ON HAUSDORFF ULTRAFILTERS

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Abstract. An ultrafilter U is Hausdorff if for any two functions $f, g \in \omega^\omega$, $f(U) = g(U)$ iff $f{\restriction}X = g{\restriction}X$ for some $X \in U$. We will show that the statement that Hausdorff ultrafilters are dense in the Rudin-Keisler order is independent of ZFC.

1. Introduction.

For $f \in \omega^\omega$ and an ultrafilter U on ω define $f(U) = \{X \subseteq \omega : f^{-1}(X) \in U\}$, and for $f, g \in \omega^\omega$ we say that $f = g \mod U$ if there is $X \in U$ such that $f(n) = g(n)$ for $n \in X$.

We say that U is Hausdorff if for any two functions $f, g \in \omega^\omega$, if $f(U) = g(U)$ then $f = g \mod U$.

Let FtO be the collection of all finite-to-one functions $f \in \omega^\omega$. Recall that an ultrafilter U is a p-point if for every function $f \in \omega^\omega$ either there is n such that $f^{-1}(\{n\}) \in U$ or there exists $g \in \mathsf{FtO}$ such that $f = g \mod U$. Similarly, U is Ramsey if for every function $f \in \omega^\omega$ either there is n such that $f^{-1}(\{n\}) \in U$ or there exists a one-to-one function $g \in \omega^\omega$ such that $f = g \mod U$.

In this paper we will assume that all ultrafilters U, and their images $f(U)$ are non-principal.

It is worth mentioning that the following appears as an exercise in [8]. If $f(U) = U$ then $f = id \mod U$. Therefore, if U is not Hausdorff, then this is witnessed by two functions, both not one-to-one mod U. It follows from it that Ramsey ultrafilters are Hausdorff.

The notion of a Hausdorff ultrafilters was reintroduced and studied by Mauro Di Nasso, Marco Forti and others in a sequence of papers ([6], [5], [2] and [4]) in context of topological extensions. They used the name Hausdorff because Hausdorff ultrafilters are precisely those ultrafilters whose ultrapowers equipped with the standard topology are Hausdorff topological spaces. In this paper we will show that it is consistent that for every ultrafilter U there exists a function $f \in \omega^\omega$ such that $f(U)$ is Hausdorff. A counterexample to this theorem is an ultrafilter called strongly non-Hausdorff.

Definition 1. *An ultrafilter U is strongly non-Hausdorff if for every $f \in \omega^\omega$, $f(U)$ is either a trivial ultrafilter or $f(U)$ is not Hausdorff.*

We will prove the following two theorems:

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Theorem 2. *Assume* CH*. There exists a strongly non-Hausdorff p-point.*

Theorem 3. *It is consistent that there are no strongly non-Hausdorff ultrafilters.*

2. A construction of a non-Hausdorff ultrafilter.

Let $I \subseteq \omega$ be a finite set and let $\Delta = \{(n,n) : n \in \omega\}$. Denote by $[I]^2 = (I \times I) \setminus \Delta$. For a set $X \subseteq [I]^2$ define

$$\|X\|_I = \min \left\{ k : \exists \{A_i, B_i : i \leq k\}\ \forall i < k\ A_i \cap B_i = \emptyset \text{ and } X \subseteq \bigcup_{i \leq k} A_i \times B_i \right\}.$$

We will drop the subscript I if it is clear from the context what it is.

Lemma 4. (1) $\left\|[I]^2\right\|_I \longrightarrow \infty$ *as* $|I| \to \infty$.
(2) $\|X \cup Y\|_I \leq \|X\|_I + \|Y\|_I$,
(3) *if* $Z \subseteq I$ *and* $X \subseteq [I]^2$, $\|X\|_I > 2$, *then either* $\left\|[Z]^2 \cap X\right\|_I \geq \|X\|_I/2 - 1$ *or* $\left\|[I \setminus Z]^2 \cap X\right\|_I \geq \|X\|_I/2 - 1$.

Proof. If (1) fails then there is $k \in \omega$ and sets $\{A^n_j, B^n_j : n, j \leq k\}$ such that $A^n_j \cap B^n_j = \emptyset$ for $j \leq k$ and $[n]^2 = \bigcup_{j \leq k} A^n_j \times B^n_j$. By compactness we get sets $\{A^\omega_j, B^\omega_j : j \leq k\}$ such that $A^\omega_j \cap B^\omega_j = \emptyset$ for $j \leq k$ and $[\omega]^2 = \bigcup_{j \leq k} A^\omega_j \times B^\omega_j$, which is not possible.

A more direct argument shows that $\left\|[I]^2\right\|_I \geq |I| - 2$.

(2) is obvious.

(3) Note that

$$\|X\|_I \leq \left\| \left([Z]^2 \cup [I \setminus Z]^2 \cup (Z \times (I \setminus Z)) \cup ((I \setminus Z) \times Z)\right) \cap X \right\|_I \leq \left\|[Z]^2 \cap X\right\|_I + \left\|[I \setminus Z]^2 \cap X\right\|_I + 1 + 1.$$

Thus

$$\left\|[Z]^2 \cap X\right\|_I + \left\|[I \setminus Z]^2 \cap X\right\|_I \geq \|X\|_I - 2.$$

□

For $I \in [\omega]^{<\omega}$ let $\pi_0, \pi_1 : [I]^2 \longrightarrow I$ be projections onto first and second coordinate respectively.

Lemma 5. *Suppose that* $X \subseteq [I]^2$, *and* $\|X\|_I > 1$. *Then* $\pi_0(X) \cap \pi_1(X) \neq \emptyset$.

Proof. Since $X \subseteq \pi_0(X) \times \pi_1(X)$, it follows that if $\pi_0(X) \cap \pi_1(X) = \emptyset$ then $\|X\|_I \leq 1$. □

Next we define functions $f^0, g^0 \in$ FtO that will witness that ultrafilter V_0 that we are about to construct is not Hausdorff.

Let $\{I_k, J_k : k \in \omega\}$ be two sequences of disjoint consecutive intervals such that for $k \in \omega$,

(1) $\left\|[I_k]^2\right\|_{I_k} \geq 2^{2^k}$,
(2) $|J_k| = |[I_k]^2|$.

Bijection implicit in (2) allows us to define projections $\pi^k_0, \pi^k_1 : J_k \longrightarrow I_k$. Let $f^0 = \bigcup_k \pi^k_0$ and $g^0 = \bigcup_k \pi^k_1$. Note that $f^0(x) \neq g^0(x)$ for any $x \in J_k = [I_k]^2$, $k \in \omega$.

As a warm-up let us use these definitions to show the following:

Lemma 6. *Assume* CH. *There exists a p-point that is not Hausdorff.*

Proof. We will need the following easy observation:

Lemma 7. *If $f, g \in \mathsf{FtO}$ and U is an ultrafilter then the following conditions are equivalent:*

(1) $f(U) \neq g(U)$,
(2) $f[X] \cap g[X] = \emptyset$ *for some* $X \in U$. □

We will build an ultrafilter V_0 on the set $\bigcup_k [I_k]^2$ which we identified with ω. Let $\{Z_\alpha : \alpha < \omega_1\}$ be enumeration of $[\omega]^\omega$.

We will build by induction a sequence $\{X_\alpha : \alpha < \omega_1\}$ so that

(1) $\forall \beta < \alpha \; X_\alpha \subseteq^\star X_\beta$,
(2) $X_{\alpha+1} \cap Z_\alpha = \emptyset$ or $X_{\alpha+1} \subseteq Z_\alpha$ for all α.
(3) for every $\alpha < \omega_1$, $f^0[X_\alpha] \cap g^0[X_\alpha] \neq \emptyset$.
(4) for every $\alpha < \omega_1$, $\limsup_k \|X_\alpha \cap J_k\|_{I_k} = \infty$.

Let $V_0 = \{X : \exists \alpha \; X_\alpha \subseteq^\star X\}$. Note that the conditions (1) and (2) guarantee that V_0 is a p-point, and lemma 7 and (3) implies that $f^0(V_0) = g^0(V_0)$. Finally, (4) is the requirement that (by lemma 5) implies (3).

Successor step. Suppose that X_α is given. Find a strictly increasing sequence $\{l_k : k \in \omega\}$ such that the set $A = \{k : \|X_\alpha \cap J_k\|_{I_k} = l_k\}$ is infinite. Let $A_0 = \{k : \|X_\alpha \cap Z_\alpha \cap J_k\|_{I_k} \geq l_k/2 - 1\}$ and $A_1 = \{k : \|(X_\alpha \setminus Z_\alpha) \cap J_k\|_{I_k} \geq l_k/2 - 1\}$. By lemma 4(2), one of these sets, say A_0, is infinite. Let $X_{\alpha+1} = \bigcup_{k \in A_0} X_\alpha \cap Z_\alpha \cap J_k$. The other case is the same.

Limit step. Given $\{X_\beta : \beta < \alpha < \omega_1\}$ let $\{\beta_k : k \in \omega\}$ be an increasing sequence cofinal in α. By finite modifications we can assume that $X_{\beta_{k+1}} \subseteq X_{\beta_k}$ for all k. Build by recursion a strictly increasing sequence $\{u_k : k \in \omega\}$ such that

$$\forall k \; \forall j \leq k \; \exists i \in [u_k, u_{k+1}) \; \left\|X_{\beta_j} \cap J_i\right\|_{I_i} \geq k,$$

and let

$$X_\alpha = \bigcup_k \left(X_{\beta_k} \cap \bigcup_{i \in [u_k, u_{k+1})} J_i \right).$$

It is clear that X_α satisfies (1) and (4). □

Observe that CH was only needed in the limit step. If we do not require that that V_0 is a p-point then we have the following:

Theorem 8. *There exists an ultrafilter that is not Hausdorff.*

Proof. As in lemma 6, we will build an ultrafilter on the set $\bigcup_k [I_k]^2$. Let

$$\mathcal{I} = \left\{ X \subseteq \bigcup_k [I_k]^2 : \limsup_k \|X \cap J_k\|_{I_k} < \infty \right\}.$$

Note that $\mathcal{I}$ is an ideal, and let U be any ultrafilter disjoint with $\mathcal{I}$. Functions f^0, g^0 witness that U is not Hausdorff. □

These constructions are optimal. Suppose that $f_0, f_1 \in \omega^\omega$ witness that some ultrafilter U is not Hausdorff. Without loss of generality we can assume that $f_0(n) \neq f_1(n)$ for all n. Let $\mathcal{I}_{f_0,f_1}$ be the ideal generated by $[\omega]^{<\omega} \cup \{X : f_0[X] \cap f_1[X] = \emptyset\}$. Clearly $U \cap \mathcal{I}_{f_0,f_1} = \emptyset$. Moreover, if V is another ultrafilter which is disjoint with

$\mathcal{I}_{f_0,f_1}$, then V is not Hausdorff as witnessed by the same functions. If U is a p-point then $\mathcal{I}_{f_0,f_1}$ is defined as $\mathcal{I}$ above. To see it, let us start with the following observation.

Lemma 9. *Suppose that U is a p-point. The following conditions are equivalent.*

1. *U is not Hausdorff,*
2. *there exist functions $f_0, f_1 \in \omega^\omega$ and a sequence of disjoint intervals $\langle I_n : n \in \omega\rangle$ such that*
   (a) *$\forall n\ f_0(n) \neq f_1(n)$,*
   (b) *$f_0(U) = f_1(U)$,*
   (c) *$\forall n\ \forall i = 0, 1\ f_i{\restriction} I_n : I_n \longrightarrow I_n$.*

Proof. One implication is obvious we will prove the other. Suppose that U is not Hausdorff and let functions $\bar{f}_i$, $i = 0, 1$ witness that. We can assume that sets $A^i_n = \bar{f}_i^{-1}(\{n\}) \notin U$ for all n and $i = 0, 1$. Moreover, by restriction to an element of U we can assume that $A^0_n \cap A^1_n = \emptyset$ for all n. Since U is a p-point there is a set $B \in U$ such that $B \cap A^i_n$ is finite for all n and i. Let $k_0 = 0$ and define $k_{n+1} = \max\{A^i_n \cap B : A^i_n \cap B \cap [0, k_n] \neq \emptyset,\ i = 0, 1\}$. Without loss of generality $C = \bigcup_{n\in\omega}[k_{2n}, k_{2n+1}) \in U$. Put $I_n = [k_{2n}, k_{2n+2})$ for $n \in \omega$ and let $A_i = \bigcup_n\{A^i_n \setminus k_{2n} : A^i_n \cap B \cap [k_{2n}, k_{2n+1}) \neq \emptyset\}$. Finally, put

$$f_0(n) = \begin{cases} \bar{f}_0(n) & \text{if } n \in A_0 \\ \min(I_k) & \text{if } n \in I_k \setminus A_0 \end{cases} \quad \text{and } f_1(n) = \begin{cases} \bar{f}_1(n) & \text{if } n \in A_1 \\ \max(I_k) & \text{if } n \in I_k \setminus A_1 \end{cases}.$$

Clearly for $i = 0, 1$, $\bar{f}_i = f_i \mod U$, as exemplified by $B \cap C$, and f_0, f_1 have the other required properties as well. □

Suppose that $f_0, f_1, \langle I_n : n \in \omega\rangle$ are as in Lemma 9. For a set $X \subseteq I_n$ define

$$\|X\|_n = \min\{k : X = X_1 \cup \cdots \cup X_k \ \&\ \forall i \leq k\ f_0[X_i] \cap f_1[X_i] = \emptyset\}.$$

Then $\mathcal{I}_{f_0,f_1} = \{X \subseteq \omega : \exists k\ \forall n\ \|X \cap I_n\|_n \leq k\}$.

3. A construction of a strongly non-Hausdorff ultrafilter under CH.

Now we are ready to prove Theorem 2 and to construct a p-point ultrafilter U_0 whose all finite-to-one images are not Hausdorff.

Let $\langle h_\alpha, Z_\alpha : \alpha < \omega_1\rangle$ be enumeration of $\mathsf{FtO} \times [\omega]^\omega$. We will construct a sequence $\langle X_\alpha, e_\alpha : \alpha < \omega_1\rangle$ such that $U_0 = \{X \in [\omega]^\omega : \exists \alpha\ X_\alpha \subseteq^\star X\}$ is the ultrafilter that we are looking for. We will require that

1. $\forall \beta < \alpha\ X_\alpha \subseteq^\star X_\beta$,
2. for all α either $X_\alpha \subseteq Z_\alpha$ or $X_\alpha \cap Z_\alpha = \emptyset$,
3. $f^0 \circ e_\alpha, g^0 \circ e_\alpha$ witness that $h_\alpha(U_0)$ is not Hausdorff.

As before, (1) and (2) guarantees that U_0 is a p-point, and (3) implies that U_0 is strongly non-Hausdorff.

Definition 10. *A finite set $Y \subseteq \omega$ is a (n, β)-witness if there exists $k \in \omega$ such that $\|e_\beta \circ h_\beta[Y] \cap J_k\|_{I_k} \geq n$.*

To satisfy (3), we demand that,

(4) $\forall \beta \leq \alpha < \omega_1\ \limsup_k \|e_\beta \circ h_\beta[X_\alpha] \cap J_k\|_{I_k} = \infty$, or equivalently $\forall n\ \forall \beta \leq \alpha\ \exists Y \in [X_\alpha]^{<\omega}\ Y$ is a (n, β)-witness.

The last condition concerns the inductive requirement for the function e_α:

(5) for all $\beta < \alpha$,

$$\lim_{n\in\omega} \max\{k : (e_\beta \circ h_\beta)^{-1}(\{n\}) \cap X_\alpha \text{ contains a } (k,\beta)\text{-witness}\} = \infty.$$

SUCCESSOR STEP.

Suppose that $\{X_\beta : \beta < \alpha\}$ satisfying (1)-(5) are already defined and we want to define $X_{\alpha+1}$ satisfying (1)-(5).

CASE 1. If for some $\beta \leq \alpha$, $\limsup_k \|e_\beta \circ h_\beta[Z_\alpha \cap X_\alpha] \cap J_k\|_{I_k} < \infty$ then let $X_{\alpha+1} = X_\alpha \setminus Z_\alpha$.

CASE 2. If for some $\beta \leq \alpha$, $\limsup_k \|e_\beta \circ h_\beta[X_\alpha \setminus Z_\alpha] \cap J_k\|_{I_k} < \infty$ then let $X_{\alpha+1} = X_\alpha \cap Z_\alpha$.

In all other cases let $X_{\alpha+1} = X_\alpha \cap Z_\alpha$.

We have to check that cases 1 and 2 are exclusive. By the inductive hypothesis,

$$\forall \beta \leq \alpha \ \limsup_k \|e_\beta \circ h_\beta[X_\alpha] \cap J_k\|_{I_k} = \infty.$$

Assume that for the two possible choices for $X_{\alpha+1}$: $A = X_\alpha \cap Z_\alpha$ and $B = X_\alpha \setminus Z_\alpha$ one of them is rejected. Let $\beta_0 \leq \alpha$ be the minimal ordinal witnessing this. Without loss of generality we can assume that for some $\beta_0 \leq \alpha$,

$$\limsup_k \|e_{\beta_0} \circ h_{\beta_0}[A] \cap J_k\|_{I_k} < \infty.$$

The following lemma will complete the construction:

Lemma 11. *For every $\beta \leq \alpha$,*

$$\limsup_k \|e_\beta \circ h_\beta[B] \cap J_k\|_{I_k} = \infty.$$

Proof. By minimality of β_0, the statement is true for $\beta < \beta_0$. By the induction hypothesis it is also true for $\beta = \beta_0$. In particular, we must have $\beta_0 < \alpha$.

Suppose that the Lemma is false and let $\beta_1 > \beta_0$ be the first ordinal such that for some $N \in \omega$,

$$\limsup_k \|e_{\beta_1} \circ h_{\beta_1}[B] \cap J_k\|_{I_k} < N.$$

Since by the induction hypothesis, $X_\alpha = A \cup B$ contains a sequence of (k_n, β_1)-witnesses where $k_n \longrightarrow \infty$, it follows that A contains a sequence of $(k_n - N, \beta_1)$-witnesses. Since $e_{\beta_1} \circ h_{\beta_1}[A \cup B] = e_{\beta_1} \circ h_{\beta_1}[A] \cup e_{\beta_1} \circ h_{\beta_1}[B]$, it means that

$$e_{\beta_1} \circ h_{\beta_1}[A] \setminus e_{\beta_1} \circ h_{\beta_1}[B] \text{ is infinite.}$$

By the induction hypothesis, for all but finitely many $n \in e_{\beta_1} \circ h_{\beta_1}[A] \setminus e_{\beta_1} \circ h_{\beta_1}[B]$, $(h_{\beta_1} \circ e_{\beta_1})^{-1}(\{n\}) \cap A = (h_{\beta_1} \circ e_{\beta_1})^{-1}(\{n\}) \cap (A \cup B)$ contains an (l_n, β_0)-witness, where $l_n \longrightarrow \infty$. This means that $\limsup_k \|e_{\beta_0} \circ h_{\beta_0}[A] \cap J_k\|_{I_k} = \infty$, a contradiction.

□

Finally we define the function $e_{\alpha+1}$. Let $\{\gamma_k : k \in \omega\}$ be an enumeration of α.

Define strictly increasing sequence $\langle u_k : k \in \omega\rangle$ such that

(1) $\forall k\ \forall j \leq k\ [u_k, u_{k+1}) \cap X_{\alpha+1}$ contains a (k, γ_j)-witness,

(2) $\forall k\ [u_k, u_{k+1}) \cap h_{\alpha+1}[X_{\alpha+1}] \neq \emptyset$.

Define $e_{\alpha+1}(l) = k \iff j \in [u_k, u_{k+1})$. It is easy to see that the inductive conditions are satisfied.

LIMIT STEP. Suppose that $\{X_\beta : \beta < \alpha\}$ are defined and α is a limit ordinal. Let $\{\beta_k : k \in \omega\}$ be an increasing sequence cofinal in α, and let $\{\gamma_k : k \in \omega\}$

be an enumeration of α such that $\gamma_j \leq \beta_k$ for $j \leq k$. By finite modifications we can assume that $X_{\beta_{k+1}} \subseteq X_{\beta_k}$ for all k. Build by recursion a strictly increasing sequence $\langle u_k : k \in \omega\rangle$ such that

$$\forall k\ \forall j \leq k\ [u_k, u_{k+1}) \cap X_{\beta_k} \text{ contains a } (k, \beta_j)\text{-witness},$$

and let

$$X_\alpha = \bigcup_k X_{\beta_k} \cap [u_k, u_{k+1}),$$

and $e_\alpha(l) = k \iff l \in [u_k, u_{k+1})$. It is clear that X_α satisfies (1)-(5).

4. A model where there are no strongly non-Hausdorff ultrafilters

In the next two sections we will show that:

Theorem 12. *It is consistent with* ZFC *that for every ultrafilter U there is $h \in \mathsf{FtO}$ such that $h(U)$ is a Hausdorff p-point. In particular, it is consistent that there are no strongly non-Hausdorff ultrafilters.*

In order to prove this theorem we will show that there exists a proper forcing notion $\mathbb{M}$ which has the following properties:

1. If U is an ultrafilter in $\mathbf{V}$ then $\mathbf{V}^{\mathbb{M}} \models \exists h \in \mathsf{FtO}\ h(U)$ is a p-point ultrafilter,
2. If U is a p-point in $\mathbf{V}$ then $\mathbf{V}^{\mathbb{M}} \models \exists h \in \mathsf{FtO}\ h(U)$ is a Hausdorff p-point,
3. $\mathbb{M}$ preserves p-points,
4. $\mathbb{M}$ preserves Hausdorff p-points.

We will show that this suffices for the proof. Let $\mathbb{M}_{\omega_2}$ be a countable support support iteration of $\mathbb{M}$. We will show that $\mathbf{V}^{\mathbb{M}_{\omega_2}}$ is the model we are looking for. Suppose that $U \in \mathbf{V}^{\mathbb{M}_{\omega_2}}$ is an ultrafilter. By the standard Skolem-Lowenheim argument we can find $\delta < \omega_2$ such that $\mathbf{V}^{\mathbb{M}_\delta} \models U \cap \mathbf{V}^{\mathbb{M}_\delta}$ is an ultrafilter. By (1) there is $h_1 \in \mathbf{V}^{\mathbb{M}_{\delta+1}} \cap \mathsf{FtO}$ such that $\mathbf{V}^{\mathbb{M}_{\delta+1}} \models\ h_1(U)$ is a p-point. By (2) there is $h_2 \in \mathbf{V}^{\mathbb{M}_{\delta+2}} \cap \mathsf{FtO}$ such that $\mathbf{V}^{\mathbb{M}_{\delta+2}} \models\ h_2(h_1(U))$ is a Hausdorff p-point. The rest follows from the following theorem that we will prove in the next section.

Theorem 13. *Suppose that $\langle \mathcal{P}_\alpha, \mathcal{Q}_\alpha : \alpha < \delta\rangle$ is a countable support iteration of proper forcing notions such that $\Vdash_\alpha \mathcal{Q}_\alpha$ preserves Hausdorff p-points. Then $\mathcal{P}_\delta$ preserves Hausdorff p-points.*

We will show that the rational perfect set forcing $\mathbb{M}$ defined below has the required properties. The rational perfect forcing $\mathbb{M}$ is the following forcing notion:

$p \in \mathbb{M}$ if $p \subseteq \omega^{<\omega}$ is a perfect tree of finite increasing sequences of natural numbers and $\forall s \in p\ \exists t \in p\ \big(s \subseteq t\ \&\ |\mathsf{succ}_p(t)| = \aleph_0\big)$. For $p, q \in \mathbb{M}$, $p \geq q$ if $p \subseteq q$. Without loss of generality we can assume that $|\mathsf{succ}_p(s)| = 1$ or $|\mathsf{succ}_p(s)| = \aleph_0$ for all $p \in \mathbb{M}$ and $s \in p$. Conditions of this type form a dense subset of $\mathbb{M}$.

Let $\mathsf{split}(p) = \{s \in p : |\mathsf{succ}_p(s)| > 1\} = \bigcup_{n\in\omega} \mathsf{split}_n(p)$, where $\mathsf{split}_n(p) = \Big\{s \in \mathsf{split}(p) : \Big|\big\{t \subsetneq s : t \in \mathsf{split}(p)\big\}\Big| = n\Big\}$.

For $p, q \in \mathbb{M}$, $n \in \omega$, we let $p \geq_n q \iff p \geq q\ \&\ \forall i, j \leq n$ if s is j-th element of $\mathsf{split}_i(q)$ then $s \in \mathsf{split}_i(p)$.

We have to verify that $\mathbb{M}$ has property (2) and (4) above, property (1) was proved in [3] and property (3) in [9]. Property (2) follows from the following result of Dow that we present here for completeness.

Lemma 14 (Dow). *Suppose that $U \in \mathbf{V}$ is a p-point. Then $\mathbf{V}^{\mathbb{M}} \models \exists h \in \omega^\omega\ h(U)$ is a Ramsey ultrafilter. In particular, $h(U)$ is Hausdorff in $\mathbf{V}^{\mathbb{M}}$.*

Proof. Define $\mathbf{h}(n) = k$ if $m_k \leq n < m_{k+1}$, where $\langle m_k : k \in \omega\rangle$ is the generic sequence added by the Miller real. Since Miller forcing preserves p-points $\mathbf{V}^{\mathbb{M}} \models \mathbf{h}(U)$ is a p-point. Thus it remains to check that if $p \Vdash_{\mathbb{M}} \dot{f} \in \mathsf{FtO}$ then there exists $q \geq p$ and $X \in U$ such that $q \Vdash_{\mathbb{M}} \dot{f}\restriction\mathbf{h}[X]$ is one-to-one or constant. By shrinking we can assume that for every $t \in \mathsf{split}(p)$ and every $n \in \mathsf{succ}_p(t)$, there exists $k_{t^\frown n} \in \omega$ such that $p_{t^\frown n} \Vdash \dot{f}(|t|) = k_{t^\frown n}$. In particular, $p_{t^\frown n} \Vdash \dot{f}(\mathbf{h}(j)) = k_{t^\frown n}$ for $t(|t|-1) \leq j < n$. By further shrinking, we can assume that for every $t \in \mathsf{split}(p)$ and $n, m \in \mathsf{split}(p(t)$, either $k_{t^\frown n} = k_{t^\frown m}$ or that $k_{t^\frown n} < k_{t^\frown m}$ if $n < m$. This defines a partition of $\mathsf{split}(p) = A \cup B$ where $A = \{t \in \mathsf{split}(p) : k_{t^\frown n} = k_{t^\frown m} = k_t$ for all $n, m \in \mathsf{succ}_p(t)\}$ and $B = \mathsf{split}(p) \setminus A$. By passing to a stronger condition we can assume that one of these sets is empty (see [9]). Specifically, one of the following cases holds:

(1) $\mathsf{split}(p) \subseteq A$ and all k_t are distinct for $t \in \mathsf{split}(p)$,
(2) $\mathsf{split}(p) \subseteq A$ and $k_t = \bar{k}$ for all $t \in \mathsf{split}(p)$,
(3) $\mathsf{split}(p) \cap A = \emptyset$ and all $k_{t^\frown n}$ are distinct for all $t \in \mathsf{split}(p)$ and $n \in \mathsf{succ}_p(t)$.

For $t \in \mathsf{split}_l(p)$ and $n \in \mathsf{succ}_p(t)$ let $I_{t^\frown n} = [n, s(|s|-1))$, where $t^\frown n \subseteq s$ and $s \in \mathsf{split}_{l+1}(p)$. By shrinking p we can find $X \in U$ such that for every $t \in \mathsf{split}(p)$, $X \cap \bigcup\{I_{t^\frown n} : n \in \mathsf{succ}_p(t)\} = \emptyset$. We proceed as follows, given $t \in \mathsf{split}(p)$ we first thin out $\mathsf{succ}_p(t)$ so that intervals $I_{t^\frown n}$ are pairwise disjoint. Next we thin out $\mathsf{succ}_p(t)$ so that $B_t = \bigcup\{I_{t^\frown n} : n \in \mathsf{succ}_p(t)\} \notin U$. Since U is a p-point there is $X \in U$ such that $X \setminus B_t$ is finite for each t. Finally we shrink the set $\mathsf{succ}_p(t)$ so that $X \cap B_t = \emptyset$ for all t. This is the condition q that we are looking for. Suppose that $n, m \in X$ and $r \geq q$ is such that $r \Vdash_{\mathbb{M}} \dot{f}(\mathbf{h}(n)) = \dot{f}(\mathbf{h}(m))$. Without loss of generality we can assume that $r = q_{\bar{s}}$ where $n, m < \bar{s}(|\bar{s}|-1)$. Let $t_0 \subseteq t_1 \subseteq \cdots \subseteq t_k \subseteq \bar{s}$ be the list of all splitting nodes of q shorter than $\bar{s}$. Since $n, m \notin \bigcup_{j \leq k} B_{t_j}$ it follows that $n, m \in \bigcup_{j<k}[\bar{s}(|t_j|-1), \bar{s}(|t_j|))$. It follows that for some $j \leq k$, $q_{\bar{s}} \Vdash_{\mathbb{M}} \dot{f}(\mathbf{h}(n)) = \dot{f}(\mathbf{h}(m)) = k_{t_j}$ (or $k_{t_j^\frown \bar{s}(|t_j|)}$ or $\bar{k}$). In the first two cases it follows that $n, m \in [\bar{s}(|t_j|-1), \bar{s}(|t_j|))$, so $q_{\bar{s}} \Vdash_{\mathbb{M}} \mathbf{h}(n) = \mathbf{h}(m) = |t_j| - 1$ (so $\dot{f} \circ \mathbf{h}$ is forced to be one-to-one on X) and in the second case it means that $q_{\bar{s}} \Vdash_{\mathbb{M}} \dot{f}(\mathbf{h}(n)) = \dot{f}(\mathbf{h}(m)) = \bar{k}$ (so $\dot{f} \circ \mathbf{h}$ is forced to be constant on X). □

In the remainder of this section we will show that Miller forcing preserves Hausdorff p-points.

We will start with the following observation.

Lemma 15. *The following conditions are equivalent for an ultrafilter U:*

(1) *U is not Hausdorff,*
(2) *there are functions $f_0, f_1 \in \omega^\omega$ such that*
  (a) *$f_0(U) = f_1(U)$,*
  (b) *$\forall n\ f_0(n) \neq f_1(n)$,*
  (c) *$\forall n\ f_0(n), f_1(n) \leq 2n+2$.*

Proof. We will show that (1) implies (2), the other implication is obvious.

Let $g_0, g_1 \in \omega^\omega$ be such that $g_0(U) = g_1(U)$ and $g_0 \neq g_1 \mod U$. Let $A = \{n : g_0(n) \neq g_1(n)\} \setminus \{0, 1\}$. By the assumption $A \in U$. Define $h \in \omega^\omega$ as follows: $\mathsf{dom}(h) = \mathsf{range}(g_0\restriction A) \cup \mathsf{range}(g_1\restriction A)$ and let $h(n) = \min(2k+i : i \in 2,\ g_i(k) = n\}$. Next put for $i = 0, 1$, $f_i(n) = h(g_i(n))$ for $n \in A$ and $f_i(n) = i$ if $n \notin A$. Note that $f_0(U) = h(g_0(U)) = h(g_1(U)) = f_1(U)$, the other two properties are equally easy to see. □

Lemma 16. *Suppose that $U \in \mathbf{V}$ is a Hausdorff p-point. Then $\mathbf{V}^{\mathbb{M}} \models U$ is a Hausdorff p-point.*

Proof. Let $U \in \mathbf{V}$ be a Hausdorff p-point. First of all recall that U remains a p-point in $\mathbf{V}^{\mathbb{M}}$ ([9], Proposition 4.2). Thus we only have to show that U remains Hausdorff. Let $p \in \mathbb{M}$ and $\dot{f}_0, \dot{f}_1$ be such that

(1) $p \Vdash_{\mathbb{M}} \forall n \ \dot{f}_0(n) \neq \dot{f}_1(n)$,
(2) $p \Vdash_{\mathbb{M}} \forall n \ \dot{f}_0(n), \dot{f}_1(n) \leq 2n+2$.

It suffices to find $q \geq p$ and $X \in U$ such that $q \Vdash_{\mathbb{M}} \dot{f}_0[X] \cap \dot{f}_1[X] = \emptyset$.

Define a game $G(U)$ played by players I and II. Player I on his n-th move plays a set $X_n \in U$ and player II responds with a finite set $a_n \subseteq X_n$. Together they construct a sequence $X_1, a_1, X_2, a_2, \ldots$. Player I wins if $\bigcup_{n\in\omega} a_n \notin U$; otherwise player II wins. It is well known, [1], that player I has a winning strategy in $G(U)$ if and only if U is not a p-point. We will be simultaneously playing game $G(U)$ and constructing sequences of $\langle q_n : n \in \omega\rangle$, $\langle X_n : n \in \omega\rangle$ and $\langle a_n : n \in \omega\rangle$ such that

(1) $\langle X_n, a_n : n \in \omega\rangle$ is a play in $G(U)$,
(2) $q_0 = p$ and $q_{n+1} \geq_n q_n$,
(3) $q_{n+1} \Vdash_{\mathbb{M}} \dot{f}_0[\bigcup_{j\leq n} a_j] \cap \dot{f}_1[\bigcup_{j\leq n} a_j] = \emptyset$.

Since U is a p-point, player II does not have a winning strategy in $G(U)$. Therefore there exists a play for player II such that $X = \bigcup_n a_n \in U$. At the same time, if $q \geq q_n$ for all n then, by (3) above, $q \Vdash_{\mathbb{M}} \dot{f}_0[X] \cap \dot{f}_1[X] = \emptyset$.

Without loss of generality we can assume that for every $t \in \mathsf{split}(p)$ and $n \in \mathsf{succ}_p(t)$, there exists $(f_0^{t^\frown n}, f_1^{t^\frown n}) \in (\omega^{<\omega})^2$ such that $p_{t^\frown n} \Vdash_{\mathbb{M}} \forall i = 0,1 \ \dot{f}_i{\restriction}|t|+n = f_i^{t^\frown n}{\restriction}|t|+n$. For given t, sequences $\{(f_0^{t^\frown n}, f_1^{t^\frown n}) : n \in \omega\}$ form a finitely branching tree. Let (f_0^t, f_1^t) be one of its infinite branches. Now prune the tree so that (f_0^t, f_1^t) is the unique branch for every $t \in \mathsf{split}(p)$. Since these functions need not be finite-to-one for each t we have one of the following four cases: either there is $i \in \{0,1\}$, $a_i^t \in \omega$ and $X_i^t \in U$ such that $f_i^t{\restriction}X_i^t$ is constant with value a_i^t, or for $i \in \{0,1\}$ there is $X^t \in U$ such that $f_0^t, f_1^t{\restriction}X^t$ are both finite-to-one. By passing to a stronger condition we can assume that exactly one of these cases holds. Furthermore, if it is one of the first three cases then we can assume that for the relevant $i \in \{0,1\}$ the values a_i^t are all the same or all distinct.

The construction follows the cases described above. Suppose that $\langle q_k, X_k, a_k : k \leq n\rangle$ are already constructed. Let $S_n \subseteq \mathsf{split}(q_n)$ be the set of nodes that need to be contained in $\mathsf{split}(q_{n+1})$.

Case 1 $f_0^t, f_1^t{\restriction}X^t$ are finite-to-one for $t \in \mathsf{split}(p)$.

Since U is Hausdorff and functions are finite-to-one, for each $t \in S$ there is $X_t \in U$ such that $f_0^t[X_t \cap X^t] \cap f_1^t[X_t \cap X^t] = \emptyset$, and $f_i^t[X_t \cap X^t] \cap f_{1-i}^t[\bigcup_{k\leq n} a_k] = \emptyset$ for $i = 0,1$. Define $X_{n+1} = \bigcap_{t\in S} X_t \cap X^t$, this is the move played in the ultrafilter game. Player II responds with $a_{n+1} \subseteq X_{n+1}$. Finally, let q_{n+1} be obtained by keeping only thise nodes $s \in \mathsf{split}(q_n) \setminus \bigcup_{k\leq n} S_k$ such that $|s| + n \geq \max(a_{n+1})$ for $n \in \mathsf{succ}_{q_n}(s)$. It is easy to verify that q_{n+1}, X_{n+1} have the required properties.

Case 2 $f_0^t{\restriction}X^t$ is constant with value a_0^t and a_0^t are distinct for $t \in \mathsf{split}(p)$ and $f_1^t{\restriction}X^t$ is finite-to-one.

Assume inductively that for each $s \in \mathsf{split}(q_n) \setminus S_{n-1}$, $a_0^s \notin f_1^t[\bigcup_{k\leq n} a_k]$. Since U is Hausdorff, for each $t \in S$ there is $X_t \in U$ such that $f_0^t[X_t \cap X^t] \cap f_1^t[X_t \cap X^t] = \emptyset$, and $f_1^t[X_t \cap X^t] \cap f_0^t[\bigcup_{k\leq n} a_k] = \emptyset$.

Define $X_{n+1} = \bigcap_{t\in S} X_t \cap X^t$, this is the move played in the ultrafilter game. Player II responds with $a_{n+1} \subseteq X_{n+1}$. Let q'_{n+1} be obtained by keeping only thise nodes $s \in \mathsf{split}(q_n) \setminus \bigcup_{k\leq n} S_k$ such that $|s| + n \geq \max(a_{n+1})$ for $n \in \mathsf{succ}_{q_n}(s)$. Finally let q_{n+1} be the condition obtained from q'_{n+1} by removing all splitting nodes $s \notin \bigcup_{k\leq n} S_k$ with $a^s_0 \in \bigcup_{t\in S_n} f^t_1[a_{n+1}]$. Again, q_{n+1}, X_{n+1} have the required properties.

Remaining cases are handled similarly. □

It is possible to show directly that an iteration of Miller forcing of countable (and thus arbitrary) length preserves Hausdorff p-points as well. This can be done by introducing $\leq_{F,n}$ order on the iteration and modifying the proof of Lemma 16. Instead we will prove much more general Theorem 13.

5. Preserving Hausdorff p-points

In this section we will prove a general theorem concerning the preservation of Hausdorff p-points. The theorem is a specific application of a general preservation theorem due to Shelah ([10], chapter XVIII).

Let $\mathbf{C} = \{(f_0, f_1) \in \omega^\omega \times \omega^\omega : \forall n\ f_0(n) \neq f_1(n)\}$. Consider the following relation $\sqsubseteq = \bigcup_n \sqsubseteq_n$ on $\mathbf{C} \times P(\omega)$ defined as

$$(f_0, f_1) \sqsubseteq_n X \iff f_0[X \setminus n] \cap f_1[X \setminus n] = \emptyset.$$

Note that if U is a Hausdorff filter then U is a $\sqsubseteq$-dominating family, that is, for every $(f_0, f_1) \in \mathbf{C}$ there is $X \in U$ such that $(f_0, f_1) \sqsubseteq X$.

Suppose that U is a family of subsets of ω. Let $\bar{U} = \{Y \subseteq \omega : \exists n\ \exists X \in U\ X \setminus n \subseteq Y\}$. For a Hausdorff p-point U, let $S \subseteq [U]^{\leq\aleph_0}$ be the stationary set consisting of sets of form $N \cap U$ where N is a countable elementary submodel of $\mathbf{H}(\chi)$. Let $\mathbf{g} = \{X_a : a \in S\}$, where $X_a \in U$ is such that $X_a \subseteq^\star X$ for $X \in a$.

Shelah's preservation theorem says:

Theorem 17. *Suppose that $(\sqsubseteq, S, \mathbf{g})$ strongly covers and $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\alpha : \alpha < \delta\rangle$ is a countable support iteration of forcing notions such that for every $\alpha < \delta$, $\Vdash_\alpha \dot{\mathcal{Q}}_\alpha$ preserves $(\sqsubseteq, S, \mathbf{g})$. Then $\mathcal{P}_\delta$ preserves $(\sqsubseteq, S, \mathbf{g})$.*

We will define all the relevant notions below and we will show that Theorem 13 follows from it.

Definition 18. *A countable model N is good if every $(f_0, f_1) \in N \cap \mathbf{C}$, $(f_0, f_1) \sqsubseteq X_{N\cap U}$. We say that $(\sqsubseteq, S, \mathbf{g})$ covers (in $\mathbf{V}$) if every countable model $N \preceq \mathbf{H}(\chi)$ containing $(\sqsubseteq, S, \mathbf{g})$ is good.*

Note that this corresponds to the settings $\alpha^\star = \omega$, $\bar{R} = \sqsubseteq$ in Definition 3.2 page 889 in [10].

Lemma 19. *Suppose that $\mathcal{P}$ is a proper forcing notion which preserves p-points. The following conditions are equivalent for a Hausdorff p-point U in $\mathbf{V}$:*

1. *$\bar{U}$ is a Hausdorff p-point in $\mathbf{V}^{\mathcal{P}}$,*
2. *$(\sqsubseteq, S, \mathbf{g})$ covers in $\mathbf{V}^{\mathcal{P}}$.*

Proof. (1) → (2) Suppose that $N \preceq \mathbf{H}(\chi)^{\mathbf{V}^{\mathcal{P}}}$ is a countable model containing $(\sqsubseteq, S, \mathbf{g})$. Let $(f_0, f_1) \in N \cap \mathbf{C}$. Since $\bar{U}$ is Hausdorff in $\mathbf{V}^{\mathcal{P}}$ there is $Y \in U$ such that $(f_0, f_1) \sqsubseteq Y$. Since N is an elementary submodel containing all relevant objects we can assume that $Y \in N$. Since $X_{U\cap N} \subseteq^\star Y$ we are done.

(2) → (1) If $\bar{U}$ is not Hausdorff in $\mathbf{V}^{\mathcal{P}}$ then there are functions $f_0, f_1 \in \mathbf{V}^{\mathcal{P}} \cap \omega^\omega$ witnessing this. Since $\mathcal{P}$ preserves p-points, without loss of generality we can assume by Lemma 9, that $(f_0, f_1) \in \mathbf{C}$. Therefore $|f_0[X] \cap f_1[X]| = \aleph_0$ for all $X \in U$. It follows that if N contains f_0, f_1 then $(f_0, f_1) \not\sqsubseteq X_{N\cap U}$. □

The definition below is a special case of the Definition 3.3 on page 899 in [10] (Possibility A, case $\oplus'_k$).

Definition 20. *We say that* $(\sqsubseteq, S, \mathbf{g})$ *strongly covers if*

1. *if* $(\sqsubseteq, S, \mathbf{g})$ *covers in* $\mathbf{V}$,
2. *for each* n, *relation* $\sqsubseteq_n$ *is a closed subset of* $\mathbf{C} \times P(\omega)$,
3. *the following holds in every proper forcing extension* $\mathbf{V}^{\mathcal{P}}$ *in which* $(\sqsubseteq, S, \mathbf{g})$ *covers: IF* $N \prec \mathbf{H}(\chi)$ *is a countable model,* $k \in \omega$ *and* $\langle (f_0^{n,j}, f_1^{n,j} : j \leq k, n \in \omega\rangle \in N \cap \mathbf{C}$ *such that*
   (a) $\lim_n (f_0^{n,j}, f_1^{n,j}) = (f_0^j, f_1^j)$,
   (b) *for each* $j \leq k$, *there is* $n_j \in \omega$ *such that* $(f_0^j, f_1^j) \sqsubseteq_{n_j} X_{N\cap U}$,
   THEN there exists $n^\star$ *such that*
$$\forall j \leq k\ (f_0^{n^\star,j}, f_1^{n^\star,j}) \sqsubseteq_{n_j} X_{N\cap U}.$$

The next lemma shows that for the forcing notions preserving Hausdorff p-points both notions coincide.

Lemma 21. *If* $(\sqsubseteq, S, \mathbf{g})$ *covers then* $(\sqsubseteq, S, \mathbf{g})$ *strongly covers.*

Proof. Let $\mathcal{P}$ be a proper forcing notion that preserves Hausdorff p-points. Work in $\mathbf{V}^{\mathcal{P}}$. Suppose that $N \prec \mathbf{H}(\chi)^{\mathbf{V}^{\mathcal{P}}}$ is a countable model and let $k \in \omega$, $\langle (f_0^{n,j}, f_1^{n,j} : j \leq k, n \in \omega\rangle, \langle (f_0^j, f_1^j) : j \leq k\rangle$, $\langle n_j : j \leq k\rangle$ be as required. We have to show that there exists $n^\star$ such that
$$\forall j \leq k\ (f_0^{n^\star,j}, f_1^{n^\star,j}) \sqsubseteq_{n_j} X_{N\cap U}.$$
Since U is a p-point in N for each $n \in \omega, j \leq k$ there is a set $A^{n,j} \in U \cap N$ such that $(f_0^{n,j}, f_1^{n,j}) \sqsubseteq_0 A^{n,j}$. Let $A \in N \cap U$ be such that $A \subseteq^\star A^{n,j}$ for $n \in \omega, j \leq k$. Work in N and define sequence $\langle m_l : l \in \omega\rangle$ such that

1. $m_0 = 0$, $m_l < m_{l+1}$,
2. $\forall n \geq m_{l+1}\ \forall j \leq k\ \forall i = 0,1\ f_i^{n,j}\restriction m_l = f_i^j \restriction m_l$,
3. $A \setminus m_{l+1} \subseteq \bigcap_{n \leq m_l, j \leq k} A^{n,j}$,
4. $\forall i = 0,1\ \forall n \leq m_l\ \forall j \leq k\ \forall u \leq m_l\ f_i^{n,j}(u) < m_{l+1}$,
5. $\forall i = 0,1\ \forall n \leq m_l\ \forall j \leq k\ \forall u \geq m_{l+1}\ f_i^{n,j}(u) \geq m_l$.

Without loss of generality the set $B = \bigcup_j [m_{3l+1}, m_{3l+2}) \in U \cap N$. Let $l^\star$ be such that $X_{N\cap U} \setminus m_{3l^\star} \subseteq A \cap B$ and put $n^\star = m_{3l^\star}$. We have to show that for all $j \leq k$, $(f_0^{n^\star,j}, f_1^{n^\star,j}) \sqsubseteq_{n_j} X_{N\cap U}$, that is $f_0^{n^\star,j}[X_{N\cap U} \setminus n_j] \cap f_1^{n^\star,j}[X_{N\cap U} \setminus n_j] = \emptyset$. Fix $j \leq k$ and suppose that $x, y \in X_{N\cap U} \setminus n_j$.

CASE 1 $x, y < m_{3l^\star}$.

In this case $x, y \leq m_{3l^\star-1} = m_{3(l^\star-1)+2}$. Since for $i = 0,1$, $f_i^{m_{3l^\star},j}\restriction m_{3(l^\star-1)+2} = f_i^j \restriction m_{3(l^\star-1)+2}$ it follows that $f_0^{n^\star,j}(x) \neq f_1^{n^\star,j}(y)$ since $f_0^j(x) \neq f_1^j(y)$.

CASE 2 $x, y \geq m_{3l^\star}$. In this case $x, y \geq m_{3l^\star+1}$ and since $A \setminus m_{3l^\star+1} \subseteq A^{3l^\star,j}$ it follows that $x, y \in A^{3l^\star,j}$. Thus $f_0^{n^\star,j}(x) \neq f_1^{n^\star,j}(y)$ by the choice of $A^{n^\star,j}$.

CASE 3 $x < m_{3l^\star} \leq y$.

In this case $x < m_{3l^\star-1} < m_{3l^\star+1} \leq y$, and $f_0^{n^\star,j}(x) \neq f_1^{n^\star,j}(y)$ by the property of sequence $\langle m_l : l \in \omega\rangle$. $\square$

Definition 22. *We say that a forcing notion $\mathcal{P}$ is $(\sqsubseteq, S, \mathbf{g})$-preserving if whenever $N \prec \mathbf{H}(\chi)$ is a countable model containing $\mathcal{P}$ and $\sqsubseteq$ whenever $\langle p_n : n \in \omega\rangle \in N$ is an increasing sequence of conditions interpreting $\langle(\dot f_0^0, \dot f_1^0), \dots, (\dot f_0^k, \dot f_1^k)\rangle \in N$ as $\langle(f_0^0, f_1^0), \dots, (f_0^k, f_1^k)\rangle$ then there is an N-generic condition $q \geq p_0$ such that:*

1. *$q \Vdash_{\mathcal{P}} N[\dot G]$ is $(\sqsubseteq, S, \mathbf{g})$-good and*
2. *$\forall n \in \omega\ \forall j \leq k\ q \Vdash_{\mathcal{P}} \big((f_0^j, f_1^j) \sqsubseteq_n X_{N\cap U} \to (\dot f_0^j, \dot f_1^j) \sqsubseteq_n X_{N\cap U}\big)$.*

Lemma 23. *Suppose that $\mathcal{P}$ is a forcing notion which preserves p-points and U is a Hausdorff p-point ultrafilter in $\mathbf{V}$. The following conditions are equivalent:*

1. *$\bar U$ is a Hausdorff p-point in $\mathbf{V}^{\mathcal{P}}$,*
2. *$\mathcal{P}$ is $(\sqsubseteq, S, \mathbf{g})$-preserving.*

Proof. (2) $\to$ (1) If $\mathcal{P}$ is $(\sqsubseteq, S, \mathbf{g})$-preserving then $(\sqsubseteq, S, \mathbf{g})$ covers in $\mathbf{V}^{\mathcal{P}}$. In particular, U is a Hausdorff ultrafilter in $\mathbf{V}^{\mathcal{P}}$.

(1) $\to$ (2) Suppose that $N \prec \mathbf{H}(\chi)$ is a countable model containing $\mathcal{P}$ and $\sqsubseteq$. Since U is Hausdorff, N is $(\sqsubseteq, S, \mathbf{g})$-good. Let $\langle p_n : n \in \omega\rangle \in N$ be an increasing sequence of conditions interpreting $\langle(\dot f_0^0, \dot f_1^0), \dots, (\dot f_0^k, \dot f_1^k)\rangle \in N$ as $\langle(f_0^0, f_1^0), \dots, (f_0^k, f_1^k)\rangle$. In other words, for every n,

1. $p_{n+1} \geq p_n$,
2. $p_n \Vdash \forall j \leq k\ \forall i = 0,1\ \dot f_i^j\restriction n = f_i^j\restriction n$.

We have to show that there exists $q \geq p_0$ such that

$$q \Vdash_{\mathcal{P}} \forall n\ \forall j \leq k\ \big((f_0^j, f_1^j) \sqsubseteq_n X_{N\cap U} \to (\dot f_0^j, \dot f_1^j) \sqsubseteq_n X_{N\cap U}\big).$$

Since $\mathcal{P}$ preserves p-points we can assume that for every $j \leq k$ $i = 0,1$, p_0 decides whether $\dot f_i^j$ is constant or finite-to-one mod U. For each n try to choose $q_n \geq p_n$ such that

1. q_n is N-generic,
2. there exists $X_n \in N \cap U$ such that $q_n \Vdash \forall j \leq k\ (\dot f_0^j, \dot f_1^j) \sqsubseteq_0 X_n\ \&\ \dot f_i^j[X_n] \cap f_{1-i}^j[n] = \emptyset$.

Given n we proceed as follows: First find $q' \geq p_n$, $X \in N \cap U$, $l_i^j \in \omega$ for $j \leq k$, $i = 0,1$ such that

1. $q' \Vdash \forall j \leq k\ (\dot f_0^j, \dot f_1^j) \sqsubseteq_0 X$,
2. for every $j \leq k$, $i = 0,1$ either
   - (a) $q' \Vdash \dot f_i^j$ is finite-to-one on X or
   - (b) $q' \Vdash \dot f_i^j$ is constant on X with value l_i^j.

Next choose $q'' \geq q'$ and m so that either

1. $q'' \Vdash \forall i = 0,1\ \forall j \leq k\ \min\big(\dot f_i^j[X \setminus m]\big) > \max\big(f_i^j(k) : k \leq n\big)$ (finite-to-one case) or
2. $l_i^j \in f_i^j[n]$ (constant case).

Finally, let $q_n \geq q''$ be N-generic and let $X_n = X \setminus m$. Note that this construction will succeed for all sufficiently large n, that is when $l \in f_i^j[n]$. Since U is a p-point there is infinitely many n such that $X_{N\cap U} \setminus n \subseteq X_n$. Choose $n^\star$ to be such an n and let $q = q_{n^\star}$.

Fix $j \leq k$ and let $n_j = \min\big(n : (f_0^j, f_1^j) \sqsubseteq_n X_{N\cap U}\big)$. We have to show that $q \Vdash (\dot{f}_0^j, \dot{f}_1^j) \sqsubseteq_{n_j} X_{N\cap U}$. Let $x, y \in X_{N\cap U} \setminus n_j$.

Case 1 $x, y \leq n$. In this case $q \Vdash \dot{f}_i^j \restriction n = f_i^j \restriction n$ and we know that $f_i^j(x) \neq f_{1-i}^j(y)$ for $i = 0, 1$.

Case 2 $x, y > n$. In this case $x, y \in X_n$ and since $q \Vdash (\dot{f}_0^j, \dot{f}_1^j) \sqsubseteq_0 X_n$, it follows that $q \Vdash \dot{f}_{1-i}^j(x) \neq \dot{f}_i^j(y)$

Case 3 $x \leq n < y$. In this case we have two possibilities: either one of the functions $\dot{f}_0^j, \dot{f}_1^j$ is forced to be constant on X_n or both are forced to be finite-to-one on X_n. In the first case $q \Vdash \exists k \leq n\ \dot{f}_i^j(y) = f_i^j(k)$. Thus $q \Vdash \dot{f}_{1-i}^j(x) \neq \dot{f}_i^j(y)$ since $f_i^j(k) \neq f_{1-i}^j(x)$. If both functions are finite-to-one then $q \Vdash \forall i = 0, 1\ f_i^j(x) \leq \max\big(f_i^j(k) : k \leq n\big) < f_{1-i}^j(y)$. □

Now Theorem 13 follows readily from Lemma 16 and the results of this section.

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References

[1] Tomek Bartoszynski and Haim Judah. *Set Theory: on the structure of the real line.* A.K. Peters, 1995.

[2] Viero Benci, Marco Forti, and Mauro di Nasso. Hausdorff nonstandard extensions. *Bol. Soc. Parana. Mat.*, 20:9–20, 2002.

[3] Andreas Blass and Saharon Shelah. Near coherence of filters. III. A simplified consistency proof. *Notre Dame Journal of Formal Logic*, 30(4):530–538, 1989.

[4] Mauro di Nasso. Ultrafilter semirings and nonstandard submodels of the Stone-Cech compactification of the natural numbers. submitted for the volume "Logic and its application in algebra and geometry", Contemporary Mathematics AMS, edited by Yi Zhang.

[5] Mauro di Nasso and Marco Forti. Hausdorff ultrafilters. to appear.

[6] Mauro di Nasso and Marco Forti. Topological and Nonstandard extensions. to appear.

[7] Martin Goldstern. Tools for your forcing constructions. In Haim Judah, editor, *Set theory of the reals*, Israel Mathematical Conference Proceedings, pages 305–360. Bar Ilan University, 1992.

[8] Thomas Jech. *Set Theory.* Academic Press, 1978.

[9] Arnold W. Miller. Rational perfect set forcing. In *Axiomatic set theory*, volume 31 of *Contemp. Math.*, pages 143–159, Providence, R.I., 1984. Boulder, Colo., 1983, Amer. Math. Soc.

[10] Saharon Shelah. *Proper and Improper Forcing.* Perspectives in Logic. Springer-Verlag, 1998.

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