PCF WITHOUT CHOICE
SH835

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Abstract. We mainly investigate models of set theory with restricted choice, e.g., ZF + DC + the family of countable subsets of λ is well ordered for every λ (really local version for a given λ). We think that in this frame much of pcf theory, (and combinatorial set theory in general) can be generalized. We prove here that there is a class of regular cardinals, every large enough successor of singular is not measurable and we can prove cardinal inequalities.

Anotated Content

§0 Introduction

§(0A) Background, aims and results

§(0B) Preliminaries

[Include quoting [Sh:497], (0.1,0.2); hrtg(Y), wlor(Y), (0.6); defining $\text{rk}_D(f)$, (0.7 + 0.8) on $J[f,D]$, (0.9, 0.7, 0.11); $\mathscr{H}_{<\kappa,\gamma}(Y)$, (0.13 and observation 0.14); and on closure operations (0.15).]

§1 Representing ${}^\kappa\lambda$

[We define Fil^ℓ_κ and prove a representation of ${}^\kappa\lambda$. Essentially under "reasonable choice" the set ${}^\kappa\lambda$ is the union of few well ordered set, i.e., their number depends on κ only". We end by a claim on $\Pi\mathfrak{a}$.]

§2 No decreasing sequence of subalgebras

[As suggested in the title we weaken the axioms. We deal with ${}^\kappa\lambda$ with λ^+ not measurable, existence of ladder $\bar{C}$ witnessing cofinality and prove that many λ^+ are regular (2.13).]

§3 Concluding remarks

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§ 0. Introduction

§ 0(A). **Background, aims and results.**

The thesis of [Sh:497] was that pcf theory without full choice exists. Two theorems supporting this thesis were proved. The first ([Sh:497, 4.6,pg.117]).

{0.1} **Theorem 0.1.** [DC] *If $\mathscr{H}(\mu)$ is well ordered, μ strong limit singular of uncountable cofinality then μ^+ is regular not measurable (and 2^μ is an $\aleph$, i.e. $\mathscr{P}(M)$ can be well ordered and no $\lambda \in (\mu, 2^\mu]$ is measurable).*

Note that before this Apter and Magidor [AM95] had proved the consistency of "$\mathscr{H}(\mu)$ well ordered, $\mu = \beth_\omega, (\forall \kappa < \mu)\mathrm{DC}_\kappa$ and μ^+ is measurable" so 0.1 says that this consistency result cannot be fully lifted to uncountable cofinalities. Generally without full choice, a successor cardinal being not measurable is worthwhile information.
A second theorem ([Sh:497, §5]) was

{0.2} **Theorem 0.2.** *Assume*

(a) $\mathrm{DC} + \mathrm{AC}_\kappa + \kappa$ *regular uncountable*
(b) $\langle \mu_i : i < \kappa \rangle$ *is increasing continuous with limit $\mu, \mu > \kappa, \mathscr{H}(\mu)$ well ordered, μ strong limit, (we need just a somewhat weaker version, the so-called $i < \kappa \Rightarrow Tw_{\mathscr{D}_\kappa}(\mu_i) < \mu$).*

Then , we cannot have two regular cardinals θ such that for some stationary $S \subseteq \kappa$, the sequence $\langle \mathrm{cf}(\mu_i^+) : i \in S \rangle$ is constantly θ.

A dream was to prove that there is a class of regular cardinals from a restricted version of choice (see more [Sh:497] and a little more in [Sh:E37]).

Our original aim here is to improve those theorems. As for 0.1 we replace "$\mathscr{H}(\mu)$ well ordered" by "$[\mu]^{\aleph_0}$ is well ordered" and then by weaker statements.

We know that if, e.g., $\neg\exists 0^\#$ or there is no inner model with a measurable then though $\langle 2^\kappa : \kappa \text{ regular} \rangle$ is quite arbitrary, the size of $[\lambda]^\kappa, \lambda >> \kappa$ is strictly controlled (by Easton forcing [Eas70], and Jensen and Dodd [DJ82]). It seemed that the situation here is parallel in some sense; under the restricted choice we assume, we cannot say much on the cardinality of $\mathscr{P}(\kappa)$ but can say something on the cardinality of $[\lambda]^\kappa$ for $\lambda >> \kappa$.

In the proofs we fulfill a promise from [Sh:589, §5] about using $J[f,D]$ from Definition 0.10 instead of the nice filters used in [Sh:497] and, to some extent, in early versions of this work. This work is continued in Larson-Shelah [LrSh:925] and will be continued in [Sh:F1078]. On a different line with weak choice (say $\mathrm{DC}_{\aleph_0} + \mathrm{AC}_\mu, \mu$ fixed): see [Sh:938], [Sh:955] and [Sh:F1039]. The present work fits the thesies of [Sh:g] which in particular says: it is better to look at $\langle \lambda^{\aleph_0} : \lambda$ a cardinality$\rangle$ then at $\langle 2^\lambda : \lambda$ a cardinal$\rangle$. Here instead well ordering $\mathscr{P}(\lambda)$ we well order $[\lambda]^{\aleph_0}$, this is enough for much.

A simply stated conclusion is

{0n.2} **Conclusion 0.3.** *[DC] Assume every $[\lambda]^{\aleph_0}$ is well ordered for every λ.*
1) If 2^{2^κ} is well ordered then for every $\lambda, [\lambda]^\kappa$ is well ordered.
2) For any set Y, there is a derived set Y_* so called $\mathrm{Fil}^4_{\aleph_1}(Y)$ of power near $\mathscr{P}(\mathscr{P}(Y))$ such that $\Vdash_{\mathrm{Levy}(\aleph_0, Y_*)}$ "for every $\lambda, {}^Y\lambda$ is well ordered".

{0n.9}

Thesis 0.4. 1) If $\mathbf{V} \models$ "ZF + DC" and "every $[\lambda]^{\aleph_0}$ is well orderable "<u>then</u> $\mathbf{V}$ looks like the result of starting with a model of ZFC and using $\aleph_1$-complete forcing notions like Easton forcing, Levy collapsed, and more generally, iterating of κ-complete forcing for $\kappa > \aleph_0$.
2) This approach is dual to investigating $\mathbf{L}[\mathbb{R}]$ - here we assume ω-sequences are understood (or weaker versions) and we try to understand $\mathbf{V}$ (over this), there over the reals everything is understood.

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§ 0(B). **Preliminaries.**

Convention 0.5. We assume just $\mathbf{V} \models$ ZF if not said otherwise. {0n.2.7}

Definition 0.6. 1) $\mathrm{hrtg}(A) = \mathrm{Min}\{\alpha$: there is no function from A onto $\alpha\}$. {0.3}
2) $\mathrm{wlor}(A) = \mathrm{Min}\{\alpha$: there is no one-to-one function from α into A or $\alpha = 0 \wedge A \neq \emptyset\}$ so $\mathrm{wlor}(A) \leq \mathrm{hrtg}(A)$.

Definition 0.7. 1) For D an $\aleph_1$-complete filter on Y and $f \in {}^Y\mathrm{Ord}$ and $\alpha \in \mathrm{Ord} \cup \{\infty\}$ we define when $\mathrm{rk}_D(f) = \alpha$, by induction on α: {0.A}

> ⊛ For $\alpha < \infty$, $\mathrm{rk}_D(f) = \alpha$ iff $\beta < \alpha \Rightarrow \mathrm{rk}_D(f) \neq \beta$ and for every $g \in {}^Y\mathrm{Ord}$ satisfying $g <_D f$ there is $\beta < \alpha$ such that $\mathrm{rk}_D(g) = \beta$.

2) We can replace D by the dual ideal. If $f \in {}^Z\mathrm{Ord}$ and $Z \in D$ then we let $\mathrm{rk}_D(f) = \mathrm{rk}(f \cup 0_{Y \setminus Z})$.

Galvin-Hajnal [GH75] use the rank for the club filter on ω_1. This was continued in [Sh:71] where varying D was extensively used.

Claim 0.8. [DC] *In 0.7,* $\mathrm{rk}_D(f)$ *is always an ordinal and if* $\alpha \leq \mathrm{rk}_D(f)$ <u>*then*</u> *for some* $g \in \prod_{y \in Y}(f(y)+1)$ *we have* $\alpha = \mathrm{rk}_D(g)$, *(if* $\alpha < \mathrm{rk}_D(f)$ *we can add* $g <_D f$*; if* $\mathrm{rk}_D(f) < \infty$ *then* DC *is not necessary; if* $\mathrm{rk}_D(f) = \alpha$ *this is trivial, as we can choose* $g = f$*).* {0.B}

Claim 0.9. *1)* [DC] *If D is an $\aleph_1$-complete filter on Y and $f \in {}^Y\mathrm{Ord}$ and* $Y = \cup\{Y_n : n < \omega\}$ <u>*then*</u> $\mathrm{rk}_D(f) = \mathrm{Min}\{\mathrm{rk}_{D+Y_n}(f) : n < \omega$ *and* $Y_n \in D^+\}$, *(*[Sh:71]*).* {0.B1}
*2) [*DC + AC_{α^*}*] If D is a κ-complete filter on Y, κ a cardinal $> \aleph_0$ and $f \in {}^Y\mathrm{Ord}$ and* $Y = \cup\{Y_\alpha : \alpha < \alpha^*\}, \alpha^* < \kappa$ <u>*then*</u> $\mathrm{rk}_D(f) = \mathrm{Min}\{\mathrm{rk}_{D+Y_\alpha}(f) : \alpha < \alpha^*$ *and* $Y_\alpha \in D^+\}$.

Definition 0.10. For Y, D, f in 0.7 let $J[f,D] =: \{Z \subseteq Y : Y \setminus Z \in D$ or $\mathrm{rk}(f)_{D+(Y \setminus Z)} > \mathrm{rk}_D(f)\}$. {0.C}

Claim 0.11. *[*DC+$\mathrm{AC}_{<\kappa}$] Assume D is a κ-complete filter on $Y, \kappa > \aleph_0$. {0.D}
1) If $f \in {}^Y\mathrm{Ord}$ <u>then</u> $J[f,D]$ is a κ-complete ideal on Y.
2) If $f_1, f_2 \in {}^Y\mathrm{Ord}$ and $J = J[f_1,D] = J[f_2,D]$ then $\mathrm{rk}_D(f_1) < \mathrm{rk}_D(f_2) \Rightarrow f_1 < f_2$ mod J and $\mathrm{rk}_D(f_1) = \mathrm{rk}_D(f_2) \Rightarrow f_1 = f_2$ mod J.

Proof. Straight or see [Sh:589, §5] and the reference there to [Sh:497] (and [Sh:71]). $\square_{0.11}$

{0n.D}

Definition 0.12. 1) $|Y| \leq \text{qu}(Z)$ or $Y \leq_{\text{qu}} Z$ or $|Y| \leq_{\text{qu}} |Z|$ or $|Y| \leq_{\text{qu}} Z$ or $Y \leq_{\text{qu}} |Z|$ means that $Y = \emptyset$ or there is a function from Z (equivalently a subset of Z) onto Y.
2) $\text{reg}(\alpha) = \text{Min}\{\partial : \partial \geq \alpha \text{ is a regular cardinal}\}$.

{0.E}

Definition 0.13. For a set Y, cardinal κ and ordinal γ we define $\mathscr{H}_{<\kappa,\gamma}(Y)$ by induction on γ: if $\gamma = 0, \mathscr{H}_{<\kappa,\gamma}(Y) = Y$, if $\gamma = \beta+1$ then $\mathscr{H}_{<\kappa,\gamma}(Y) = \mathscr{H}_{<\kappa,\beta}(Y) \cup \{u : u \subseteq \mathscr{H}_{<\kappa,\beta}(Y) \text{ and } |u| < \kappa\}$ and if γ is a limit ordinal then $\mathscr{H}_{<\kappa,\gamma}(Y) = \cup\{\mathscr{H}_{<\kappa,\beta}(Y) : \beta < \gamma\}$.

{x.5}

Observation 0.14. *1) If λ is the disjoint union of $\langle W_z : z \in Z\rangle$ and $z \in Z \Rightarrow |W_z| < \lambda$ and* $\text{wlor}(Z) \leq \lambda$ *<u>then</u>* $\lambda = \sup\{\text{otp}(W_z) : z \in Z\}$ *hence* $\text{cf}(\lambda) < \text{hrtg}(Z)$.
2) If $\lambda = \cup\{W_z : z \in Z\}$ and $\text{wlor}(\mathscr{P}(Z)) \leq \lambda$ *<u>then</u>* $\sup\{\text{otp}(W_z) : z \in Z\} = \lambda$.
3) If $\lambda = \cup\{W_z : z \in Z\}$ and $|Z| < \lambda$ <u>then</u> $\lambda = \sup\{\text{otp}(W_z) : z \in Z\}$.

Proof. 1) Let $Z_1 = \{z \in Z : W_z \neq \emptyset\}$, so the mapping $z \mapsto \text{Min}(W_z)$ exemplifies that Z_1 is well ordered hence by the definition of $\text{wlor}(Z_1)$ the power $|Z_1|$ is an aleph $< \text{wlor}(Z_1) \leq \text{wlor}(Z)$ and by assumption $\text{wlor}(Z) \leq \lambda$. Now if the desirable conclusion fails then $\gamma^* = \sup(\{\text{otp}(W_z) : z \in Z_1\} \cup \{|Z_1|\})$ is an ordinal $< \lambda$, so we can find a sequence $\langle u_\gamma : \gamma < \gamma^*\rangle$ such that $\text{otp}(u_\gamma) \leq \gamma^*, u_\gamma \subseteq \lambda$ and $\lambda = \cup\{u_\gamma : \gamma < \gamma^*\}$, so $\gamma^* < \lambda \leq |\gamma^* \times \gamma^*|$, easy contradiction.
2) For $x \subseteq Z$ let $W^*_x = \{\alpha < \lambda : (\forall z \in Z)(\alpha \in W_z \equiv z \in x)\}$ hence λ is the disjoint union of $\{W^*_x : x \in \mathscr{P}(Z)\backslash\{\emptyset\}\}$. So the result follows by (1).
3) So let $<_*$ be a well ordering of Z and let $W'_z = \{\alpha \in W_z\text{: if } y <_* z \text{ then } \alpha \notin W_y\}$, so $\langle W'_z : z \in Z\rangle$ is a well defined sequence of pairwise disjoint sets with union equal to $\cup\{W_z : z \in Z\} = \lambda$ and $\text{otp}(W'_z) \leq \text{otp}(W_z)$. Hence if $|W_z| = \lambda$ for some $z \in Z$ the desirable conclusion is obvious, otherwise the result follows by part (1). $\square_{0.14}$

{0.F}

Definition 0.15. 1) We say that $c\ell$ is a very weak closure operation on λ of character (μ, κ) <u>when</u>:

- (a) $c\ell$ is a function from $\mathscr{P}(\lambda)$ to $\mathscr{P}(\lambda)$
- (b) $u \in [\lambda]^{\leq\kappa} \Rightarrow |c\ell(u)| \leq \mu$
- (c) $u \subseteq \lambda \Rightarrow u \cup \{0\} \subseteq c\ell(u)$, the 0 for technical reasons.

1A) We say that $c\ell$ is a weak closure[1] operation on λ of character (μ, κ) <u>when</u> (a),(b),(c) above and

- (d) $u \subseteq v \subseteq \lambda \Rightarrow u \subseteq c\ell(u) \subseteq c\ell(v)$
- (e) $c\ell(u) = \cup\{c\ell(v) : v \subseteq u, |v| \leq \kappa\}$.

1B) Let "... character $(< \mu, \kappa)$ or $(\mu, < \kappa)$, or $(< \mu, < \kappa)$" has the obvious meaning but if μ is an ordinal not a cardinal, then "$< \mu$" means of order type $< \mu$; similarly for "$< \kappa$". Let "... character (μ, Y)" means "character $(< \mu^+, < \text{hrtg}(Y))$"
1C) We omit the weak if in addition

- (f) $c\ell(u) = c\ell(c\ell(u))$ for $u \subseteq \lambda$.

[1]so by actually only $c\ell\restriction[\lambda]^{\leq\kappa}$ count

2) λ is f-inaccessible when $\delta \in \lambda \cap \mathrm{Dom}(f) \Rightarrow f(\delta) < \lambda$.
3) $c\ell : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is well founded <u>when</u> for no sequence $\langle \mathscr{U}_n : n < \omega \rangle$ of subsets of λ do we have $c\ell(\mathscr{U}_{n+1}) \subset \mathscr{U}_n$ for $n < \omega$.
4) For $c\ell$ a partial function from $\mathscr{P}(\alpha)$ to $\mathscr{P}(\alpha)$ (for simplicity assume $\alpha = \cup\{u : u \in \mathrm{Dom}(c\ell)\}$) let $c\ell^1_{\varepsilon,<\kappa}$ a function from $\mathscr{P}(\alpha)$ to $\mathscr{P}(\alpha)$ be defined by induction on the ordinal ε as follows:

(a) $c\ell^1_{0,<\kappa}(u) = u$
(b) $c\ell^1_{\varepsilon+1,<\kappa}(u) = \{0\} \cup c\ell^1_{\varepsilon,<\kappa}(u) \cup \bigcup\{c\ell(v) : v \subseteq c\ell^1_{\varepsilon,<\kappa}(u) \text{ and } v \in \mathrm{Dom}(c\ell), |v| < \kappa\}$
(c) for limit ε let $c\ell^1_{\varepsilon,<\kappa}(u) = \cup\{c\ell^1_{\zeta,<\kappa}(u) : \zeta < \varepsilon\}$.

4A) Instead "$< \kappa$" we may write "$\leq \kappa$".
5) For any function $F : [\lambda]^{\aleph_0} \to \lambda$ and countable $u \subseteq \lambda$ we define $c\ell^2_\varepsilon(u, F)$ by induction on $\varepsilon \leq \omega_1$

(a) $c\ell^2_0(u, F) = u \cup \{0\}$
(b) $c\ell^2_{\varepsilon+1}(u, F) = c\ell^2_*(u, F) \cup \{F(c\ell^2_\varepsilon(u, F))\}$
(c) $c\ell^2_\varepsilon(u, F) = \cup\{c\ell^2_\zeta(u, F) : \zeta < \varepsilon\}$ when $\varepsilon \leq \omega_1$ is a limit ordinal.

6) For countable u and F as in part (5) let $c\ell^3_F(u) = c\ell^3(u, F) := c\ell^2_{\omega_1}(u, F)$ and for any $u \subseteq \lambda$ let $c\ell^4_F(u) := u \cup \bigcup\{c\ell^3_F(v) : v \subseteq \mathrm{Dom}(F) \text{ is countable}\}$.
7) For a cardinal ∂ we say that $c\ell : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is ∂-well founded <u>when</u> for no $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial \rangle$ of subsets of λ do we have $\varepsilon < \zeta < \partial \Rightarrow c\ell(\mathscr{U}_\zeta) \not\supseteq \mathscr{U}_\varepsilon$.
8) If $F : [\lambda]^{\leq\kappa} \to \lambda$ and $u \subseteq \lambda$ then we let $c\ell_F(u) = c\ell^1_F(u)$ be the minimal subset v of λ such that $w \in [v]^{\leq\kappa} \Rightarrow F(w) \in v$ and $u \subseteq v$ (exists).

Observation 0.16. *For $F : [\lambda]^{\aleph_0} \to \lambda$, the operation $u \mapsto c\ell^3_F(u)$ is a very weak closure operation of character $(\aleph_0, \aleph_1)$.* {0n.F}

Remark 0.17. So for any very weak closure operation, $\aleph_0$-well founded is a stronger property than well founded, but if $u \subseteq \lambda \Rightarrow c\ell(c\ell(u)) = c\ell(u)$ which is reasonable, they are equivalent. {0.G}

Observation 0.18. *$[\alpha]^\partial$ is well ordered iff ${}^\partial\alpha$ is well ordered when $\alpha \geq \partial$.* {0.H}

Proof. Use a pairing function on α for showing $|{}^\partial\alpha| \leq [\alpha]^\partial$, so $\Rightarrow$ holds. If ${}^\partial\alpha$ is well ordered by $<_*$ map $u \in [\alpha]^\partial$ to the $<_*$-first $f \in {}^\partial\alpha$ satisfying $\mathrm{Rang}(f) = u$. $\square_{0.18}$

§ 1. Representing ${}^{\kappa}\lambda$

Here we give a simple case to illustrate what we do (see latter on improvements in the hypothesis and conclusion). Specifically, if Y is uncountable and $[\lambda]^{\aleph_0}$ is well ordered, then the set ${}^{Y}\lambda$ can be an analysis modulo countable union over few (i.e., their number depends on Y but not on λ) well ordered sets.

{r.1} **Definition 1.1.** 1)

(a) $\mathrm{Fil}_{\aleph_1}(Y) = \mathrm{Fil}^1_{\aleph_1}(Y) = \{D : D$ an $\aleph_1$-complete filter on $Y\}$, so Y is defined from D as $\cup\{X : X \in D\}$

(b) $\mathrm{Fil}^2_{\aleph_1}(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are $\aleph_1$-complete filters on Y, $(\emptyset \notin D_2$, of course$)\}$; in this context $Z \in \bar{D}$ means $Z \in D_2$

(c) $\mathrm{Fil}^3_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h) : (D_1, D_2) \in \mathrm{Fil}^2_{\aleph_1}(Y)$ and $h : Y \to \alpha$ for some $\alpha < \mu\}$, if we omit μ we mean $\mu = \mathrm{hrtg}(Y) \cup \omega$

(d) $\mathrm{Fil}^4_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h, Z) : (D_1, D_2, h) \in \mathrm{Fil}^3_{\aleph_1}(Y, \mu), Z \in D_2\}$; omitting μ means as above.

2) For $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y, \mu)$ let $Y = Y^{\mathfrak{y}}, \mathfrak{y} = (D^{\mathfrak{y}}_1, D^{\mathfrak{y}}_2, h^{\mathfrak{y}}, Z^{\mathfrak{y}}) = (D_1[\mathfrak{y}], D_2[\mathfrak{y}], h[\mathfrak{y}], Z[\mathfrak{y}])$; similarly for the others and let $D^{\mathfrak{y}} = D[\mathfrak{y}]$ be $D^{\mathfrak{y}}_1 + Z^{\mathfrak{y}}$.
3) We can replace $\aleph_1$ by any $\kappa > \aleph_1$ (the results can be generalized easily assuming DC + $\mathrm{AC}_{<\kappa}$, used in §2).

{r.2} **Theorem 1.2.** *$[\mathrm{DC}_{\aleph_0}]$ Assume*

(a) $[\lambda]^{\aleph_0}$ *is well ordered.*

Then we can find a sequence $\langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\rangle$ satisfying

(α) $\mathscr{F}_{\mathfrak{y}} \subseteq {}^{Z[\mathfrak{y}]}\lambda$

(β) *$\mathscr{F}_{\mathfrak{y}}$ is a well ordered set by $f_1 <_{\mathfrak{y}} f_2 \Leftrightarrow \mathrm{rk}_{D[\mathfrak{y}]}(f_1) < \mathrm{rk}_{D[\mathfrak{y}]}(f_2)$ so $f \mapsto \mathrm{rk}_{D[\mathfrak{y}]}(f)$ is a one-to-one mapping from $\mathscr{F}_{\mathfrak{y}}$ into the ordinals*

(γ) *if $f \in {}^{Y}\lambda$ then we can find a sequence $\langle \mathfrak{y}_n : n < \omega\rangle$ with $\mathfrak{y}_n \in \mathrm{Fil}^4_{\aleph_1}(Y)$ such that $n < \omega \Rightarrow f \restriction Z^{\mathfrak{y}_n} \in \mathscr{F}_{\mathfrak{y}_n}$ and $\cup\{Z^{\mathfrak{y}_n} : n < \omega\} = Y$.*

An immediate consequence of 1.2 is

{r.1x} **Conclusion 1.3.** *1) [DC + ${}^{\omega}\alpha$ is well-orderable for every ordinal α].*

For any set Y and cardinal λ there is a sequence $\langle \mathscr{F}_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y))\rangle$ such that

(a) ${}^{Y}\lambda = \cup\{\mathscr{F}_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y))\}$

(b) *$\mathscr{F}_{\bar{\mathfrak{x}}}$ is well orderable for each $\bar{\mathfrak{x}} \in {}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y))$*

$(b)^+$ *moreover, uniformly, i.e., there is a sequence $\langle <_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y)\rangle$ such that $<_{\bar{\mathfrak{x}}}$ is a well order of $\mathscr{F}_{\bar{\mathfrak{x}}}$*

(c) *there is a function F with domain $\mathscr{P}({}^{Y}\alpha)\backslash\{\emptyset\}$ such that: if $S \subseteq {}^{Y}\alpha$ then $F(S)$ is a non-empty subset of S of cardinality $\leq_{qu} {}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y)))$ recalling Definition 0.12. In fact, some sequence $\langle \mathscr{U}_\alpha : \alpha < \alpha(*)\rangle$ is a partition of ${}^{Y}\alpha$, we have $F(S) = \mathscr{U}_{f(S)}$ where $f(S) = \mathrm{Min}\{\alpha : \mathscr{U}_\alpha \cap S \neq \emptyset\}$ (and $|\mathscr{U}_\alpha| < \mathrm{hrtg}({}^{\omega}(\mathrm{Fil}^4_{\aleph_1}(Y))))$.*

2) [DC] If $[\alpha(*)]^{\aleph_0}$ is well ordered where $\alpha(*) = \cup\{\mathrm{rk}_D(f)+1 : f \in {}^Y\lambda$ and $D \in \mathrm{Fil}^1_{\aleph_1}(Y)\}$ then ${}^Y\lambda$ satisfies the conclusion of part (1).

Remark 1.4. 1) So clause (c) of 1.3(1) is a weak form of choice.

Proof. Proof of 1.3 1) Let $\langle \mathscr{F}_\mathfrak{y} : \mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)\rangle$ be as in 1.2.

For each $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ (so $\bar{\mathfrak{x}} = \langle \mathfrak{x}_n : n < \omega\rangle$) let

$$\mathscr{F}'_{\bar{\mathfrak{x}}} = \{f : \ f \text{ is a function from } Y \text{ to } \lambda \text{ such that } n<\omega \Rightarrow f \restriction Z^{\mathfrak{x}_n} \in \mathscr{F}_{\mathfrak{x}_n} \text{ and } Y = \cup\{Z^{\mathfrak{x}_n} : n<\omega\}\}.$$

Now

$(*)_1$ ${}^Y\lambda = \cup\{\mathscr{F}'_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\}$.

[Why? By clause (γ) of 1.2.]

Let $\alpha(*) = \cup\{\mathrm{rk}_D(f)+1 : f \in {}^Y\lambda$ and $D \in \mathrm{Fil}^1_{\aleph_1}(Y)\}$. For $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ we define the function $G_{\bar{\mathfrak{x}}} : \mathscr{F}'_{\bar{\mathfrak{x}}} \to {}^\omega\alpha(*)$ by $G_{\bar{\mathfrak{x}}}(f) = \langle \mathrm{rk}_{D_1[\mathfrak{x}_n]}(f) : n<\omega\rangle$.

Clearly

$(*)_2$ (α) $\bar{G} = \langle G_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\rangle$ exists

(β) $G_{\bar{\mathfrak{x}}}$ is a function from $\mathscr{F}'_{\bar{\mathfrak{x}}}$ to ${}^\omega\alpha(*)$

(γ) $G_{\bar{\mathfrak{x}}}$ is one to one.

Let $<_*$ be a well ordering of ${}^\omega\alpha(*)$ and for $\bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ let $<_{\bar{\mathfrak{x}}}$ be the following two place relation on $\mathscr{F}'_{\bar{\mathfrak{x}}}$:

$(*)_3$ $f_1 <_{\bar{\mathfrak{x}}} f_2$ iff $G_{\bar{\mathfrak{x}}}(f_1) <_* G_{\bar{\mathfrak{x}}}(f_2)$.

Obviously

$(*)_4$ (α) $\langle <_{\bar{\mathfrak{x}}} : \bar{\mathfrak{x}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))\rangle$ exists

(β) $<_{\bar{\mathfrak{x}}}$ is a well ordering of $\mathscr{F}'_{\bar{\mathfrak{x}}}$.

By $(*)_1 + (*)_4$ we have proved clauses (a),(b),(b)$^+$ of the conclusion. Now clause (c) follows: for non-empty $S \subseteq {}^Y\alpha$, let $f(S)$ be $\min\{\mathrm{otp}(\{g : g <_{\bar{\mathfrak{y}}} f\}, <_{\bar{\mathfrak{y}}}) : \bar{\mathfrak{y}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ and $f \in \mathscr{F}'_{\bar{\mathfrak{y}}} \cap S\}$. Also for any ordinal γ let $\mathscr{U}^1_\gamma := \{f$: for some $\bar{\mathfrak{y}} \in {}^\omega(\mathrm{Fil}^4_{\aleph_1}(Y))$ we have $\gamma = \mathrm{otp}(\{g : g <_{\bar{\mathfrak{y}}} f\}, <_{\bar{\mathfrak{y}}})\}$ and $\mathscr{U}_\gamma = \mathscr{U}^1_\gamma \setminus \cup\bigcup\{\mathscr{U}^1_\beta : \beta < \gamma\}$.

Lastly, we let $F(S) = \mathscr{U}_{f(S)} \cap S$. Now check.

2) Similarly. $\square_{1.3}$

Proof. Proof of 1.2 First

$\circledast_1$ there are a cardinal μ and a sequence $\bar{u} = \langle u_\alpha : \alpha < \mu\rangle$ listing $[\lambda]^{\aleph_0}$.

[Why? by assumption (a).]

Second, we can deduce

$\circledast_2$ there are $\mu_1 \le \mu$ and a sequence $\bar{u} = \langle u_\alpha : \alpha < \mu_1\rangle$ such that:

(a) $u_\alpha \in [\lambda]^{\aleph_0}$

(b) if $u \in [\lambda]^{\le \aleph_0}$ then for some finite $w \subseteq \mu_1, u \subseteq \cup\{u_\beta : \beta \in w\}$

(c) u_α is not included in $u_{\alpha_0} \cup \ldots \cup u_{\alpha_{n-1}}$ when $n < \omega, \alpha_0, \ldots, \alpha_{n-1} < \alpha$.

[Why? Let $\bar{u}^0$ be of the form $\langle u_\alpha : \alpha < \alpha^* \rangle$ such that $(a) + (b)$ holds and $\ell g(\bar{u}^0)$ is minimal; it is well defined and $\ell g(\bar{u}^0) \le \mu$ by $\circledast_1$. Let $W = \{\alpha < \ell g(\bar{u}^0) : u^0_\alpha \nsubseteq \cup\{u^0_\beta : \beta \in w\}$ when $w \subseteq \alpha$ is finite$\}$. Let $\mu_1 = |W|$ and let $f : \mu_1 \to W$ be a one-to-one onto, let $u_\alpha = u^0_{f(\alpha)}$ so $\langle u_\alpha : \alpha < \mu_1 \rangle$ satisfies $(a) + (b)$ and $\mu_1 = |W| \le \ell g(\bar{u}^0)$. So by the choice of $\bar{u}^0$ we have $\ell g(\bar{u}^0) = \mu_1$. So we can choose f such that it is increasing hence $\bar{u}$ is as required.]

$\circledast_3$ we can define $\mathbf{n} : [\lambda]^{\le \aleph_0} \to \omega$ and partial functions $F_\ell : [\lambda]^{\le \aleph_0} \to \mu_1$ for $\ell < \omega$ (so $\langle F_\ell : \ell < \omega \rangle$ exists) as follows:
- (a) u infinite $\Rightarrow F_0(u) =$ Min$\{\alpha$: for some finite $w \subseteq \alpha, u \subseteq u_\alpha \cup \bigcup\{u_\beta : \beta \in w\}$ mod finite$\}$
- (b) u finite $\Rightarrow F_0(u)$ undefined
- (c) $F_{\ell+1}(u) := F_0(u \backslash (u_{F_0(u)} \cup \ldots \cup u_{F_\ell(u)}))$ for $\ell < \omega$ when $F_\ell(u)$ is defined
- (d) $\mathbf{n}(u) :=$ Min$\{\ell : F_\ell(u)$ undefined$\}$.

Then

$\circledast_4$
- (a) $F_{\ell+1}(u) < F_\ell(u) < \mu_1$ when they are well defined
- (b) $\mathbf{n}(u)$ is a well defined natural number and $u \backslash \cup \{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\}$ is finite and $k < \mathbf{n}(u) \Rightarrow (u \backslash \cup \{u_{F_\ell(u)} : \ell < k\}) \cap u_{F_k(u)}$ is infinite
- (c) if $u_1, u_2 \in [\lambda]^{\aleph_0}, u_1 \subseteq u_2$ and $u_2 \backslash u_1$ is finite then $F_\ell(u_1) = F_\ell(u_2)$ for $\ell < \mathbf{n}(u_1)$ and $\mathbf{n}(u_1) = \mathbf{n}(u_2)$

$\circledast_5$ define $F_* : [\lambda]^{\aleph_0} \to \lambda$ by $F_*(u) =$ Min$(\cup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\} \cup \{0\} \backslash u)$ if well defined, zero otherwise

$\circledast_6$ if $u \in [\lambda]^{\aleph_0}$ then
- (α) $c\ell^3(u, F_*) = c\ell^3_{F_*}(U)$ is $F'(u) := u \cup \bigcup\{u_{F_\ell(u)} : \ell < \mathbf{n}(u)\} \cup \{0\}$
- (β) $c\ell^3_{F_*}(u) = c\ell^2_{\varepsilon(u)}(F)$ for some $\varepsilon(u) < \omega_1$
- (γ) there is $\bar{F} = \langle F'_\varepsilon : \varepsilon < \omega_1 \rangle$ such that: for every $u \in [\lambda]^{\aleph_0}, c\ell^3_{F_*}(u) = \{F'_\varepsilon(u) : \varepsilon < \varepsilon(u)\}$ and $F'_\varepsilon(u) = 0$ if $\varepsilon \in [\varepsilon(u), \omega_1)$
- (δ) in fact $F'_\varepsilon(u)$ is the ε-th member of $c\ell^3_{F_*}(u)$ if $\varepsilon < \varepsilon(u)$.

[Why? Define w^ε_u by induction on ε by $w^0_u = u, w^{\varepsilon+1}_u = w^\varepsilon_u \cup \{F_*(w^\varepsilon_u)\}$ and for limit $\varepsilon, w^\varepsilon_u = \cup\{w^\zeta_u : \zeta < \varepsilon\}$. We can prove by induction on ε that $w^\varepsilon_u \subseteq F'(u)$ which is countable. The partial function g with domain $F'(u) \backslash u$ to Ord, $g(\alpha) =$ Min$\{\varepsilon : \alpha \in w^{\varepsilon+1}_u\}$ is one to one onto an ordinal call it $\varepsilon(*)$, so $w^{\varepsilon(*)}_u \subseteq F'(u)$ and if they are not equal that $F_*(w^{\varepsilon(*)}_u) \in F'(u) \backslash w^{\varepsilon(*)}_u$ hence $w^{\varepsilon(*)}_u \subsetneqq w^{\varepsilon(*)+1}_u$ contradicting the choice of $\varepsilon(*)$. So clause (α) holds. In fact, $c\ell^3(u, F_*) = w^{\varepsilon(*)}_u$ and clause (β) holds.]

$\circledast_7$ there is no sequence $\langle \mathscr{U}_n : n < \omega \rangle$ such that
- (a) $\mathscr{U}_{n+1} \subseteq \mathscr{U}_n \subset \lambda$
- (b) $\mathscr{U}_n$ is closed under F_*, i.e. $u \in [\mathscr{U}_n]^{\aleph_0} \Rightarrow F_*(u) \in \mathscr{U}_n$
- (c) $\mathscr{U}_{n+1} \neq \mathscr{U}_n$.

[Why? Assume toward contradiction that $\langle \mathscr{U}_n : n < \omega\rangle$ satisfies clauses (a),(b),(c). Let $\alpha_n = \text{Min}(\mathscr{U}_n \backslash \mathscr{U}_{n+1})$ for $n < \omega$ hence the sequence $\bar\alpha = \langle \alpha_n : n < \omega\rangle$ is with no repetitions and let $\beta_{m,\ell} := F_\ell(\{\alpha_n : n \ge m\})$ for $m < \omega$ and $\ell < \mathbf{n}_m := \mathbf{n}(\{\alpha_n : n \in [m,\omega)\})$. As $\bar\alpha$ is with no repetition, $\mathbf{n}_m > 0$ and by $\circledast_4(c)$ clearly $\mathbf{n}_m = \mathbf{n}_0$ for $m < \omega$ and $\beta_{m,\ell} = \beta_{m,0}$ for $m < \omega, \ell < \mathbf{n}_0$. So letting $v_m = \cup\{u_{F_\ell(\{\alpha_n : n \in [m,\omega)\})} : \ell < \mathbf{n}_m\}$, it does not depend on m so $v_m = v_0$, and by the choice of F_*, as $\{\alpha_n : n \in [m,\omega)\} \subseteq \mathscr{U}_m$ and $\mathscr{U}_m$ is closed under F_* clearly $v_m \subseteq \mathscr{U}_m$. Together $v_0 = v_m \subseteq \mathscr{U}_m$ so $v_0 \subseteq \cap\{\mathscr{U}_m : m < \omega\}$. Also, by the definition of the F_ℓ's, $\{\alpha_n : n < \omega\}\backslash v_0$ is finite so for some $k < \omega, \{\alpha_m : n \in [k,\omega)\} \subseteq v_0$ but $v_0 \subseteq \mathscr{U}_{k+1}$ contradicting the choice of α_k.]

Moreover, recalling Definition 0.15(6):

$\circledast'_7$ there is no sequence $\langle \mathscr{U}_n : n < \omega\rangle$ such that
 (a) $\mathscr{U}_{n+1} \subseteq \mathscr{U}_n \subseteq \lambda$
 (b) $c\ell^4_{F_*}(\mathscr{U}_{n+1})\backslash \mathscr{U}_n \ne \emptyset$.

[Why? As above but letting $\alpha_n = \text{Min}(\mathscr{U}_n \backslash c\ell^3_{F_*}(\mathscr{U}_n))$, recalling Definition 0.16(6).]

Now we define for $(D_1, D_2, h, Z) \in \text{Fil}^4_{\aleph_1}(Y)$ and ordinal α the following, recalling Definition 0.15(6) for clauses (e),(f):

$\circledast_8$ $\mathscr{F}_{(D_1 D_2,h,Z),\alpha} =: \{f :$
 (a) f is a function from Z to λ
 (b) $\text{rk}_{D_1+Z}(f \cup 0_{(Y\backslash Z)}) = \alpha$
 (c) $D_2 = \{Y\backslash X : X \subseteq Y$ satisfies $X = \emptyset \mod D_1$ or $X \in D^+_1$ and $\text{rk}_{D_1+X}(f \cup 0_{(Y\backslash Z)}) > \alpha$ that is $\text{rk}_{D_1+X}(f) > \alpha\}$
 (d) $Z \in D_2$, really follows
 (e) if $Z' \subseteq Z \wedge Z' \in D_2$ then $c\ell^3_{F_*}(\text{Rang}(f \restriction Z')) = c\ell^3_{F_*}(\text{Rang}(f))$
 (f) $y \in Z \Rightarrow f(y) =$ the $h(y)$-th member of $c\ell^3_{F_*}(\text{Rang}(f))\}$.

So we have:

$\circledast_9$ $\mathscr{F}_{(D_1,D_2,h,Z),\alpha}$ has at most one member; called it $f_{(D_1,D_2,h,Z),\alpha}$ (when defined; pedantically we should write $f_{(D_1,D_2,h,Z),c\ell,\alpha}$)

$\circledast_{10}$ $\mathscr{F}_{(D_1,D_2,h,Z)} =: \cup\{\mathscr{F}_{(D_1,D_2,h,Z),\alpha} : \alpha$ an ordinal$\}$ is a well ordered set.

[Why? Define $<_{(D_1,D_2,h,Z)}$ by the α's.]

$\circledast_{11}$ if $f : Y \to \lambda$ and $Z \subseteq Y$ then $c\ell^4_{F_*}(\text{Rang}(f \restriction Z))$ has cardinality $< \text{hrtg}(Z)$.

[Why? By the definition of hrtg($-$) we are done.]

$\circledast_{12}$ if $f : Z \to \lambda$ and $Z \subseteq Y$ <u>then</u> $c\ell^4_{F_*}(\text{Rang}(f)) \subseteq \lambda$ has cardinality $< \text{hrtg}([Z]^{\aleph_0})$ or is finite.

[Why? If Rang(f) is countable more hold by 0.16. Otherwise, by $\circledast_6(\beta)$ we have $c\ell^4_{F_*}(\text{Rang}(f)) = \text{Rang}(f) \cup \{F'_\varepsilon(u) : u \in [\text{Rang}(f)]^{\aleph_0}$ and $\varepsilon < \omega_1\}$.

Let $\alpha(*)$ be minimal such that $\text{Rang}(f) \cap \alpha(*)$ has order type ω_1. Let $h_1, h_2 : \omega_1 \to \omega_1$ be such that $h_\ell(\varepsilon) < \max\{\varepsilon, 1\}$ and for every $\varepsilon_1, \varepsilon_2 < \omega_1$ there is $\zeta \in [\varepsilon_1 + \varepsilon_2 + 1, \omega_1)$ such that $h_\ell(\zeta) = \varepsilon_\ell$ for $\ell = 1, 2$. Define $F : [Z]^{\aleph_0} \to \lambda$ as follows: if $u \in [\text{Rang}(f)]^{\aleph_0}$, let $\varepsilon_\ell(u) = h_\ell(\text{otp}(u \cap \alpha(*))$ for $\ell = 1, 2$ and $F(u) = F'(\{\alpha \in u$: if $\alpha < \alpha(*)$ then $\text{otp}(u \cap \alpha) < \varepsilon_1(u)\})$.]

Now

$\bullet_1$ if $u \in [\text{Rang}(f)]^{\aleph_0}$ then $F(u)$ is $F_\varepsilon(v)$ for some $v \in [Z]^{\aleph_0}$ and $\varepsilon < \omega_1$.

[Why? As $F(u) \in \text{Rang}(F'_{\varepsilon_2(u)} \restriction [\text{Rang}(f)]^{\aleph_0})$.]

$\bullet_2$ $\{F(u) : u \in [\text{Rang}(f)]^{\aleph_0}\} \subseteq c\ell^4_{F_*}(\text{Rang}(f))$.

[Why? By $\bullet_1$ recalling $\circledast_6$.]

$\bullet_3$ if $u \in [\text{Rang}(f)]^{\aleph_0}$ and $\varepsilon < \omega_1$ then $F'_\varepsilon(u)$ is $F(u)$ for some $v \in [\text{Rang}(f)]^{\aleph_0}$.

[Why? Let $\varepsilon_1 = \text{otp}(u \cap \alpha(*)), \varepsilon_2 = \varepsilon$ and $\zeta < \omega_1$ such that $h_\ell(\zeta) = \varepsilon_\ell$ for $\ell = 1,2$. Let $v = u \cup \{\alpha : \alpha \in \text{Rang}(f) \cap \alpha(*)$ and $\alpha \geq \sup(u \cap \alpha(*)) + 1$ and $\text{otp}(\text{Rang}(f) \cap \alpha \backslash (\sup(u \cap \alpha(*) + 1)) < (\zeta - \varepsilon_1))\}$,
so $F(u) = F'_\varepsilon(u)$.]

$\bullet_4$ in $\bullet_2$ we have equality.

Together $c\ell^4_{F_*}(\text{Rang}(f)) = \{F(u) : u \in [\text{Rang}(f)]^{\aleph_0}\} \cup \text{Rang}(f)$ so it is the union of two sets; the first of cardinality $< \text{hrtg}([Z]^{\aleph_0})$ and the second of cardinality $< \text{hrtg}[Z]$, so we are easily done.

$\circledast_{13}$ if $f : Y \to \lambda$ then for some $\mathfrak{y}_n \in \text{Fil}^4_{\aleph_1}(Y)$ and $\alpha_n \in \text{Ord}$ for $n < \omega$ we have $f = \cup\{f_{\mathfrak{y}_n,\alpha_n} : n < \omega\}$.

[Why? Let

$$\mathscr{I}^0_f = \{Z \subseteq Y : \quad \text{for some } \mathfrak{y} \in \text{Fil}^4_{\aleph_1}(Y) \text{ satisfying } Z^{\mathfrak{y}} = Z \text{ and ordinal } \alpha, f_{\mathfrak{y},\alpha} \text{ is well defined and equal to } f \restriction Z\}$$

$$\mathscr{I}_f = \{Z \subseteq Y : Z \text{ is included in a countable union of members of } \mathscr{I}^0_f\}.$$

So it is enough to show that $Y \in \mathscr{I}_f$.

Toward contradiction assume not. Let $D_1 = \{Y \backslash Z : Z \in \mathscr{I}_f\}$, clearly it belongs to $\text{Fil}_{\aleph_1}(Y)$, e.g. noting that $\emptyset \in \mathscr{I}_f$. So $\alpha(*) = \text{rk}_{D_1}(f)$ is well defined (by 0.8) recalling that only $\text{DC}_{\aleph_0}$ is needed.

Let

$$D_2 = \{X \subseteq Y : X \in D_1 \text{ or } \text{rk}(f)_{D_1 + (Y \backslash X)}(f) > \alpha(*)\}.$$

By 0.10 + 0.11 clearly D_2 is an $\aleph_1$-complete filter on Y extending D_1.

Now we try to choose $Z_n \in D_2$ for $n < \omega$ such that $Z_{n+1} \subseteq Z_n$ and $c\ell^4_{F_*}(\text{Rang}(f \restriction Z_{n+1}))$ does not include $\text{Rang}(f \restriction Z_n)$.

For $n = 0, Z_0 = Y$ is O.K.

By $\circledast'_7$ we cannot have such ω-sequence $\langle Z_n : n < \omega \rangle$; so by DC for some (unique) $n = n(*), Z_n$ is defined but not Z_{n+1}.

Let $h : Z_n \to \text{hrtg}([Y]^{\aleph_0}) \cup \omega_1$ be:

$$h(y) = \text{otp}(f(y) \cap c\ell^4_{F_*}(\text{Rang}(f \restriction Z_n))).$$

Now h is well defined by $\circledast_{12}$. Easily

$$f \restriction Z_n \in \mathscr{F}_{(D_1 + Z_n, D_2, h, Z_n), \alpha(*)}$$

hence $Z_n \in \mathscr{I}^0_f \subseteq \mathscr{I}_f$, contradiction to $Z_n \in D_2, D_1 \subseteq D_2$.

So we are done proving $\circledast_{13}$.]

Now clause (β) of the conclusion holds by the definition of $\mathscr{F}_{\mathfrak{y}}$, clause (α) holds by $\circledast_{10}$ recalling $\circledast_8, \circledast_9$ and clause (γ) holds by $\circledast_{12}$. $\square_{1.2}$

{r.2y}

Remark 1.5. We can improve 1.2 in some way the choice of $\bar{u}$.

Assume $(a) + (b)$ where

- (a) let $\langle u_\alpha : \alpha < \alpha^* \rangle$ be a sequence of members of $[\lambda]^{\aleph_0}$ such that $(\forall u \in [\lambda]^{\aleph_0})(\exists \alpha)(u \cap u_\alpha$ infinite), and $\alpha < \alpha^* \Rightarrow \mathrm{otp}(u_\alpha) = \omega$.

[Why? We define $F_0(v) :=$ Min$\{\alpha < \alpha^* : v \cap u_\alpha$ infinite$\}$. Let $F_*(u) =$ min$(u_{F_0(\alpha)} \backslash u)$. Now we use only F'_ℓ for $\ell < \omega$ such that $F'_\ell(u)$ is the ℓ-th member of u_F or of $u_{F_0(\alpha)} \backslash u$; so $\varepsilon(u) = \omega$.]

{1p.4}

Observation 1.6. *1) The power of* $\mathrm{Fil}^4_{\aleph_1}(Y, \mu)$ *is smaller or equal to the power of the set* $(\mathscr{P}(\mathscr{P}(Y)))^2 \times \mathscr{P}(Y) \times \mu^{|Y|}$*; if* $\aleph_0 \le |Y|$ *this is equal to the power of* $\mathscr{P}(\mathscr{P}(Y)) \times {}^Y\mu$.
2) The power of $\mathrm{Fil}^4_{\aleph_1}(Y)$ *is smaller or equal to the power of the set* $(\mathscr{P}(\mathscr{P}(Y)))^2 \times \mathscr{P}(Y) \times \cup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}$.
3) In part (2), if $\aleph_0 \le |Y|$ *this is equal to* $|\mathscr{P}(\mathscr{P}(Y)) \times \cup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}$*; also* $\alpha < \mathrm{hrtg}([Y]^{\aleph_0}) \Rightarrow |\mathscr{P}(\mathscr{P}(Y)) \times {}^Y\alpha| = |\mathscr{P}(\mathscr{P}(Y))|$.

Remark 1.7. 1) As we are assuming DC, the case $\aleph_0 \not\le |Y|$ is Y finite, so degenerated; even $|Y| = \aleph_0$ is $|\mathrm{Fil}^4_{\aleph_0}(Y)| \le_{\mathrm{qu}} |\mathscr{P}(\mathscr{P}(Y \times Y))|$.

Proof. 1) Reading the definition of $\mathrm{Fil}^4_{\aleph_1}(Y, \mu)$ clearly its power is $\le$ the power of $\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y) \times \mu^\lambda$. If $\aleph_0 \le |Y|$ then $|\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y)| \le |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y))| = 2^{|\mathscr{P}(Y))} \times 2^{|\mathscr{P}(Y)|} \le 2^{|\mathscr{P}(Y)| + |\mathscr{P}(Y)|} = 2^{|\mathscr{P}(Y)|} = |\mathscr{P}(\mathscr{P}(Y))| \le |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(Y) \times \mu^\lambda|$ as $\mathscr{P}(Y) + \mathscr{P}(Y) = 2^{|Y|} \times 2 = 2^{|Y|+1} = 2^{|Y|}$.
2) Read the definitions.
3) If $\alpha < \mathrm{hrtg}([Y]^{\aleph_0})$ then let f be a function from $[Y]^{\aleph_0}$ onto α and for $\beta < \alpha$ let $A_{f,\beta} = \{u \in [Y]^{\aleph_0} : f(u) < \beta\}$. So $\beta \mapsto A_{f,\beta}$ is a one-to-one function from α onto $\{A_{f,\gamma} : \gamma < \alpha\} \subseteq \mathscr{P}(\mathscr{P}(Y))$ so $|{}^Y\alpha| \le \mathscr{P}(\mathscr{P}(Y)), \mathscr{P}(\mathscr{P}(Y)) \times |{}^Y\alpha| \le \mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y)) \le 2^{|\mathscr{P}(Y)|} = 2^{|P(Y)|} = 2^{|P(Y)| + |P)Y)|} = 2^{|P(Y)|}$. Better, for f a function from $[Y]^{\aleph_0}$ onto $\alpha < \vartheta(Y)$ let $A_f = \{(y_1, y_2) \equiv f(y_1) < f(y_2)\} \subseteq Y_1 \times Y$. Define $F : \mathscr{P}(Y \times Y) \to \mathrm{hrtg}(Y)$ by $F(A) = \alpha$ if $A = A_f$ and f, α are as above, and $F(A) = 0$ otherwise.

So $|\mathscr{P}(\mathscr{P}(u)) \cup \bigcup\{{}^Y\alpha : \alpha < \mathrm{hrtg}([Y]^{\aleph_0})\}| \le_{\mathrm{qu}} |\mathscr{P}(\mathscr{P}(Y)) \times \mathscr{P}(\mathscr{P}(Y \times Y))) = |\mathscr{P}(\mathscr{P}(Y) \times Y)|$. By part (2) we easily get $|\mathrm{Fil}^4_{\aleph_1}(Y)| \le_{\mathrm{qu}} \mathscr{P}(\mathscr{P}(Y \times Y))$. $\square_{1.6}$

{6.1}

Claim 1.8. [$\mathrm{DC}_{\aleph_0}$] *Assume*

- (a) $\mathfrak{a}$ *is a countable set of limit ordinals*
- (b) $<_*$ *is a well ordering of* $\Pi\mathfrak{a}$
- (c) $\theta \in \mathfrak{a} \Rightarrow \mathrm{cf}(\theta) \ge \mathrm{hrtg}(\mathscr{P}(\omega))$ *or just* $\Pi\mathfrak{a}/[\mathfrak{a}]^{<\aleph_0}$ *is* $\mathrm{hrtg}(\mathscr{P}(\omega))$*-directed.*

<u>*Then*</u> *we can define* $(\bar{J}, \bar{\mathfrak{b}}, \bar{\mathbf{f}})$ *such that*

- (α) (i) $\bar{J} = \langle J_i : i \le i(*) \rangle$
  - (ii) J_i *is an ideal on* $\mathfrak{a}$

(iii) J_i *is increasing continuous with* $i, J_0 = \{\emptyset\}, J_{i(*)} = \mathscr{P}(\mathfrak{a})$

(iv) $\bar{\mathfrak{b}} = \langle \mathfrak{b}_i : i < i(*)\rangle, \mathfrak{b}_i \subseteq \mathfrak{a}$ *and* $J_{i+1} = J_i + \mathfrak{b}_i$

(v) *so* J_i *is the ideal on* $\mathfrak{a}$ *generated by* $\{\mathfrak{b}_j : j < i\}$

(β) (i) $\bar{\mathbf{f}} = \langle \bar{f}^i : i < i(*)\rangle$

(ii) $\bar{f}^i = \langle f^i_\alpha : \alpha < \alpha_i\rangle$

(iii) $f^i_\alpha \in \prod \mathfrak{a}$ *is* $<_{J_i}$*-increasing with* $\alpha < \alpha_i$

(iv) $\{f^i_\alpha : \alpha < \alpha_i\}$ *is cofinal in* $(\prod \mathfrak{a}, <_{J_i+(\mathfrak{a}\backslash\mathfrak{b}_i)})$

(γ) (i) $\mathrm{cf}(\prod \mathfrak{a}) \le \sum\limits_{i<i(*)} \alpha_i$

(ii) *for every* $f \in \Pi\mathfrak{a}$ *for some* n *and finite set* $\{(i_\ell, \gamma_\ell) : \ell < n\}$ *such that* $i_\ell < i(*), \gamma_\ell < \alpha_{i_\ell}$ *we have* $f < \max_{\ell<n} f^{i_\ell}_{\gamma_\ell}$, *i.e.,* $(\forall\theta \in \mathfrak{a})(\exists\ell < n)[f(\theta) < f^{i_\ell}_{\gamma_\ell}(\theta)]$.

Remark 1.9. Note that no harm in having more than one occurance of $\theta \in \mathfrak{a}$. See more in [Sh:F1078], e.g. on uncountable $\mathfrak{a}$.

Proof. Note that clause (γ) follows from (α) + (β).
[Why? Easily $(\gamma)(ii) \Rightarrow (\gamma)(i)$. Now let $g \in \Pi\mathfrak{a}$ and let $I_g = \{\mathfrak{b}$: we can find $n < \omega$ and $i_\ell < i(*)$ and $\beta_\ell < \alpha_{i_\ell}$ for $\ell < n$ such that $\theta \in \mathfrak{b} \Rightarrow (\exists\ell < n)(g(\theta) < f^{i_\ell}_{\beta_\ell}(\theta))$.

Easily I_g is an ideal on $\mathfrak{a}$; note that if $\mathfrak{a} \in I_g$ we are done. So assume $\mathfrak{a} \notin I_g$. Note that $J_g \subseteq J_{i(*)}$ hence $j_g = \min\{i \le i(*)$: some $\mathfrak{c} \in \mathscr{P}(\mathfrak{a})\backslash I_g$ belongs to $J_i\}$ is well defined (as $\mathfrak{a} \in \mathscr{P}(\mathfrak{a})\backslash I_g \wedge \mathfrak{a} \in J_{i(*)}$). As $J_0 = \{\emptyset\}$ and $\langle J_i : i \le i(*)\rangle$ is $\subseteq$-increasing continuous, necessairly j_g is a successor ordinal say $j_g = i_g + 1$ and let $\mathfrak{c} \in \mathscr{P}(\mathfrak{a})\backslash I_g$ belongs to J_i. By clause $(\beta)(iv)$ we can finish easily.]

By induction on i we try to choose $(\bar{J}^i, \bar{\mathfrak{b}}^i, \bar{\mathbf{f}}^i)$ where $\bar{J}^i = \langle J_j : j \le i\rangle, \bar{\mathfrak{b}}^i = \langle \mathfrak{b}^i_j : j < i\rangle, \bar{\mathbf{f}}^i = \langle \bar{f}^j : j < i\rangle$ which satisfies the relevant parts of the conclusion and do it uniformly from $(\mathfrak{a}, <_*)$. Once we arrive to i such that $J_i = \mathscr{P}(\mathfrak{a})$ we are done.

For $i = 0$ recalling $J_0 = \{\emptyset\}$ there is no problem.

For i limit recalling that $J_i = \cup\{J_j : j < i\}$ there is no problem and note that if $j < i \Rightarrow \mathfrak{a} \notin J_j$ then $\mathfrak{a} \notin J_i$.

So assume that $(\bar{J}^i, \bar{\mathbf{b}}^i, \bar{\mathbf{f}}^i)$ is well defined and we shall define for $i + 1$.

We try to choose $\bar{g}^{i,\varepsilon} = \langle g^{i,\varepsilon}_\alpha : \alpha < \delta_{i,\varepsilon}\rangle$ and $\mathfrak{b}_{i,\varepsilon}$ by induction on $\varepsilon < \omega_1$ and for each ε we try to choose $g^{i,\varepsilon}_\alpha \in \Pi\mathfrak{a}$ by induction on α (in fact $\alpha < \mathrm{hrtg}(\Pi\mathfrak{a})$ suffice, we shall get stuck earlier) such that:

$\circledast_{i,\varepsilon}$ (a) if $\beta < \alpha$ then $g^{i,\varepsilon}_\beta <_{J_i} g^{i,\varepsilon}_\alpha$

(b) if $\zeta < \varepsilon$ and $\alpha < \delta_{i,\zeta}$ then $g^{i,\zeta}_\alpha \le g^{i,\varepsilon}_\alpha$

(c) if $\mathrm{cf}(\alpha) = \aleph_1$ then $g^{i,\varepsilon}_\alpha$ is defined by

$$\theta \in \mathfrak{a} \Rightarrow g^{i,\varepsilon}_\alpha(\theta) = \mathrm{Min}\{\bigcup_{\beta\in C} g^{i,\varepsilon}_\beta(\theta) : C \text{ is a club of } \alpha\}$$

(d) if $\mathrm{cf}(\alpha) \ne \aleph_1, \alpha \ne 0$ then $g^{i,\varepsilon}_\alpha$ is the $<_*$-first $g \in \Pi\mathfrak{a}$ satisfying clauses (a) + (b) + (c)

(e) if we have $\langle g^{i,\varepsilon}_\beta : \beta < \alpha\rangle$, $\mathrm{cf}(\alpha) > \aleph_1$, moreover $\mathrm{cf}(\alpha) \ge \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$ and there is no g as required in clause (d) then $\delta_{i,\varepsilon} = \alpha$

(f) if $\alpha = 0$, then $g^{i,\varepsilon}_\alpha$ is the $<_*$-first $g \in \Pi\mathfrak{a}$ such that
$\zeta < \varepsilon \wedge \alpha < \delta_{i,\zeta} \Rightarrow g^{i,\zeta}_\alpha \le g$
and $\varepsilon = \zeta + 1 \Rightarrow (\forall \beta < \delta_{i,\zeta})[\neg(g \le_{J_i} g^{i,\zeta}_\beta)]$

(g) J_i is the ideal on $\mathscr{P}(\mathfrak{a})$ generated by $\{\mathfrak{b}_j : j < i\}$

(h) $\mathfrak{b}_{i,\varepsilon} \in (J_i)^+$ so $\mathfrak{b}_{i,\varepsilon} \subseteq \mathfrak{a}$

(i) $\bar{g}^{i,\varepsilon}$ is increasing and cofinal in $(\Pi(\mathfrak{a}), <_{J_i + (\mathfrak{a} \backslash \mathfrak{b}_{i,\varepsilon})})$

(j) $\mathfrak{b}_{i,\varepsilon}$ is such that under clauses $(h) + (i)$ the set $\{\mathrm{otp}(\mathfrak{a} \cap \theta) : \theta \in \mathfrak{b}_{i,\varepsilon}\}$ is $<_*$-minimal

(k) if $\zeta < i$ then $g^{i,\varepsilon}_0$ is a $<_{J_i + \mathfrak{b}_{i,\zeta}}$-upper bound of $\bar{g}^{i,\zeta}$,
hence $\mathfrak{b}_{i,\zeta} \subseteq \mathfrak{b}_{i,\varepsilon} \bmod J_i$.

The rest should be clear.

We now give details on some points

$(*)_1$ Clause (c) is O.K.

[Why? We already have $\langle g^{i,\varepsilon}_\alpha : \alpha < \delta\rangle$ and $\langle g^{i,\zeta}_\alpha : \alpha < \delta_{i,\zeta}, \zeta < \varepsilon\rangle$, and we define $g^{i,\varepsilon}_\delta$ as there. Now $g^{i,\varepsilon}_\alpha(\theta)$ is well defined as the "Min" is taken on a non-empty set as we are assuming $\mathrm{cf}(\delta) = \aleph_1$. The value is $< \theta$ because for some club C of θ, $\mathrm{otp}(C) = \omega_1$, so $g^{i,\varepsilon}_\delta(\theta) \le \cup\{g^{i,\varepsilon}_\beta(\theta) : \beta \in C\}$ but this set is $\subseteq \theta$ while $\mathrm{cf}(\theta) > \aleph_1$ by clause (c) of the assumption. By $\mathrm{AC}_{\aleph_0}$ we can find a sequence $\langle C_\theta : \theta \in \mathfrak{a}\rangle$ such that: C_θ is a club of δ of order type ω_1 such that $g^{i,\varepsilon}_\delta(\theta) = \cup\{g^{i,\varepsilon}_\alpha(\theta) : \alpha \in C_\theta\}$ hence for every club C of δ included in C_θ we have $g^{i,\varepsilon}_\delta(\theta) = \cup\{g^{i,\varepsilon}_\alpha(\theta) : \alpha \in C_\theta\}$. Now $\theta \in \mathfrak{a} \Rightarrow g^{1,\varepsilon}_\delta(\theta) = \bigcup\limits_{\alpha \in C} g^{1,\varepsilon}_\alpha(\theta)$ when $C := \cap\{C_\sigma : \sigma \in \mathfrak{a}\}$, because C too is a club of δ recalling $\mathfrak{a}$ is countable. So if $\alpha < \delta$ then for some β we have $\alpha < \beta \in C$ hence the set $\mathfrak{c} := \{\theta \in \mathfrak{a} : g^{i,\varepsilon}_\alpha(\theta) \ge g^{i,\varepsilon}_\beta(\theta)\}$ belongs to J_i and $\theta \in \mathfrak{a} \backslash \mathfrak{b} \Rightarrow g^{i,\varepsilon}_\alpha(\theta) < g^{i,\varepsilon}_\beta(\theta) \le g^{i,\varepsilon}_\delta(\theta)$, so indeed $g^{i,\varepsilon}_\alpha <_{J_i} g^{i,\varepsilon}_\delta$.

Lastly, why $\zeta < \varepsilon \Rightarrow g^{i,\zeta}_\delta \le g^{i,\varepsilon}_\delta$? As we can find a club C of δ which is as above for both $g^{i,\zeta}_\delta$ and $g^{i,\varepsilon}_\delta$ and recall that clause (b) of $\circledast_{i,\varepsilon}$ holds for every $\beta \in C$. Together $g^{i,\varepsilon}_\delta$ is as required.]

$(*)_2$ $\mathrm{cf}(\delta_{i,\zeta}) > \aleph_1$ and even $\mathrm{cf}(\delta_{\ell,t}) \ge \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$.

[Why? We have to prove that arriving to α, if $\mathrm{cf}(\alpha) < \min\{\mathrm{cf}(\theta) : \theta \in \mathfrak{a}\}$ then we can choose $g^{i,\zeta}_\alpha$ as required. The cases $\mathrm{cf}(\alpha) = \aleph_1, \alpha = 0$ are covered, otherwise let $u \subseteq \alpha$ be unbounded of order type $\mathrm{cf}(\alpha)$, and define a function g from $\mathfrak{a}$ to the ordinals by $g(\theta) = \sup\{\{g^{i,\zeta}_\beta(\theta) : \beta \in u\} \cup \{g^{i,\zeta}_\alpha(\theta) : \zeta < \varepsilon\})$. This is a subset of θ of cardinality $< |\mathfrak{a}| + \mathrm{cf}(\alpha)$ which is $< \theta = \mathrm{cf}(\theta)$ hence $g \in \Pi\mathfrak{a}$, easily is as required, i.e. satisfies clauses (a) + (b) holds, and the $<_*$-first such g is $g^{i,\zeta}_\alpha$.]

Note that clause (e) of $\circledast_{1,\varepsilon}$ follows.

$(*)_3$ if $\zeta < \varepsilon$ then $\delta_{i,\varepsilon} \le \delta_{i,\zeta}$.

[Why? Otherwise $g^{i,\varepsilon}_{\delta_{i,\zeta}}$ contradict clause (e) of $\circledast_{i,\zeta}$.]

Case 1: We succeed to carry the induction, i.e. choose $\bar{g}^{i,\varepsilon}$ for every $\varepsilon < \omega_1$.

Let $\varepsilon(*) < \omega_1$ be minimal such that $\varepsilon \in [\varepsilon(*), \omega_1) \Rightarrow \delta_{i,\varepsilon} = \delta_{i,\varepsilon(*)}$, exists by $(*)_3$ and let $\delta_* := \delta_{i,\varepsilon(*)}$. For each $\theta \in \mathfrak{a}$ the sequence $\langle g^{i,\varepsilon}_0(\theta) : \varepsilon < \omega_1\rangle$ is non-decreasing hence $C_\theta := \{\varepsilon < \omega_1 : (\forall \zeta)(\varepsilon \le \zeta < \omega_1 \to g^{i,\zeta}_0(\theta) = g^{i,\varepsilon}_0(\theta))$ or $(\forall \varepsilon < \zeta)(g^{i,\zeta}_0(\theta) < g^{i,\varepsilon}_0(\theta)\}$ is a club of ω_1 (and even an end-segment), hence also $C = \cap\{C_\theta : \theta \in \mathfrak{a}\}$ is a club of ω_1.

Let $\mathfrak{c} = \{\theta$: for every $\zeta < \varepsilon$ from $C, g^{i,\zeta}_0(\theta) < g^{i,\varepsilon}_0(\theta)\}$ we get contradiction to the demand on $\mathfrak{b}_{i,\zeta}, \mathfrak{b}_{i,\varepsilon}$.

Case 2: We are stuck in $\varepsilon < \omega_1$.

For $\varepsilon = 0$ there is no problem to define $g^{i,\varepsilon}_\alpha$ by induction on α till we are stuck, say in α, necessarily α is of large enough cofinality, and so $\bar{g}^{i,\varepsilon}$ is well defined. We then prove $\mathfrak{b}_{i,\varepsilon}$ exists again using $<_*$.

For ε limit we can also choose $\bar{g}^\varepsilon$.

For $\varepsilon = \zeta + 1$, if $g^{i,\varepsilon}_0$ as required cannot be chosen, we can prove that $\bar{g}^{i,\varepsilon}, \mathfrak{b}_{i,\varepsilon}$ exists as above. $\square_{1.8}$

{6.2p} *Remark* 1.10. From 1.8 we can deduce bounds on hrtg$({}^Y(\aleph_\delta))$ when $\delta < \aleph_1$ and more like the one on $\aleph_\omega^{\aleph_0}$ (better the bound on pp$(\aleph_\omega)$).

§ 2. No decreasing sequence of subalgebras

We concentrate on weaker axioms. We consider Theorem 1.2 under weaker assumptions than "$[\lambda]^{\aleph_0}$ is well orderable". We are also interested in replacing ω by ∂ in "no decreasing ω-sequence of $c\ell$-closed sets", but the reader may consider $\partial = \aleph_0$ only. Note that for the full version, Ax^4_α, i.e., $[\alpha]^\partial$ is well orderable, case $\partial = \aleph_0$ is implied by the $\partial > \aleph_0$ version and suffices for the results. But for other versions the axioms for different ∂'s seems incomparable.

Note that if we add many Cohens (not well ordering them) then Ax^4_λ fails below even for $\partial = \aleph_0$, whereas the other axioms are not affected. But forcing by $\aleph_1$-complete forcing notions preserve Ax_4.

Hypothesis 2.1. DC_∂ and let $\partial(*) = \partial + \aleph_1$. We fix a regular cardinal ∂ (below we can use $\mathrm{DC}_{\aleph_0}$ + AC_∂ only). {22.0}

Definition 2.2. Below we should, e.g. write $\mathrm{Ax}^{\ell,\partial}$ instead of Ax^ℓ and assume $\alpha > \mu > \kappa \geq \partial$. If $\kappa = \partial$ we may omit it. {22.1}
1) $\mathrm{Ax}^1_{\alpha,\mu,\kappa}$ means that there is a weak closure operation on λ of character (μ, κ), see Definition 0.15, such that there is no $\subseteq$-decreasing ∂-sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ of subsets of α with $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$. We may here and below replace κ by $< \kappa$; similarly for μ; let $< |Y|^+$ means $|Y|$.
2) Let $\mathrm{Ax}^0_{\alpha,\mu,\kappa}$ mean there is $c\ell : [\alpha]^{\leq\kappa} \to [\alpha]^{<\mu}$ of members of $[\alpha]^{\leq\kappa}$ such that $u \cup \{0\} \subseteq c\ell(u)$ and there is no $\subseteq$-decreasing sequence $\langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ of members of $[\alpha]^{\leq\kappa}$ such that $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \not\supseteq \mathscr{U}_\varepsilon$.

Writing Y instead of κ means $c\ell : [\alpha]^{<\mathrm{hrtg}(Y)} \to [\alpha]^\mu$. Let $c\ell_{[\varepsilon]} : \mathscr{P}(\alpha) \to \mathscr{P}(\alpha)$ be $c\ell^1_{\varepsilon,<\mathrm{reg}(\kappa^+)}$ as defined in 0.15(4) recalling $\mathrm{reg}(\gamma) = \mathrm{Min}\{\chi : \chi$ a regular cardinal $\geq \gamma\}$.
3) Ax^2_α means that there is $\mathscr{A} \subseteq [\alpha]^\partial$ which is well orderable and for every $u \in [\alpha]^\partial$ for some $v \in \mathscr{A}, u \cap v$ has power $\geq \partial$.
4) Ax^3_α means that $\mathrm{cf}([\alpha]^{\leq\partial}, \subseteq)$ is below some cardinal, i.e., some cofinal $\mathscr{A} \subseteq [\alpha]^\partial$ (under $\subseteq$) is well orderable.
5) Ax^4_α means that $[\alpha]^{\leq\partial}$ is well orderable.
6) Above omitting α (or writing ∞) means "for every α", omitting μ we mean "$< \mathrm{hrtg}(\mathscr{P}(\partial))$".
7) Lastly, let $\mathrm{Ax}_\ell = \mathrm{Ax}^\ell$ for $\ell = 1, 2, 3$.

So easily (or we have shown in the proof of 1.2):

Claim 2.3. *1)* Ax^4_α *implies* Ax^3_α, Ax^3_α *implies* Ax^2_α, Ax^2_α *implies* Ax^1_α *and* Ax^1_α *implies* Ax^0_α *. Similarly for* $\mathrm{Ax}^\ell_{\alpha,<\mu,\kappa}$. {r.2b}
2) In Definition 2.2(2), the last demand, if $c\ell$ has monotonicity, then only $c\ell \restriction [\alpha]^{\leq\partial}$ is relevant, in fact, an equivalent demand is that if $\langle \beta_\varepsilon : \varepsilon < \partial\rangle \in {}^\partial\alpha$ then for some $\varepsilon, \beta_\varepsilon \in c\ell\{\beta_\zeta : \zeta \in (\varepsilon, \partial)\}$.
3) If $\mathrm{Ax}^0_{\alpha,<\mu_1,<\theta}$ *and* $\theta \leq \mathrm{hrtg}(Y)$ *and* [2] $\mu_2 = \sup\{\mathrm{hrtg}(\mu_1 \times [\beta]^\theta) : \beta < \mathrm{hrtg}(Y)\}$
then $Ax^0_{\alpha,<\mu_2,<\mathrm{hrtg}(Y)}$.

Proof. 1) Easy; least obvious is $\mathrm{Ax}^2_{\alpha,<\mu,\kappa} \Rightarrow \mathrm{Ax}^1_{\alpha,<\mu,\kappa}$ which holds by 1.5.
2) First assume that we have a $\subseteq$-decreasing sequence $\langle u_\varepsilon : \varepsilon < \partial\rangle$ such that $\varepsilon < \partial \Rightarrow c\ell(\mathscr{U}_{\varepsilon+1}) \nsubseteq \mathscr{U}_\varepsilon$. Let $\beta_\varepsilon = \min(\mathscr{U}_\varepsilon \backslash c\ell(\mathscr{U}_{\varepsilon+1}))$ for $\varepsilon < \partial$ so clearly

[2] can do somewhat better

$\bar\beta = \langle \beta_\varepsilon : \varepsilon < \partial\rangle$ exist so by monotonicity $c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1,\gamma\}) \subseteq c\ell(\mathscr{U}_{\varepsilon+1})$ hence $\beta_\varepsilon \notin c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1,\gamma)\}$.

Second, assume that $\bar\beta = \langle \beta_\varepsilon : \varepsilon < \gamma\rangle \in {}^\partial\alpha$ satisfies $\beta_\varepsilon \notin c\ell(\{\beta_\zeta : \zeta \in [\varepsilon+1,\partial)\}$ for $\varepsilon < \partial$. Now letting $\mathscr{U}'_\varepsilon = \{\beta_\zeta : \zeta < \partial$ satisfies $\varepsilon \le \zeta\}$ for $\varepsilon < \partial$ clearly $\langle \mathscr{U}'_\varepsilon : \varepsilon < \partial\rangle$ exists, is $\subseteq$-decreasing and $\varepsilon < \partial \Rightarrow \beta_\varepsilon \notin c\ell(\mathscr{U}'_{\varepsilon+1}) \cap \beta_\varepsilon \in \mathscr{U}'_\varepsilon$. So we have shown the equivalence.

3) Let $c\ell(-)$ witness $\mathrm{Ax}^0_{\alpha,<\mu_1,<\theta}$. We define $c\ell' : [\alpha]^{<\mathrm{hrtg}(Y)} \to [\alpha]^{<\mu_2}$ by $c\ell'(u) = \cup\{c\ell(v) : v \subseteq u$ has cardinality $< \theta\}$. It is enough to note:

$(*)_1$ if $u \in [\alpha]^{<\mathrm{hrtg}(Y)}$ then $c\ell'(u)$ has cardinality $< \mu_2 := \mathrm{hrtg}(\mu_1 \times [\mu_1]^{<\theta})$.

[Why? Let $C_u = \{(v,\varepsilon) : v \subseteq u$ has cardinality $< \theta$ and $\varepsilon <$ otp$(c\ell(v))$ which is $< \mu_1\}$. Clearly $|c\ell'(u)| < \mathrm{hrtg}(C_u)$ and $|C_u| = |\mu_1 \times [\mathrm{otp}(u)]^{<\theta}|$, so $(*)_1$ holds. Note that if $\alpha_* < \mu_1^+$ we can replace the demand $v \in [u]^{<\theta} \Rightarrow |c\ell(v)| < \mu_1$ by $v \in [u]^{<\theta} \Rightarrow$ otp$|c\ell(v)| < \alpha_*$.]

$(*)_2$ If $\langle u_\varepsilon : \varepsilon < \partial\rangle$ is $\subseteq$-decreasing then $u_\varepsilon \subseteq c\ell'(u_{\varepsilon+1})$ for some $\varepsilon < \partial$.

[Why? If not by AC_∂ we can choose a sequence $\langle \beta_\varepsilon : \varepsilon < \partial\rangle$ such that $\varepsilon < \partial \Rightarrow \beta_\varepsilon = \min(u_\varepsilon \backslash c\ell'(u_{\varepsilon+1}))$. Let $u'_\varepsilon = \{\beta_\zeta : \zeta \in [\varepsilon,\partial)\}$, and note that as u_ε is $\subseteq$-decreasing also $c\ell'(u_\varepsilon)$ is $\subseteq$-decreasing. So by the choice of $c\ell(-)$ for some $\varepsilon, \beta_\varepsilon \in c\ell\{\beta_\zeta : \zeta < [\varepsilon,\partial)\}$, but this set is $\subseteq c\ell'(u_{\varepsilon+1})$ by the definition of $c\ell'(-)$, so we are done.] $\square_{2.3}$

{r.B.3} **Claim 2.4.** *Assume $c\ell$ witness $\mathrm{Ax}^0_{\alpha,<\mu,\kappa}$ so $\partial \le \kappa < \mu$ and $c\ell : [\alpha]^{\le\kappa} \to [\lambda]^{<\mu}$ and recall $c\ell^1_{\varepsilon,\le\kappa} : \mathscr{P}(\lambda) \to \mathscr{P}(\lambda)$ is from 2.2(2), 0.15(4).*

1) $c\ell^1_{1,\le\kappa}$ is a weak closure operation, it has character (μ_κ,κ) whenever $\partial \le \kappa \le \lambda$ and $\mu_\kappa = \mathrm{hrtg}(\mu \times \mathscr{P}(\kappa))$, see Definition 0.15.

2) $c\ell^1_{\mathrm{reg}(\kappa^+),\le\kappa}$ is a closure operation, has characteristic ∂ and it has character $(<\mu'_\kappa,\kappa)$ when $\partial \le \kappa \le \lambda$ and $\mu'_\kappa = \mathrm{hrtg}(\mathscr{H}_{<\partial^+}(\mu \times \kappa))$.

Proof. 1) By its definition $c\ell^1_{1,\le\kappa}$ is a weak closure operation.

Assume $u \subseteq \alpha, |u| \le \kappa$; non-empty for simplicity. Clearly $\mu \times [|u|]^{\le\partial}$ has the same power as $\mu \times [u]^{<\partial}$. Define [3] the function G with domain $\mu \times [u]^{<\partial}$ as follows: if $\alpha < \mu$ and $v \in [u]^{\le\partial}$ then $G((\alpha,v))$ is the α-th member of $c\ell(v)$ if $\alpha <$ otp$(c\ell(v))$ and $G((\alpha,v)) =$ min(u) otherwise.

So G is a function from $\mu \times [u]^{\le\partial}$ onto $c\ell^1_{1,\le\kappa}(u)$. This proves that $c\ell^1_{1,\le\kappa}$ has character $(<\mu_\kappa,\kappa)$ as $\mu_\kappa = \mathrm{hrtg}(\mu \times \mathscr{P}(\kappa))$.

2) If $\langle u_\varepsilon : \varepsilon \le$ reg$(\kappa^+)\rangle$ is an increasing continuous sequence of sets <u>then</u> $[u_{\partial^+}]^{\le\partial} = \cup\{[u_\varepsilon]^{\le\partial} : \varepsilon <$ reg$(\kappa^+)\}$ as reg(κ^+) is regular (even of cofinality $> \partial$ suffice) by its definition, note reg$(\partial^+) = \partial^+$ because as AC_∂ holds as DC_∂ holds.

Second, let $u \subseteq \alpha, |u| \le \kappa$ and let $u_\varepsilon = c\ell^1_{\varepsilon,\kappa}(u)$ for $\varepsilon \le \partial^+$; it is enough to show that $|u_{\partial^+}| < \mu'_\kappa$. The proof is similar to earlier one. $\square_{2.4}$

{r.4} **Definition/Claim 2.5.** Let $c\ell$ exemplify $\mathrm{Ax}^0_{\lambda,<\mu,Y}$ and Y be an uncountable set of cardinality $\ge \partial(*)$.

1) Let $\mathscr{F}_\mathfrak{y}, \mathscr{F}_{\mathfrak{y},\alpha}$ be as in the proof of Theorem 1.2 for $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ and ordinal α (they depend on λ and $c\ell$ but note that $c\ell$ determine λ; so if we derive $c\ell$ by

[3] clearly we can replace $< \mu$ by $< \gamma$ for $\gamma \in (\mu,\mu^+)$

Ax^4_λ then they depend indirectly on the well ordering of $[\lambda]^{\partial}$) so we may write $\mathscr{F}_{\mathfrak{y},\alpha} = \mathscr{F}_{\mathfrak{y}}(\alpha, c\ell)$, etc.

That is, fully

$(*)_1$ for $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ and ordinal α let $\mathscr{F}_{\mathfrak{y},\alpha}$ be the set of f such that:
- (a) f is a function from $Z^{\mathfrak{y}}$ to λ
- (b) $\mathrm{rk}_{D[\mathfrak{y}]}(f) = \alpha$ recalling that this means $\mathrm{rk}_{D^{\mathfrak{y}}_1 + Z^{\mathfrak{y}}}(f \cup 0_{Y \backslash Z^{\mathfrak{y}}})$ by Definition 0.7(2)
- (c) $D^{\mathfrak{y}}_2 = D^{\mathfrak{y}}_1 \cup \{Y \backslash A : A \in J[f, D^{\mathfrak{y}}_1]\}$, see Definition 0.10
- (d) $Z^{\mathfrak{y}} \in D^{\mathfrak{y}}_2$
- (e) if $Z \in D^{\mathfrak{y}}_2$ and $Z \subseteq Z^{\mathfrak{y}}$ then $c\ell(\{f(y) : y \in Z\}) \supseteq \{f(y) : y \in Z^{\mathfrak{y}}\}$
- (f) $h^{\mathfrak{y}}$ is a function with domain $Z^{\mathfrak{y}}$ such that $y \in Z^{\mathfrak{d}} \Rightarrow h^{\mathfrak{h}}(y) = \mathrm{otp}(f(y) \cap \{c\ell(\{f(z) : z \in Z^{\mathfrak{y}}\})$

$(*)_2$ $\mathscr{F}_{\mathfrak{y}} = \cup\{\mathscr{F}_{\mathfrak{y},\alpha} : \alpha$ an ordinal$\}$.

2) Let $\Xi_{\mathfrak{y}} = \Xi_{\mathfrak{y}}(c\ell) = \Xi_{\mathfrak{y}}(\lambda, c\ell) = \{\alpha : \mathscr{F}_{\mathfrak{y},\alpha} \neq \emptyset\}$ and $f_{\mathfrak{y},\alpha}$ is the function $f \in \mathscr{F}_{\mathfrak{y},\alpha}$ when $\alpha \in \Xi_{\mathfrak{y}}$; it is well defined.
3) If $D \in \mathrm{Fil}_{\partial(*)}(Y)$, $\mathrm{rk}_D(f) = \alpha$ and $f \in {}^Y\lambda$ <u>then</u> $\alpha \in \Xi_D(\lambda, c\ell)$ and $f \restriction Z^{\mathfrak{y}} = f_{\mathfrak{y},\alpha}$ for some $\mathfrak{y} \in \mathrm{Fil}^4_{\aleph_1}(Y)$ where $\Xi_D(\lambda, c\ell) := \cup\{\Xi_{\mathfrak{y}} : \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y)$ and $D^{\mathfrak{y}}_1 = D + Z$ for some $Z \equiv Y \mod J[f, D]\}$.
4) If $D \in \mathrm{Fil}_{\partial(*)}(Y), f \in {}^Y\lambda, Z \in D^+$ and $\mathrm{rk}_{D+Z}(f) \geq \alpha$ <u>then</u> for some $g \in \prod\limits_{y \in Y}(f(y)+1) \subseteq {}^Y(\lambda+1)$ we have $\mathrm{rk}_D(g) = \alpha$ hence $\alpha \in \Xi_D(\lambda, c\ell)$.
5) So we should write $\mathscr{F}_{\mathfrak{y}}[c\ell], \Xi_{\mathfrak{y}}[\lambda, c\ell], f_{\mathfrak{y},\alpha}[c\ell]$.

Proof. As in the proof of 1.2 recalling "$c\ell$ exemplifies $\mathrm{Ax}^0_{\lambda,<\mu,\mathrm{hrtg}(Y)}$" holds, this replaces the use of F_* there; and see the proof of 2.11 below. $\square_{2.5}$

Claim 2.6. *We have ξ_2 is an ordinal and $\mathrm{Ax}^0_{\xi_2,<\mu_2,Y}$ holds <u>when</u>,(note that μ_2 is not much larger than μ_1):* {y.21}

- (a) *$Ax^0_{\xi_1,<\mu_1,Y}$ so $\partial < \mathrm{hrtg}(Y)$*
- (b) *$c\ell$ witnesses clause (a)*
- (c) *$D \in \mathrm{Fil}_{\partial(*)}(Y)$*
- (d) *$\xi_2 = \{\alpha : f_{\mathfrak{y},\alpha}[c\ell]$ is well defined for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ which satisfies $D^{\mathfrak{y}}_1 = D$ and necessarily $\mathrm{Rang}(f_{\mathfrak{y},\alpha}[c\ell]) \subseteq \xi_1\}$*
- (e) *μ_2 is defined as $\mu_{2,3}$ where*
  - (α) *let $\mu_{2,0} = \mathrm{hrtg}(Y)$*
  - (β) *$\mu_{2,1} = \sup_{\beta<\mu_{2,0}} \mathrm{hrtg}(\beta \times \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1))$*
  - (γ) *$\mu_{2,2} = \sup_{\alpha<\mu_{2,1}} \mathrm{hrtg}(\mu_1 \times [\alpha]^{\leq\partial})$*
  - (δ) *$\mu_{2,3} = \sup\{\mathrm{hrtg}({}^Y\beta \times \mathrm{Fil}_{\gamma(*)}(Y)) : \beta < \mu_{2,\mu}\}$ (this is an overkill).*

Proof.

$\oplus_1$ ξ_2 is an ordinal.

[Why? To prove that ξ_2 is an ordinal we have to assume $\alpha < \beta \in \xi_2$ and prove $\alpha \in \xi_2$. As $\beta \in \xi_2$ clearly $\beta \in \Xi_{\mathfrak{y}}[c\ell]$ for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ for which $D^{\mathfrak{y}}_1 = D$ so there is $f \in {}^Y(\xi_1)$ such that $f \restriction Z^{\mathfrak{y}} \in \mathscr{F}_{\mathfrak{y},\beta}$. So $\mathrm{rk}_{D+Z[\mathfrak{y}]}(f) = \beta$ hence by 0.7 there is $g \in {}^Y\lambda$ such that $g \le f$, i.e., $(\forall y \in Y)(g(y) \le f(y))$ and $\mathrm{rk}_{D+Z[\mathfrak{y}]}(g) = \alpha$. By 2.5(4) there is $\mathfrak{z} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu)$ such that $D^{\mathfrak{z}}_1 = D$ and $g \restriction Z^{\mathfrak{z}} \in \mathscr{F}_{\mathfrak{z},\alpha}$ so we are done proving ξ_2 is an ordinal.]

We define the function $c\ell'$ with domain $[\xi_2]^{<\mathrm{hrtg}(Y)}$ as follows:

$\oplus_2$ $c\ell'(u) = u \cup \{0\} \cup \{\alpha$: there is $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ such that $f_{\mathfrak{y},\alpha}[c\ell]$ is well defined [4] and $\mathrm{Rang}(f_{\mathfrak{y},\alpha}[c\ell]) \subseteq c\ell(\mathbf{v}[u])\}$.

where

$\oplus_3$ $\mathbf{v}[u] := \cup\{c\ell(v) : v \subseteq \xi_1$ is of cardinality $\le \partial$ and is $\subseteq \mathbf{w}(v)\}$.

where

$\oplus_4$ for $v \subseteq \xi_1$ we let $\mathbf{w}(v) = \cup\{\mathrm{Rang}(f_{\mathfrak{z},\beta}[c\ell]) : \mathfrak{z} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ and $\beta \in u$ and $f_{\mathfrak{z},\beta}[c\ell]$ is well defined$\}$.

Note that

$\oplus_5$ $c\ell'(u) = \{\mathrm{rk}_D(f) : D \in \mathrm{Fil}_{\partial(*)}(Y), Z \in D^+$ and $f \in {}^Y\mathbf{v}(u)\}$.

Note:

$\boxtimes_1$ for each $u \subseteq \xi_1$ and $\mathfrak{x} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ the set $\{\alpha < \xi_2 : f_{\mathfrak{x},\alpha}[c\ell]$ is a well defined function into $u\}$ has cardinality $< \mathrm{wlor}(T_{D^{\mathfrak{y}}_2}(u))$, that is, $\langle f_{\mathfrak{x},\alpha}[c\ell] : \alpha \in \Xi_{\mathfrak{x}} \cap \xi_2\rangle$ is a sequence of functions from $Z^{\mathfrak{x}}$ to $u_1 \subseteq \xi_1$, any two are equal only on a set $= \emptyset \mod D^{\mathfrak{x}}_2$ (with choice it is $\le {}^{|Y|}|u|)$), call this bound $\mu'_{|u|}$.

Note

$\boxtimes_2$ if $u_1 \subseteq u_2 \subseteq \xi_2$ <u>then</u>

(α) $\mathbf{w}(u_1) \subseteq \mathbf{w}(u_2)$ and $\mathbf{v}(u_1) \subseteq \mathbf{v}(u_2) \subseteq \xi_1$

(β) $c\ell'(u_1) \subseteq c\ell'(u_2)$ for $\varepsilon \le \partial^+$, see Definition 0.15(4)

(γ) if $c\ell'(u_1) \subset c\ell'(u_2)$ then $c\ell(\mathbf{v}(u_1)) \subset c\ell(\mathbf{v}(u_2))$

$\boxtimes_3$ if $u \subseteq \xi_2, |u| < \mathrm{hrtg}(Y)$ then $\mathbf{w}(u) = \{f_{\mathfrak{y},\alpha}(z) : \alpha \in u, \mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)$ and $z \in Z^{\mathfrak{y}}\}$ is a subset of ξ_1 of cardinality $< \mathrm{hrtg}(|u| \times \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)) \le \sup\{\mathrm{hrtg}(\beta) \times \mathrm{Fil}^4_{\partial(*)}(Y,\mu_1)) : \beta < \mathrm{hrtg}(Y)\}$ which is named $\mu_{2,1}$

$\boxtimes_4$ if $u \subseteq \xi_1$ and $|u| < \mu_{2,1}$ then $\cup\{c\ell(v) : v \in [u]^{\le\partial}\}$ is a subset of μ_1 of cardinality $< \mathrm{hrtg}(\mu_1 \times [u]^{\le\partial}) \le \sup_{\alpha<\mu_{2,1}} \mathrm{hrtg}(\mu_1 \times [\alpha]^{\le\partial})$ which we call $\mu_{2,2}$

$\boxtimes_5$ if $u \subseteq \xi_2$ and $|u| < \mathrm{hrtg}(Y)$ then $\mathbf{v}(u)$ has cardinality $< \mu_{2,2}$.

[Why? By $\oplus_3$ and $\boxtimes_3$ and $\boxtimes_4$.]

$\boxtimes_6$ if $u \subseteq \xi_2$ and $|u| < \mathrm{hrtg}(Y)$ then $c\ell'(u) \subseteq \xi_2$ and has cardinality $< \mu_{2,3}$ which we call μ_2.

[4] we could have used $\{t \in Y : f_{\eta,\alpha}[c\ell](t) \in c\ell(\mathbf{v}(u))\} \ne \emptyset \mod D^{\mathfrak{y}}_2)$

[Why? Without loss of generality $\mathbf{v}(u) \neq \emptyset$. By $\oplus_5$ we have $|c\ell'(u)| < \text{hrtg}({}^Y\mathbf{v}(Y)) \times \text{Fil}_{\partial(*)}(Y))$ and by $\boxplus_5$ the latter is $\leq \sup\{\text{hrtg}({}^Y\beta \times \text{Fil}_{\partial(*)}(Y)) : \beta < \mu_{2,2}\} = \mu_{2,3}$ recalling clause $(e)(\gamma)$ of the claim, so we are done.]

$\boxtimes_7$ $c\ell'$ is a very weak closure operation on λ and has character $(< \mu_2, Y)$.

[Why? In Definition 0.15(1), clause (a) holds by the Definition of $c\ell'$, clause (b) holds by $\boxplus_6$ and as for clause (c), $0 \in c\ell'(u)$ by the definition of $c\ell'$ and $u \subseteq c\ell'(u)$ by the definition of $c\ell'$.]

Now it is enough to prove

$\boxtimes_8$ $c\ell'$ witnessed $\text{Ax}^0_{\xi_2,<\mu_2,Y}$.

Recalling $\boxtimes_7$, toward contradiction assume $\bar{\mathscr{U}} = \langle \mathscr{U}_\varepsilon : \varepsilon < \partial\rangle$ is $\subseteq$-decreasing, $\mathscr{U}_\varepsilon \in [\xi_1]^{<\text{hrtg}(Y)}$ and $\varepsilon < \partial \Rightarrow \mathscr{U}_\varepsilon \nsubseteq c\ell(\mathscr{U}_\varepsilon)$. We define $\bar{\gamma} = \langle \gamma_\varepsilon : \varepsilon < \partial\rangle$ by

$$\gamma_\varepsilon = \text{ Min}(\mathscr{U}_\varepsilon \backslash c\ell(\mathscr{U}_{\varepsilon+1})).$$

As AC_∂ follows from DC_∂, we can choose $\langle \mathfrak{y}_\varepsilon : \varepsilon < \partial\rangle$ such that $f_{\mathfrak{y}_\varepsilon,\gamma_\varepsilon}[c\ell]$ is well defined for $\varepsilon < \partial$.

We can choose $\langle z_\varepsilon : \varepsilon < \partial\rangle$ such that $z_\varepsilon \in Z^{\mathfrak{y}_\varepsilon}$ and $\beta_\varepsilon := f_{\mathfrak{y}_\varepsilon,\gamma_\varepsilon}(z_\varepsilon) \notin \mathbf{v}(\mathscr{U}_{\varepsilon+1})$. [Why? By AC_∂ it is enough to show that for each $\varepsilon < \partial$ there is such β_ε. If there is no such β_ε then $\text{Rang}(f_{\mathfrak{y}_\varepsilon,\gamma_\varepsilon}) \subseteq \mathbf{v}(\mathscr{U}_{\varepsilon+1})$ which by the definition of $c\ell'(\mathscr{U}_{\varepsilon+1})$ means that $\gamma_\varepsilon \in c\ell'(\mathscr{U}_{\varepsilon+1})$.]

Let for $\varepsilon < \partial$

$$u_\varepsilon = \{\beta_\zeta : \zeta \in [\varepsilon, \partial)\}.$$

So

$(*)_1$ $u_\varepsilon \in [\xi_1]^{\leq\partial} \subseteq [\xi_1]^{<\text{hrtg}(Y)}$.

[Why? By clause (a) of the assumption of 2.6.]

$(*)_2$ u_ε is $\subseteq$-decreasing with ε.

[Why? By the definition.]

$(*)_3$ $\beta_\varepsilon \in u_\varepsilon \backslash c\ell(u_{\varepsilon+1})$ for $\varepsilon < \partial$.

[Why? $\beta_\varepsilon \in u_\varepsilon$ by the definition of u_ε.]

Also $u_\varepsilon \subseteq \mathbf{v}(\mathscr{U}_\varepsilon)$ by the definition of $\mathbf{v}(\mathscr{U}_\varepsilon)$.

Lastly, $\beta_\varepsilon \notin \mathbf{v}(\mathscr{U}_{\varepsilon+1})$ by the choice of β_ε. So $\langle u_\varepsilon : \varepsilon < \partial\rangle$ contradict the assumption on $(\xi_1, c\ell)$. From the above the conclusion should be clear. $\square_{2.6}$

{r.4A}

Claim 2.7. *Assume $\aleph_0 < \kappa = \text{cf}(\lambda) < \lambda$ hence κ is regular $\geq \partial$ of course, and D is the club filter on κ and $\bar{\lambda} = \langle \lambda_i : i < \kappa\rangle$ is increasing continuous with limit λ.*
1) $\lambda^+ \leq \{\text{rk}_{D_\kappa}(f) : f \in \prod_{i<\kappa^+} \lambda_i^+\}$.
2) Assume further $c\ell$ is a weak closure operation on λ of character (μ, κ) <u>then</u> in part (1) we can restrict ourselves to a function from $\prod_{i\in Z} \lambda_i^+, Z \subseteq \kappa$.

Proof. For each $\alpha < \lambda^+$ there is a one to one [5] function g from α into $|\alpha| \le \lambda$ and we let $f : \kappa \to \prod\limits_{i<\kappa} \lambda_i$ be

$$f(i) = \text{ otp}(\{\beta < \alpha : f(\beta) < \lambda_i\}.$$

Let

$$\begin{aligned}\mathscr{F}_\alpha = \{f : \quad & f \text{ is a function with domain } \kappa \text{ satisfying } i < \kappa \Rightarrow f(i) < \lambda_i^+ \\ & \text{such that for some one to one function } g \text{ from } \alpha \text{ into } \lambda \\ & \text{for each } i < \kappa \text{ we have } f(i) = \text{ otp}(\{\beta < \alpha : f(\beta) < \lambda_i\})\}.\end{aligned}$$

Now

$(*)_1$ (α) $\mathscr{F}_\alpha \ne \emptyset$ for $\alpha < \lambda^+$

(β) $\langle \mathscr{F}_\alpha : \alpha < \lambda^+ \rangle$ exists as it is well defined

(γ) if $f \in \mathscr{F}_\alpha$ then, for part (2), for some $\mathfrak{y} \in \text{ Fil}^4_{\partial(*)}(\lambda, c\ell)$ we have $f{\restriction}Z[\mathfrak{y}] \in \mathscr{F}_{\mathfrak{y},\alpha}$.

[Why? For clause (α) let $g : \alpha \to \lambda$ be one to one and so the f defined above belongs to $\mathscr{F}_\alpha$. For clause (β) see the definition of $\mathscr{F}_\alpha$ (for $\alpha < \lambda^+$). Lastly, clause (γ) use 2.5(3).]

$(*)_2$ (α) if $f \in \mathscr{F}_\beta, \alpha < \beta < \lambda^+$ <u>then</u> for some $f' \in \mathscr{F}_\alpha$ we have $f' <_{J^{\rm bd}_\kappa} f$

(β) $\langle \min\{\text{rk}_D(f) : f \in \mathscr{F}_\alpha\} : \alpha < \lambda^+\rangle$ is strictly increasing hence $\min\{\text{rk}_D(f) : f \in \mathscr{F}_\alpha\} \ge \alpha$.

[Why? For clause (α), let g witness "$f \in \mathscr{F}_\beta$" and define the function f' by: $f' \in \prod\limits_{i<\kappa} \lambda_i^+$ by $f'(i) = \text{ otp}\{\gamma < \alpha : g(\gamma) < \lambda_i\}$. So $g{\restriction}\alpha$ witness $f' \in \mathscr{F}_\alpha$, and letting $i(*) = \min\{i : g(\alpha) < \lambda_i\}$ we have $i \in [i(*), \kappa) \Rightarrow f'(i) < f(i)$ hence $f' <_{J^{\rm bd}_\kappa} f$ as promised. For clause (β) it follows.]

So we have proved 2.7. $\square_{2.7}$

{r.5} **Conclusion 2.8.** *1) Assume*

(a) $\text{Ax}^0_{\lambda,<\mu,\kappa}$

(b) $\lambda > \text{cf}(\lambda) = \kappa$ *(not really needed in (1)).*

<u>*Then*</u> *for some* $\mathscr{F}_* \subseteq {}^\kappa\lambda =: \{f : f$ *a partial function from* κ *to* $\lambda\}$ *we have*

(α) *every* $f \in {}^\kappa\lambda$ *is a countable union of members of* $\mathscr{F}_*$

(β) $\mathscr{F}_*$ *is the union of* $|\text{Fil}^4_{\partial(*)}(\kappa, <\mu)|$ *well ordered sets:* $\{\mathscr{F}^*_{\mathfrak{y}} : \mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa, \mu)\}$

(γ) *moreover there is a function giving for each* $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa)$ *a well ordering of* $\mathscr{F}^*_{\mathfrak{y}}$.

2) Assume in addition that $\text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa, <\mu)) < \lambda$ *and* $\text{hrtg}({}^\kappa\mu) < \lambda$ <u>*then*</u> *for some* $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa)$ *we have* $|\mathscr{F}_{\mathfrak{y}}| > \lambda$.

[5]but, of course, possibly there is no such sequence $\langle f_\alpha : \alpha < \lambda^+\rangle$

Proof. 1) By the proof of 1.2.
2) Assume that this fails, let $\langle \mu_i : i < \kappa \rangle$ be increasing continuous with limit λ such that $\mu_0 > \text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa, < \mu))$.

We choose $f \in {}^\kappa\lambda$ by

$$f(i) = \text{Min}(\lambda \backslash \{f_{\mathfrak{y},\alpha}(i) : \begin{array}{l} \mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa, \mu) \\ f_{\mathfrak{y},\alpha}(i) \text{ is well defined, i.e.} \\ i \in Z[\mathfrak{y}] \text{ and } \alpha \in \Xi_{\mathfrak{y}} \text{ and} \\ \text{otp}(\alpha \cap \Xi_{\mathfrak{y}}) < \mu_i\}). \end{array}$$

Now $f(i)$ is well defined as the minimum is taken over a non-empty set, this holds as we substruct from λ a set which as cardinality $\leq \text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa, \mu)) + \text{hrtg}({}^\kappa\mu) + \mu_i$ which is $< \lambda$. But f contradicts part (1). $\square_{2.8}$

{r.6}

Conclusion 2.9. *Assume $Ax^0_{\lambda,<\mu,\kappa}$.*
λ^+ is not measurable (which implies regular [6]*) when*

$\boxtimes$ (a) $\lambda > \text{cf}(\lambda) = \kappa > \aleph_0$
(b) $\lambda > \text{hrtg}((\text{Fil}^4_{\partial(*)}(\kappa, \mu))$.

Proof. Naturally we fix a witness $c\ell$ for $\text{Ax}^0_{\lambda,<\mu,\kappa}$. Let $\mathscr{F}_{\mathfrak{y}}, \Xi_{\mathfrak{y}}, f_{\mathfrak{y},\alpha}, \mathscr{F}^\lambda_{\mathfrak{y},\alpha}$ be defined as in 2.5 so by claim 2.5, 2.7 we have $\cup\{\Xi_{\alpha,\mathfrak{y}} : \mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa)\} \supseteq \lambda^+$.

Let $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa, \mu)$ be such that $|\mathscr{F}_{\mathfrak{y}}| > \lambda$, we can find such $\mathfrak{y}$ by 2.8, even easier as without loss of generality we can assume λ^+ is regular (or even measurable, toward contradiction). Let $Z = Z[\mathfrak{y}]$. So $\Xi_{\mathfrak{y}}$ is a set of ordinals of cardinality $> \lambda$. For $\zeta < \text{otp}(\Xi_{\mathfrak{y}})$ let α_ζ be the ζ-th member of $\Xi_{\mathfrak{y}}$, so $f_{\mathfrak{y},\alpha_\zeta}$ is well defined. Toward contradiction let D be a (non-principal) ultrafilter on λ^+ which is λ^+-complete. For $i \in Z$ let $\gamma_i < \lambda$ be the unique ordinal γ such that $\{\zeta < \lambda^+ : f_{\mathfrak{y},\alpha_\zeta}(i) = \gamma\} \in D$. As $|Z| \leq \kappa < \lambda^+$ and D is κ^+-complete clearly $\{\zeta : \bigwedge_{i \in Z} f_{\mathfrak{y},\alpha_\zeta}(i) = \gamma_i\} \in D$, so as D is a non-principal ultrafilter, for some $\zeta_1 < \zeta_2, f_{\mathfrak{y},\alpha_{\zeta_1}} = f_{\mathfrak{y},\alpha_{\zeta_2}}$, contradiction. So there is no such D. $\square_{2.9}$

{r.7n}

Remark 2.10. Similarly if D is κ^+-complete and weakly λ^+-saturated and $\text{Ax}^0_{\lambda^+,<\mu}$, see [Sh:F1078].

{r.9}

Claim 2.11. *If $\text{Ax}^0_{\lambda,<\mu,\kappa}$, then we can find $\bar{C}$ such that:*

(a) $\bar{C} = \langle C_\delta : \delta \in S \rangle$
(b) $S = \{\delta < \lambda : \delta$ *is a limit ordinal of cofinality* $\geq \partial(*)\}$
(c) C_δ *is an unbounded subset of* δ*, even a club*
(d) *if* $\delta \in S$, $\text{cf}(\delta) \leq \kappa$ *then* $|C_\delta| < \mu$
(e) *if* $\delta \in S$, $\text{cf}(\delta) > \kappa$ *then* $|C_\delta| < \text{hrtg}(\mu \times [\text{cf}(\delta)]^\kappa)$.

Remark 2.12. 1) Recall that if we have Ax^4_λ then trivially there is $\langle C_\delta : \delta < \lambda, \text{cf}(\delta) \leq \partial \rangle, C_\delta$ a club of δ of order type $\text{cf}(\delta)$ as if $<_*$ well order $[\lambda]^{\leq \partial}$ we let $C_\delta :=$ be the $<_*$-minimal C which is a closed unbounded subset of δ of order type $\text{cf}(\delta)$.
2) $\text{Ax}^0_{\lambda,<\xi,\kappa}$ suffices if $\kappa < \xi < \lambda$.

[6]the regular holds many times by 2.13

Proof. The "even a club" is not serious as we can replace C_δ by its closure in δ.

Let $c\ell$ witness $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$. For each $\delta \in S$ with $\mathrm{cf}(\delta) \in [\partial(*), \kappa]$ we let

$$C_\delta = \cap\{\delta \cap c\ell(C) : C \text{ a club of } \delta \text{ of order type } \mathrm{cf}(\delta)\}.$$

Now $\bar{C}' = \langle C_\delta : \delta \in S \text{ and } \mathrm{cf}(\delta) \in [\partial(*), \kappa]\rangle$ is well defined and exist. Clearly C_δ is a subset of δ.

For any club C of δ of order type $\mathrm{cf}(\delta) \in [\partial(*), \kappa]$ clearly $\delta \cap c\ell(C) \subseteq c\ell(C)$ which has cardinality $< \mu$.

The main point is to show that C_δ is unbounded in δ, otherwise we can choose by induction on $\varepsilon < \partial$, a club $C_{\delta,\varepsilon}$ of δ of order type $\mathrm{cf}(\delta)$, decreasing with ε such that $C_{\delta,\varepsilon} \nsubseteq c\ell(C_{\delta,\varepsilon+1})$, we use DC_∂. But this contradicts the choice of $c\ell$ recalling Definition 2.2(1).

If $\delta < \lambda$ and $\mathrm{cf}(\delta) > \kappa$ we let

$$C^*_\delta = \cap\{\cup\{\delta \cap c\ell(u) : u \subseteq C \text{ has cardinality } \le \partial\} : C \text{ is a club of } \delta \text{ of order type } \mathrm{cf}(\delta)\}.$$

A problem is a bound of $|C^*_\delta|$. Clearly for C a club of δ of order type $\mathrm{cf}(\delta)$ the order-type of the set $\cup\{\delta \cap c\ell(v) : v \subseteq C$ has cardinality $\le \partial\}$ is $< \mathrm{hrtg}(\mu \times [\mathrm{cf}(\delta)]^\kappa)$. As for "$C^*_\delta$ is a club" it is proved as above. $\square_{2.11}$

{s.2} The following lemma gives the existence of a class of regular successor cardinals.

Lemma 2.13. *1) Assume*

(a) *δ is a limit ordinal $< \lambda_*$ with $\mathrm{cf}(\delta) \ge \partial$*
(b) *λ^*_i is a cardinal for $i < \delta$ increasing with i*
(c) $\lambda_* = \Sigma\{\lambda^*_i : i < \delta\}$
(d) *$\lambda^*_{i+1} \ge \mathrm{hrtg}(\mu \times {}^\kappa(\lambda^*_i))$ for $i < \delta$ and $(\alpha) \vee (\beta)$ where*
  (α) *Ax^4_λ <u>or</u>*
  (β) *$\lambda^*_{i+1} \ge \mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\lambda^*_i, \mu))$ and $\mathrm{hrtg}([\lambda^*_i]^{\le\kappa} \le \lambda^*_{i+1}$*
(e) *$\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$ and $\mu < \lambda^*_0$*
(f) $\lambda = \lambda^+_*$

<u>Then</u> λ is a regular cardinal.
2) Assume $\mathrm{Ax}^4_\lambda, \lambda = \lambda^+_*, \lambda_*$ singular and $\chi < \lambda_* \Rightarrow \mathrm{hrtg}({}^\partial\chi) \le \lambda_*$ <u>then</u> λ is regular.

Remark 2.14. 1) This says the successor of a strong limit singular is regular.
2) We can separate the proof of $\boxtimes_1$ below to a claim.

Question 2.15. 1) Is $\mathrm{hrtg}(\mathscr{P}(\mathscr{P}(\lambda^*_i))) \ge \mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\lambda^*_i))$?
2) Is $|c\ell(f \restriction B)| \le \mathrm{hrtg}([B]^{<\aleph_0})$ for the natural $c\ell$?

Proof. 1) We can replace δ by $\mathrm{cf}(\delta)$ so without loss of generality δ is a regular cardinal.

So fix $c\ell : [\lambda]^{\le\kappa} \to \mathscr{P}(\lambda)$ a witness to $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$ and let $\langle C_\xi[c\ell] : \xi < \lambda, \mathrm{cf}(\xi) \ge \partial\rangle$ be as in the proof of 2.11, so $\xi < \lambda \wedge \partial \le \mathrm{cf}(\xi) < \lambda \Rightarrow |C_\xi[c\ell]| < \lambda$.

First, we shall use just $\lambda > \lambda_* \wedge (\forall\delta < \lambda)(\mathrm{cf}(\delta) < \lambda_*)$, a weakening of assumption (f).

Now

$\boxtimes_1$ for every $i < \delta$ and $A \subseteq \lambda$ of cardinality $\leq \lambda^*_i$, we can find $B \subseteq \lambda$ of cardinality $\leq \lambda_*$ satisfying $(\forall \alpha \in A)[\alpha$ is limit $\wedge\, \mathrm{cf}(\alpha) \leq \lambda^*_i \Rightarrow \alpha = \sup(\alpha \cap B)]$.

Why? By 2.11 the only problem is for $Y := \{\alpha : \alpha \in A, \alpha > \sup(A \cap \alpha), \alpha$ a limit ordinal of cofinality $< \partial + \aleph_1\}$; so $|Y| \leq \lambda^*_i$. Note: if we assume Ax^4_λ this would be immediate.

We define D as the family of $A \subseteq Y$ such that

$\circledast_1$ for some set $C \subseteq \lambda$ of $\leq \partial$ ordinals, the set $B_C =: \cup\{\mathrm{Rang}(f_{\mathfrak{x},\zeta}) : \mathfrak{x} \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i)$ and $\zeta \in C$ <u>or</u> for some $\xi \in C$, we have $\lambda^*_i \geq \mathrm{cf}(\xi) > \partial$ and $\zeta \in C_\xi[c\ell]\}$ satisfies $\alpha \in Y \backslash A \Rightarrow \alpha = \sup(\alpha \cap B_C)$.

Clearly

$\circledast_2$ (a) $Y \in D$
(b) D is upward closed
(c) D is closed under intersection of $\leq \partial$ hence of $< \partial(*)$ sets.

[Why? For clause (a) use $C = \emptyset$, for clause (b), note that if C witness a set $A \subseteq Y$ belongs to D then it is a witness for any $A' \subseteq A$. Lastly, for clause (c) if $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial^+$, as we have AC_∂, there is a sequence $\langle C_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ such that C_ε witnesses $A_\varepsilon \in D$ for $\varepsilon < \varepsilon(*) < \partial^+$, then $C := \cup\{C_\varepsilon : \varepsilon < \varepsilon(*)\}$ witnesses $A := \cap\{A_\varepsilon : \varepsilon < \varepsilon(*)\} \in D$ and, again by AC_∂, we have $|C| \leq \partial$.]

$\circledast_3$ if $\emptyset \in D$ then we are done.

[Why? As B_C is as required; its cardinality $\leq \lambda^*_{i+1}$ by 2.11.]

So assume $\emptyset \notin D$, so D is an ∂^+-complete filter on Y. As $1 \leq |Y| \leq \lambda^*_i$, let g be a one to one function from $|Y| \leq \lambda^*_i$ onto Y and let

$\circledast_3$ (a) $D_1 := \{B \subseteq \lambda^*_i : \{g(\alpha) : \alpha \in B \cap |Y|\} \in D\}$
(b) $\zeta := \mathrm{rk}_{D_1}(g)$
(c) $D_2 := \{B \subseteq \lambda^*_i : \mathrm{rk}_{D_1+(\lambda^*_i \backslash B)}(g) > \zeta$ and $B \notin D_1$ of course$\}$.

So D_2 is an ∂^+-complete filter on λ^*_i extending D_1.

Let $B_* \in D_2$ be such that $(\forall B')[B' \in D_2 \wedge B' \subseteq B \Rightarrow c\ell(\mathrm{Rang}(g \restriction B')) \supseteq (\mathrm{Rang}(g \restriction B)]$. Let $\mathscr{U} = \cap\{c\ell(\mathrm{Range}(g \restriction B') : B' \in D_2\}$, so $\mathrm{Rang}(g \restriction B^*) \subseteq \mathscr{U}$.

Let h be the function with domain B_* defined by $\alpha \in B_* \Rightarrow h(\alpha) = \mathrm{otp}(g(\alpha) \cap \mathscr{U})$.

So $\mathfrak{x} := (D_1, D_2, B_*, h) \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i)$ and for some ζ we have $g \restriction B_* = f_{\mathfrak{x},\zeta}[c\ell]$.

Now ζ is a limit ordinal (as each $g(\alpha), \alpha < |Y|$, is) so it suffices to consider the following two subcases.

<u>Subcase 1a</u>: $\mathrm{cf}(\zeta) > \partial$.

So $C_\zeta[c\ell]$ is well defined and let $C := \{\zeta\}$ hence $B_C = \cup\{\mathrm{Rang}(f_{\mathfrak{x},\varepsilon}[c\ell] : \varepsilon \in C_\zeta[c\ell]\}$ so C exemplifies that the set $A := \{\alpha \in Y : \alpha > \sup(\alpha \cap B_C)\}$ belongs to D hence $A_* = \{\alpha < |Y| : g(\alpha) \in A\}$ belongs to D_1.

Now define g', a function from λ^*_i to Ord by $g'(\alpha) = \sup(g(\alpha) \cap B_C) + 1$ if $\alpha \in A_*$ and $g'(\alpha) = 0$ otherwise. Clearly $g' < g \bmod D_1$ hence $\mathrm{rk}_{D_1}(g') < \zeta$, hence there is $g'', g' <_{D_1} g'' <_{D_1} g$ such that $\xi := \mathrm{rk}_{D_1}(g'') \in C_\zeta[c\ell]$.

Now for some $\mathfrak{y} \in \mathrm{Fil}^4_{\partial(*)}(\lambda^*_i)$ we have $D^{\mathfrak{y}} = D_2$ and $g'' = f_{\mathfrak{y},\xi} \bmod D^{\mathfrak{y}}_2$.

So $B =: \{\varepsilon < |Y| : g''(\varepsilon) = f_{\mathfrak{y},\xi}\} \in D_2^{\mathfrak{y}}$ hence $B \in D_2^+$. So $B \cap B_* \cap A_* \in D_2^+$ but if $\varepsilon \in B \cap B_* \cap A_*$ then $f_{\mathfrak{y},\xi}(\varepsilon) \in B_C$ and $f_{\mathfrak{y},\xi}(\varepsilon) \in (\sup(B_C \cap g(\varepsilon)), g(\varepsilon))$.

This gives contradiction.

Subcase 1b: $\mathrm{cf}(\zeta) \leq \partial$.

We choose a $C \subseteq \zeta$ of order type $\leq \partial$ unbounded in ζ and proceed as in subcase 1a.

As we have covered both subcases, we have proved $\boxtimes_1$.

Recall we are assuming $\delta \leq \partial$ so

- $\boxtimes_2$ for every $A \subseteq \lambda$ of cardinality $\leq \lambda_*$ there is $B \subseteq \lambda$ of cardinality $\leq \lambda_*$ such that:
  - (a) $A \subseteq B, [\alpha + 1 \in A \Rightarrow \alpha \in B]$ and $[\alpha \in A \wedge \aleph_0 \leq \mathrm{cf}(\alpha) < \lambda_* \Rightarrow \alpha = \mathrm{supp}(B \cap \alpha)]$.

[Why? Choose $\langle A_j :< \delta\rangle$ such that $A = \cup\{A_i : i < \delta\}$ and $j < \delta \Rightarrow |A_j| \leq \lambda_j^*$, possible as $|A| \leq \lambda_*$. For each j there $\bar{B}_j = \langle B_{j,i} : i < \delta\rangle$ as in $\boxplus_1$ so as AC_δ holds (as $\delta \leq \partial$) there is a sequence $\langle \bar{B}_j : j < \delta\rangle$, each $\bar{B}_j$ as above.

Lastly, let $B = \cup\{B_{j,i} : j, i < \delta\}$, it is as required.]

- $\boxtimes_3$ for every $A \subseteq \lambda$ of cardinality $\leq \lambda_*$ we can find $B \subseteq \lambda$ of cardinality $\leq \lambda_*$ such that $A \subseteq B, [\alpha + 1 \in B \Rightarrow \alpha \in B]$ and $[\alpha \in B$ is a limit ordinal $\wedge\ \mathrm{cf}(\alpha) < \lambda_* \Rightarrow \alpha = \sup(B \cap \alpha)]$.

[Why? We choose B_i by induction on $i < \omega \leq \partial$ such that $|B_i| \leq \lambda_*$ by $B_0 = B, B_{2i+1} = \{\alpha : \alpha \in B_{2i}$ or $\alpha + 1 \in B_{2i+1}\}$ and B_{2i+2} is chosen as B was chosen in $\boxtimes_2$ for i with B_{2i+1} here in the role of A there. There is such $\langle B_i : i < \omega\rangle$ as DC holds. So easily $B = \cup\{B_i : i < \omega\}$ is as required.]

Now return to our main case $\lambda = \lambda_*^+$

- $\boxtimes_4$ λ_*^+ is regular.

[Why? Otherwise $\mathrm{cf}(\lambda_*^+) < \lambda_*^+$ hence $\mathrm{cf}(\lambda_*^+) \leq \lambda_*$, but λ_* is singular so $\mathrm{cf}(\lambda_*^+) < \lambda_*$ hence there is a set A of cardinality $\mathrm{cf}(\lambda_*^+) < \lambda_*$ such that $A \subseteq \lambda_*^+ = \sup(A)$. Now choose B as in $\boxtimes_3$. So $|B| \leq \lambda_*, B$ is an unbounded subset of $\lambda_*^+, \alpha+1 \in B \Rightarrow \alpha \in B$ and if $\delta \in B$ is a limit ordinal then $\mathrm{cf}(\delta) \leq |\delta| \leq \lambda_*$, but $\mathrm{cf}(\delta)$ is regular so $\mathrm{cf}(\delta) < \lambda_*$ hence $\delta = \sup(B \cap \delta)$. But this trivially implies that $B = \lambda_*^+$, but $|B| \leq \lambda_*$, contradiction.]

2) Similar, just easier. $\square_{2.13}$

{sp.3}

Remark 2.16. Of course, if we assume Ax_λ^4 then the proof of 2.13 is much simpler: if $<_*$ is a well ordering of $[\lambda]^{\leq \partial}$ for $\delta < \lambda$ of cofinality $\leq \partial$ let $C_\delta =$ the $<_*$-first closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$, see 3.3.

{sp.4}

Claim 2.17. *Assume*

- (a) $\langle \lambda_i : i < \kappa\rangle$ *is an increasing continuous sequence of cardinals* $> \kappa$
- (b) $\lambda = \lambda_\kappa = \Sigma\{\lambda_i : i < \kappa\}$
- (c) $\kappa = \mathrm{cf}(\kappa) > \partial$
- (d) $\mathrm{Ax}^0_{\lambda,<\mu,\kappa}$
- (e) $\mathrm{hrtg}(\mathrm{Fil}^4_{\partial(*)}(\kappa,\mu)) < \lambda$ *and* $\kappa, \mu < \lambda_0$

(f) $S := \{i < \kappa : \lambda_i^+$ *is a regular cardinal*$\}$ *is a stationary subset of* κ

(g) *let* $D := D_\kappa + S$ *where* D_κ *is the club filter on* κ

(h) $\gamma(*) = \text{rk}_D(\langle \lambda_i^+ : i < \kappa \rangle)$.

Then $\gamma(*)$ *has cofinality* $> \lambda$, *so* $(\lambda, \gamma(*)] \cap \text{Reg} \neq \emptyset$.

Proof. Recall 2.7 which we shall use. Toward contradiction assume that $\text{cf}(\gamma(*)) \leq \lambda_\kappa$, but λ_κ is singular hence for some $i(*) < \kappa$, $\text{cf}(\gamma(*)) \leq \lambda_{i(*)}$. Let $c\ell$ witness $\text{Ax}^0_{\lambda,<\mu,\kappa}$.

Let B be an unbounded subset of $\gamma(*)$ of order type $\text{cf}(\gamma(*)) < \lambda_{i(*)}$. By renaming without loss of generality $i(*) = 0$.

For $\alpha < \gamma(*)$ let

$$\mathscr{U}_\alpha = \cup\{\text{Rang}(f_{\mathfrak{y},\alpha}) : \quad f_{\mathfrak{y},\alpha}[c\ell] \text{ is well defined } \in \Pi\{\lambda_i : i \in Z^{\mathfrak{y}}\} \text{ and } \mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa) \text{ and } D_1^{\mathfrak{y}} = D\}.$$

Clearly $\mathscr{U}_\alpha$ is well defined by 2.7 and $\langle \mathscr{U}_\alpha : \alpha < \gamma(*)\rangle$ exists and $|\mathscr{U}_\alpha| \leq \text{hrtg}(\kappa \times \text{Fil}^4_{\partial(*)}(\kappa,\mu)) = \text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa,\mu))$, even $<$. Let $\mathscr{U} = \cup\{\mathscr{U}_\alpha : \alpha \in B\}$ so $|\mathscr{U}| \leq \text{hrtg}(\text{Fil}^4_{\partial(*)}(\kappa,\mu)) + |B|$.

We define $f \in \prod\limits_{i<\kappa} \lambda_i^+$ by

(α) $f(i)$ is: $\sup(\mathscr{U} \cap \lambda_i^+) + 1$ if $\text{cf}(\lambda_i^+) > |\mathscr{U}|$
zero otherwise.

So

(β) $f \in \prod\limits_{i<\kappa} \lambda_i^+$.

Clearly

(γ) $\{i < \kappa : f(i) = 0\} = \emptyset \mod D$ moreover $\{i < \kappa : f(i) \text{ is a limit ordinal}\} \in D$.

Let $\alpha(*) = \text{rk}_D(f)$, it is $< \text{rk}_D(\langle \lambda_i^+ : i < \kappa\rangle) = \gamma(*)$, so by clause (γ) there is $\beta(*) \in B$ such that $\alpha(*) < \beta(*) < \gamma(*)$ hence for some $g \in \prod\limits_{i<\kappa} \lambda_i^+$ we have $\text{rk}_D(g) = \beta(*)$ and $f < g \mod D$, so for some $\mathfrak{y} \in \text{Fil}^4_{\partial(*)}(\kappa), D_1^{\mathfrak{y}} = D_\kappa + S$ and $g \in \mathscr{F}_{\mathfrak{y},\beta(*)}$, hence $f(i) < g(i) < f_{\mathfrak{y},\alpha(*)}(i) \in \mathscr{U} \cap \lambda_i^+$ for every $i \in Z^{\mathfrak{y}} \cap S$.

So we get easy contradiction to the choice of g. $\square_{2.17}$

Claim 2.18. *Assume* $c\ell$ *witness* $\text{Ax}^0_{\alpha,<\mu}$. *The ordinals* $\gamma_\ell, \ell = 0,1,2$ *are nearly equal see, i.e.* ⊛ *below holds where:* {sp.6}

⊠ (a) $\gamma_0 = \text{hrtg}({}^Y\alpha)$, *a cardinal*

(b) $\gamma_1 = \cup\{\|\alpha\|_D : D \in \text{Fil}_{\partial(*)}(Y)\}$

(c) $\gamma_2 = \sup\{\text{otp}(\Xi_{\mathfrak{y}}[c\ell] + 1 : \mathfrak{y} \in \text{Fil}^4_{\partial(*)}(Y)\}$

⊛ (α) $\gamma_2 \leq \gamma_1 \leq \gamma_0$

(β) γ_0 *is the union of* $\text{Fil}^4_{\partial(*)}(Y)$ *sets each of order type* $< \gamma_2$

(γ) γ_0 *is the disjoint union of* $< \text{hrtg}(\mathscr{P}(\text{Fil}^4_{\partial(*)}(Y)))$ *sets each of order type* $< \gamma_2$

(δ) *if* $\gamma_0 > \mathrm{hrtg}(\mathscr{P}(\mathrm{Fil}^4_{\partial(*)}(Y)))$ *and* $\gamma_0 \geq |\gamma_0|^+$ *then* $|\gamma_0| \leq |\gamma_2|^{++}$ *and* $\mathrm{cf}(|\gamma_2|^+) < \mathrm{hrtg}(\mathscr{P}(\mathrm{Fil}^4_{\partial(*)}(Y)))$.

Proof. Straight. $\square_{2.18}$

§ 3. Concluding Remarks

In May 2010, David Aspero asked where it is true that I have results along the following lines (or that it follows from such a result):

If GCH holds and λ is a singular cardinal of uncountable cofinality, then there is a well-order of $\mathscr{H}(\lambda^+)$ definable in $(\mathscr{H}(\lambda^+), \in)$ using a parameter.

The answer is yes by [Sh:497, 4.6,pg.117] but we elaborate this below somewhat more generally.

Also though our original motivation was to look at consequences of Ax_4, this was shadowed by the try to use weaker relatives; see more in [Sh:F1078].

{c2}

Claim 3.1. [DC] *Assume that μ is a limit singular of cofinality $\kappa > \aleph_0$ (no* GCH *needed), the parameter $X \subseteq \mu$ codes in particular the tree $\mathscr{T} = {}^{\kappa>}\lambda$ and the set $\mathscr{P}(\mathscr{P}(\kappa))$, in particular from it well orderings of those sets are definable.* <u>*Then*</u> *(with this paramter) we can define a well ordering of the set of κ-branches of this tree.*

{c4}

Conclusion 3.2. *[DC] Assume μ is a singular cardinal of uncountable cofinality and $\mathscr{H}(\mu)$ is well orderable of cardinality μ and $X \subseteq \mu$ code $\mathscr{H}(\mu)$ and a well ordering of $\mathscr{H}(\mu)$. Then we can (with this X as parameter) define a well ordering of $\mathscr{P}(\mu)$.*

Proof. <u>Proof of 3.1</u>:

Let $\langle \text{ cd}_i : i < \kappa \rangle$ satisfies

- $\boxplus_1$ cd_i is a one-to-one from ${}^i\mu$ into μ, (definable from X uniformly (in i))
- $\boxplus_2$ let $<_\kappa$ be a well ordering of $\text{Fil}^4_\kappa(\kappa)$ definable from X.

For $\eta \in {}^\kappa\mu$ let $f_\eta : \kappa \to \mu$ be defined by $f_\eta(i) = \text{ cd}_i(\eta{\restriction}i)$, so $\bar{f} = \langle f_\eta : \eta \in {}^\kappa\mu \rangle$ is well defined.

Let $\bar{\mathscr{F}} = \langle \mathscr{F}_{\mathfrak{y}} : \mathfrak{y} \in \text{ Fil}^4_\kappa(\kappa) \rangle$ be as in Theorem 1.2 with μ, κ here standing for λ, Y there; there is such $\bar{\mathscr{F}}$ definable from X as X codes also $[\mu]^{\aleph_0}$.

So for every $\eta \in {}^\kappa\mu$ there is $\mathfrak{y} \in \text{ Fil}^4_\kappa(\kappa)$ such that $f{\restriction}Z_{\mathfrak{y}} \in \mathscr{F}_{\mathfrak{y}}$ so let $\mathfrak{y}(\eta)$ be the $<_\kappa$-first such $\mathfrak{y}$. Now we define a well ordering $<_*$ of ${}^\kappa\mu$: for $\eta, \nu \in {}^\kappa\mu$ let $\eta <_* \nu$ <u>iff</u> $\text{rk}_{D_1[\mathfrak{y}(\eta)]}(f_\eta{\restriction}Z_{\mathfrak{y}(\eta)}) < \text{ rk}_{D_1(\mathfrak{y}(\nu))}(f_\nu{\restriction}Z_{\mathfrak{y}(\nu)})$ or equality holds and $\mathfrak{y}(\eta) < \mathfrak{y}(\nu)$.

This is O.K. because

- $(*)$ if $\eta \neq \nu \in {}^\kappa\mu$ then $f_\eta(i) \neq f_\nu(i)$ for every large enough $i < \kappa$ (i.e. $i \geq \min\{j : \eta(j) \neq \nu(j)\}$.

$\square_{3.1}$

Proof. <u>Proof of 3.2</u>:

Let $\langle \mu_i : i < \kappa \rangle$ be an increasing sequence of cardinals $< \mu$ with limit μ.

Let $\langle \text{cd}^*_i : i < \kappa \rangle$ satisfies

- $\boxplus_2$ cd_i is a one-to-one function from $\mathscr{P}(\mu_i)$ into μ, (definable uniformly from X).

So $\text{cd}_* : \mathscr{P}(\mu) \to {}^\kappa\mu$ defined by $(\text{cd}_*(A))(i) = \text{ cd}^*_i(A \cap \mu_i)$ for $A \subseteq \mu, i < \kappa$, is a one-to-one function from $\mathscr{P}(\mu)$ into ${}^\kappa\mu$. Now use 3.1. $\square_{3.2}$

We return to 2.13(2)

{c11}

Claim 3.3. [DC] *1) The cardinal λ^+ is regular* <u>*when*</u>*:*

$\boxplus$ (a) $\text{Ax}^4_{\lambda^+}$, *i.e.* $[\lambda^+]^{\aleph_0}$ *is well orderable*

(b) $|\alpha|^{\aleph_0} < \lambda$ *for* $\alpha < \lambda$

(c) λ *is singular.*

2) Also there is $\bar{e} = \langle e_\delta : \delta < \lambda^+\rangle, e_\delta \subseteq \delta = \sup(e_\delta), |e_\delta| \le \text{cf}(\delta)^{\aleph_0}$.

Remark 3.4. Compare with 2.13. We use here more choice, but cover more cardinals.

Proof. Let $<_*$ be a well ordering of the set $[\lambda^+]^{\aleph_0}$.

Let $F : {}^\omega(\lambda^+) \to \lambda^+$ be such that there is no $\subset$-decreasing sequence $\langle c\ell_F(u_n) : n < \omega\rangle$ with $u_n \subseteq \lambda^+$. Let $\Omega = \{\delta \le \lambda^+ : \delta$ a limit ordinal, $\delta < \lambda^+ \wedge \text{cf}(\delta) < \lambda\}$, so $\Omega \in \{\lambda^+, \lambda^+ + 1\}$.

We define $\bar{e} = \langle e_\delta : \delta \in \Omega\rangle$ as follows.

Case 1: $\text{cf}(\delta) = \aleph_0, e_\delta$ is the $<_*$-minimal member of $\{u \subseteq \delta : \delta = \sup(u)$ and $\text{otp}(u) = 0\}$.

Case 2: $\text{cf}(\delta) > \aleph_0$.

Let $e_\delta = \cap\{c\ell_F(C) : C$ a club of $\delta\}$.

So

$(*)_1$ e_δ is an unbounded subset of δ of order type $< \lambda$.

[Why? If $\text{cf}(\delta) = \aleph_0$ then e_δ has order type ω which is $< \lambda$ by clause (b) of the assumption.

If $\text{cf}(\delta) > \aleph_0$ then for some club C of $\delta, e_\delta = c\ell_F(C)$ has $\text{otp}(e_\delta) \le |c\ell_F(C)| \le (\text{cf}(\delta)^{\aleph_0} < \lambda$. The last inequality holds as $\text{cf}(\delta) \le \lambda$ as $\delta < \lambda^+$, $\text{cf}(\delta) \ne \lambda$ as λ is singular by clause (c) of the assumption, and lastly $((\text{cf}(\delta)^{\aleph_0}) < \lambda$ by clause (b) of the assumption.]

This is enough for part (2). Now we shall define a one-to-one function f_α from α into λ by induction on $\alpha \in \Omega$ as follows: $\text{pr}_\lambda : \lambda \times \lambda \to \lambda$ is the pairing function so one to one (can add "onto λ"); if we succeed then f_{λ^+} cannot be well defined so $\lambda^+ \notin \Omega$ hence $\text{cf}(\lambda^+) \ge \lambda$, but λ is singular so $\text{cf}(\lambda^+) = \lambda^+$, i.e. λ^+ is not singular so we shall be done proving part (1).

The inductive definition is:

$\boxplus$ (a) if $\alpha \le \lambda$ then f_α is the identity

(b) if $\alpha = \beta + 1 \in [\lambda, \lambda^+)$ then for $i < \alpha$ we let $f_\alpha(i)$ be
- $1 + f_\beta(i)$ if $i < \beta$
- 0 if $i = \beta$

(c) if $\alpha \in \Omega$ so α is a limit ordinal, $e_\alpha \subseteq \alpha = \sup(e_\alpha), e_\alpha$ of cardinality $< \lambda$ and we let f_α be defined by: for $i < \alpha$ we let $f_\alpha(i) = \text{pr}_\lambda(f_{\min(e_\alpha \backslash (i+1))}(i), \min(e_\alpha \backslash (i+1)))$.

$\square_{3.3}$

§ 4. Private Appendix

Moved 2010.8.09 from Remark **??**, pgs.10,11:

This works as well from weaker assumptions (see more later).

(*b*) we can define functions $\mathbf{i}_n : [\lambda]^{\aleph_0} \to \lambda$ such that $u = \{\mathbf{i}_n(u) : n < \omega\}$ (enough to define it for the countable ordinals). This holds if

α) $[\omega_1]^{\aleph_0}$ is well ordered or just

(β) there is $\langle h_\alpha : \alpha \in [\omega, \omega_1)\rangle$ such that h_α is one-to-one from α onto ω. Then it suffices to apply F_* just once.

[Why? For countable $v \subseteq \lambda$, let $F_*(v)$ be $\mathbf{i}_n(u_{\alpha(v)}) : n$ is minimal such that $\mathbf{i}_n(u_{\alpha(v)}) \notin v$ if there is such n and $n = 0$ otherwise. Why can we apply F_* just once? Toward contradiction assume $\langle \mathscr{U}_n : n < \omega\rangle$ is strictly decreasing. So choose $\alpha_n \in \mathscr{U}_n \backslash \mathscr{U}_{n+1}$. For some infinite $w \subset \omega, \langle \alpha_n : n \in w\rangle$ is strictly increasing and included in some u_β. Let $\beta = \alpha(v)$. Let $\gamma_n = \mathbf{i}(u_\beta)$ for $n < \omega$ hence $\langle \gamma_n : n < \omega\rangle$ list u_β. Without loss of generality there is an increasing function h from w to ω such that $\alpha_n = \mathbf{i}_{h(n)}(u_\beta)$. Let $n_* < n_{**}$ be the first two members of u.

Let $u = \{\gamma_n : n_{**} \le n \in w\}$ so $u \subseteq \mathscr{U}_{n_{**}}, n_* \le n_{**} - 1$ and clearly $\alpha_{n_*} \in \{\gamma_\ell : \ell \le n_{**}\}$ and we can prove by downward induction on $\ell \le n_{**}$ that $\gamma_\ell \in \mathscr{U}_{n_{**}}$ because $\gamma_\ell = F_*(u \cup \{\gamma_k : \ell < n_{**}, k \le \ell(*)\})$, contradiction.]

Moved 2010.8.09 from Claim 2.3, pg.14:

2) If there is a ∂-well founded [weak] closure operation on λ of character (μ, κ) then there is a well founded [weak] closure operation of character (μ, ∂).
3) Assume $\theta < \text{hrtg}(\mathscr{H}_{<\aleph_1}(\mu)), \aleph_0$. There is a well founded weak closure operation of character $(\mu, \aleph_0)$ on λ and κ_2 there is a ∂-well founded weak closure operation of character $(< \aleph_0)$ on λ.

Moved 2010.8.09 from Proof, pgs.15:

2) Assume $c\ell : [\lambda]^{\le\kappa} \to [\lambda]^{\le\mu}$ witness the assumption; let $c\ell' : [\lambda]^{\le\partial} \to [\lambda]^{\le\mu}$ be $c\ell'(u) = \cup\{c\ell(v) : v \subseteq u, |v| \le u\}$ which is $\subseteq c\ell(u)$ recalling Definition 0.15(1A)(d). So $c\ell'$ witness the desired conclusion using proved part (4) below.
3) Let $c\ell : [\lambda]^{\le\partial} \to [\lambda]^{\le\mu}$ witness the assumption and we define $c\ell' : [\lambda]^{\le\kappa} \to [\lambda]^{\le\mu}$ by $c\ell'(u) = \cup\{c\ell(v) : v \subseteq u, |v| \le \kappa\}$ and continue as above.

Moved 2010.8.09 from Claim 2.7, pg.18:

Remark 4.1. 1) Assuming $\text{Ax}^0_{\lambda,\mu,\kappa}$ but not AC_κ, still λ^+ is regular (see later) or just has cofinality $\ge \text{hrtg}(\text{Fil}^4_\kappa(\kappa, \mu)$ we can find the limit of the $\mathscr{F}_\alpha$'s.
2) See [Sh:497].

Moved 2010.8.09 from Remark 2.11, pg.21:

Remark 4.2. In 2.11, we are given $S' \subseteq S$ such that $\langle \text{cf}(\delta) : \delta \in S'\rangle$ is constant then we can choose $\langle C'_\delta : \delta \in S'\rangle, C'_\delta$ is a club of δ of order type $\text{cf}(\delta)$. {r.9a}

Moved 2010.8.09 from end of §2, pg.24:

{sp.5}

Claim 4.3. *Assume(a)-(e) from 2.17.*
1) If θ *a cardinal* $< \lambda$ *and* $S := \{i < \kappa : \mathrm{cf}(\lambda_i^+) > \theta\}$ *is a stationary subset of* κ *then* $\mathrm{cf}(\lambda_\kappa^+) > \theta$.
2) If AC_κ *and* $S \subseteq \kappa$ *is stationary then* $\mathrm{cf}(\lambda_\kappa^+) < \mathrm{hrtg}(\prod_{i \in S} \mathrm{cf}(\lambda_i^+))$.

Proof. 1) Like 2.17.
2) Let D_κ be the club filter on κ and let $D = D_\kappa + S$. By AC_κ we can choose $\bar{B} = \langle B_i : i \in S \rangle, B_i$ a club of λ_i^+ and $\mathrm{otp}(B_i) = \mathrm{cf}(\lambda_i^+)$ and let $B_i = \{0\}$ for $i \in \kappa \backslash S$. Let $B = \{\mathrm{rk}_D(f) : f \in \prod_{i \in S} B_i\}$, it is an unbounded of λ^+, so $f \mapsto \mathrm{rk}_D(f)$ exemplify $\mathrm{hrtg}(\prod_{i \in S} B_i) >\ \mathrm{cf}(\lambda_\kappa^+)$, as $S \in D$ so we are done. $\square_{4.3}$

Implicit above is
END OF LATEX REVISIONS

References

[AM95] Arthur Apter and Menahem Magidor, *Instances of Dependent Choice and the Measurability of* $\aleph_{\omega+1}$, Annals of Pure and Applied Logic **74** (1995), 203–219.

[DJ82] Tony Dodd and Ronald B. Jensen, *The covering lemma for K*, Ann. of Math Logic **22** (1982), 1–30.

[Eas70] William B. Easton, *Powers of regular cardinals*, Annals of Math. Logic **1** (1970), 139–178.

[GH75] Fred Galvin and Andras Hajnal, *Inequalities for cardinal powers*, Annals of Mathematics **101** (1975), 491–498.

[Sh:g] Saharon Shelah, *Cardinal Arithmetic*, Oxford Logic Guides, vol. 29, Oxford University Press, 1994.

[Sh:E37] ______, *What majority decisions are possible*, arxiv:math.CO/0303323.

[Sh:71] ______, *A note on cardinal exponentiation*, The Journal of Symbolic Logic **45** (1980), 56–66.

[Sh:497] ______, *Set Theory without choice: not everything on cofinality is possible*, Archive for Mathematical Logic **36** (1997), 81–125, A special volume dedicated to Prof. Azriel Levy. arxiv:math.LO/9512227.

[Sh:589] ______, *Applications of PCF theory*, Journal of Symbolic Logic **65** (2000), 1624–1674, arxiv:math.LO/9804155.

[LrSh:925] Paul Larson and Saharon Shelah, *Splitting stationary sets from weak forms of Choice*, Mathematical Logic Quarterly **55** (2009), 299–306.

[Sh:938] Saharon Shelah, *PCF arithmetic without and with choice*, Israel Journal of Mathematics **191** (2012), 1–40, arxiv:0905.3021.

[Sh:955] ______, *Pseudo pcf*, Israel Journal of Mathematics **201** (2014), 185–231, arxiv:1107.4625.

[Sh:F1039] ______, *Bounds on pcf with weak choice using ranks and normal filters.*

[Sh:F1078] ______, *Pcf with narrow choice or ZF + DC +*AX_4.

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