PCF WITHOUT CHOICE

SH835

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Abstract. We mainly investigate models of set theory with restricted choice, e.g., ZF + DC + the family of countable subsets of $\lambda$ is well ordered for every $\lambda$ (really local version for a given $\lambda$). We think that in this frame much of pcf theory, (and combinatorial set theory in general) can be generalized. We prove here, in particular, that there is a class of regular cardinals, every large enough successor of singular is not measurable and we can prove cardinal inequalities.
Anotated Content

§0 Introduction, pg. 3
§(0A) Background, aims and results, pg. 4
§(0B) Preliminaries, pg. 4

[Include quoting [Sh:497], ([3.1, 3.2], hrtg(Y), wlor(Y), ([3.3]), defining rkD(f), ([3.4, 3.5]) on J[f, D], ([3.6, 3.7), hrtg(Y), ([3.8, 3.9]), and observation ([3.10]); and on closure operations ([3.11].]

§1 Representing \( ^{\kappa}\lambda \), pg. 8

[We define Fil\( _{\kappa} \) and prove a representation theorem for \( ^{\kappa}\lambda \). Essentially under “reasonable choice” the set \( ^{\kappa}\lambda \) is the union of few well ordered sets, i.e., their number depends on \( \kappa \) only”. We end by a claim on \( \Pi_{\alpha} \).

§2 No decreasing sequence of subalgebras, pg. 8

[As suggested in the title we weaken the axioms. We deal with \( ^{\kappa}\lambda \) with \( \lambda^+ \) not measurable, existence of ladder \( \bar{C} \) witnessing cofinality and prove that many \( \lambda^+ \) are regular ([2.13]).]

§3 Concluding remarks, pg. 10

[We prove that if \( \mu > \kappa = \text{cf}(\mu) > \aleph_0 \), then from a well ordering of \( \mathcal{P}(\mathcal{P}(\kappa)) \cup ^{\kappa^+}\mu \) we can define a well ordering of \( ^{\kappa}\mu \), see ([3.1]). If e.g. \( \mu \) is strong limit singular of uncountable cofinality, using a well order of \( \mathcal{H}(\mu) \) we can define a well ordering of \( \mathcal{P}(\mu) \) hence of \( H(\mu^+) \), see ([3.2]). Lastly, we give sufficient conditions (in ZF + DC) for singular \( \mu \), that \( \mu^+ \) is regular, see ([3.3]). Actually if \( \mu = \kappa^{\aleph_0} + 2^{\aleph_0}, \kappa = \kappa^{\aleph_0} \) and \( X \subseteq \mu \) codes \( \mathcal{P}(\mathcal{P}(\kappa)) \) and \( \omega \), then using \( X \) as a parameter we can define a well ordering of \( ^{\kappa}\mu \), see ([3.4]).]
§ 0. Introduction

0A

§ 0(A). Background, aims and results.

The thesis of [She97] was that pcf theory without full choice exists. Two theorems supporting this thesis were proved. The first ([She97, 4.6, pg.117], we shall not mention ZF) is:

0.1 Theorem 0.1. [DC] If \( H(\mu) \) is well ordered, \( \mu \) strong limit singular of uncountable cofinality then \( \mu^+ \) is regular not measurable (and \( 2^\mu \) is an \( \aleph_0 \), i.e. \( P(\mu) \) can be well ordered and no \( \lambda \in (\mu, 2^\mu) \) is measurable).

Note that before this Apter and Magidor [AM95] had proved the consistency of \( H(\mu) \) well ordered, \( \mu = \beth_\omega \), \( (\forall \kappa < \mu) \text{DC}_\kappa \) and \( \mu^+ \) is measurable\) so 0.1 says that this consistency result cannot be fully lifted to uncountable cofinalities. Generally without full choice, a successor cardinal being not measurable is worthwhile information.

A second theorem ([She97, §5]) was

0.2 Theorem 0.2. Assume

(a) \( \text{DC + AC}_\kappa + \kappa \) regular uncountable
(b) \( \langle \mu_i : i < \kappa \rangle \) is increasing continuous with limit \( \mu, \mu > \kappa, H(\mu) \) is well ordered, \( \mu \) strong limit, (we need just a somewhat weaker version, the so-called \( i < \kappa \Rightarrow Tw_{\mathcal{D}}(\mu_i) < \mu \)).

Then, we cannot have two regular cardinals \( \theta \) such that for some stationary \( S \subseteq \kappa \), the sequence \( \langle \text{cf}(\mu_i^+) : i \in S \rangle \) is constantly \( \theta \).

A dream was to prove that there is a class of regular cardinals from a restricted version of choice (see more [She97] and a little more in [Shec]).

Our original aim here is to improve those theorems. As far [Sh:497] we replace \( H(\mu) \) well ordered\) by \( \langle \mu \rangle^{\aleph_0} \) is well ordered\) and then by weaker statements.

We know (assuming full choice) that if, e.g., \( \neg \exists \emptyset \# \) or there is no inner model with a measurable cardinal then \( 2^\kappa : \kappa \) regular\) is quite arbitrary, the size of \( |\lambda|, \lambda >> \kappa \) is strictly controlled (by Easton forcing [Eas70], and Jensen and Dodd [DJ82] respectively). It seemed that the situation here is parallel in some sense; under the restricted choice we assume, we cannot say much on the cardinality of \( P(\kappa) \) but can say something on the cardinality of \( |\lambda|^\kappa \) for \( \lambda >> \kappa \).

In the proofs we fulfill a promise from [She97, §5] about using \([f, D] \) from Definition 0.12 instead of the nice filters used in [She97] and, to some extent, in early versions of this work which require going through inner models to prove their existence. This work is continued in Larson-Shelah [LSh99] and will be continued in [Sh16]. On a different line with weak choice (say \( \text{DC}_{\aleph_0} + \text{AC}_{\mu +} \mu \) fixed): see [She12], [She02] and [Shea]. The present work fits the theses of [She94] which in particular says: it is better to look at \( \lambda^{\aleph_0} : \lambda \) a cardinality then at \( \langle 2^\lambda : \lambda \rangle \) a cardinal. Here instead well ordering \( P(\lambda) \) we well order \( |\lambda|^\kappa \), this is enough for much.

A simply stated conclusion is (see [Sh12])
Conclusion 0.3. [DC] Assume $|\lambda|^\kappa$ is well ordered for every $\lambda$.
1) If $2^\kappa$ is well ordered then for every $\lambda, |\lambda|^\kappa$ is well ordered.
2) For any set $Y$, there is a derived set $Y_*$ so called $\text{Fil}_1^\kappa(Y)$ of power near $\mathcal{P}(\mathcal{P}(Y))$ such that $\models_{\text{Levy}(\kappa, Y)}$ “for every $\lambda, Y^\lambda$ is well ordered”.

Thesis 0.4. 1) If $V \models “ZF + DC$” and “every $|\lambda|^\aleph_0$ is well orderable” then $V$ looks like the result of starting with a model of ZFC and using $\aleph_1$-complete forcing notions like Easton forcing, Levy collapses, and more generally, iterating of $\aleph_1$-complete forcing for $\kappa > \aleph_0$.
2) This approach is dual to investigating $L[\mathbb{R}]$ - here we assume $\omega$-sequences are understood (or weaker versions) and we try to understand $V$ (over this), there over the reals everything is understood.

Also though our original motivation was to look at consequences of $\text{Ax}_3$, this was shadowed here by the try to use weaker relatives; see more in [Shel16].

Explanation 0.5. How do we analyze $[\mu]^{\kappa}$ or equivalently $\kappa^\mu$ here? We use $\aleph_1$-complete filters on $\kappa$ and a well ordering of $[\alpha]^{\aleph_0}$ for appropriate $\alpha$ or less. We will consider $f : \kappa \to \mu$; now for every $\aleph_1$-complete filter $D$ on $\kappa$, the ordinal $\text{rk}_D(f)$ gives us some information on $\alpha$, but if $A, \kappa \setminus A \in D^+$ and $\text{rk}_A = 0$, then $\alpha = 0$ but we have no information on $f\mid (\kappa \setminus A)$, then $\alpha = 0$ but we have no information on $f\mid (\kappa \setminus A)$. Trying to correct this we consider the ideal $J[f, D] = \{ A \subseteq \kappa : A = \emptyset \mod D$ or $A \in D^+$ but $\text{rk}_{D^+(A)}(f) > \alpha)\}$, this is an $\aleph_1$-complete ideal and so we may consider the pair $\tilde{D} = (D_1, D_2) = (D, \text{dual}(J[f, D]))$. Now $\alpha$ and the pair $\tilde{D}$ gives more information on $f$; they determine $f$ modulo $D_2$. This is not enough so we use an algebra $\mathcal{B}$ on $\mu$ with no infinite decreasing sequence of sub-algebras built using the assumption “$[\mu]^{\aleph_0}$ is well ordered”. So there is $Z \in D_2$ such that $A = \mathcal{B}(\text{Rang}(f\mid Z))$ is $\subseteq$-minimal.

Now the triple $(D_1, D_2, Z)$ and the ordinal $\alpha$ almost determines $f$, we need one more piece of information with domain $\kappa : h(i) = \text{otp}(\alpha \cap Z)$, hence an ordinal $< h_{\text{rng}}(\text{Rang}(f))$. So we need a bound on it which depends on the choice of $\mathcal{B}$, usually it is $h_{\text{rng}}([\kappa]^{\aleph_0})$, natural by the construction of $\mathcal{B}$.

So $f\mid Z$ is uniquely determined by the ordinal $\text{rk}_D(f)$ and the quadruple $(D_1, D_2, Z, h)$, which belongs to a set defined from $\kappa$, independently of $\mu$.

Lastly, considering all such filters $D$ (recalling we are assuming DC) we can find countably many quadruple $(D^n_1, D^n_2, Z^n, h^n)$ which together are enough as $\bigcup Z^n = \kappa$.

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§ 0(B). Preliminaries.

Convention 0.6. We assume just $V \models \text{ZF}$ if not said otherwise.
Notation 0.7. Let
1) \( \alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j \) denote ordinals.
2) \( \kappa, \lambda, \mu, \chi \) denotes cardinals, infinite if not said otherwise.
3) \( n, m, k, \ell \) denotes natural numbers.
4) \( D \) denotes a filter (on some set), \( I, J \) denote ideals on some set.

Definition 0.8. 1) \( \text{hrg}(A) = \min\{\alpha : \text{there is no function from } A \text{ onto } \alpha\} \).
2) \( \text{wloc}(A) = \min\{\alpha : \text{there is no one-to-one function from } \alpha \text{ into } A \text{ or } \alpha = 0 \land A = \emptyset\} \) so \( \text{wloc}(A) \leq \text{hrg}(A) \).

Definition 0.9. 1) For \( D \) an \( \aleph_1 \)-complete filter on \( Y \) and \( f \in Y^{\text{Ord}} \) and \( \alpha \in \text{Ord} \cup \{\infty\} \) we define when \( \text{rk}_D(f) = \alpha \), by induction on \( \alpha \):

+ For \( \alpha < \infty \), \( \text{rk}_D(f) = \alpha \) iff \( \beta < \alpha \Rightarrow \text{rk}_D(f) \neq \beta \) and for every \( g \in Y^{\text{Ord}} \) satisfying \( g \leq f \) there is \( \beta < \alpha \) such that \( \text{rk}_D(g) = \beta \).

2) We can replace \( D \) by the dual ideal. If \( f \in Z^{\text{Ord}} \) and \( Z \in D \) then we let \( \text{rk}_D(f) = \text{rk}_{D^+}(f \cup 0_{Y \setminus Z}) \).

Shelah-Hajnal [GH75] use the rank for the club filter on \( \omega_1 \). This was continued in [Sh80] where varying \( D \) was extensively used.

Claim 0.10. [DC] In Definition 0.9, \( \text{rk}_D(f) \) is always an ordinal and if \( \alpha \leq \text{rk}_D(f) \) then for some \( g \in \prod_{y \in Y} (f(y) + 1) \) we have \( \alpha = \text{rk}_D(g) \), (if \( \alpha < \text{rk}_D(f) \) we can add \( g < \alpha f \); if \( \text{rk}_D(f) < \infty \) then DC is not necessary; if \( \text{rk}_D(f) = \alpha \) this is trivial, as we can choose \( g = f \)).

Claim 0.11. 1) [DC] If \( D \) is an \( \aleph_1 \)-complete filter on \( Y \) and \( f \in Y^{\text{Ord}} \) and \( Y = \cup\{Y_n : n < \omega\} \) then \( \text{rk}_D(f) = \min\{\text{rk}_D(Y_n) : n < \omega \text{ and } Y_n \in D^+\} \) [Sh71].
2) [DC + \( \text{AC}_{\kappa\omega} \)] If \( D \) is a \( \kappa \)-complete filter on \( Y, \kappa \) a cardinal \( > \aleph_0 \) and \( f \in Y^{\text{Ord}} \) and \( Y = \cup\{Y_\alpha : \alpha < \kappa^*\}, \alpha^* < \kappa \) then \( \text{rk}_D(f) = \min\{\text{rk}_D(Y_\alpha) : \alpha < \alpha^* \text{ and } Y_\alpha \in D^+\} \).

Proof. 1) By [Sh71], in fact, \( \text{AC}_{\kappa\omega} \) suffice.
2) By [Sh80], in fact, DC is not necessary. \( \square \).

Definition 0.12. For \( Y, D, f \) as in 0.9 let \( J[f, D] := \{z \subseteq Y : Y \setminus z \in D \lor Y \setminus z \in D^+ \text{ and } \text{rk}_D(f)(D + \{z\}) > \text{rk}_D(f)\} \).

Claim 0.13. [DC + \( \text{AC}_{\kappa\omega} \)] Assume \( D \) is a \( \kappa \)-complete filter on \( Y, \kappa > \aleph_0 \).
1) If \( f \in Y^{\text{Ord}} \) then \( J[f, D] \) is a \( \kappa \)-complete ideal on \( Y \).
2) If \( f_1, f_2 \in Y^{\text{Ord}} \) and \( J = J[f_1, D] = J[f_2, D] \) then \( \text{rk}_D(f_1) < \text{rk}_D(f_2) \Rightarrow f_1 < f_2 \) \( \mod J \) and \( \text{rk}_D(f_1) = \text{rk}_D(f_2) \Rightarrow f_1 = f_2 \mod J \).

Proof. Straightforward or see [Sh80], [4] and the reference there to [Sh80]. \( \square \).

Definition 0.14. 1) Here \( Y \leq_{\text{qu}} Z \) or \( |Y| \leq_{\text{qu}} |Z| \) or \( |Y| \leq_{\text{qu}} |Z| \) means that \( Y = \emptyset \) or there is a function from \( Z \) (equivalently from a subset of \( Z \)) onto \( Y \).
2) \( \text{reg}(\alpha) = \min\{\beta : \beta \geq \alpha \text{ is a regular cardinal}\} \).

Definition 0.15. For a set \( Y \), cardinal \( \kappa \) and ordinal \( \gamma \) we define \( \mathcal{H}_{\kappa, \gamma}(Y) \) by induction on \( \gamma \): if \( \gamma = 0 \), \( \mathcal{H}_{\kappa, \gamma}(Y) = Y \); if \( \gamma = \beta + 1 \) then \( \mathcal{H}_{\kappa, \gamma}(Y) = \mathcal{H}_{\kappa, \beta}(Y) \cup \{u : u \subseteq \mathcal{H}_{\kappa, \beta}(Y) \text{ and } |u| < \kappa\} \) and if \( \gamma \) is a limit ordinal then \( \mathcal{H}_{\kappa, \gamma}(Y) = \cup\{\mathcal{H}_{\kappa, \beta}(Y) : \beta < \gamma\} \).
Observation 0.16. 1) If $\lambda$ is the disjoint union of $\langle W_z : z \in Z \rangle$ and $z \in Z \Rightarrow |W_z| < \lambda$ and $\text{wlor}(Z) \leq \lambda$ then $\lambda = \sup\{\text{otp}(W_z) : z \in Z\}$ hence $\text{cf}(\lambda) < \text{hrtg}(Z)$.
2) If $\lambda = \cup\{W_z : z \in Z\}$ and $\text{wlor}(\mathcal{P}(Z)) \leq \lambda$ then $\sup\{\text{otp}(W_z) : z \in Z\} = \lambda$.
3) If $\lambda = \cup\{W_z : z \in Z\}$ and $|Z| < \lambda$ then $\lambda = \sup\{\text{otp}(W_z) : z \in Z\}$.
4) If $Z \subseteq \text{Ord}, W = (W_\alpha : \alpha \in Z), W_\alpha \subseteq \text{Ord}$ and $\lambda \geq \aleph_0, |Z|, |W_\alpha|$ for $\alpha \in Z$ then $\cup\{W_\alpha : \alpha \in Z\}$ has cardinality $\leq \lambda$.

Proof. 1) Let $Z_1 = \{z \in Z : W_z \neq \emptyset\}$, so the mapping $z \mapsto \text{Min}(W_z)$ exemplifies that $Z_1$ is well ordered hence by the definition of $\text{wlor}(Z_1)$ the power $|Z_1|$ is an aleph $< \text{wlor}(Z_1) \leq \text{wlor}(Z)$ and by assumption $\text{wlor}(Z) \leq \lambda$. Now if the desirable conclusion fails then $\gamma^* = \sup\{\text{otp}(W_z) : z \in Z_1\} \cup \{|Z_1|\}$ is an ordinal $< \lambda$, so we can find a sequence $(u_\gamma : \gamma < \gamma^*)$ such that $\text{otp}(u_\gamma) \leq \gamma^*, u_\gamma \subseteq \lambda$ and $\lambda = \cup\{u_\gamma : \gamma < \gamma^*\}$, so $\gamma^* < \lambda \leq |\gamma^* \times \gamma^*|$, easy contradiction.
2) For $x \subseteq Z$ let $W^*_x = \{\alpha < \lambda : \forall z \in Z_1(\alpha \in W_z \equiv z \in x)\}$ hence $\lambda$ is the disjoint union of $\langle W^*_x : x \in \mathcal{P}(Z)\setminus\emptyset\rangle$. So the result follows by part (1).
3) So let $<_\mathcal{P}$ be a well ordering of $Z$ and let $W'_x = \{\alpha \in W_z : y < _\mathcal{P} z \text{ then } \alpha \notin W_y\}$, so $(W'_z : z \in Z)$ is a well defined sequence of pairwise disjoint sets with union equal to $\cup\{W_z : z \in Z\} = \lambda$ and $\text{otp}(W'_z) \leq \text{otp}(W_z)$. Hence if $|W_z| = \lambda$ for some $z \in Z$ the desirable conclusion is obvious, otherwise the result follows by part (1).
4) Should be clear.

Definition 0.17. 1) We say that $\text{c}\ell$ is a very weak closure operation on $\lambda$ of character $(\mu, \kappa)$ when:

(a) $\text{c}\ell$ is a function from $\mathcal{P}(\lambda)$ to $\mathcal{P}(\lambda)$
(b) $u \in [\lambda]^{<\kappa} \Rightarrow |\text{c}\ell(u)| \leq \mu$
(c) $u \subseteq \lambda \Rightarrow u \cup \{0\} \subseteq \text{c}\ell(u)$ for technical reasons.

1A) We say that $\text{c}\ell$ is a weak closure$^1$ operation on $\lambda$ of character $(\mu, \kappa)$ when (a),(b),(c) above and:

(d) $u \subseteq v \subseteq \lambda \Rightarrow u \subseteq \text{c}\ell(u) \subseteq \text{c}\ell(v)$
(e) $\text{c}\ell(u) = \cup\{\text{c}\ell(v) : v \subseteq u, |v| \leq \kappa\}$.

1B) Let “... character $(< \mu, \kappa)$ or $(\mu, < \kappa)$, or $(< \mu, < \kappa)$” have the obvious meaning but if $\mu$ is an ordinal not a cardinal, then “$< \mu^+$ means of order type $< \mu$; similarly for “$< \kappa$”. Let “... character $(\mu, Y)$” means “character $(< \mu^+, < \text{hrtg}(Y))$”

1C) We omit the weak when in addition:

(f) $\text{c}\ell(u) = \text{c}\ell(\text{c}\ell(u))$ for $u \subseteq \lambda$.

2) We say $\lambda$ is $f$-inaccessible when $\delta \in \lambda \cap \text{Dom}(f) \Rightarrow f(\delta) < \lambda$.

3) We say $\text{c}\ell : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda)$ is well founded when for no sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of subsets of $\lambda$ do we have $\text{c}\ell(\mathcal{U}_{n+1}) \subset \mathcal{U}_n$ for $n < \omega$.

4) For $\text{c}\ell$ a partial function from $\mathcal{P}(\alpha)$ to $\mathcal{P}(\alpha)$ (for simplicity assume $\alpha = \cup\{u : u \in \text{Dom}(\text{c}\ell)\}$ we let $\text{c}\ell_{\lambda<\kappa}^1$ be the function from $\mathcal{P}(\alpha)$ to $\mathcal{P}(\alpha)$ defined by induction on the ordinal $\varepsilon$ as follows:

(a) $\text{c}\ell_{0,\varepsilon}^1(u) = u$

$^1$so by actually only $\text{c}\ell[\lambda]^{<\kappa}$ count
(b) $c^{1}_{1, \kappa}(u) = \{0\} \cup \bigcup_{v \subseteq c^{1}_{1, \kappa}(u) \cap \text{Dom}(c^{1})} \{c^{1}_{1}(v) : v \in \text{Dom}(c^{1}) \}.$

(c) for limit $\varepsilon$ let $c^{1}_{1, \kappa}(u) = \bigcup \{c^{1}_{1, \kappa}(u) : \varepsilon < \kappa \}.$

4A) Instead “$< \kappa$” we may use “$\leq \kappa$”.

5) For any function $F : [\lambda]^{\aleph_{0}} \rightarrow \lambda$ and countable $u \subseteq \lambda$ we define $c^{2}_{1}(u, F)$ by induction on $\varepsilon \leq \omega_{1}$(a) $c^{2}_{1}(u, F) = u \cup \{0\}.$

(b) $c^{2}_{1}(u, F) = c^{2}_{1}(u, F) \cup \{F(c^{2}_{1}(u, F))\}.$

(c) $c^{2}_{1}(u, F) = \bigcup \{c^{2}_{1}(u, F) : \varepsilon < \kappa \}$ when $\varepsilon \leq \omega_{1}$ is a limit ordinal.

6) For countable $u$ and $F$ as in part (5) let $c^{2}_{0}(u, F) = c^{2}_{1}(u, F) : \varepsilon \leq \omega_{1}.$

7) For a cardinal $\partial$ we say that $c^{1} : \mathcal{P}(\lambda) \rightarrow \mathcal{P}(\lambda)$ is $\omega_{1}$-well founded when for no $\subseteq$-decreasing sequence $\langle \mathcal{U}_{\varepsilon} : \varepsilon < \omega_{1} \rangle$ of subsets of $\lambda$ do we have $\varepsilon < \zeta < \omega_{1} \Rightarrow c^{1}(\mathcal{U}_{\zeta}) \supseteq \mathcal{U}_{\varepsilon}.$

8) If $F : [\lambda]^{\aleph_{0}} \rightarrow \lambda$ and $u \subseteq \lambda$ then we let $c^{1}_{F}(u) = c^{1}_{F}(u) : \varepsilon \leq \omega_{1}$ be the minimal subset $v$ of $\lambda$ such that $w \in [v]^{\leq \kappa} \Rightarrow F(w) \in v$ and $u \subseteq v$ (exists).

**Observation 0.18.** For $F : [\lambda]^{\aleph_{0}} \rightarrow \lambda$, the operation $u \mapsto c^{1}_{F}(u)$ is a very weak closure operation of character $(\aleph_{1}, \aleph_{0}).$

**Remark 0.19.** So for any very weak closure operation, $\aleph_{0}$-well founded is a stronger property than well founded, but if $u \subseteq \lambda \Rightarrow c^{1}(c^{1}(u)) = c^{1}(u)$ which is reasonable, they are equivalent.

**Observation 0.20.** $[\alpha]^{\partial}$ is well ordered iff $\partial^{\alpha}$ is well ordered when $\alpha \geq \partial.$

**Proof.** Use a pairing function on $\alpha$ for showing $|\partial^{\alpha}| \leq [\alpha]^{\partial}$, so $\Rightarrow$ holds. If $\partial^{\alpha}$ is well ordered by $<_{\ast}$ map $u \in [\alpha]^{\partial}$ to the $<_{\ast}$-first $f \in \partial^{\alpha}$ satisfying $\text{Rang}(f) = u.$
§ 1. Representing $\kappa\lambda$

Here we give a simple case to illustrate what we do (see later on improvements in the hypothesis and the conclusion). Specifically, if $Y$ is uncountable and $[\lambda]^\aleph_0$ is well ordered, then the set $Y\lambda$ can be analyzed modulo countable union over few (i.e., their number depends on $Y$ but not on $\lambda$) well ordered sets.

Definition 1.1. 1)  

(a) $\Fil^1_{\aleph_1}(Y) = \Fil^1_{\aleph_1}(Y) = \{D : D$ is an $\aleph_1$-complete filter on $Y\}$, so $Y$ is defined from $D$ as $\cup\{X : X \in D\}$

(b) $\Fil^2_{\aleph_1}(Y) = \{(D_1, D_2) : D_1 \subseteq D_2$ are $\aleph_1$-complete filters on $Y$, $(\emptyset \notin D_2$, of course)\}; in this context, $Z \in D$ means $Z \subseteq D_2$

(c) $\Fil^3_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h) : (D_1, D_2) \in \Fil^2_{\aleph_1}(Y)$ and $h : Y \rightarrow \alpha$ for some $\alpha < \mu$, if we omit $\mu$ we mean $\mu = \text{hrtg}(Y) \cup \omega$

(d) $\Fil^4_{\aleph_1}(Y, \mu) = \{(D_1, D_2, h, Z) : (D_1, D_2, h) \in \Fil^3_{\aleph_1}(Y, \mu), Z \in D_2\}$ omitting $\mu$ means as above.

2) For $\eta \in \Fil^4_{\aleph_1}(Y, \mu)$ let $Y = Y^{[\eta]} = Y[\eta]$ and $\eta = (D^0, D^1, h^\eta, Z^\eta) = (D_1[\eta], D_2[\eta], h[\eta], Z[\eta])$; similarly for the others and let $D^0 = D[\eta]$ be $D^0 + Z^\eta$.

3) We can replace $\aleph_1$ by any $\kappa > \aleph_1$ (the results can be generalized easily assuming DC + AC_{<\kappa}, used in §2).

Theorem 1.2. $[\text{DC}]$ Assume $[\lambda]^\aleph_0$ is well ordered.

Then we can find a sequence $\langle \mathcal{F}_\eta : \eta \in \Fil^4_{\aleph_1}(Y) \rangle$ satisfying

\begin{enumerate}
\item[(a)] $\mathcal{F}_\eta \subseteq Z[\eta]_\lambda$
\item[(b)] $\mathcal{F}_\eta$ is a well ordered set by $f_1 <_\eta f_2$ $\iff$ $\text{rk}_{D[\eta]}(f_1) < \text{rk}_{D[\eta]}(f_2)$ so $f \mapsto \text{rk}_{D[\eta]}(f)$ is a one-to-one mapping from $\mathcal{F}_\eta$ into the ordinals
\item[(c)] if $f \in Y^\lambda$ then we can find a sequence $\langle \eta_n : n < \omega \rangle$ with $\eta_n \in \Fil^4_{\aleph_1}(Y)$ such that $n < \omega \Rightarrow f \upharpoonright Z^n \in \mathcal{F}_{\eta_n}$ and $\cup\{Z^n : n < \omega\} = Y$.
\end{enumerate}

An immediate consequence of 1.2 is

Conclusion 1.3. 1) $[\text{DC} + \aleph_\omega$ is well-orderable for every ordinal $\alpha]$

For any set $Y$ and cardinal $\lambda$ there is a sequence $\langle \mathcal{F}_\bar{\alpha} : \bar{\alpha} \in \text{"(Fil}^4_{\aleph_1}(Y)\text{)"} \rangle$ such that

\begin{enumerate}
\item[(a)] $Y^\lambda = \cup\{\mathcal{F}_\bar{\alpha} : \bar{\alpha} \in \text{"(Fil}^4_{\aleph_1}(Y)\text{)"}\}$
\item[(b)] $\mathcal{F}_\bar{\alpha}$ is well orderable for each $\bar{\alpha} \in \text{"(Fil}^4_{\aleph_1}(Y)\text{)"}$
\item[(b)'] moreover, uniformly, i.e., there is a sequence $\langle <_\bar{\alpha} : \bar{\alpha} \in \text{"(Fil}^4_{\aleph_1}(Y)\text{)"} \rangle$ such that $<_\bar{\alpha}$ is a well order of $\mathcal{F}_\bar{\alpha}$
\item[(c)] there is a function $F$ with domain $\mathcal{P}(Y^\lambda)\setminus\{\emptyset\}$ such that: if $S \subseteq Y^\lambda$ is non-empty then $F(S)$ is a non-empty subset of $S$ of power $\leq_{\text{qu}} \text{"(Fil}^4_{\aleph_1}(Y)\text{)"}$ recalling Definition 2.1.3. In fact, some ordinal $\alpha(*)$ and $\bar{u}$ we have:
\item[(a)] $\bar{u} = \langle \mathcal{U}_\alpha : \alpha < \alpha(*) \rangle$ is a partition of $Y^\lambda$
\item[(b)] if $S \subseteq Y^\lambda$ then $F(S) = \mathcal{U}_f(S) \cap S$ where $f(S) = \text{Min}\{\alpha : \mathcal{U}_\alpha \cap S \neq \emptyset\}$
\item[(c)] if $\alpha < \alpha(*)$ then $\mathcal{U}_\alpha < \text{hrtg}(\text{"(Fil}^4_{\aleph_1}(Y)\text{)"})$.
\end{enumerate}
2) \([\text{DC}]\) For any \(Y, \lambda\) above, if \(\lfloor \alpha(*) \rfloor_{\aleph_0}^\lambda\) is well ordered where \(\alpha(*) = \cup \{\text{rk}_D(f) + 1 : f \in Y \lambda \text{ and } D \in \text{Fil}_{\aleph_1}(Y)\}\) then \(Y \lambda\) satisfies the conclusion of part (1).

Remark 1.4. So clause (c) of \([\text{DC}]\) is a weak form of choice.

Proof. Proof of \([\text{DC}]\) Let \(\langle \mathcal{F}_0 : \eta \in \text{Fil}_{\aleph_1}(Y) \rangle\) be as in \([\text{DC}]\).

For each \(\bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y))\) (so \(\bar{r} = \langle r_n : n < \omega \rangle\)) let

\[\mathcal{F}_{\bar{r}} = \{f : f \text{ is a function from } Y \text{ to } \lambda \text{ such that } n < \omega \Rightarrow f \upharpoonright Z^n \in \mathcal{F}_{r_n} \text{ and } Y = \cup \{Z^n : n < \omega\}\}.\]

Now

\[(*)_1 \ Y \lambda = \cup \{\mathcal{F}_{\bar{r}} : \bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y))\}.

[Why? By clause (\(\gamma\)) of \([\text{DC}]\).]

Let \(\alpha(*) = \cup \{\text{rk}_D(f) + 1 : f \in Y \lambda \text{ and } D \in \text{Fil}_{\aleph_1}(Y)\}\). For \(\bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y))\) we define the function \(G_{\bar{r}} : \mathcal{F}_{\bar{r}} \rightarrow \omega(\alpha(*))\) by \(G_{\bar{r}}(f) = \langle \text{rk}_{D_{r_n}}(f) : n < \omega \rangle\).

Next

\[(*)_2 (\alpha) \ \bar{G} = \{G_{\bar{r}} : \bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y))\}\) exists
\n(\(\beta\)) \(G_{\bar{r}}\) is a function from \(\mathcal{F}_{\bar{r}}\) to \(\omega(\alpha(*))\)
\n(\(\gamma\)) \(G_{\bar{r}}\) is one to one.

[Should be clear, e.g. for \((*)_2(\gamma)\) read the definition of \(\mathcal{F}_{\bar{r}}\) and clause (\(\beta\)) of Theorem \([\text{DC}]\).]

Let \(<_*\) be a well ordering of \(\omega(\alpha(*))\) and for \(\bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y))\) let \(<_\bar{r}\) be the following two place relation on \(\mathcal{F}_{\bar{r}}:\)

\[\text{(*)}_3 \ f_1 <_\bar{r} f_2 \iff G_{\bar{r}}(f_1) <_* G_{\bar{r}}(f_2).\]

Obviously

\[(*)_4 (\alpha) \ \langle <_\bar{r} : \bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y)) \rangle\) exists
\n(\(\beta\)) \(<_\bar{r}\) is a well ordering of \(\mathcal{F}_{\bar{r}}\).

By \((*)_1 + (*)_4\) we have proved clauses (a),(b),(b)+ of the conclusion. Now clause (c) follows: for non-empty \(S \subseteq Y, \lambda\), let \(f(S) = \min\{\text{otp}\{g : g <_\bar{r} f\}, <_* : \bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y))\} \in \omega(\text{Fil}_{\aleph_1}(Y))\) and \(f \in \mathcal{F}_0 \cap S\). Also for any ordinal \(\gamma\) let \(\mathcal{W}_\gamma := \{f : \text{for some } \bar{r} \in \omega(\text{Fil}_{\aleph_1}(Y)) \text{ we have } \gamma = \text{otp}\{g : g <_\bar{r} f\}, <_*\}\) and \(\mathcal{W}_\gamma = \mathcal{W}_\gamma \cap \cup \{\mathcal{W}_\beta : \beta < \gamma\}\).

Lastly, we let \(F(S) = \mathcal{W}_f(S) \cap S\). Now check.

2) Similarly.

Proof. Proof of Theorem \([\text{DC}]\) First

\(\oplus_1\) there are a cardinal \(\mu\) and a sequence \(\bar{u} = \langle u_\alpha : \alpha < \mu \rangle\) listing \([\lambda]^{\aleph_0}\).

[Why? By the assumption.]

Second, we can deduce

\(\oplus_2\) there are \(\mu_1 < \mu\) and a sequence \(\bar{u} = \langle u_\alpha : \alpha < \mu_1 \rangle\) such that:

(a) \(u_\alpha \in [\lambda]^{\aleph_0}\)

(b) if \(u \in [\lambda]^{\aleph_0}\) then for some finite \(w \subseteq \mu_1, u \subseteq \cup \{u_\beta : \beta \in w\}\)

(c) \(u_\alpha\) is not included in \(u_{\alpha_0} \cup \ldots \cup u_{\alpha_n} \) when \(n < \omega, \alpha_0, \ldots, \alpha_n < \alpha\).
[Why? Let $\hat{u}^0$ be of the form $(u_\alpha : \alpha < \alpha^*)$ such that $(a) + (b)$ holds and $\ell g(\hat{u}^0)$ is minimal; it is well defined and $\ell g(\hat{u}^0) \leq \mu$ by $\preceq_1$. Let $W = \{ \alpha < \ell g(\hat{u}^0) : u_\alpha \notin \cup \{u_\beta : \beta \in w\} \text{ when } w \subseteq \alpha \text{ is finite} \}$. Let $\mu_1 = |W|$ and let $f : \mu_1 \to W$ be one-to-one onto, let $u_\alpha = u^{(\alpha)}$ so $(u_\alpha : \alpha < \mu_1)$ satisfies $(a) + (b)$ and $\mu_1 = |W| \leq \ell g(\hat{u}^0)$. So by the choice of $\hat{u}^0$ we have $\ell g(\hat{u}^0) = \mu_1$. So we can choose $f$ such that it is increasing hence $\hat{u}$ is as required.]

$\odot_3$ we can define $n : [\lambda]^{R_0} \to \omega$ and partial functions $F_\ell : [\lambda]^{R_0} \to \mu_1$ for $\ell < \omega$ (so $(F_\ell : \ell < \omega)$ exists) as follows:

(a) $u$ infinite $\Rightarrow F_0(u) = \min \{ \alpha : \text{for some finite } w \subseteq \alpha, u \subseteq u_\alpha \cup \{ u_\beta : \beta \in w \} \mod \text{finite} \}$

(b) $u$ finite $\Rightarrow F_0(u)$ undefined

(c) $F_{\ell+1}(u) := F_0(u \setminus (u_{F_\ell(u)} \cup \ldots \cup u_{F_\ell(u)}))$ for $\ell < \omega$ when $F_\ell(u)$ is defined

(d) $n(u) := \min \{ \ell : F_\ell(u) \text{ undefined} \}$.

Then

$\odot_4$ (a) $F_{\ell+1}(u) < F_\ell(u) < \mu_1$ when they are well defined

(b) $n(u)$ is a well defined natural number and $u \setminus \{u_{F_\ell(u)} : \ell < n(u)\}$ is finite and $k < n(u) \Rightarrow (u \setminus \{u_{F_\ell(u)} : \ell < k\}) \cap u_{F_\ell(u)}$ is infinite

(c) if $u_1, u_2 \in [\lambda]^{R_0}, u_1 \subseteq u_2$ and $u_2 \setminus u_1$ is finite then $F_\ell(u_1) = F_\ell(u_2)$ for $\ell < n(u_1)$ and $n(u_1) = n(u_2)$

$\odot_5$ define $F_* : [\lambda]^{R_0} \to \lambda$ by $F_*(u) = \min \{ \cup \{u_{F_\ell(u)} : \ell < n(u)\} \cup \{0\} \setminus u \}$ if well defined, zero otherwise

[Note: the reader may wonder: if you add $\{0\}$ then $\min(\cdot) = 0$ in all cases. However, if $0 \in u$ then by "$\cup u", zero does not belong to the set from which which we choose a minimal ordinal.]

$\odot_6$ if $u \in [\lambda]^{R_0}$ then

(a) $cF^0(u, F_*^0) = \ell F_\ell(u)$ is $F'(u) := u \cup \{u_{F_\ell(u)} : \ell < n(u)\} \cup \{0\}$

(b) $cF^0_\ell(u) = \ell F_\ell(u)$ for some $\varepsilon(u) < \omega_1$

(c) there is $\tilde{F} = \langle F'_\ell : \varepsilon \in \omega_1 \rangle$ such that: for every $u \in [\lambda]^{R_0}, cF^\varepsilon_\ell(u) = \{F'_\ell(u) : \varepsilon < \varepsilon(u)\}$ and $F'_\ell(u) = 0$ if $\varepsilon \in \varepsilon(u), \omega_1$

(d) in fact $F'_\ell(u)$ is the $\varepsilon$-th member of $cF^\varepsilon_\ell(u)$ if $\varepsilon < \varepsilon(u)$.

[Why? Define $w^\varepsilon_u$ by induction on $\varepsilon$ by $w^0_u = u, w^{\varepsilon+1}_u = w^\varepsilon_u \cup \{F_*(w^\varepsilon_u)\}$ and for limit ordinal $\varepsilon$ we let $w^\varepsilon_u = \cup \{w^\zeta_u : \zeta < \varepsilon\}$. We can prove by induction on $\varepsilon$ that $w^\varepsilon_u \subseteq F'(u)$ which is countable. The partial function $g$ with domain $F'(u)\setminus u$ to Ord, $g(\alpha) = \min \{ \varepsilon : \alpha < w^{\varepsilon+1}_u \}$ is one to one onto an ordinal call it $\varepsilon(\ast)$, so $w^{\varepsilon(\ast)}_u \subseteq F'(u)$ and if they are not equal that $F_*(w^{\varepsilon(\ast)}_u) \in F'(u)\setminus w^{\varepsilon(\ast)}_u$ hence $w^{\varepsilon(\ast)}_u \not\subseteq w^{\varepsilon(\ast)+1}_u$ contradicting the choice of $\varepsilon(\ast)$. So clause (a) holds. In fact, $cF^0(u, F_*) = w^{\varepsilon(\ast)}_u$ and clause (b) holds. C.Lauses (c), (d) should be clear.]

$\odot_7$ there is no sequence $\langle \mathcal{U}_n : n < \omega \rangle$ such that:

(a) $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n \subseteq \lambda$

(b) $\mathcal{U}_n$ is closed under $F_*$, i.e. $u \in [\mathcal{U}_n]^{|\mathcal{U}_n|} \Rightarrow F_*(u) \in \mathcal{U}_n$

(c) $\mathcal{U}_{n+1} \neq \mathcal{U}_n$.}
[Why? Assume toward contradiction that $\langle \mathcal{U}_n : n < \omega \rangle$ satisfies clauses (a),(b),(c). Let $\alpha_n = \operatorname{Min}(\mathcal{U}_n \setminus \mathcal{U}_{n+1})$ for $n < \omega$ hence the sequence $\alpha = \langle \alpha_n : n < \omega \rangle$ is well defined with no repetitions and let $\beta_{m,\ell} := F_\ell(\alpha_n : n \geq m\}$ for $m < \omega$ and $\ell < n_m := n(\alpha_n : n \in [m,\omega))$. As $\alpha$ is with no repetition, $n_m > 0$ and by $\oplus_4(c)$ clearly $n_m = n_0$ for $m < \omega$ and $\beta_{m,\ell} = \beta_{m,0}$ for $m < \omega, \ell < n_0$. So letting $v_m = \cup \{u_{F_\ell(\alpha_n, n \in [m,\omega))} : \ell < n_m \}$, it does not depend on $m$ so $v_m = v_0$, and by the choice of $F_\ell$, as $\{\alpha_n : n \in [m,\omega]\} \subseteq \mathcal{U}_m$ and $\mathcal{U}_m$ is closed under $F_\ell$ clearly $v_m \subseteq \mathcal{U}_m$. Together $v_0 = v_m \subseteq \mathcal{U}_m$ so $v_0 \subseteq \cap \{\mathcal{U}_m : m < \omega\}$. Also, by the definition of the $F_\ell$'s, $\{\alpha_n : n < \omega\}\setminus v_0$ is finite so for some $k < \omega$, $\{\alpha_n : n \in [k,\omega]\} \subseteq v_0$ but $v_0 \subseteq \mathcal{U}_{k+1}$ contradicting the choice of $\alpha_k$.]

Moreover, recalling Definition $\oplus_7(6)$:

$\oplus_7$ there is no sequence $\langle \mathcal{U}_n : n < \omega \rangle$ such that
(a) $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n \subseteq \lambda$
(b) $\mathcal{U}_n \not= \emptyset$.

[Why? As above but letting $\alpha_n = \operatorname{Min}(\mathcal{U}_n \setminus \mathcal{U}_n \cup \mathcal{U}_m \setminus \mathcal{U}_n)]$.

Now we define for $(D_1, D_2, h, Z) \in \mathcal{F}(Y)$ and ordinal $\alpha$ the following, recalling Definition $\oplus_7(6)$ for clauses (e),(f):

$\oplus_8 \mathcal{F}(D_1, D_2, h, Z, \alpha) = \{ f : (a) f$ is a function from $Z$ to $\lambda$
(b) $\rho_{D_1+Z}(f \cup 0_{Y \setminus Z}) = \alpha$
(c) $D_2 = \{ Y \setminus X : X \subseteq Y$ satisfies $X = \emptyset$ mod $D_1$
or $X \in D_1^f$ and $\rho_{D_1+X}(f \cup 0_{Y \setminus Z}) > \alpha$
that is $\rho_{D_1+X}(f) > \alpha$,
(d) $Z \subseteq D_2$, really follows
(c) if $Z' \subseteq Z$ and $Z' \subseteq D_2$ then
$\rho_{D_1^f}(\operatorname{Rang}(f \setminus Z')) = \rho_{D_1^f}(\operatorname{Rang}(f))$
(f) $y \in Z \Rightarrow f(y) = the h(y)-th member of \rho_{D_1^f}(\operatorname{Rang}(f))$.]

So we have:

$\oplus_9 \mathcal{F}(D_1, D_2, h, Z, \alpha)$ has at most one member; call it $f(D_1, D_2, h, Z, \alpha$ (when defined; pedantically we should write $f(D_1, D_2, h, Z, \alpha))$ a well ordered set.

$\oplus_{10} \mathcal{F}(D_1, D_2, h, Z) := \cup \{ \mathcal{F}(D_1, D_2, h, Z, \alpha) : \alpha$ an ordinal $\} is a well ordered set.

[Why? Define $\langle(D_1, D_2, h, Z)$ by the $\alpha$'s, i.e. $f_1 < f_2$ if there are $\alpha_1 < \alpha_2$ such that
$f_\ell = f(D_1, D_2, h, Z, \alpha)$ for $\ell = 1, 2$.,]

$\oplus_{11}$ if $f : Y \to \lambda$ and $Z \subseteq Y$ then the set $\operatorname{Rang}(f \upharpoonright Z)$ has cardinality $< hrtg(Z)$.

[Why? By the definition of hrtg(–) this should be clear.]

$\oplus_{12}$ if $f : Z \to \lambda$ and $Z \subseteq Y$ then $\rho_{D_1^f}(\operatorname{Rang}(f)) \subseteq \lambda$ has cardinality $< hrtg([Z]^{<\omega})$ or is finite.

Why? If $\operatorname{Rang}(f)$ is countable more holds by $\oplus_1\beta$. Otherwise, by $\oplus_6(\beta)$ recalling Definition $\oplus_7(6)$ we have $\rho_{D_1^f}(\operatorname{Rang}(f)) = \operatorname{Rang}(f) \cup \{ F_\ell(u) : u \in [\operatorname{Rang}(f)]^{<\omega}\}$ and $\varepsilon < \omega_1$.

Let $\alpha(*)$ be minimal such that $\operatorname{Rang}(f) \cap \alpha(*)$ has order type $\omega_1$. Let $h_1, h_2 : \omega_1 \to \omega_1$ be such that $h_1(\varepsilon) < \max\{\varepsilon, 1\}$ and for every $\varepsilon_1, \varepsilon_2 < \omega_1$ there is $\zeta \in [\varepsilon_1 + \varepsilon_2 + 1, \omega_1)$ such that $h_1(\zeta) = \varepsilon_\ell$ for $\ell = 1, 2$. Define $F : [Z]^{<\omega} \to \lambda$ as follows: if
\( u \in [\text{Rang}(f)]^{\aleph_0}, \) let \( \varepsilon_\ell(u) = h_\ell(\text{otp}(u \cap \alpha *)) \) for \( \ell = 1, 2 \) and \( F(u) = F'_\varepsilon(u) \{ \alpha \in u: \) if \( \alpha < \alpha(*) \) then \( \text{otp}(u \cap \alpha) < \varepsilon_1(u) \} \).

Now

- \( u \in [\text{Rang}(f)]^{\aleph_0} \) then \( F(u) = F_\varepsilon(v) \) for some \( v \in [Z]^{\aleph_0} \) and \( \varepsilon < \omega_1 \).

[Why? As \( F(u) \in \text{Rang}(F'_\varepsilon(u) \{ [\text{Rang}(f)]^{\aleph_0} \}) \)]

- \( \{ F(u) : u \in [\text{Rang}(f)]^{\aleph_0} \} \subseteq c\ell_{\aleph_1}(\text{Rang}(f)) \).

[Why? By \( \bullet_1 \) recalling \( \aleph_6 \).]

- \( u \in [\text{Rang}(f)]^{\aleph_0} \) and \( \varepsilon < \omega_1 \) then \( F'_\varepsilon(u) = F(u) \) for some \( v \in [\text{Rang}(f)]^{\aleph_0} \).

[Why? Let \( \varepsilon_1 = \text{otp}(u \cap \alpha(*)), \varepsilon_2 = \varepsilon \); now let \( \zeta < \omega_1 \) be such that \( h_\ell(\zeta) = \varepsilon_\ell \) for \( \ell = 1, 2 \). Let \( v = u \cup \{ \alpha : \alpha \in \text{Rang}(f) \cap \alpha(\varepsilon) \) and \( \alpha \geq \sup(u \cap \alpha(\varepsilon)) + 1 \) and \( \text{otp}(\text{Rang}(f) \cap \alpha)(\text{sup}(u \cap \alpha(\varepsilon) + 1)) < (\zeta - \varepsilon_1) \} \].

So \( F(u) = F'_\varepsilon(u) \). By \( \bullet_2 + \bullet_3 \) we can conclude:

- \( \bullet_4 \) in \( \bullet_2 \) we have equality.

Together \( c\ell_{\aleph_1}(\text{Rang}(f)) = \{ F(u) : u \in [\text{Rang}(f)]^{\aleph_0} \} \cup \text{Rang}(f) \) so it is the union of two sets; by the definition of \( \text{hrtg}(-) \) the first is of cardinality \( < \text{hrtg}([Z]^{\aleph_0}) \) and the second is of cardinality \( < \text{hrtg}[Z] \), so we are easily done proving \( \odot_{12} \)

\( \odot_{13} \) if \( f : Y \rightarrow \lambda \) then for some sequence \( \langle (\eta_n, \alpha_n) : n < \omega \rangle \) we have \( \eta_n \in \text{Fil}^4_{\aleph_1}(Y) \) and \( \alpha_n \in \text{Ord} \) for \( n < \omega \) and \( f = \cup \{ (\eta_n, \alpha_n) : n < \omega \} \).

[Why? Let:

\[ \mathcal{I}_f^0 = \{ Z \subseteq Y : \) for some \( \eta \in \text{Fil}^4_{\aleph_1}(Y) \) satisfying \( Z^\eta = Z \) and ordinal \( \alpha, f_{\eta, \alpha} \) is well defined and equal to \( f \mid Z \} \]

\[ \mathcal{I}_f = \{ Z \subseteq Y : Z \) is included in a countable union of members of \( \mathcal{I}_f^0 \}. \]

So recalling we are assuming DC it is enough to show that \( Y \in \mathcal{I}_f \).

Toward contradiction assume not. Let \( D_1 = \{ Y \setminus Z : Z \in \mathcal{I}_f \} \), clearly it belongs to \( \text{Fil}_{\aleph_1}(Y) \), noting that \( \emptyset \in \mathcal{I}_f \). So \( \alpha(*) := \text{rk}_{D_1}(f) \) is well defined (by \( \odot_{10} \)) recalling that only DC = DC_{\aleph_0} is needed.

Let

\( D_2 = \{ X \subseteq Y : X \in D_1 \text{ or rk}_{D_1 \cup (Y \setminus X)}(f) > \alpha(*) \} \).

By \( \odot_{12} \) clearly \( D_2 \) is an \( \aleph_1 \)-complete filter on \( Y \) extending \( D_1 \).

Now we try to choose \( Z_n \in D_2 \) for \( n < \omega \) such that \( Z_{n+1} \subseteq Z_n \) and \( c\ell_{\aleph_1}(\text{Rang}(f \mid Z_{n+1})) \) does not include \( \text{Rang}(f \mid Z_n) \).

For \( n = 0, Z_0 = Y \) is O.K.

By \( \odot_7 \) we cannot have such \( \omega \)-sequence \( \langle Z_n : n < \omega \rangle \); so by DC for some (unique) \( n = n(*) \), \( Z_n \) is chosen but not \( Z_{n+1} \).

Let \( h : Z_n \rightarrow \text{hrtg}([Y]^{\aleph_0}) \cup \omega_1 \) be:

\[ h(y) = \text{otp}(f(y) \cap c\ell_{\aleph_1}(\text{Rang}(f \mid Z_n))). \]
Now $h$ is well defined by $\otimes_{12}$. Easily
\[ f \upharpoonright Z_n \in \mathcal{F}(D_1+Z_n,D_2,h),Z_a,\alpha(\ast) \]
hence $Z_n \in \mathcal{F}^\alpha \subseteq \mathcal{F}_f$, contradiction to $Z_n \in D_2, D_1 \subseteq D_2$.
So we are done proving $\otimes_{13}$.

Now clause $(\beta)$ of the conclusion holds by the definition of $\mathcal{F}_b$, clause $(\alpha)$ holds by $\oplus_{10}$ recalling $\otimes_8, \otimes_9$ and clause $(\gamma)$ holds by $\otimes_{12}$. \[ \square \]

**Remark 1.5.** We can improve $\frac{1}{6}$.2 in some way by weakening the demands on $u$.

We may replace the assumption “$[\lambda]^{\aleph_0}$ is well ordered” by:

$(\ast)$ there is $(u_\alpha : \alpha < \alpha^*)$, a sequence of members of $[\lambda]^{\aleph_0}$ such that $(\forall u \in [\lambda]^{\aleph_0})(\exists \alpha)(u \cap u_\alpha$ infinite).

[Why? We define $F_z : [\lambda]^{\aleph_0} \to \alpha^*$ by induction on $\varepsilon < \omega_1$ by $F_z(v) := \min\{\alpha < \alpha^* : (v \setminus \varepsilon) \cap u_\alpha$ infinite $\}$ if well defined and let $F : [\lambda]^{\aleph_0} \to [\lambda]^{\aleph_0}$ be defined by $F(v) = \cup F_z(v) : \varepsilon < \omega_1, F_z(v)$ well defined$].$

Lastly, let $F_z(u) = \min(F(u) \setminus u)$.]

**Observation 1.6.**

1) The power of $\text{Fil}_{\aleph_1}^4(Y,\mu)$ is smaller or equal to the power of the set $(\mathcal{P}(\mathcal{P}(Y)))^2 \times \mathcal{P}(Y) \times \mu^{|Y|}$; if $\aleph_0 \leq |Y|$ this is equal to the power of $\mathcal{P}(\mathcal{P}(Y)) \times Y \mu$.

2) The power of $\text{Fil}_{\aleph_2}^4(Y)$ is smaller or equal to the power of the set $(\mathcal{P}(\mathcal{P}(Y)))^2 \times \mathcal{P}(Y) \times \mu^{|Y|}$.

3) In part (2), if $\aleph_0 \leq |Y|$ this is equal to $|\mathcal{P}(\mathcal{P}(Y))| \times \mu^{|Y|}$ if $\text{hrt}_{\mathcal{P}}(Y)$.

also $\alpha < \text{hrt}_{\mathcal{P}}(Y)$, $\Rightarrow |\mathcal{P}(\mathcal{P}(Y))| \times \mu^{|Y|}$ and $|\text{Fil}_{\aleph_2}^4(Y)| \leq_{\mu} \mathcal{P}(\mathcal{P}(Y) \times Y)$.

**Remark 1.7.**

1) As we are assuming DC, the case $\aleph_0 \not\leq |Y|$ means that $Y$ is finite, so degenerated. Also if $|Y| = \aleph_0$ then $\text{Fil}_{\aleph_1}^4(Y) = \{|X \subseteq Y : X \supseteq X\} : X \subseteq Y\}$ hence $|\text{Fil}_{\aleph_1}^4(Y)| = \mathcal{P}(Y)$ hence $\text{FIL}_{\aleph_1}^4(Y,\mu)$ has the same power as $\mathcal{P}(Y) \times \omega \mu$ again this is a dull case.

**Proof.**

1) Reading the definition of $\text{Fil}_{\aleph_1}^4(Y,\mu)$ clearly its power is $\leq$ the power of $\mathcal{P}(\mathcal{P}(Y)) \times \mu^{|Y|}$. If $\aleph_0 \leq |Y|$ then $|\mathcal{P}(\mathcal{P}(Y)) \times \mathcal{P}(Y)| \leq |\mathcal{P}(\mathcal{P}(Y)) \times \mu^{|Y|}| \leq 2^{|\mathcal{P}(Y)|} \times 2^{|\mathcal{P}(Y)|} \leq 2^{|\mathcal{P}(Y)|+|\mathcal{P}(Y)|} = 2^{|\mathcal{P}(Y)|}$, better, for $f$ a function from $[\lambda]^{\aleph_0}$ onto $\alpha$ let $\mathcal{P}(Y) \times \mathcal{P}(Y) \times \mu^{|Y|}$ as $\mathcal{P}(Y) + \mathcal{P}(Y) = 2^{|Y|} \times 2 = 2^{|Y|+1} = 2^{|Y|}$; so the second conclusion follows.

2) Read the definitions.

3) If $\alpha < \text{hrt}_{\mathcal{P}}(Y)$ then let $f$ be a function from $[\lambda]^{\aleph_0}$ onto $\alpha$ and for $\beta < \alpha$ let $A_{f,\beta} = \{u \in [\lambda]^{\aleph_0} : f(u) < \beta\}$. So $\beta \mapsto A_{f,\beta}$ is a one-to-one function from $\alpha$ onto $\{A_{f,\gamma} : \gamma < \alpha\} \subseteq \mathcal{P}(\mathcal{P}(Y))$ so $|\mathcal{P}(\mathcal{P}(Y)) \times \mu^{|Y|}| \leq \mathcal{P}(\mathcal{P}(Y)) \times |\mathcal{P}(\mathcal{P}(Y)) \times \mu^{|Y|}| \leq 2^{|\mathcal{P}(Y)|+|\mathcal{P}(Y)|} = 2^{|2^{|\mathcal{P}(Y)|}|}$. Better, for $f$ a function from $[\lambda]^{\aleph_0}$ onto $\alpha < \mathcal{P}(Y)$ let $A_f = \{(y_1,y_2) : f(y_1) < f(y_2)\} \subseteq Y \times Y$. Define $F : \mathcal{P}(Y) \times Y \to \text{hrt}_{\mathcal{P}}(Y)$ by $F(A) = \alpha$ if $A = A_f$ and $f, \alpha$ are as above, and $F(A) = 0$ otherwise.

So $|\mathcal{P}(\mathcal{P}(Y)) \cup \cup \{1^{|Y|} \alpha : \alpha < \text{hrt}_{\mathcal{P}}(Y)\}| \leq_{\mu} |\mathcal{P}(\mathcal{P}(Y)) \times \mathcal{P}(\mathcal{P}(Y) \times Y)| = |\mathcal{P}(\mathcal{P}(Y) \times Y)|$. By the above proof we easily get $|\text{Fil}_{\aleph_1}^4(Y)| \leq_{\mu} |\mathcal{P}(\mathcal{P}(Y) \times Y)|$. \[ \square \]
6.1 Claim 1.8. \[DC\] Assume

(a) \(a\) is a countable set of limit ordinals
(b) \(<_a\) is a well ordering of \(\Pi a\)
(c) \(\theta \in a \Rightarrow cf(\theta) \geq \kappa\) where \(\kappa = hrtg(\mathcal{P}(\omega))\) or just \(\Pi a/\kappa^{<\kappa}\) is \(<\kappa\)-directed.

Then we can define \((J, b, f)\) such that

(a) \(i J = \langle J_i : i \leq i(*)\rangle\) where \(i(*) < hrtg(\mathcal{P}(\omega))\)
(ii) \(J_i\) is an ideal on \(a\) (though not necessarily a proper ideal)
(iii) \(J_i\) is increasing continuous with \(i, J_0 = \{\emptyset\}, J_i(\ast) = \mathcal{P}(a)\)
(iv) \(b = \{b_i : i < i(*)\}, b_i \subseteq a\) and \(J_{i+1} = J_i + b_i\)
(v) so \(J_i\) is the ideal on \(a\) generated by \(\{b_j : j < i\}\)

(\(\beta\)) (i) \(f = \langle f^i : i < i(*)\rangle\)
(ii) \(f^i = \langle f^i_\alpha : \alpha < \alpha_i\rangle\)
(iii) \(f^i_\alpha \subseteq \prod a\) is \(<_{J_i}\)-increasing with \(\alpha < \alpha_i\)
(iv) \(\langle f^i_\alpha : \alpha < \alpha_i\rangle\) is cofinal in \((\prod a, <_{J_i(a \setminus b_i)}\rangle\)

(\(\gamma\)) (i) \(cf(\prod a) \leq \sum_{i < i(*)} \alpha_i\)
(ii) for every \(f \in \Pi a\) for some \(n\) and finite set \(\{i_\ell, \gamma_\ell\} : \ell < n\) such that \(i_\ell < i(*)\) and \(\gamma_\ell < \alpha_i\) for \(\ell < n\) we have \(f < \max_{i < n} f^i_{\gamma_\ell}\) i.e., \((\forall \theta \in a)(\exists \ell < n)[f(\theta) < f^i_{\gamma_\ell}(\theta)]\).

Remark 1.9. Note that there is no harm in having more than one occurrence of \(\theta \in a\). See more in [She16], e.g. on uncountable \(a\).

Proof. Note that:

\(\odot_1\) clause \((\gamma)\) follows from \((\alpha) + (\beta)\).

[Why? Easily \((\gamma)(ii) \Rightarrow (\gamma)(i)\). Now let \(g \in \Pi a\) and let \(I_g = \{b \subseteq a : \text{we can find } n < \omega \text{ and } i_\ell < i(*)\) and \(\beta_\ell < \alpha_i\) for \(\ell < n\) such that \(\theta \in b \Rightarrow (\exists \ell < n)(g(\theta) < f^i_{\gamma_\ell}(\theta))\).

Easily \(I_g\) is an ideal on \(a\) though not necessarily a proper ideal. Note that if \(a \in I_g\) we are done. So assume \(a \notin I_g\). Note that \(I_g \subseteq J_i(\ast)\) hence \(j_g = \min\{i \leq i(*)\) some \(c \in \mathcal{P}(a) \setminus I_g\) belongs to \(J_i\) is well defined (as \(a \in \mathcal{P}(a) \setminus I_g \land c \in J_i(a)\). As \(J_0 = \{\emptyset\}\) and clearly if \(\emptyset \in J_g\) we have \(j_0 > 0\). As \(J_i : i < i(*)\) is \(<\_\ast\)-increasing continuous, necessarily \(j_g\) is a successor ordinal say \(j_g = i_g + 1\) and let \(i(g) = i_g\) and choose \(c \in J_{i(g)} \setminus I_g\) clearly \(J_{i(g)} \subseteq I_g\) so \(c \in J_{i(g)} \setminus I_g\) by clause \((\beta)(iv)\) there is \(\alpha < \alpha_{i(g)}\) such that \(g < f^i_{\gamma_\ell}\) mod \((J_{i(g)} + (a \setminus b_{i(g)})))\)

Now let \(d = \{\theta \in a : g(\theta) < f^i_{\gamma_\ell}(\theta)\}\) so by the choice of \(\alpha\) we have \(d = a\) mod \((J_{i(g)} + (a \setminus b_{i(g)}))\) which means that \(b_{i(g)} \subseteq d\) mod \(J_{i(g)}\) so as \(J_{i(g)+1} = J_{i(g)} + b_{i(g)}\) and \(c \in J_{i(g)+1} \setminus J_{i(g)}\) clearly \(c \subseteq b_{i(g)}\) mod \(J_{i(g)}\).

But by the definition of the ideal \(J_{i(g)}\) and of \(d\) necessarily \(d \in J_{i(g)}\) and recall \(J_{i(g)} \subseteq J_{i(g)}\) (contradicting the conclusion of the last sentence.)

Since \((\gamma)\) follows from \((\alpha) + (\beta)\), it suffices to prove these parts. By induction on \(i < \kappa\) we try to choose \((J, b, f)\) where \(J = \langle J_j : j \leq i\rangle, b^j = \{b^j_i : j < i\}, f^j = \langle j^j : j < i\rangle\) which satisfies the relevant parts of the conclusion and do it uniformly from \((a, <, \ast)\). Once we arrive at \(i\) such that \(J_i = \mathcal{P}(a)\) we are done.
For $i = 0$ recalling $J_0 = \{0\}$ there is no problem.

For $i$ limit recalling that $J_i = \bigcup \{J_j : j < i\}$ there is no problem and note that if $j < i \Rightarrow a \notin J_j$ then $a \notin J_i$.

So assume that $(J^*, b^*, \bar{\theta}^*)$ is well defined and $a \notin J_i$ and we shall define for $i + 1$.

We try to choose $g_{\beta \xi}^i = \langle g_{\alpha \xi}^i : \alpha < \delta_{i, \xi} \rangle$ and $b_{i, \xi}$ by induction on $\xi < \omega_1$ and for each $\xi$ we try to choose $g_{\alpha \xi}^i \in \Pi \alpha$ by induction on $\alpha$ (in fact $\alpha < \hrtg(\Pi \alpha)$ suffice, we shall get stuck earlier) such that:

\[ \circ \quad (a) \quad \text{if } \beta < \alpha \text{ then } g_{\beta \xi}^i \subseteq J_i, g_{\alpha \xi}^i \]

\[ \circ \quad (b) \quad \text{if } \zeta < \varepsilon \text{ and } \alpha < \delta_{i, \zeta} \text{ then } g_{\alpha \zeta}^i \subseteq g_{\alpha \varepsilon}^i \]

\[ \circ \quad (c) \quad \text{if } \cf(\alpha) = \aleph_1 \text{ then } g_{\alpha \xi}^i \text{ is defined by } \]

\[ \theta \in a \Rightarrow g_{\alpha \xi}^i(\theta) = \min\{ \bigcup_{\beta \in C} g_{\beta \xi}^i(\theta) : C \text{ is a club of } \alpha \} \]

\[ \circ \quad (d) \quad \text{if } \alpha \text{ is a limit ordinal and } \cf(\alpha) \neq \aleph_1, \alpha \neq 0 \text{ then } g_{\alpha \varepsilon}^i \text{ is the } \varepsilon\text{-first } \in \Pi \alpha \text{ satisfying clauses (a) + (b)} \]

\[ \circ \quad (e) \quad \text{if we have } \langle g_{\beta \xi}^i : \beta < \alpha \rangle, \cf(\alpha) > \aleph_1, \text{ moreover } \cf(\alpha) \geq \min\{ \cf(\theta) : \theta \in a \} \text{ and there is no } g \text{ as required in clause (d) then } \delta_{i, \varepsilon} = \alpha \]

\[ \circ \quad (f) \quad \text{if } \alpha = 0 \text{ or } \alpha \text{ is a successor, then } g_{\alpha \varepsilon}^i \text{ is the } \varepsilon\text{-first } \in \Pi \alpha \text{ such that: } \]

\[ \circ \quad 1. \quad \zeta < \varepsilon \land \alpha < \delta_{i, \zeta} \Rightarrow g_{\alpha \zeta}^i \subseteq g \]

\[ \circ \quad 2. \quad \beta < \alpha \Rightarrow g_{\beta \xi}^i < g_{\alpha \xi}^i \text{ mod } J_i \]

\[ \circ \quad 3. \quad \varepsilon = \zeta + 1 \Rightarrow (\forall \beta < \delta_{i, \zeta}) \langle \beta \leq \xi, g_{\beta \xi}^i \rangle, \text{ follows if } \alpha > 0 \]

\[ \circ \quad (g) \quad J_i \text{ is the ideal on } \mathcal{P}(\alpha) \text{ generated by } \{ b_j : j < i \} \]

\[ \circ \quad (h) \quad b_{i, \xi} \in (J_i)^+ \text{ so } b_{i, \xi} \subseteq a \]

\[ \circ \quad (i) \quad g_{\xi}^i \text{ is increasing and cofinal in } (\Pi \alpha, <_{J_i + (a \setminus b_{i, \xi})}) \]

\[ \circ \quad (j) \quad b_{i, \xi} \text{ is such that under clauses (h) + (i) the set } \{ \otp(a \cap \theta) : \theta \in b_{i, \xi} \} \text{ is } \varepsilon\text{-minimal} \]

\[ \circ \quad (k) \quad b_{i, \zeta} \subseteq b_{i, \xi} \text{ mod } J_i \text{ (follows by } "\text{if } \zeta < i \text{ then } g_{\alpha \zeta}^i \text{ is a } <_{J_i + b_{i, \zeta}} \text{-upper bound of } g_{\zeta}^{\varepsilon, i}\).} \]

Clearly in stage $\varepsilon$ we first choose $g_{\alpha \varepsilon}^i$ by induction on $\alpha$. As $\beta < \alpha \Rightarrow g_{\beta \varepsilon}^i \neq g_{\alpha \varepsilon}^i$ we are stuck in some $\delta_{i, \varepsilon}$ and then choose $b_{i, \varepsilon}$.

We now give details on some points:

\[ \ast \quad \text{if } \alpha = 0 \text{ then we can choose } g_{\alpha \varepsilon}^i. \]

[Why? Trivial.]

\[ \ast_1 \quad \text{Clause (c) is O.K., that is: if we arrive to } (\varepsilon, \alpha), \cf(\alpha) = \aleph_1 \text{ then we can define } g_{\alpha \varepsilon}^i. \]

[Why? We already have $\langle g_{\alpha \varepsilon}^i : \alpha < \delta \rangle$ and $\langle g_{\alpha \varepsilon}^i : \alpha < \delta_{i, \xi} \rangle$, and we define $g_{\alpha \varepsilon}^i$ as there. Now $g_{\delta \varepsilon}^i(\theta)$ is well defined as the “Min” is taken on a non-empty set of ordinals as we are assuming $\cf(\delta) = \aleph_1$. The value is $< \theta$ because for some club $C$ of $\delta$, $\otp(C) = \omega_1$, so $g_{\delta \varepsilon}^i(\theta) \subseteq \bigcup\{ g_{\beta \varepsilon}^i(\theta) : \beta \in C \}$ but this set is $\subseteq \theta$ while $\cf(\theta) > \aleph_1$ by clause (c) of the assumption. By $\AC_{\aleph_0}$ we can find a sequence $\langle C_\theta : \theta \in a \rangle$ such that: $C_\theta$ is a club of $\delta$ of order type $\omega_1$ satisfying $g_{\delta \varepsilon}^i(\theta) = \bigcup\{ g_{\alpha \varepsilon}^i(\theta) : \alpha \in C_\theta \}$]
We succeed to carry the induction, i.e. choose \( g^{i,\varepsilon}_\delta(\theta) = \bigcup_{\alpha \in C} g^{i,\varepsilon}_\alpha(\theta) \) when \( C := \cap\{C_\sigma : \sigma \in a\} \), because \( C \) too is a club of \( \delta \) recalling \( a \) is countable. So if \( \alpha < \delta \) then for some \( \beta \) we have \( \alpha < \beta \in C \) hence the set \( \varepsilon := \{\theta \in a : g^{i,\varepsilon}_\alpha(\theta) \geq g^{i,\varepsilon}_\delta(\theta)\} \) belongs to \( J_i \) and \( \theta \in a \setminus \varepsilon \Rightarrow g^{i,\varepsilon}_\alpha(\theta) < g^{i,\varepsilon}_\delta(\theta) \leq g^{i,\varepsilon}_\beta(\theta) \), so indeed \( g^{i,\varepsilon}_\alpha <_J g^{i,\varepsilon}_\delta \).

Lastly, why \( \zeta < \varepsilon \Rightarrow g^{i,\varepsilon}_\delta \leq g^{i,\varepsilon}_\zeta \)? As we can find a club \( C \) of \( \delta \) which is as above for both \( g^{1,\zeta}_\delta \) and \( g^{1,\zeta}_\delta \) and recall that clause (b) of \( \otimes_{i,\varepsilon} \) holds for every \( \beta \in C \). Together \( g^{i,\varepsilon}_\delta \) is as required.]

\[(*)_2\] \( \text{cf}(\delta_{i,\varepsilon}) > N_1 \) and even \( \text{cf}(\delta_{i,\varepsilon}) \geq \min\{\text{cf}(\theta) : \theta \in a\} \).

[Why? We have to prove that arriving to \( \alpha > 0 \), if \( \text{cf}(\alpha) < \min\{\text{cf}(\theta) : \theta \in a\} \) then we can choose \( g^{i,\varepsilon}_\alpha \) as required. The cases \( \text{cf}(\alpha) = N_1, \alpha = 0 \) are covered by \((*)_1, (*)_0\) respectively, otherwise let \( u \subseteq \alpha \) be unbounded of order type \( \text{cf}(\alpha) \), and define a function \( g \) from \( a \) to the ordinals by \( g(\theta) = \sup\{\{g^{i,\varepsilon}_\beta(\theta) : \beta \in u\} \cup \{g^{i,\varepsilon}_\alpha(\theta) : \zeta < \varepsilon\}\}. \)

This is a subset of \( \theta \) of cardinality \( <|a| + \text{cf}(\alpha) \) which is \( < \theta = \text{cf}(\theta) \) hence \( g \in \Pi a \), easily is as required, i.e. satisfies clauses (a) + (b) and the \( \vartriangleleft_{s, \text{first}} \) such \( g \) is \( g^{i,\varepsilon}_\alpha \).

Note that clause (e) of \( \otimes_{i,\varepsilon} \) follows.

\[(*)_3\] if \( \zeta < \varepsilon \) then \( \delta_{i,\varepsilon} \leq \delta_{i,\zeta}. \)

[Why? Otherwise \( g^{i,\varepsilon}_{\delta_{i,\zeta}} \) contradict clause (e) of \( \otimes_{i,\zeta} \).]

\[(*)_4\] if \( g^{i,\varepsilon}_\alpha = \langle g^{i,\varepsilon}_{\delta_{i,\varepsilon}} : \alpha < \delta_{i,\varepsilon} \rangle \) is well defined and \( \text{cf}(\delta_{i,\varepsilon}) \) then \( g^{i,\varepsilon}_{\delta_{i,\varepsilon}} = \text{cf}(\delta_{i,\varepsilon}) \).

[Why? Clearly it suffices to prove that there is \( b \) as required on \( b_{i,\varepsilon} \) (in clauses (b),(i)). So toward contradiction assume that for every \( b \in J_i^* \), \( g^{i,\varepsilon}_b \) is not \( <_{J_i} \)-cofinal in \( \Pi a \) hence there is \( h \in \Pi a \) such that \( \alpha < \delta_{i,\varepsilon} \Rightarrow h \not\in g^{i,\varepsilon}_b \) and let \( h_\alpha \) be the \( <_{s, \text{minimal}} \) such \( h \). Let \( h_\alpha \) be the function with domain \( a \) such that \( h(\theta) = \cup\{h_\theta(\theta) + 1 : b \in J_i^*\}. \)

As \( \text{hrtg}(J_i^*) \leq \text{hrtg}(\mathcal{P}(\alpha)) < \min\{\text{cf}(\theta) : \theta \in a\} \), clearly \( h_\alpha \in \Pi a \). Now for \( \alpha < \delta_{i,\varepsilon} \) let \( \mathcal{A}_{i,\varepsilon,\alpha} = \{\theta \in a : g^{i,\varepsilon}_\alpha(\theta) \leq h_\alpha(\theta)\} \). So \( \mathcal{A}_{i,\varepsilon,\alpha}/J_i : \alpha < \delta_{i,\varepsilon} \) is \( \leq \)-increasing in the Boolean Algebra \( \mathcal{P}(\alpha)/J_i \), so for some \( \beta_{i,\varepsilon} \leq \delta_{i,\varepsilon} \) we have \( \alpha \in \langle \beta_{i,\varepsilon}, \delta_{i,\varepsilon} \rangle \Rightarrow \mathcal{A}_{i,\varepsilon,\alpha} = \mathcal{A}_{i,\varepsilon,\beta_{i,\varepsilon}} \) \( \text{mod} J_i \). This implies \( \mathcal{A}_{i,\varepsilon} \) can serve as \( b_{i,\varepsilon} \).

To finish consider the following two cases.

Case 1: We succeed to carry the induction, i.e. choose \( g^{i,\varepsilon}_\delta \) for every \( \varepsilon < \kappa \).

So \( b_{i,\varepsilon} : \varepsilon < \kappa \) is a sequence of subsets of \( a \), pairwise distinct (by \( \otimes_{i,0}^2 \) clauses (g) + (b)), but \( \kappa \geq \text{hrtg}(\mathcal{P}(\omega)) \) and \( a \) is countable; contradiction.

Case 2: We are stuck in \( \varepsilon < \kappa \).

For \( \varepsilon = 0 \) there is no problem to define \( g^{i,0}_\alpha \) by induction on \( \alpha \) till we are stuck, say in \( \alpha \), necessarily \( \alpha \) is of large enough cofinality \( \geq \kappa \) by \((*)_2\), and so \( g^{i,\varepsilon}_\alpha \) is well defined. We then prove \( b_{i,\varepsilon} \) exists by \((*)_4\) again using \( \vartriangleleft_{s, \text{first}} \).

For \( \varepsilon \) limit we can also choose \( g^{i,\varepsilon}_\delta \).

For \( \varepsilon = \zeta + 1 \), if \( a \in J_i \), then we are done; otherwise \( g^{i,\varepsilon}_\alpha \) as required can be chosen by \((*)_0\), and then we can prove that \( g^{i,\varepsilon}_\delta, b_{i,\varepsilon} \) exists as above.
Remark 1.10. From §6.1 we can deduce bounds on $hrtg(Y(R_\delta))$ when $\delta < R_1$ and more like the one on $R_\omega^{\aleph_0}$ (better the bound on $pp(N_\omega)$).
§ 2. No decreasing sequence of subalgebras

In this section we concentrate on weaker axioms. We consider Theorem \( \text{II}^2 \) under weaker assumptions than \( \langle \lambda \rangle_{\aleph_0} \) is well orderable. We are also interested in replacing \( \omega \) by \( \theta \) in “no decreasing \( \omega \)-sequence of \( cl \)-closed sets”, but the reader may consider \( \theta = \aleph_0 \) only. Note that for the full version, \( Ax^4_\alpha \), i.e., \( \langle \alpha \rangle^0 \) is well orderable, the case of \( \theta = \aleph_0 \) is implied by the \( \theta > \aleph_0 \) version and suffices for the results. But for other versions, the axioms for different \( \theta \)’s seem incomparable.

Note that if we add many Cohens (not well ordering them) then \( Ax^1_\alpha \) fails below even for \( \theta = \aleph_0 \), whereas the other axioms are not affected. But forcing by \( \aleph_1 \)-complete forcing notions preserve \( Ax_4 \).

**Hypothesis 2.1.** DC\( \theta \) and let \( \theta(*) = \theta + \aleph_1 \). Actually we use only DC in \( \text{II}^2(1) \) and DC\( \theta \) in \( \text{II}^2(3) \) and the later claims. We fix a regular cardinal \( \theta \).

**Definition 2.2.** Below we should, e.g. write \( Ax^\ell \beta \) instead of \( Ax^\ell \) and assume \( \alpha > \mu > \kappa \geq \theta \). If \( \kappa = \theta \) we may omit it.

1) \( Ax^\ell_{\alpha, \mu, \kappa} \) means that there is a weak closure operation on \( \ell \) of character \( (\mu, \kappa) \), see Definition 7.17(1A), such that there is no \( \subseteq \)-decreasing \( \theta \)-sequence \( \langle \mathcal{U}_\varepsilon : \varepsilon < \theta \rangle \) of subsets of \( \alpha \) with \( \varepsilon < \theta \Rightarrow cl(\mathcal{U}_{\varepsilon+1}) \not\subseteq \mathcal{U}_\varepsilon \). We may here and below replace \( \kappa \) by \( < \kappa \); similarly for \( \mu \) let \( \varepsilon < |Y|^{+} \) means \( |Y| \).

2) Let \( Ax^\ell_{\alpha, < \mu, \kappa} \) mean there is a function \( cl : [\alpha]^{< \kappa} \to [\alpha]^{< \mu} \) such that \( \omega \cup \{0\} \subseteq cl(u) \) and there is no \( \subseteq \)-decreasing sequence \( \langle \mathcal{U}_\varepsilon : \varepsilon < \theta \rangle \) of members of \( [\alpha]^{< \kappa} \) such that \( \varepsilon < \theta \Rightarrow cl(\mathcal{U}_{\varepsilon+1}) \not\subseteq \mathcal{U}_\varepsilon \).

2A) Writing \( Y \) instead of \( \kappa \) means \( cl : [\alpha]^{< hrtg(Y)} \to [\alpha]^{< \mu} \). Let \( cl(\beta) : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha) \) be \( cl^1_{\varepsilon, \varepsilon < \reg(\gamma^+)} \) as defined in 7.17(4) recalling \( \reg(\gamma) = \min\{\chi : \chi \text{ a regular cardinal } \geq \gamma\} \).

3) \( Ax^2_\alpha \) means that there is \( A \subseteq [\alpha]^{<0} \) which is well orderable and for every \( u \in [\alpha]^0 \) for some \( v \in A \), \( u \cap v \) has power = \( \theta \).

4) \( Ax^3_\alpha \) means that \( cl([\alpha]^{< \beta}, \subseteq) \) is below some cardinal, i.e., some cofinal \( A \subseteq [\alpha]^{< \beta} \) (under \( \subseteq \)) is well orderable.

5) \( Ax^4_\alpha \) means that \( [\alpha]^{< \beta} \) is well orderable.

6) Above omitting \( \alpha \) (or writing \( \infty \)) means “for every \( \alpha \)”, omitting \( \mu \) we mean \( \langle < hrtg(\mathcal{P}(\theta)) \rangle^\uparrow \).

7) Lastly, let \( Ax_\ell = Ax^\ell \) for \( \ell = 1, 2, 3 \).

So easily (or we have shown in the proof of \( \text{II}^2 \)):

**Claim 2.3.** 1) \( Ax^3_\alpha \) implies \( Ax^3_\alpha \), \( Ax^3_\alpha \) implies \( Ax^2_\alpha \), \( Ax^2_\alpha \) implies \( Ax^1_\alpha \) and \( Ax^4_\alpha \) implies \( Ax^0_\alpha \). Similarly for \( Ax^1_{\alpha, < \mu, \kappa} \).

2) In Definition 7.17(2), the last demand, if \( cl \) has monotonicity, then only \( cl \mid [\alpha]^{< \beta} \) is relevant, in fact, an equivalent demand is that if \( \langle \beta_\varepsilon : \varepsilon < \theta \rangle \subseteq \alpha \) then for some \( \varepsilon, \beta_\varepsilon \in cl\{\beta_\zeta : \zeta \in (\varepsilon, \theta)\} \).

3) If \( Ax^0_{\alpha, < \mu_1, < \theta} \) and \( \theta \leq hrtg(Y) \) and \( 2 \mu_2 = sup\{hrtg(\mu_1 \times [\beta]^0) : \beta < hrtg(Y)\} \) then \( Ax^0_{\alpha, < \mu_2, < hrtg(Y)} \).

**Proof.** 1) Clearly \( Ax^2_{\alpha, < \mu, \kappa} \Rightarrow Ax^1_{\alpha, < \mu, \kappa} \) holds similarly to the proof of \( \text{II}^2 \); the other implications hold by inspection.

2) First assume that we have a \( \subseteq \)-decreasing sequence \( \langle \mathcal{U}_\varepsilon : \varepsilon < \theta \rangle \) such that \( \varepsilon < \theta \Rightarrow cl(\mathcal{U}_{\varepsilon+1}) \not\subseteq \mathcal{U}_\varepsilon \). Let \( \beta_\varepsilon = min\{\mathcal{U}_\varepsilon \cap cl(\mathcal{U}_{\varepsilon+1})\} \) for \( \varepsilon < \theta \) so clearly

\[ \text{Can do somewhat better; we can replace } [\alpha]^{< \mu_1} \text{ by } \{v \subseteq \alpha : otp(v) \subseteq \mu_1\} \]
Claim 2.4. Assume $\epsilon < \partial$ exists; so by monotonicity $\cl\{\beta_\zeta : \zeta \in [\epsilon + 1, \partial]\} \subseteq \cl([\mathcal{U}_\zeta + 1])$ hence $\beta_\zeta \notin \cl\{\beta_\zeta : \zeta \in [\epsilon + 1, \partial]\}$.

Second, assume that $\beta = \langle \beta_\zeta : \zeta < \partial \rangle \in \partial\alpha$ satisfies $\beta_\zeta \notin \cl\{\beta_\zeta : \zeta \in [\epsilon + 1, \partial]\}$ for $\epsilon < \partial$. Now letting $\mathcal{U}_\zeta' = \{\beta_\zeta : \zeta < \partial \text{ satisfies } \epsilon \leq \zeta\}$ for $\epsilon < \partial$ clearly $\langle \mathcal{U}_\zeta' : \epsilon < \partial \rangle$ exists, is $\subseteq$-decreasing and $\epsilon < \partial \Rightarrow \beta_\zeta \notin \cl([\mathcal{U}_\zeta + 1]) \wedge \beta_\zeta \in \mathcal{U}_\zeta'$. So we have shown the equivalence.

3) Let $\cl(-)\alpha^{\mathfrak{A}}_{\mu,\gamma,\theta}$. We define the function $\cl'$ with domain $[\alpha]^{\mathfrak{A}}_{\mathfrak{H}(\mathcal{V})}$ by $\cl'(u) = \bigcup\{\cl(v) : v \subseteq u \text{ has cardinality } \theta\}$.

Now

$(*)$ $\cl'$ is a function from $[\alpha]^{\mathfrak{A}}_{\mathfrak{H}(\mathcal{V})}$ into $[\alpha]^{\mathfrak{A}}_{\mathfrak{H}(\mathcal{V})\mathfrak{A}}$.

For this it is enough to note:

$(*)_1$ if $u \in [\alpha]^{\mathfrak{A}}_{\mathfrak{H}(\mathcal{V})}$ then $\cl'(u)$ has cardinality $\mu_2 := \sup\{\mathfrak{H}(\mu_1 \times [\beta]^{\mathfrak{A}}_{\mathfrak{H}(\mathcal{V})} : \beta < \mathfrak{H}(\mathcal{V})\})$.

[Why? Let $C_u = \{(v, \epsilon) : v \subseteq u \text{ has cardinality } \theta \text{ and } \epsilon < \mathfrak{H}(\mathcal{V})\}$ which is $\subseteq<\mu_1$. Clearly $\cl'(u) < \mathfrak{H}(C_u)$ and $|C_u| = |\mu_1 \times [\mathfrak{H}(\mathcal{V})^{\mathfrak{B}}|\mu_1|$, so $(*)_1$ holds.]

Note that if $\alpha_\epsilon < \mu_1^+$ we can replace the demand $v \in [u]^{\mathfrak{A}}_{\mathfrak{H}(\mathcal{V})} \Rightarrow \cl'(v) < \mu_1$ by $\epsilon < \mu_1$ by

$(*)_2$ If $(u, \epsilon) < \partial$ is $\subseteq$-decreasing where $u_\epsilon \subseteq \alpha$ then $u_\epsilon \subseteq \cl'(u_{\epsilon+1})$ for some $\epsilon < \partial$.

[Why? If not we can choose a sequence $(\beta_\zeta : \zeta < \partial)$ by letting $\epsilon < \partial \Rightarrow \beta_\zeta = \min(u_\epsilon \cl'(u_{\epsilon+1})).$ Let $u_\epsilon' = \{\beta_\zeta : \zeta \in [\epsilon, \partial]\}$. $u_\epsilon' < \partial$ is $\subseteq$-decreasing by the choice of $\cl(-)$ for some $\epsilon, \beta_\zeta < \cl\{\beta_\zeta : \zeta \in (\epsilon + 1, \partial]\}$, but this set is $\subseteq \cl'(u_{\epsilon+1})$ by the definition of $\cl'(\epsilon)$, so we are done.]

Claim 2.4. Assume $\cl$ witness $\mathfrak{A}_{\mathfrak{A}}^{\mathfrak{A}}_{\gamma,\mu,\kappa}$ so $\partial \leq \kappa < \mu$ and so $\cl : [\alpha]^{\kappa} \rightarrow [\alpha]^{\mu}$ and recall $\cl_1^{\kappa} : \mathcal{P}(\alpha) \rightarrow \mathcal{P}(\alpha)$ is from (2.57). 1) $\cl_1^{\kappa}$ is a weak closure operation, it has character $(\mu_\kappa, \kappa)$ whenever $\partial \leq \kappa \leq \alpha$ and $\mu_\kappa = \mathfrak{H}(\mu \times \mathcal{P}(\kappa)),$ see Definition (2.7).

2) $\cl_1^{\mathfrak{A}(\gamma^\mathfrak{A}) \kappa}$ is a closure operation and it has character $(< \mu_\kappa, \kappa)$ when $\partial \leq \kappa \leq \alpha$ and $\mu_\kappa = \mathfrak{H}(\mathcal{H}_{<\mathfrak{A}(\gamma)\kappa}(\mu \times \kappa))$.

Proof. 1) By its definition $\cl_1^{\kappa}$ is a weak closure operation.

Assume $u \subseteq \alpha, |u| \leq \kappa$; non-empty for simplicity. Clearly $\mu \times |u|^{\mathfrak{A}}\gamma$ has the same power as $\mu \times |u|^{\mathfrak{A}}\gamma$ def 3 the function $G$ with domain $\mu \times |u|^{\mathfrak{A}}\gamma$ as follows: if $\alpha < \mu$ and $v \in [u]^{\mathfrak{A}}\gamma$ then $G(\alpha, v)$ is the $\alpha$-th member of $\cl(\alpha)$ if $\alpha < \mathfrak{H}(\mathcal{V}(v))$ and $G(\alpha, v) = \min(u)$ otherwise.

So $G$ is a function from $\mu \times [u]^{\mathfrak{A}}\gamma$ onto $\cl_1^{\kappa}(u)$. This proves that $\cl_1^{\kappa}$ has character $(< \mu_\kappa, \kappa)$ as $\mu_\kappa = \mathfrak{H}(\mu \times \mathcal{P}(\kappa))$.

2) $\mu_\kappa = \mathfrak{H}(\mu \times \mathcal{P}(\kappa))$ is an increasing continuous sequence of sets then $|u_{\gamma^\mathfrak{A}}|^{\mathfrak{A}}\gamma = \cup\{[u_\gamma|^{\mathfrak{A}}\gamma : \gamma < \mathfrak{H}(\gamma^\mathfrak{A})\}$ as $\mathfrak{H}(\gamma^\mathfrak{A})$ is regular (even of cofinality $> \partial$ suffice) by its definition, note $\mathfrak{H}(\gamma^\mathfrak{A}) = \partial^+$. This proves that $\cl_1^{\kappa}$ is a closure operation.

Second, let $u \subseteq \alpha, |u| \leq \kappa$ and let $u_{\gamma^\mathfrak{A}} = \cl_1^{\kappa}(u)$ for $\epsilon \leq \partial^+$; it is enough to show that $|u_{\gamma^\mathfrak{A}}| < \mu_\kappa$. The proof is similar to earlier one.
Definition/Claim 2.5. Let $\ell$ exemplify $\text{Ax}_{\lambda, <\mu, Y}$ and $Y$ be an uncountable set such that $\partial(f) \leq_{\text{st}} Y$.

1) Let $\mathcal{F}_\sigma, \mathcal{F}_{\sigma, \alpha}$ be as in the proof of Theorem \[\ref{t.4}\] for $\eta \in \text{Fil}_{\sigma(\alpha)}(Y, \mu)$ and ordinal $\alpha$ (they depend on $\lambda$ and $\ell$ but note that $\ell$ determines $\lambda$, so if we derive $\ell$ by $\text{Ax}_{\lambda}^4$ then they depend indirectly on the well ordering of $[\lambda]^\eta$) so we may write $\mathcal{F}_{\sigma, \alpha} = \mathcal{F}_\sigma(\alpha, \ell)$, etc.

That is, fully

\[\ast\] for $\eta \in \text{Fil}_{\sigma(\alpha)}(Y, \mu)$ and ordinal $\alpha$ let $\mathcal{F}_{\sigma, \alpha}$ be the set of $f$ such that:

(a) $f$ is a function from $Z^\eta$ to $\lambda$
(b) $\text{rk}_{D[\alpha]}(f) = \alpha$ recalling that this means $\text{rk}_{D_1^\alpha + Z^\eta}(f \cup 0 \setminus Z^\eta) = \alpha$ by Definition \[\ref{r.4}\]
(c) $D_2^\eta = D_1^\eta \cup \{Y \setminus A : A \in J[f, D_1^\eta]\}$, see Definition \[\ref{t.2}\]
(d) $Z^\eta \in D_2^\eta$
(e) if $Z \in D_1^\eta$ and $Z \subseteq Z^\eta$ then $\ell(f(y) : y \in Z) \geq \{f(y) : y \in Z^\eta\}$
(f) $h^\eta$ is a function with domain $Z^\eta$ such that $y \in Z^\eta \Rightarrow h^\eta(y) = \text{otp}(f(y) \cap \{f(z) : z \in Z^\eta\})$

(\[\ast\]) $\mathcal{F}_\sigma = \bigcup\{\mathcal{F}_{\sigma, \alpha} : \alpha \text{ an ordinal}\}$.

2) Notice that $\mathcal{F}_{\sigma, \alpha}$ is a singleton or the empty set. Let $\Xi_\sigma = \Xi_\sigma(\ell) = \Xi_\sigma(\lambda, \ell) = \{\alpha : \mathcal{F}_{\sigma, \alpha} \neq \emptyset\}$ and $f_{\sigma, \alpha}$ is the function $f \in \mathcal{F}_{\sigma, \alpha}$ when $\alpha \in \Xi_\sigma$; it is well defined.

3) If $D \in \text{Fil}_{\sigma(\alpha)}(Y)$, $\text{rk}_{D}(f) = \alpha$ and $f \in \Upsilon_\lambda$ then $\alpha \in \text{Fil}_{\sigma(\lambda, \ell)}(f)$ and $f \upharpoonright Z^\eta = f_{\sigma, \alpha}$ for some $\eta \in \text{Fil}_{\sigma(\lambda)}(Y)$; moreover, $(D_1^\eta, D_2^\eta) = (D, \text{dual}(J[f, D]))$ where $\Xi_{D}(\lambda, \ell) := \bigcup\{\Xi_\sigma : \eta \in \text{Fil}_{\sigma(\alpha)}(Y) \text{ and } D_1^\eta = D\}$.

4) If $D \in \text{Fil}_{\sigma(\alpha)}(Y), f \in \Upsilon_\lambda, Z \in D^+$ and $\text{rk}_{D+Z}(f) \geq \alpha$ then for some $g \in \prod_{Y \in Y} (f(g) + 1) \subseteq Y(\lambda + 1)$ we have $\text{rk}_{D}(g) = \alpha$ hence $\alpha \in \Xi_{D}(\lambda, \ell)$.

5) So we should write $\mathcal{F}_{\sigma, \ell}, \Xi_\sigma[\ell], f_{\sigma, \alpha}[\ell]$.

Proof. As in the proof of \[\ref{t.4}\] recalling “$\ell$ exemplifies $\text{Ax}_{\lambda, <\mu, hrtg(Y)}$ ” holds, this replaces the use of $F_\alpha$ there; and see the proof of \[\ref{t.2}\] below in part (3), for this we need:

\[\Xi\] if $D \in \text{Fil}_{\sigma}(Y)$ and $f \in \text{def}(\partial)$, then for some $Z \in D$ we have:

\begin{itemize}
  \item if $\Upsilon \subseteq Z$ belongs to $D$ then $\ell(\text{Rang}(f|\Upsilon)) = \ell(\text{Rang}(f|Z))$.
\end{itemize}

[Why $\Xi$ holds? By Definition \[\ref{r.2}\] using the axiom DC$_\partial$.]

Claim 2.6. We have $\xi_2$ is an ordinal and $\text{Ax}_{\xi_2, <\mu_2, Y}$ holds (note that $\mu_2$ is not much larger than $\mu_1$):

(a) $\text{Ax}_{\xi_2, <\mu_1, Y}$ so $\partial < \text{hrtg}(Y)$
(b) $\ell$ witnesses clause (a)
(c) $D \in \text{Fil}_{\sigma(\alpha)}(Y)$
(d) $\xi_2 = \{\alpha : f_{\sigma, \alpha}[\ell] \text{ is well defined for some } \eta \in \text{Fil}_{\sigma(\alpha)}(Y, \mu_1) \text{ which satisfies } D_1^\eta = D \text{ and necessarily } \text{Rang}(f_{\sigma, \alpha}[\ell]) \subseteq \xi_1\}$
(e) $\mu_2$ is defined as $\mu_{2,3}$ where:

\begin{itemize}
  \item (a) Let $\mu_{2,0} = \text{hrtg}(Y)$
Proof.

\(\xi\)

\(\alpha\)

\(\mu_2 = \sup_{\beta < \mu_2, 0} \text{hrtg}(\beta \times \text{Fil}^4_{[\alpha]}(Y, \mu_1))\)

(\(\gamma\)) \(\mu_2 = \sup_{\alpha < \mu_2, 1} \text{hrtg}(\mu_1 \times [\alpha] \leq \delta)\)

(\(\delta\)) \(\mu_2 = \sup \{\text{hrtg}(\gamma, \beta \times \text{Fil}_{[\beta]}(Y)) : \beta < \mu_2, 2\}\)

(this is an overkill).

\(\oplus_1\) \(\xi_2\) is an ordinal.

[Why? To prove that \(\xi_2\) is an ordinal we have to assume \(\alpha < \beta \in \xi_2\) and prove \(\alpha \in \xi_2\). As \(\beta \in \xi_2\) clearly \(\beta \in \Xi_\eta[\epsilon]\) for some \(\eta \in \text{Fil}^4_{[\alpha]}(Y, \mu_1)\) for which \(D^\eta_1 = D\) so there is \(f \in Y(\xi_1)\) such that \(f|Z^\eta \in \mathcal{F}_{\beta, \beta}\). So \(\text{rk}_{D^\eta + Z[\beta]}(f) = \beta\) hence by \(\Xi^{\beta, \beta}(4)\) there is \(g \in Y^\lambda\) such that \(g \leq f\), i.e., \((\forall y \in Y)(g(y) \leq f(y))\) and \(\text{rk}_{D^\eta + Z[\beta]}(g) = \alpha\). By \(\Xi^{\beta, \beta}(4)\) there is \(\mathcal{F} \in \text{Fil}^4_{[\beta]}(Y, \mu_1)\) such that \(D^\mathcal{F}_1 = D + Z[\mathcal{F}]\) and \(g|Z^\mathcal{F} \in \mathcal{F}_{\beta, \alpha}\) so we are done proving \(\xi_2\) is an ordinal.]

We define the function \(\epsilon\) with domain \([\xi_2]^{-\text{hrtg}}(Y)\) as follows:

\(\oplus_2\) \(\epsilon(u) = \{0\} \cup \{\alpha : \text{there is } \eta \in \text{Fil}^4_{[\epsilon]}(Y, \mu_1) \text{ such that } f_{\eta, \alpha}[\epsilon] \text{ is well defined and } \text{Rang}(f_{\eta, \alpha}[\epsilon]) \subseteq \epsilon(v[u])\}\)

where

\(\oplus_3\) \(v[u] := \cup\{\epsilon(v) : v \subseteq \xi_1\) is of cardinality \(\leq \partial\) and is \(\subseteq w(v)\}\).

Note that

\(\oplus_4\) for \(v \subseteq \xi_1\) we let \(w(v) = \cup\{\text{Rang}(f_{\mathcal{F}, \beta}[\epsilon]) : \mathcal{F} \in \text{Fil}^4_{[\beta]}(Y, \mu_1) \text{ and } \beta \in u \text{ and } f_{\mathcal{F}, \beta}[\epsilon] \text{ is well defined}\}\).

Note that (by \(\Xi^{\beta, \beta}(1)\)):

\(\Xi_1\) for each \(u \subseteq \xi_1\) and \(r \in \text{Fil}^4_{[\mathcal{F}]}(Y, \mu_1)\) the set \(\{\alpha < \xi_2 : f_{\mathcal{F}, \alpha}[\epsilon] \text{ is a well defined function into } u\} \) has cardinality \(<\text{wlor}(T_{D^\mathcal{F}_2}(u))\), that is, \(\langle f_{\mathcal{F}, \alpha}[\epsilon] : \alpha \in \Xi_\eta \cap \xi_2\rangle\) is a sequence of functions from \(Z^\mathcal{F}\) to \(u \subseteq \xi_1\) any two are equal only on a set \(= \emptyset \) mod \(D^\mathcal{F}_2\) (with choice it has cardinality \(<|Y| |u|\)) call this bound \(\mu_{u, \mathcal{F}}\).

Note

\(\Xi_2\) if \(u_1 \subseteq u_2 \subseteq \xi_2\) then

(\(\alpha\)) \(w(u_1) \subseteq w(u_2)\) and \(v(u_1) \subseteq v(u_2) \subseteq \xi_1\)

(\(\beta\)) \(\epsilon(u_1) \subseteq \epsilon(u_2)\)

(\(\gamma\)) \(u \subseteq v(u)\) and \(w[u] \subseteq w[u]\)

(\(\delta\)) \(u_1 \subseteq \epsilon(u_1)\).

\(\Xi^{\alpha, \beta}(4)\) We could have used \(\{t \in Y : f_{\eta, \alpha}[\epsilon](t) \in \epsilon(v(u))\} \neq \emptyset \) mod \(D^\mathcal{F}_2\); also we could have added \(u\) to \(\epsilon(u)\) but not necessarily by \(\Xi_2\).
[Why? E.g. for clause (δ); assume α ∈ u and let f be a unique function from Y into {α}. Hence for some η ∈ Fil_{δ(ε)}(Y, µ_1) we have f_{u,α} is well defined. Now Rang(f_{u,α}) ⊆ w(u) by the choice of w(u) in 3 and so Rang(f_{u,α}) ⊆ v(u) by clause (γ) of 2.2 hence Rang(f_{u,α}) ⊆ v(u) by the assumption on cl, see by 2.2.(b) and 2.2.(2). So we have f_{u,δ} well defined and Rang(f_{u,α}) ⊆ cl(v(u)) so by the definition of cl(u) in 2 we have α ∈ cl(u) so we are done.]

φ_3 if u ⊆ ξ_2, |u| < hrtg(Y) then w(u) = \{f_{u,α}(z) : α ∈ u, η ∈ Fil_{δ(ε)}(Y, µ_1), f_{u,α} is well defined and z ∈ Z^0\} is a subset of ξ_1 of cardinality < hrtg(|u| × Fil_{δ(ε)}(Y, µ_1)) by \sup\{hrtg(β) × Fil_{δ(ε)}(Y, µ_1) : β < hrtg(Y)\} which was named µ_2 in 2.2.(1) [Why? By clause (a) of the assumption of γ]

φ_4 if u ⊆ ξ_1 and |u| < µ_2 then \{c(α) : v ∈ [u]^{< β}\} is a subset of µ_1 of cardinality < hrtg(µ_1 × [u]^{< β}) ≤ \sup_{α < µ_2} hrtg(µ_1 × [α]^{< β}) which we call µ_2 in 2.2.(1) [Why? Without loss of generality v(u) ∈ 0. By φ_5 we have |c(α)| < hrtg(Y, v(u)) × Fil_{δ(ε)}(Y) and by φ_5 the latter is ≤ \sup_{α < µ_2} hrtg(µ_1 × [α]^{< β}) = µ_2, recalling clause (ε)(δ) of the claim, so we are done.]

φ_7 cl' is a very weak closure operation on λ and has character (< µ_2, hrtg(Y)).

[Why? In Definition 2.2.(1), clause (a) holds by the Definition of cl', clause (b) holds by φ_6 and as for clause (c), 0 ∈ cl'(u) by the definition of cl' and u ⊆ cl'(u) by clause (δ) of 2.2.]

Now it is enough to prove

φ_8 cl' witnesses A_{ξ_2, < µ_2, Y}.

Recalling φ_7, toward contradiction assume U = (U_ε : ε < δ) is \subseteq-decreasing, U_ε ∈ [ξ_1]^{hrtg(Y)} and ε < δ ⇒ U_ε ⊆ cl(U_{ε+1}). We define γ = (γ_ε : ε < δ) by

\[ γ_ε = \text{Min}(U_ε \setminus cl(U_{ε+1})). \]

As AC_0 follows from DC_0, we can choose (η_ε : ε < δ) such that f_{u,η_ε}[cl] is well defined for ε < δ.

Let for ε < δ

\[ u_ε = \{γ_ζ : ζ ∈ [ε, δ)\}. \]

So

\[ (\ast)_1 u_ε ∈ [ξ_1]^{< δ} ⊆ [ξ_1]^{< hrtg(Y)}. \]

[Why? By clause (a) of the assumption of 2.2.]

\[ (\ast)_2 u_ε is \subseteq-decreasing with ε. \]
Claim 2.7. Assume $\kappa < \kappa = \text{cf}(\lambda) < \lambda$ hence $\kappa$ is regular $\geq \partial$ of course, and $D$ is the club filter on $\kappa$ and $\lambda = (\lambda_i : i < \kappa)$ is increasing continuous with limit $\lambda$.

Then $\lambda^+ \leq \{ \text{rk}_D(\beta) : f \in \prod_{i < \kappa^+} \lambda_i^+ \}$.

Proof. For each $\alpha < \lambda^+$ there is a one to one function $g$ from $\alpha$ into $|\alpha| \leq \lambda$ and we let $f \in \prod_{i < \kappa} \lambda_i$ be

$$f(i) = \text{otp}(\{ \beta < \alpha : g(\beta) < \lambda_i \}).$$

Let

$$\mathcal{F}_\alpha = \{ f : \text{f is a function with domain } \kappa \text{ satisfying } i < \kappa \Rightarrow f(i) < \lambda_i^+ \text{ and } \sum_{i < \kappa} f(i) < \lambda^+ \}.$$  

Now

(*)$_3$ (a) $\mathcal{F}_\alpha \neq \emptyset$ for $\alpha < \lambda^+$

(b) $\langle \mathcal{F}_\alpha : \alpha < \lambda^+ \rangle$ exists as it is well defined

[Why? For clause (a) let $g : \alpha \rightarrow \lambda$ be one to one and so the function defined above belongs to $\mathcal{F}_\alpha$. For clause (b) see the definition of $\mathcal{F}_\alpha$ (for $\alpha < \lambda^+$).]

(*)$_2$ (a) if $f \in \mathcal{F}_{\beta^+}, \alpha < \beta < \lambda^+$ then for some $f' \in \mathcal{F}_\alpha$ we have $f' <_{\text{rk}_D} f$

(b) $\langle \text{min}\{ \text{rk}_D(f) : f \in \mathcal{F}_\alpha \} : \alpha < \lambda^+ \rangle$ is strictly increasing hence $\text{min}\{ \text{rk}_D(f) : f \in \mathcal{F}_\alpha \} \geq \alpha$.

[Why? For clause (a), let $g$ witness “$f \in \mathcal{F}_{\beta^+}$” and define the function $f' \in \prod_{i < \kappa} \lambda_i^+$ by $f'(i) = \text{otp}(\gamma < \alpha : g(\gamma) < \lambda_i)$. So $g\{ \alpha \text{ witness } f' \in \mathcal{F}_\alpha \text{, and letting } i(*) = \text{min}\{ i : g(\alpha) < \lambda_i \} \text{ we have } i \in [i(*)] \text{ hence } f' <_{\text{rk}_D} f \text{ as promised. For clause (b) it follows.}]

Note that

(*)$_3$ if $f \in \mathcal{F}_\alpha$ then, for part (2), for some $\eta \in \text{Fil}_{\text{rk}(f)}(\lambda, \text{cl})$ and $\beta \geq \alpha$ we have $f|Z[\eta] \in \mathcal{F}_{\beta, \eta}$.

[Why? By (*)$_1$ + (*)$_2$.]

So we have proved

\[\text{Proof of Claim 2.7}\]

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\[\text{Proof of Claim 2.7}\]
\textbf{Conclusion 2.8.} 1) Assume
\begin{enumerate}[(a)]
  \item \textit{Ax}^0_{\lambda,<\mu,\kappa},
  \item \(\lambda > \text{cf}(\lambda) = \kappa\) (not really needed in part (1)).
\end{enumerate}

Then for some \(\mathcal{F}_* \subseteq \kappa\lambda = \{f : f\text{ a partial function from } \kappa \text{ to } \lambda\}\) we have
\begin{enumerate}[(a)]
  \item every \(f \in \kappa\lambda\) is a countable union of members of \(\mathcal{F}_*\),
  \item \(\mathcal{F}_*\) is the union of \(|\text{Fil}^4_{\delta(y)}(\kappa, \mu)|\) well ordered sets:
    \(\\{\mathcal{F}^*_{\eta} : \eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu)\}\)
  \item moreover there is a function giving for each \(\eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa)\) a well ordering of \(\mathcal{F}^*_{\eta}\).
\end{enumerate}

2) Assume in addition that \(\text{hrtg}(\text{Fil}^4_{\delta(y)}(\kappa, \mu)) < \lambda, \text{cf}(\lambda^+)\) and \(\text{hrtg}(\kappa, \mu) < \lambda\) then for some \(\eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa)\) we have \(|\mathcal{F}^*_{\eta}| > \lambda\).

3) If in part (2) we omit the assumption on \(\text{cf}(\lambda^+)\) still \(\lambda^+ = \sup\{\text{otp}(\Xi \cap \lambda^+) : \eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu)\}\).

\textit{Proof.} 1) By the proof of 2.5.
2) Assume that this fails; so for every \(\eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu)\), the set \(S_\eta = \Xi_\eta \cap \lambda^+\) has order type \(< \lambda^+\). But we are assuming \(\text{cf}(\lambda^+) \geq \text{hrtg}(\text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu))\), so there is \(\gamma < \lambda^+\) such that \(\gamma > \text{otp}(S_\eta)\) for every relevant \(\eta\), without loss of generality \(\gamma > \lambda\) and let \(g\) be a one-to-one function from \(\gamma\) onto \(\lambda\).

We choose \(f \in \kappa\lambda\) by
\[
f(i) = \text{Min}(\lambda \setminus \{f_{\eta,\alpha}(i) : \eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu), f_{\eta,\alpha}(i) \text{ is well defined, i.e. } i \in Z[\eta] \text{ and } \alpha \in \Xi_\eta \text{ and } g(\text{otp}(\alpha \cap \Xi_\eta)) < \mu_i\}).
\]

Now \(f(i)\) is well defined as the minimum is taken over a non-empty set of ordinals, this holds as we substruct from \(\lambda\) a set which has cardinality \(\leq \mu_i\) which is \(< \lambda\).

But \(f\) contradicts part (1). Note that in fact \(f \in \prod_{i} \mu_i^+\). \(\square\)

3) Same proof as in part (2).

\textbf{Conclusion 2.9.} Assume \(\text{Ax}^0_{\lambda,<\mu,\kappa}\) so \(\lambda > \mu\). \(\lambda^+\) is not measurable (even in cases it is regular\(^6\)) when
\[
\exists (a) \quad \lambda > \text{cf}(\lambda) = \kappa > \aleph_0
\]
\[
(b) \quad \lambda > \text{hrtg}((\text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu)).
\]

\textit{Proof.} Naturally we fix a witness \(\text{cf}\) for \(\text{Ax}^0_{\lambda,<\mu,\kappa}\). Let \(\mathcal{F}_{\eta}, \Xi_{\eta}, f_{\eta,\alpha}, \mathcal{F}^\lambda_{\eta,\alpha}\) be defined as in 2.5 so by claims 2.5, 4.4 we have \(\cup\{\Xi_\eta : \eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa)\} \supseteq \lambda^+\); moreover, \(\alpha \in \lambda^+ \cap \Xi_{\eta} \Rightarrow f_{\eta,\alpha} \in \kappa\lambda\).

Let \(\eta \in \text{Fil}^4_{\bar{\delta}(\kappa)}(\kappa, \mu)\) be such that \(|\mathcal{F}_{\eta}| > \lambda\), we can find such \(\eta\) by 2.5, as without loss of generality we can assume \(\lambda^+\) is regular (or even measurable, toward contradiction). Let \(Z = Z[\eta]\). So \(\Xi_\eta\) is a set of ordinals of cardinality \(> \lambda\). For \(\zeta < \text{otp}(\Xi_\eta)\) let \(\alpha_\zeta\) be the \(\zeta\)-th member of \(\Xi_\eta\), so \(f_{\eta,\alpha_\zeta}\) is well defined. Toward

\(^6\)the regular holds many times by 2.2
contradiction let $D$ be a (non-principal) ultrafilter on $\lambda^+$ which is $\lambda^+$-complete. For $i \in Z$ let $\gamma_i < \lambda$ be the unique ordinal $\gamma$ such that $\{ \zeta < \lambda^+ : f_{\gamma_i,\alpha}(i) = \zeta \} \in D$. As $|Z| \leq \kappa < \lambda^+$ and $D$ is $\kappa^+$-complete clearly $\{ \zeta : \bigwedge_{i \in Z} f_{\gamma_i,\alpha}(i) = \zeta \} \in D$, so as $D$ is a non-principal ultrafilter, for some $\zeta_1 < \zeta_2, f_{\gamma_1,\alpha_{\zeta_1}} = f_{\gamma_2,\alpha_{\zeta_2}}$, contradiction. So there is no such $D$. \hfill $\blacksquare$

**Remark 2.10.** Similarly if $D$ is $\kappa^+$-complete and weakly $\lambda^+$-saturated and $\text{Ax}^0_{\lambda^+,<\mu}$, see [Sh:1005].

**Claim 2.11.** If $\text{Ax}^0_{\lambda^+,\mu,\kappa}$, then we can find $\bar{C}$ such that:

(a) $\bar{C} = \{ C_\delta : \delta \in S \} \cup \{ C_\delta : \delta \text{ is a limit ordinal of cofinality } \leq \bar{\delta}(\ast) \}$

(b) $S = \{ \delta < \lambda : \delta \text{ is an unbounded subset of } \delta \}

(c) $C_\delta$ is an unbounded subset of $\delta$, even a club

(d) if $\delta \in S$, $\text{cf}(\delta) \leq \kappa$ then $|C_\delta| < \mu$

(e) if $\delta \in S$, $\text{cf}(\delta) > \kappa$ then $|C_\delta| < \text{hrtg}(\mu \times [\text{cf}(\delta))]^\mu)$.

**Remark 2.12.**

1) Recall that if we have $\text{Ax}^0_{\lambda^+}$ (see [Sh:221, (5)]) then trivially there is $\{ C_\delta : \delta < \lambda, \text{cf}(\delta) \leq \bar{\delta}, C_\delta \text{ a club of } \delta \text{ of order type } \text{cf}(\delta) \}$ as $\kappa$ well order $[\lambda]^\alpha$ we let $C_\delta := \text{be the } \text{<}_\ast\text{-minimal } C$ which is a closed unbounded subset of $\delta$ of order type $\text{cf}(\delta)$.

2) $\text{Ax}^0_{\lambda^+,<\kappa} \ast \text{ suffices if } \kappa < \xi < \lambda$.

**Proof.** The “even a club” is not serious as we can replace $C_\delta$ by its closure in $\delta$.

Let $\text{cf}$ witness $\text{Ax}^0_{\lambda^+,\mu,\kappa}$. For each $\delta \in S$ with $\text{cf}(\delta) \in [\bar{\delta}(\ast),\kappa]$ we let

$$C_\delta = \cap \{ \delta \cap \text{cl}(C) : C \text{ a club of } \delta \text{ of order type } \text{cf}(\delta) \}.$$ 

Now $\bar{C} = \{ C_\delta : \delta \in S \text{ and } \text{cf}(\delta) \in [\bar{\delta}(\ast),\kappa] \}$ is well defined and exist. Clearly $C_\delta$ is a subset of $\delta$.

For any club $C$ of order type $\text{cf}(\delta) \in [\bar{\delta}(\ast),\kappa]$ clearly $\delta \cap \text{cl}(C) \subseteq \text{cl}(C)$ which has cardinality $< \mu$.

The main point is to show that $C_\delta$ is unbounded in $\delta$, otherwise we can choose by induction on $\varepsilon < \bar{\delta}$, a club $C_{\delta,\varepsilon}$ of $\delta$ of order type $\text{cf}(\delta)$, decreasing with $\varepsilon$ such that $C_{\delta,\varepsilon} \subseteq \text{cl}(C_{\delta,\varepsilon+1})$, we use DC$_\bar{\delta}$. But this contradicts the choice of $\text{cf}$ recalling Definition [Sh:221, (1)].

If $\delta < \lambda$ and $\text{cf}(\delta) > \kappa$ we let

$$C^*_\delta = \cap \{ \delta \cap \text{cl}(u) : u \subseteq C \text{ has cardinality } \leq \bar{\delta} : C \text{ is a club of } \delta \text{ of order type } \text{cf}(\delta) \}.$$ 

A problem is a bound of $|C^*_\delta|$. Clearly for $C$ a club of $\delta$ of order type $\text{cf}(\delta)$ the order-type of the set $\cup \{ \delta \cap \text{cl}(v) : v \subseteq C \text{ has cardinality } \leq \bar{\delta} \}$ is $< \text{hrtg}(\mu \times [\text{cf}(\delta)]^\mu)$. As for “$C^*_\delta$ is a club” it is proved as above.

The following lemma gives the existence of a class of regular successor cardinals.

**Lemma 2.13.** Assume

(a) $\delta$ is a limit ordinal $< \lambda^+$ with $\text{cf}(\delta) = \bar{\delta}$

(b) $\lambda^+_i$ is a cardinal for $i < \delta$ increasing with $i$
Clearly cardinal so $\delta$

$\text{Ax}_2$) Assume $Y := \text{otp}(\text{Fil}_4^{\delta}(\lambda^*_i, \mu))$ and $\text{hrtg}(\lambda^*_i)^\leq \leq \lambda^*_{i+1}$

Then $\lambda$ is a regular cardinal.

2) Assume $\text{Ax}_1, \lambda = \lambda^*_i, \lambda_\ast$ singular and $\chi < \lambda_\ast \Rightarrow \text{hrtg}(\delta^\lambda \chi) \leq \lambda_\ast$ then $\lambda$ is regular.

Remark 2.14. This says that the successor of many strong limit singulars is regular.

Question 2.15. 1) Is $\text{hrtg}(\mathcal{P}(\mathcal{P}(\lambda^*_i))) \geq \text{hrtg}(\text{Fil}_4^{\delta}((\lambda^*_i)))$?

2) Is $|\text{cf}(f \upharpoonright B)| \leq \text{hrtg}([B]^{<\delta})$ for the natural $\text{cf}$ and $f, B$ as in the proof of $\text{E.7}$?

Proof. 1) We can replace $\delta$ by $\text{cf}(\delta)$ so without loss of generality $\delta$ is a regular cardinal so $\delta = \emptyset$.

So

\[ (\ast)_1 \]

(a) fix $\text{cf} : [\lambda]^{<\aleph} \rightarrow \mathcal{P}(\lambda)$ a witness to $\text{Ax}^0_{\lambda, \mu, \kappa}$

(b) let $\langle C_\xi [\text{cf}] : \xi < \lambda, \text{cf}(\xi) \geq \emptyset \rangle$ be as in the proof of $\text{E.11}$, so $\xi < \lambda \land \emptyset \leq \text{cf}(\xi) < \lambda \Rightarrow |C_\xi [\text{cf}]| < \lambda$.

[Why the last inequality? If $\delta < \lambda^+$, then there is $i$ such that $\lambda^*_i > \mu + \text{cf}(\emptyset)$ hence $\text{otp}(C_\delta) < \text{hrtg}(\mu \times [\text{cf}(\emptyset)]^{\leq \aleph}) \leq \text{hrtg}(\lambda^*_i)^{\leq \aleph} < \lambda^*_{i+1}$]

First, we shall use just $\lambda > \lambda_\ast \land (\forall \emptyset < \lambda)(\text{cf}(\emptyset) < \lambda_\ast)$, a weakening of the assumption that $\lambda = \lambda^*_i$.

Now

$\mathfrak{E}_1$ for every $i < \delta$ and $A \subseteq \lambda$ of cardinality $\leq \lambda^*_i$, we can find $B \subseteq \lambda$ of cardinality $\leq \lambda_\ast$ satisfying $(\forall \alpha \in A)[\alpha$ is limit $\land \text{cf}(\alpha) \leq \lambda^*_i \Rightarrow \alpha = \sup(\alpha \cap B)]$.

The proof of this will take some time. By $\text{E.9}$, $\text{E.11}$ and $\text{E.12}$) the only problem is for $Y := \{\alpha : \alpha \in A, \alpha \geq \sup(A \cap \alpha), \alpha$ a limit ordinal of cofinality $< \emptyset + \emptyset\}$; so $|Y| \leq \lambda^*_i$. Note: if we assume $\text{Ax}_3^1$ this would be immediate.

We define $D$ as the family of sets $A \subseteq Y$ such that:

$\mathfrak{D}_1$ for some set $C \subseteq \lambda$ of $\leq \emptyset$ ordinals, the set $B_C := \cup\{\text{Rang}(f_\xi) : \xi \in \text{Fil}_4^{\delta}(\lambda^*_i, \mu) \land \xi \in C \lor for some \xi \in C, we have \lambda^*_i \geq \text{cf}(\xi) > \emptyset \land \xi \in C_\xi [\text{cf}]\}$ satisfies $\alpha \in Y \setminus A \Rightarrow \alpha = \sup(\alpha \cap B_C)$.

Clearly

\[ \mathfrak{D}_2 \]

(a) $Y \subseteq D$

(b) $D$ is upward closed

(c) $D$ is closed under intersection of $\leq \emptyset$ hence of $\leq \emptyset(\ast)$ sets.
A witness $\varepsilon$ if $A \subseteq Y$ belongs to $D$ then it is a witness for any $A' \subseteq Y$ such that $A \subseteq A'$. Lastly, for clause (c) if $A \in D$ for $\varepsilon < \varepsilon(\ast) < \partial^\ast$, we have $AC_\theta$, there is a sequence $\{C_z : \varepsilon < \varepsilon(\ast)\}$ such that $C_z$ witnesses $A_z \in D$ for $\varepsilon < \varepsilon(\ast) < \partial^\ast$, then $C := \cup\{C_z : \varepsilon < \varepsilon(\ast)\}$ witnesses $A := \cap\{A_z : \varepsilon < \varepsilon(\ast)\}$ and, again by $AC_\theta$, we have $|C| \leq \partial$.]

$\otimes_3$ if $\emptyset \in D$ then we are done.

[Why? For $a = \emptyset \in D$ let $C_{\emptyset} \subseteq \lambda$ be as promised in $\otimes_1$ and then $BC$ is as required; its cardinality $\leq \lambda_{\ast+1}$ by $\mathbb{F}$.

So assume $\emptyset \notin D$, so $D$ is an $\partial^\ast$-complete filter on $Y$. As we have covered both subcases, we have proved

$\therefore$ recalling ($\ast$), we have $\emptyset \in D$.

So $D_2$ is an $\partial^\ast$-complete filter on $\lambda_\ast^\ast$ extending $D_1$.

Let $B \subseteq D_2$ be such that $(\forall B')[B' \subseteq D_2 \land B' \subseteq B, \Rightarrow \epsilon((\text{Rang}(g \setminus B')) \subseteq (\text{Rang}(g \setminus B))]$. Let $\mathcal{U} = \cap\{\epsilon((\text{Rang}(g \setminus B')) : B' \in D_2\}$, so $\text{Rang}(g \setminus B_\ast) \subseteq \mathcal{U}$, even equal.

Let $h$ be the function with domain $B_\ast$ defined by $a \in B_\ast \Rightarrow h(a) = \text{otp}(g(a) \cap \mathcal{U})$.

So $\gamma := (D_1, D_2, B_\ast, h) \in \text{Fil}_\gamma^D(\lambda_\ast^\ast, \mu)$ and for some $\zeta$ we have $g \setminus B_\ast = f_{\gamma, \zeta}[\epsilon]$.

It suffices to consider the following two subcases.

**Subcase 1a:** $\text{cf}(\zeta) > \partial$.

So recalling ($\ast$), $C_\zeta[\epsilon]$ is well defined and let $C := \{\epsilon\}$ hence $BC = \cup\{\text{Rang}(f_{\gamma, \epsilon}[\epsilon]) : \epsilon \in C_\zeta[\epsilon]\}$ so $C$ exemplifies that the set $X := \{\epsilon \in Y : \zeta > \text{sup}(\alpha \cap BC)\}$ belongs to $D$ hence $X_\ast := \{\epsilon \in Y : \zeta > \text{sup}(\text{Rang}(g) \cap \mathcal{U})\}$ belongs to $D_1$.

Now define $g'$, a function from $\lambda_\ast^\ast$ to Ord by $g'(a) = \text{sup}(g(a) \cap \mathcal{U}) + 1$ if $a \in X_\ast$ and $g'(a) = 0$ otherwise. Clearly $g' < g$ mod $D_1$ hence $\text{rk}_{D_1}(g') < \zeta$, hence there is $g'' < g$ mod $D_1$ hence $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$, $\text{rk}_{D_1}(g'') < \zeta$ such that $\xi := \text{rk}_{D_1}(g'') \in C_\zeta[\epsilon]$.

Now for some $\eta \in \text{Fil}_\omega^D(\lambda_\ast^\ast)$ we have $\xi$ implies $\eta$ mod $D_\infty$.

So $B := \{\epsilon \in Y : g'(\epsilon) = f_{\eta, \xi}[\epsilon] + \epsilon\} \in D_\infty$ hence $B \in D_2$. So $B \cap B \ast \subseteq Y$ but if $\epsilon \in B \cap B_\ast \cap A$, then $f_{\eta, \xi}[\epsilon] = B \in D_2$ and $f_{\eta, \xi}[\epsilon] = \text{sup}(B \cap g(\epsilon))$, $g(\epsilon)$. This gives contradiction.

**Subcase 1b:** $\text{cf}(\zeta) \leq \partial$.

We choose a $C \subseteq \zeta$ of order type $\leq \partial$ unbounded in $\zeta$ and proceed as in subcase $1a$.

As we have covered both subcases, we have proved $\exists_1$.

Recall we are assuming $\delta = \partial$; now:

$\exists_2$ for every $A \subseteq \lambda$ of cardinality $\leq \lambda$, there is $B \subseteq \lambda$ of cardinality $\leq \lambda$, such that:

$\exists_2$ for every $A \subseteq \lambda$ of cardinality $\leq \lambda$, there is $B \subseteq \lambda$ of cardinality $\leq \lambda$, such that:
[Why? Choose a \( \subseteq \)-increasing sequence \( \langle A_j : j < \delta \rangle \) such that \( A = \cup \{ A_i : i < \delta \} \) and \( j < \delta \Rightarrow |A_j| \leq \lambda_j^+ \), possible as \( |A| \leq \lambda^* \). For each \( j < \delta \) there exists \( B_j \) such that the conclusion of \( \mathbb{B}_1 \) holds with \( (A_j, B_j, \lambda_j^+) \) here standing for \( (A, B, \lambda) \) there, so \( |B_j| \leq \lambda^* \). So as AC holds (as \( \delta \leq \delta \)) there is a sequence \( \langle B_j : j < \delta \rangle \), each \( B_j \) as above.

Lastly, let \( B = \cup \{ B_j : j < \delta \} \), it is as required.]

\[ \begin{align*}
\exists_3 & \text{ for every } A \subseteq \lambda \text{ of cardinality } \leq \lambda^* \text{ we can find } B \subseteq \lambda \text{ of cardinality } \leq \lambda^* \text{ such that } A \subseteq B, [\alpha + 1 \in B \Rightarrow \alpha \in B] \text{ and } [\alpha \in B \text{ is a limit ordinal } \& \text{ cf}(\alpha) < \lambda^* \Rightarrow \alpha = \sup(B \cap \alpha)].
\end{align*} \]

[Why? We choose \( B_i \) by induction on \( i < \omega \leq \delta \) such that \( |B_i| \leq \lambda^* \) by \( B_0 = A, B_{2i+1} = \{ \alpha : \alpha \in B_{2i} \text{ or } \alpha + 1 \in B_{2i+1} \} \) and \( B_{2i+2} \) is chosen as \( B \) was chosen in \( \mathbb{B}_2 \) for \( i \) with \( B_{2i+1}, B_{2i+2} \) here in the role of \( A, B \) there. There is such \( \langle B_i : i < \omega \rangle \) as \( DC = DC_{\kappa_\delta} \) holds. So easily \( B = \cup \{ B_i : i < \omega \} \) is as required.]

Now return to our main case \( \lambda = \lambda^* \)

\[ \exists_4 \lambda^+ \text{ is regular.} \]

[Why? Otherwise \( \text{cf}(\lambda^+) < \lambda^+ \) hence \( \text{cf}(\lambda^+) \leq \lambda^* \), but \( \lambda^* \) is singular so \( \text{cf}(\lambda^+) < \lambda^* \) hence there is a set \( A \) of cardinality \( \text{cf}(\lambda^+) < \lambda^*_A \) such that \( A \subseteq \lambda^* = \sup(A) \). Now choose \( B \) as in \( \exists_3 \). So \( |B| \leq \lambda^* \), \( B \) is an unbounded subset of \( \lambda^*_A \), \( \alpha + 1 \in B \Rightarrow \alpha \in B \) and if \( \alpha \in B \) is a limit ordinal then \( \text{cf}(\alpha) \leq |\alpha| \leq \lambda^*_A \), but \( \text{cf}(\alpha) \) is regular so \( \text{cf}(\alpha) < \lambda^*_A \) hence \( \alpha = \sup(B \cap \alpha) \). But this trivially implies that \( B = \lambda^*_A \), but \( |B| \leq \lambda^*_A \), contradiction.] 2) Similar, just easier.

**Remark 2.16.** Of course, if we assume \( \text{Ax}_\lambda \) then the proof of \( \exists_{\mathbb{B}_2} \lambda^+_\lambda \) is much simpler: if \( <^* \) is a well ordering of \( [\lambda]^\leq \delta \) for \( \delta < \lambda \) of cofinality \( < \delta \) let \( C_\delta = \{ <^* \text{-first closed unbounded subset of } \delta \text{ of order type } \text{cf}(\delta) \} \), see \( \exists_{\mathbb{B}_3} \).

**Claim 2.17. Assume**

\( \langle \lambda_i : i < \kappa \rangle \) is an increasing continuous sequence of cardinals \( > \kappa \)

\( \lambda^* = \lambda = \sum \{ \lambda_i : i < \kappa \} \)

\( \kappa = \text{cf}(\kappa) > \delta \)

\( \text{Ax}_\lambda^0 < \mu, \kappa \)

\( \text{hrtg}(\text{Fil}_{\text{cf}(\kappa)}(\kappa, \mu)) < \lambda \text{ and } \kappa, \mu < \lambda_0 \)

\( S := \{ i < \kappa : \lambda_i^+ \text{ is a regular cardinal} \} \) is a stationary subset of \( \kappa \)

\( \{ \} \) \( D = D_\kappa + S \) where \( D_\kappa \) is the club filter on \( \kappa \)

\( \gamma(*) = \text{rk}_D(\langle \lambda_i^+ : i < \kappa \rangle) \).

Then \( \gamma(*) \) has cofinality \( > \lambda \), so \( \langle \lambda, \gamma(*) \rangle \cap \text{Reg} \neq \emptyset \).

**Proof.** Recall \( \exists_{\mathbb{B}_3} \) which we shall use. Toward contradiction assume that \( \text{cf}(\gamma(*)) \leq \lambda_c \), but \( \lambda^*_c \) is singular hence for some \( i(*) < \kappa, \text{cf}(\gamma(*)) \leq \lambda_i(*) \). Let \( \text{cf} \) witness \( \text{Ax}_{\lambda^*_c < \mu, \lambda} \).

Let \( B \) be an unbounded subset of \( \gamma(*) \) of order type \( \text{cf}(\gamma(*)) \leq \lambda_i(*) \). By renaming without loss of generality \( i(*) = 0 \).

For \( \alpha < \gamma(*) \) let
Claim 2.18. Assume $\text{cf} \ast$ witness $\text{Ax}_0^0$ and $\text{htrg}(Y) < \mu \in [\kappa, \mu]$. The ordinals
$
\gamma_i, \ell = 0, 1, 2
$
are nearly equal see, i.e. \cref{kp.6} below holds where:

\begin{itemize}
\item[(a)] $\gamma_0 = \text{htrg}(Y\upharpoonright\alpha)$, a cardinal
\item[(b)] $\gamma_1 = \cup\{\text{rk}_D(\gamma) : \gamma = \text{rk}_D(\alpha) \text{ for some } D \in \text{Fil}_{\beta(\ast)}(Y)\}$
\item[(c)] $\gamma_2 = \sup\{\text{otp}(\Xi_\alpha[\ell]) + 1 : \eta \in \text{Fil}_{\beta(\ast)}(Y)\}$
\end{itemize}

\begin{itemize}
\item[$\oplus$] (a) $\gamma_2 \leq \gamma_1 \leq \gamma_0$
\item[$\beta$] $\gamma_0$ is the union of $\text{Fil}_{\beta(\ast)}(Y)$ sets each of order type $< \gamma_2$
\item[$\gamma$] $\gamma_0$ is the disjoint union of $< \text{htrg}(\mathcal{P}(\text{Fil}_{\beta(\ast)}(Y)))$ sets each of order type $< \gamma_2$
\item[$\delta$] if $\gamma_0 > \text{htrg}(\mathcal{P}(\text{Fil}_{\beta(\ast)}(Y)))$ and $\gamma_0 \geq |\gamma_2|^+ + \text{then } |\gamma_0| \leq |\gamma_2|^{++}$ and $\text{cf}(|\gamma_2|^+) < \text{htrg}(\mathcal{P}(\text{Fil}_{\beta(\ast)}(Y)))$.
\end{itemize}

Proof. Straightforward, see \cref{kp.5}. \hfill $\square$
§ 3. Concluding Remarks

In May 2010, David Aspero asked whether it is true that I have results along the following lines (or that it follows from such a result):

If GCH holds and $\lambda$ is a singular cardinal of uncountable cofinality, then there is a well-order of $\mathcal{H}(\lambda^+)$ definable in $(\mathcal{H}(\lambda^+), \in)$ using a parameter.

The answer is yes by [She97, 4.6 pg.17] but we elaborate this below somewhat more generally. Much earlier Gitik [Git80] had proved (using suitable large cardinals) the consistency of “ZF + every infinite cardinal has cofinality $\aleph_0$, i.e. $\aleph_0$ is the only regular cardinal”. This naturally raises the question what suffices to have a class of regulars. Gitik told me that in Luming 2008 Woodin has conjectured:

\[ \mathbb{c} \] Let $V$ be a model of ZF + DC, suppose that $\kappa$ is a singular strong limit cardinal of cofinality $\omega_1$ and $|\mathcal{H}(\kappa)| = \kappa$. Is then $\mathcal{P}(\kappa)$ well orderable?

Now [She97] gives some information. The results here (\cite{c2}) confirm $\mathbb{c}$.

\[ \text{c2} \]

Claim 3.1. [DC] Assume that $\mu$ is a singular cardinal of cofinality $\kappa > \aleph_0$ (no GCH needed), the parameter $X \subseteq \mu$ codes in particular the tree $\mathcal{T} = {}^{< \kappa} \lambda$ and the set $\mathcal{P}(\mathcal{P}(\kappa))$ and $F: {}^\kappa \mu \rightarrow \mu$ which satisfies $\langle \mu, F \rangle$ has no infinite decreasing $\omega$-chain of subalgebras; in particular, from $X$ a well orderings of $\kappa^{< \kappa} \cup \mathcal{P}(\mathcal{P}(\kappa))$ are definable. \textbf{Then} (with this parameter) we can define a well ordering of the set of $\kappa$-branches of the tree $(< ^\kappa \lambda, \in)$.

Proof. Proof of \cite{c2}:

Let $\langle cd_i : i < \kappa \rangle$ satisfies

1. $cd_i$ is a one-to-one function from $^i \mu$ into $\mu$, (definable from $X$ uniformly (in $i$))

2. Let $< _\kappa$ be a well ordering of Fil$_\kappa(\kappa)$ definable from $X$.

For $\eta \in {}^\kappa \mu$ let $f_\eta : \kappa \rightarrow \mu$ be defined by $f_\eta(i) = cd_i(\eta[i])$, so $\bar{f} = \langle f_\eta : \eta \in {}^\kappa \mu \rangle$ is well defined.

Let $\mathcal{F} = \langle \mathcal{F}_\eta : \eta \in \text{Fil}_\kappa(\kappa) \rangle$ be as in Theorem \cite{c2} with $\mu, \kappa$ here standing for $\lambda, Y$ there; there is such $\mathcal{F}$ definable from $X$ as $X$ codes also $[\mu]^{< \kappa}$, see §1.

So for every $\eta \in {}^\kappa \mu$ there is $\eta \in \text{Fil}_\kappa(\kappa)$ such that $f|Z_\eta \in \mathcal{F}_\eta$ and $D^\eta_\theta$ contains all co-bounded subsets of $\kappa$ so let $\eta(\eta)$ be the $< _\kappa$-first such $\eta$. Now we define a well ordering $< _\kappa$ of $\kappa^{< \kappa}$: for $\eta, \nu \in \kappa^{< \kappa}$ let $\eta < _\kappa \nu$ iff $\text{rk}D_{\eta(\eta)}(f_\eta|Z_\eta(\eta)) < \text{rk}D_{\nu(\eta)}(f_\eta|Z_\eta(\eta))$ or equality holds and $\eta(\eta) < \eta(\nu)$.

This is O.K. because

\[ (*) \] if $\eta \neq \nu \in \kappa^{< \kappa}$ then $f_\eta(i) \neq f_\nu(i)$ for every large enough $i < \kappa$ (i.e. $i \geq \min \{ j : \eta(j) \neq \nu(j) \}$).

\[ \text{c4} \]

Conclusion 3.2. [DC] Assume $\mu$ is a singular cardinal of uncountable cofinality and $\mathcal{H}(\mu)$ is well orderable of cardinality $\mu$ and $X \subseteq \mu$ codes $\mathcal{H}(\mu)$ and a well ordering of $\mathcal{H}(\mu)$. \textbf{Then} we can (with this $X$ as parameter) define a well ordering of $\mathcal{P}(\mu)$; hence of $\mathcal{H}(\mu^+)$.

Proof. Proof of \cite{c4}:

Let $\langle \mu_i : i < \kappa \rangle$ be an increasing sequence of cardinals $< \mu$ with limit $\mu$. Clearly $\nu^{\mu_i} < \mu$. 

\[ \text{\hfill \cite{c4}} \]
Let \( \langle cd_\alpha^+ : i < \kappa \rangle \) satisfies

\[ \exists \beta \text{ cd}^+ \beta \text{ is a one-to-one function from } \mathcal{P}(\mu_i) \text{ into } \mu_i \text{ (definable uniformly from } X). \]

So \( \text{cd}_i : \mathcal{P}(\mu) \rightarrow ^* \mu \) defined by \( \text{cd}_i(A)(i) = \text{cd}_i^+(A \cap \mu_i) \) for \( A \subseteq \mu_i, i < \kappa \), is a one-to-one function from \( \mathcal{P}(\mu) \) into \( ^* \mu \). Now use \( \triangleleft_{\beta_7}^{2} \).

\[ \square_{\beta_7}^{2} \]

We return to \( \beta_{13}^{2} \).

\[ \square_{\beta_7}^{2} \]

Claim 3.3. \([\text{DC}]\)

\( 1) \) The cardinal \( \lambda^+ \) is regular when:

\[ \Box \]

(a) \( \text{Ax}^+_{\lambda^+} \), i.e. \( [\lambda^+]^{R_\omega} \) is well orderable

(b) \( |\alpha|^{R_\omega} < \lambda \) for \( \alpha < \lambda \)

(c) \( \lambda \) is singular.

\( 2) \) Also there is \( e = \langle e_\delta : \delta < \lambda^+ \rangle, e_\delta \subseteq \delta = \sup(e_\delta), |e_\delta| \leq \text{cf}(\delta)^{R_\omega}. \)

Remark 3.4. Compare with \( \beta_{13}^{2} \); we use here more choice, but cover more cardinals.

**Proof.** Let \( <_\omega \) be a well ordering of the set \( [\lambda^+]^{R_\omega} \).

As earlier let \( F : ^\omega(\lambda^+) \rightarrow \lambda^+ \) be such that there is no \( \subset \)-decreasing sequence \( \langle \text{cf}_F(u_n) : n < \omega \rangle \) with \( u_n \subseteq \lambda^+ \). Let \( \Omega = \{ \delta < \lambda^+ : \delta \text{ a limit ordinal, } \delta < \lambda^+ \wedge \text{cf}(\delta) < \lambda \} \), so \( \text{otp}(\Omega) \in \{ \lambda^+, \lambda^++1 \} \).

We define \( \vec{e} = \langle e_\delta : \delta \in \Omega \rangle \) as follows.

Case 1: \( \text{cf}(\delta) = \aleph_0, e_\delta \) is the \( <_\omega \)-minimal member of \( \{ u \subseteq \delta : \delta = \sup(u) \text{ and } \text{otp}(u) = 0 \} \).

Case 2: \( \text{cf}(\delta) > \aleph_0 \).

Let \( e_\delta = \cap \{ \text{cf}_F(C) : C \text{ a club of } \delta \} \).

So

\[ (\ast)_1 e_\delta \text{ is an unbounded subset of } \delta \text{ of order type } \lambda. \]

[Why? If \( \text{cf}(\delta) = \aleph_0 \) then \( e_\delta \) has order type \( \omega \) which is \( < \lambda \) by clause (b) of the assumption.

If \( \text{cf}(\delta) > \aleph_0 \) then for some club \( C \) of \( \delta, e_\delta = \text{cf}_F(C) \) has \( \text{otp}(e_\delta) \leq |\text{cf}_F(C)| \leq (\text{cf}(\delta))^{\aleph_0} < \lambda. \) The last inequality holds as \( \text{cf}(\delta) \leq \lambda \) as \( \delta < \lambda^+ \), \( \text{cf}(\delta) \neq \lambda \) as \( \lambda \) is singular by clause (c) of the assumption, and lastly \( ((\text{cf}(\delta))^{\aleph_0}) < \lambda \) by clause (b) of the assumption.]

This is enough for part (2). Now we shall define a one-to-one function \( f_\alpha \) from \( \alpha \) into \( \lambda \) by induction on \( \alpha \in \Omega \) as follows: let \( \text{pr}_\lambda : \lambda \times \lambda \rightarrow \lambda \) be a pairing function so one to one (can add “onto” \( \lambda^+ \)); if we succeed then \( f_\alpha \) cannot be well defined so \( \lambda^+ \notin \Omega \) hence \( \text{cf}(\lambda^+) \geq \lambda \), but \( \lambda \) is singular so \( \text{cf}(\lambda^+) = \lambda^+ \), i.e. \( \lambda^+ \) is not singular so we shall be done proving part (1).

The inductive definition is:

\[ \Box \]

(a) \( \text{if } \alpha < \lambda \text{ then } f_\alpha \) is the identity

(b) \( \text{if } \alpha = \beta + 1 \in [\lambda, \lambda^+] \) then for \( i < \alpha \) we let \( f_\alpha(i) \) be

- \( 1 + f_\beta(i) \) if \( i < \beta \)
- \( 0 \) if \( i = \beta \)
(c) If $\alpha \in \Omega$ so $\alpha$ is a limit ordinal, $e_\alpha \subseteq \alpha = \sup(e_\alpha), e_\alpha$ of cardinality $< \lambda$ and we let $f_\alpha$ be defined by: for $i < \alpha$ we let $f_\alpha(i) = \text{pr}_\lambda(f_{\min(e_\alpha \setminus \{i+1\}}(i), \text{otp}(e_\alpha \cap i))$. 

We later add:

**Claim 3.5.** [ZFC] Assume $\mu > \kappa = \text{cf}(\mu) > \aleph_0$ and $\mu = \mu^{\aleph_0} + 2^{2^\kappa}$.

1) From some $X \subseteq \mu$ we can define well ordering of some set $\G \subseteq \kappa \mu$ such that $\kappa \mu = \{\sup\{f_n : n < \omega\} : f_n \in \G \text{ for } n < \omega\}$.
2) If moreover $2^{2^\kappa} \leq \mu$ where $\theta = \kappa^\kappa$ then from some $X \subseteq \mu$ we can define a well ordering of $\kappa \mu$.

**Proof.**

1) Let $X \subseteq \mu$ code $\mathcal{P}(\mathcal{P}(\kappa))$, $\omega \mu \rightarrow \omega$ and $F : \omega \mu \rightarrow \omega$ which as in (cii). Unlike the proof of (cii) we do not use the $\text{cd}_i(i < \kappa)$ and we use the family of $\aleph_1$-complete filters on $\kappa$, the rest should be clear.

2) As $\theta = \theta^{\aleph_0}$ there is a one-to-one onto function $\text{cd} : \omega \theta \rightarrow \theta$ onto $\theta$, and for $i < \omega$ let $\text{cd}_i : \theta \rightarrow \theta$ be such that:

\[ (*)_1 \text{ if } \text{cd}((\eta) = \zeta, \text{ then } \text{cd}_0(\zeta) = \ell g(\eta) \text{ and } \text{cd}_{\ell g(\eta)}(\zeta) = \eta(i) \text{ for } i < \ell g(\eta). \]

Let $D$ be $\{A \subseteq \theta : \text{ for some } u \in [\theta]^{\leq \aleph_0} \text{ we have } A \supseteq \{\varepsilon < \theta : u \subseteq \{\text{cd}_i(\varepsilon) : i < \omega\}\},$

so

\[ (*)_2 \text{ } D \text{ is an } \aleph_1 \text{-complete filter on } \theta. \]

Why? Should be clear.

\[ (*)_3 \text{ for } f \in \theta \mu \text{ let } g, g_f \text{ be the unique function } g \text{ with domain } \theta \text{ such that:} \]

- if $\varepsilon < \kappa$ and $i < \text{cd}_0(\varepsilon)$, then $\text{cd}_{1+i}(\varepsilon) < \theta \Rightarrow \text{cd}_{1+i}(\varepsilon) = f(\text{cd}_{1+i}(\varepsilon))$
- and $\text{cd}_0(g(\text{cd}_i(\varepsilon))) = \text{cd}_0(\varepsilon)$ and $f(\zeta) = 0$ otherwise

Why $g_f$ exists? Just think.

\[ (*)_4 \text{ if } f \in \theta \mu, \alpha = \text{rk}_D(g_f) \text{ and } \eta = \eta_{g_f}, \text{ as in the proof of (cii) for } g_f, \text{ then:} \]

- (a) from $g_f|Z_\eta$ we can define $f$ (using some $Y \subseteq \kappa$ as a parameter)
- (b) $\text{Rang}(f) \subseteq \{\text{cd}_{1+i}(\varepsilon) : \varepsilon \in \eta \text{ and } i < \text{cd}_0(g_f(\varepsilon))\}.$

Why? Clause (a) follows clause (b). Clause (b) holds as for every $\xi < \kappa$, the set $\{\varepsilon < \theta : \xi \in \{\text{cd}_{1+i}(\varepsilon) : i < \text{cd}_0(\varepsilon)\}\} \in D.$

We continue as in the proof of (cii). 

**Conclusion 3.6.** [DC] Assume $[\lambda]^{\aleph_0}$ is well ordered for every $\lambda$.

1) If $2^{2^\kappa}$ is well ordered then for every $\lambda, [\lambda]^{\kappa}$ is well ordered.
2) For any set $Y$, there is a derived set $Y_*$ so called $\text{Fil}_{\aleph_1}(Y)$ of power near $\mathcal{P}(\mathcal{P}(Y))$ such that $\|_{\text{Levy}(\aleph_0, Y)}$ “for every $\lambda, Y \lambda$ is well ordered”.

**Proof.**

1) By (cii).
2) Follows easily.
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