $\{(\prod_{i < j} \rho_i)^{-1} : j \in \omega\}$  generates  $Q$  as an  $R$ -module.

**Lemma 2.4.** *Let  $B$  be a torsion-free  $R$ -module. Suppose  $M = \bigcup_{n \in \omega} M_n$  such that  $M_0 = 0$ , and for all  $n \in \omega$ ,  $M_{n+1}/M_n$  is isomorphic to  $B$ . Suppose that for some  $k \in \omega$  and all  $n \in \omega$ ,  $a_n + M_n$  is an element of  $M_{n+1}/M_n$  which does not belong to  $\rho_k(M_{n+1}/M_n)$ . Then the system of equations*

$$\{\rho_n v_{n+1} = v_n - a_n : n \in \omega\}$$

*in the variables  $\{v_n : n \in \omega\}$  does not have a solution in  $M$ .*

PROOF. Suppose, to the contrary, that there is a solution  $v_n = u_n \in M$ . We have  $u_0 \in M_m$  for some  $m \geq k$ . Since  $a_n \in M_m$  for  $n < m$ , and since  $B$  is torsion-free,  $u_n \in M_m$  for  $n \leq m$ . But then  $\rho_m u_{m+1} = u_m - a_m$  implies that  $u_{m+1} + M_m$  belongs to  $M_{m+1}/M_m$  (since  $M/M_{m+1}$  is torsion-free) and thus  $\rho_k$  divides  $a_m + M_m$  in  $M_{m+1}/M_m$ , which contradicts the choice of  $a_m$ .  $\square$

Recall that a module  $M$  is called a *splitter* if  $\text{Ext}_R^1(M, M) = 0$ . (See, for example, [11], [9], or [2, Chap. XVI].)

**Lemma 2.5.** *If  $\mathfrak{C}$  is a cotorsion pair which is generated and cogenerated by sets, then there is a torsion-free splitter which generates  $\mathfrak{C}$ .*

PROOF. Let  $\mathfrak{C} = (\mathcal{F}, \mathcal{C})$ . Let  $B, K$  be modules such that  $\mathcal{F} = {}^\perp K$  and  $\mathcal{C} = B^\perp$ . By [5, Theorem 10],  $K$  has a special  $\mathcal{F}$ -precover, i.e., there is an exact sequence  $0 \rightarrow M \rightarrow N \rightarrow K \rightarrow 0$  such that  $M \in \mathcal{C}$  and  $N \in \mathcal{F}$ . Since  $K \in \mathcal{C}$ , also  $N \in \mathcal{C}$ , and  $N \in \mathcal{C} \cap \mathcal{F}$  is a splitter.

We have  $\mathcal{F} = {}^\perp N$  (since clearly  $\mathcal{F} \subseteq {}^\perp N$ , and  ${}^\perp N \subseteq {}^\perp K = \mathcal{F}$ ). Let  $T$  be the torsion part of  $N$ . Then  $T$  is a direct sum of its  $p$ -components,  $T = \bigoplus_{p \in m\text{Spec}(R)} T_p$ . If  $T_p \neq 0$ , then  $\text{Ext}_R^1(R/p, N) = 0$ , so  $\text{Hom}_R(R/p, E(N)/N) = 0$ , and hence  $\text{Hom}_R(R/p, E(T_p)/T_p) = 0$ . Therefore  $T_p$  is divisible. So  $N = T \oplus L$  where  $L$  is a torsion-free splitter. Since  $T$  is divisible,  ${}^\perp L = {}^\perp N = \mathcal{F}$ .  $\square$

**Lemma 2.6.** *Suppose that  $\mathfrak{C}$  is a cotorsion pair which is cogenerated by a cotorsion module, and generated by a set. Then  $\mathfrak{C}$  is cogenerated by a cotorsion module of the form  $B \oplus T$  where  $B$  is torsion-free,  $T$  is torsion, and for every prime  $p$  such that  $R/p$  is a submodule of  $T$ ,  $pB = B$ .*

PROOF. Let  $\mathfrak{C} = (\mathcal{F}, \mathcal{C})$  and let  $K$  be a module such that  $\mathcal{F} = {}^\perp K$ . If  $K$  is cotorsion, then by [6, Thm. 16], there is a set of maximal ideals  $P$  such that  $\mathcal{F}$  is the set of all modules with zero  $p$ -torsion part for all  $p \in P$ . Then  $\mathcal{C} = B^\perp$  where  $B = Q \oplus \bigoplus_{q \notin P} R/q$ .

So we can assume that  $K$  is not cotorsion, and that, by Lemma 2.5,  $K$  is torsion-free.

Let  $C$  be a cotorsion module such that  $\mathcal{C} = C^\perp$ . We have  $C = D \oplus E$  where  $D$  is divisible and  $E$  reduced. Since  $K$  is not cotorsion,  $D$  is torsion. Denote by  $T'$  the torsion part of  $E$ . By a theorem of Harrison-Warfield, [8, XIII.8.8], we have  $E = B \oplus G$  where  $B$  is torsion-free reduced and pure-injective, and  $G$  is a cotorsion hull of  $T'$ . We claim that there is an exact sequence  $0 \rightarrow T' \rightarrow G \rightarrow Q^{(\delta)} \rightarrow 0$  for some  $\delta \geq 0$ .

Indeed, by [14, 3.4.5],  $G$  is a cotorsion envelope of  $T'$  in the sense of Enochs. Now by Theorem 2.1 there is a cotorsion preenvelope  $G'$  of  $T'$  such that  $G'/T'$  is the union of a continuous chain with successive quotients isomorphic to  $Q$ , and hence  $G'/T' \cong Q^{(\gamma)}$  for some  $\gamma$ . The claim now follows since  $G/T'$  is isomorphic to a direct summand of  $G'/T'$  by [14, 1.2.2]

Since  $K$  is torsion-free and  $G \in \mathcal{C}$ , an application of  $\text{Hom}_R(-, K)$  yields

$$0 = \text{Hom}_R(T', K) \rightarrow \text{Ext}_R^1(Q^{(\delta)}, K) \rightarrow \text{Ext}_R^1(G, K) = 0.$$

Thus,  $\text{Ext}(Q^{(\delta)}, K) = 0$ , so since  $K$  is not cotorsion,  $\delta = 0$  and  $T' = G$ . Hence  $C = B \oplus T$  where  $T = T' \oplus D$  is torsion.

By [7, 5.3.28], there is a set  $P$  of maximal ideals of  $R$  such that  $B \cong \prod_{p \in P} J_p$  where  $J_p$  is the  $p$ -adic completion of a free module over the localization of  $R$  at  $p$ . In particular,  $qB = B$  for all maximal ideals

$q \notin P$ . For each  $p \in P$ , there is an exact sequence  $0 \rightarrow J_p \rightarrow E(J_p) \rightarrow I_p \rightarrow 0$  where  $I_p$  is a direct sum of copies of  $E(R/p)$ , and  $E(J_p) = Q^{(\alpha_p)}$  for some  $\alpha_p > 0$ .

Let  $q$  be a maximal ideal such that  $R/q$  embeds in  $T$ . Assume  $q \in P$ . Then an application of  $\text{Hom}_R(-, K)$  yields

$$0 = \text{Ext}_R^1(I_q, K) \rightarrow \text{Ext}_R^1(Q^{(\alpha_q)}, K) \rightarrow \text{Ext}_R^1(J_q, K) = 0.$$

The first Ext is zero because  $R/q \hookrightarrow T$ ; so  $R/q \in \mathcal{F} = {}^\perp\mathcal{C}$  and thus  $E(R/q) \in \mathcal{F}$  by [5, Lemma 1] since  $E(R/q)$  is the union of a continuous chain of modules with successive quotients isomorphic to  $R/q$ ; the last Ext is zero because  $J_q \in \mathcal{F}$ . So  $K$  is cotorsion, a contradiction. This proves that  $q \notin P$  and hence  $qB = B$ .  $\square$

### 3. PROOF OF THEOREM 1.3

Let  $\mathfrak{C} = (\mathcal{F}, \mathcal{C})$  be a cotorsion pair cogenerated by a set, and generated by a non-cotorsion module  $K$ . We aim to produce a contradiction by constructing  $H \in {}^\perp K (= \mathcal{F})$  and  $A \in \mathcal{C}$  such that  $\text{Ext}_R^1(H, A) \neq 0$ . We do this assuming GCH plus the following principle, which is consistent with ZFC + GCH (cf. [3] or [12]):

**(UP<sup>+</sup>)** For every cardinal  $\mu$  of the form  $\tau^+$  where  $\tau$  is singular of cofinality  $\omega$  there is a stationary subset  $S$  of  $\mu$  consisting of limit ordinals of cofinality  $\omega$  and a ladder system  $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$  which has the  $\lambda$ -uniformization property for every  $\lambda < \tau$ .

Recall that if  $S$  is a subset of an uncountable cardinal  $\mu$  which consists of ordinals of cofinality  $\omega$ , a *ladder system* on  $S$  is a family  $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$  of functions  $\zeta_\delta : \omega \rightarrow \delta$  which are strictly increasing and have range cofinal in  $\delta$ . For a cardinal  $\lambda$ , we say that  $\bar{\zeta}$  has the  $\lambda$ -uniformization property if for any functions  $c_\delta : \omega \rightarrow \lambda$  for  $\delta \in S$ , there is a pair  $(f, f^*)$  where  $f : \mu \rightarrow \omega$  and  $f^* : S \rightarrow \omega$  such that for all  $\delta \in S$ ,  $f(\zeta_\delta(\nu)) = c_\delta(\nu)$  whenever  $f^*(\delta) \leq \nu < \omega$ . We refer to [2, Chap. XIII] for more details.

We consider two cases: (1)  $\mathfrak{C}$  is cogenerated by a cotorsion module; and (2) the negation of (1).

The module  $H$  will be the same in both cases. Let  $\bar{\zeta} = \{\zeta_\delta : \delta \in S\}$  be as in (UP<sup>+</sup>) for this  $\mu$ . We also use the notation from Definition 2.3. Let  $H = F/L$  where  $F$  is the free module with the basis  $\{y_{\delta,n} : \delta \in S, n \in \omega\} \cup \{x_j : j < \mu\}$  and  $L$  is the free submodule with the

basis  $\{w_{\delta,n} : \delta \in S, n \in \omega\}$  where

$$(1) \quad w_{\delta,n} = y_{\delta,n} - \rho_n y_{\delta,n+1} + x_{\zeta_\delta(n)}.$$

Then  $H$  is a module of cardinality  $\mu$  and the uniformization property of  $\bar{\zeta}$  implies that  $H \in {}^\perp K$ . (In fact,  $H \in {}^\perp K$  for any module  $K$  of cardinality  $< \tau$ . See [2, Chap. XIII] or [13].)

Assuming we are in Case (1), let  $B \oplus T$  be a cogenerator of  $\mathfrak{C}$  as given in Lemma 2.6. Let  $\kappa \geq \max(|B|, |R|, |K|)$  and let  $\mu = \tau^+ = 2^\tau$  where  $\tau > \kappa$  is a singular cardinal of cofinality  $\omega$ . Then  $\mu^\kappa = \mu$ . Let  $A = \bigcup_{\nu < \mu} A_\nu$  be as in Theorem 2.1 for this  $B$  and  $\mu$ ; so, in particular,  $A \in B^\perp$ . Note that then  $A \in (B \oplus T)^\perp = \mathfrak{C}$  because  $T^\perp$  consists of precisely those modules  $M$  such that  $pM = M$  whenever  $R/p \hookrightarrow T$ . Note that  $A/A_\delta$  is torsion-free for all  $\delta \in \mu$ , because  $B$  is torsion-free.

We need to show that  $\text{Ext}_R^1(H, A) \neq 0$ ; in other words, to define a homomorphism  $\psi : L \rightarrow A$  which does not extend to  $F$ .

Since  $B$  is reduced there is a  $k \in \omega$  such that  $\rho_k B \neq B$ ; then for all  $\delta \in S$  and  $n \in \omega$  we can choose  $a_{\delta,n} \in A_{\delta+n+1}$  such that  $a_{\delta,n} + A_{\delta+n} \notin \rho_k(A_{\delta+n+1}/A_{\delta+n})$ . We claim that

( $\boxtimes$ ) for all  $\delta \in S$ , the family of equations

$$\mathcal{E}_\delta = \{\rho_n v_{n+1} = v_n - (a_{\delta,n} + A_\delta) : n \in \omega\}$$

does not have a solution in  $A/A_\delta$ .

Supposing, for the moment, that this claim is true, we will prove that  $\text{Ext}_R^1(F/L, A) \neq 0$ . Define  $\psi : L \rightarrow A$  by  $\psi(w_{\delta,n}) = a_{\delta,n}$  for all  $\delta \in S, n \in \omega$ . Suppose, to obtain a contradiction, that  $\psi$  extends to a homomorphism  $\varphi : F \rightarrow A$ . The set of  $\delta < \mu$  such that  $\varphi(x_j) \in A_\delta$  for all  $j < \delta$  is a club,  $C$ , in  $\mu$ , so there exists  $\delta \in S \cap C$ . By applying  $\varphi$  to the relations (1), and since  $\varphi(x_j) \in A_\delta$  for all  $j < \delta$ , we have that  $v_n = \varphi(y_{\delta,n}) + A_\delta$  is a solution to the equations in  $A/A_\delta$ , a contradiction.

Thus it remains to prove ( $\boxtimes$ ). Suppose that ( $\boxtimes$ ) is false for some  $\delta \in S$ , and that for some  $\{b_n : n \in \omega\} \subseteq A$ ,  $v_n = b_n + A_\delta$  is a solution to  $\mathcal{E}_\delta$ . There are two subcases.

Suppose first that  $b_0 + A_{\delta+\omega}$  is a non-zero element of  $A/A_{\delta+\omega}$ . Then  $A/A_{\delta+\omega}$  contains a copy of  $Q$  (generated over  $R$  by the cosets of the  $b_n, n \in \omega$ ). But this contradicts Lemma 2.2 (with  $M = A/A_{\delta+\omega}, M_\nu = A_{\delta+\omega+\nu}/A_{\delta+\omega}$ ).

Otherwise we can prove by induction that  $b_n \in A_{\delta+\omega}$  for all  $n \in \omega$  because  $A/A_{\delta+\omega}$  has no torsion and  $\rho_n(b_{n+1} + A_{\delta+\omega}) = b_n + A_{\delta+\omega}$ . Thus there is a solution of

$$\{\rho_n v_{n+1} = v_n - (a_{\delta,n} + A_\delta) : n \in \omega\}$$

in  $A_{\delta+\omega}/A_\delta$ . But this contradicts Lemma 2.4 (with  $M = A_{\delta+\omega}/A_\delta$ ,  $M_n = A_{\delta+n}/A_\delta$  and  $a_n = a_{\delta,n} + A_\delta$ ).

This completes the proof in Case (1).

Now supposing we are in Case (2), let  $B$  be a module cogenerating  $\mathfrak{C}$ . Let  $\kappa \geq \max(|B|, |R|, |K|)$  and let  $\mu = \tau^+ = 2^\tau$  where  $\tau > \kappa$  is a singular cardinal of cofinality  $\omega$ . Let  $A = \bigcup_{\nu < \mu} A_\nu$  be as in Theorem 2.1 for this  $B$  and  $\mu$ ; so  $A \in B^\perp$ . Let  $H$  be as above.

Then for all  $\delta \in \mu$ ,  $A/A_\delta$  cogenerates  $\mathfrak{C}$  since the construction of  $B$  and Lemma 1 of [5] implies that  $M \in (A/A_\delta)^\perp$  whenever  $M \in B^\perp$ . Hence, since we are in Case (2),  $\text{Ext}_R^1(Q, A/A_\delta) \neq 0$  for all  $\delta \in \mu$ .

Now  $Q \cong F_\delta/L_\delta$  where  $F_\delta$  is the free module with the basis  $\{y_{\delta,n} : n \in \omega\}$  and  $L_\delta$  is the free submodule with the basis  $\{w'_{\delta,n} : \delta \in S, n \in \omega\}$  where  $w'_{\delta,n} = y_{\delta,n} - \rho_n y_{\delta,n+1}$ . Hence there is a homomorphism  $\psi_\delta : L_\delta \rightarrow A/A_\delta$  which does not extend to  $F_\delta$ .

Let  $\pi_\delta : A \rightarrow A/A_\delta$  be the canonical projection. Define  $\psi : L \rightarrow A$  so that  $\pi_\delta \psi(w_{\delta,n}) = \psi_\delta(w'_{\delta,n})$ . In order to prove  $\text{Ext}_R^1(H, A) \neq 0$ , we will show that  $\psi$  does not extend to a homomorphism  $\varphi : F \rightarrow A$ . If it did, there would exist  $\delta \in S \cap C$  where  $C$  is the club of all  $\delta < \mu$  such that  $\varphi(x_j) \in A_\delta$  for all  $j < \delta$ . But then  $\pi_\delta \circ (\varphi \upharpoonright F_\delta)$  would be an extension of  $\psi_\delta$ , a contradiction.

This completes the proof of Theorem 1.3.  $\square$

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