On topological properties of ultraproducts of finite sets

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Abstract

In [3] a certain family of topological spaces was introduced on ultraproducts. These spaces have been called ultratopologies and their definition was motivated by model theory of higher order logics. Ultratopologies provide a natural extra topological structure for ultraproducts. Using this extra structure in [3] some preservation and characterization theorems were obtained for higher order logics.

The purely topological properties of ultratopologies seem interesting on their own right. We started to study these properties in [2], where some questions remained open. Here we present the solutions of two such problems. More concretely we show that

(1) there are sequences of finite sets of pairwise different cardinalities such that in their certain ultraproducts there are homeomorphic ultratopologies and

(2) if A is an infinite ultraproduct of finite sets then every ultratopology on A contains a dense subset D such that $|D| < |A|$.

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1 Introduction

In first order model theory the ultraproduct construction can often be applied. This is because ultraproducts preserve the validity of first order formulas. It is natural to ask, what connections can be proved between certain higher order formulas and ultraproducts of models of them.

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In [3] we answer related questions in terms of topological spaces which can be naturally associated to ultraproducts. These spaces are called ultratopologies and their definition can be found in [3] and also at the beginning of [2].

Although ultratopologies were introduced from logical (model theoretical) reasons, these spaces can be interesting on their own right. In [2] a systematic investigation has been started about topological properties of these spaces. However, in [2] some problems remained open. In the present note we are dealing with two such problems.

In Section 2 we give a positive answer for Problem 5.2 of [2]: there are sequences of finite sets of pairwise different cardinalities such that certain ultraproducts of them are still homeomorphic with respect to some carefully chosen ultratopologies. In fact, in Theorem 2.1 below we show that if U is a good ultrafilter and A is any infinite ultraproduct of finite sets modulo U then there is an ultratopology on A in which the family of closed sets consists of just the finite subsets of A and A itself. The affirmative answer for Problem 5.2 of [2] can immediately be deduced from this observation.

In Section 3 we investigate the possible cardinalities of dense sets in ultratopologies, again on ultraproducts of finite sets. In Corollary 3.3 we show that if $\mathcal{C}$ is any ultratopology on an infinite ultraproduct A then the density of $\mathcal{C}$ is smaller than $|A|$, that is, one can always find a dense set whose cardinality is less than $|A|$.

Throughout we use the following conventions. I is a set and, for every $i \in I$, A_i is a set. Moreover, $A = \Pi_{i\in I} A_i/U$ denotes the ultraproduct of A_i's modulo an ultrafilter U (on I).

Every ordinal is the set of smaller ordinals and natural numbers are identified with finite ordinals. Throughout, ω denotes the smallest infinite ordinal and cf denotes the cofinality operation.

In order to simplify notation, sometimes we will identify ${}^k(\Pi_{i\in I} A_i)$ with $\Pi_{i\in I}({}^k A_i)$ the natural way, that is, k–tuples of sequences are identified with single sequences whose terms are k–tuples.

If X is a topological space and $A \subseteq X$ then $cl(A)$ denotes the closure of A. Suppose $k \in \omega$, $\langle A_i : \ i \in I\rangle$ is a sequence of sets, U is an ultrafilter on I, and $R_i \subseteq {}^k A_i$ is a relation, for every $i \in I$. Then the *ultraproduct* relation $\Pi_{i\in I} R_i/U$ is defined as follows:

$$\Pi_{i\in I} R_i/U = \{s/U \in ({}^k\Pi_{i\in I} A_i/U) : \ \{i \in I : \ s(i) \in R_i\} \in U\}.$$

We assume that the reader is familiar with the notions of "choice function", "ultratopology", "a point is close to a relation", "$T(R, a)$", etc. These notions were introduced in [3] and a short (but fairly complete) survey can be found at the beginning of [2].

2 Homeomorphisms between different ultraproducts

In [2], Problem 5.2 asks whether it is possible to choose ultrafilters U_1 and U_2 (over I and J, respectively), sequences of natural numbers $s = \langle n_i,\ i \in I \rangle$, and $z = \langle m_j,\ j \in J \rangle$ such that

(*i*) n_i and m_j are different for all $i \in I, j \in J$ and

(*ii*) for every $k \in \omega$ there are k–dimensional ultratopologies $\mathcal{C}_k$ in $\Pi_{i \in I} n_i / U_1$ and $\mathcal{D}_k$ in $\Pi_{j \in J} m_j / U_2$ such that $\mathcal{C}_k$ and $\mathcal{D}_k$ are homeomorphic.

To this question we will give an affirmative answer. In fact, we prove the following theorem from which this answer can easily be deduced.

Theorem 2.1 *Suppose $\langle n_i,\ i \in I \rangle$ is an infinite sequence of natural numbers and U is a good ultrafilter on I such that $A = \Pi_{i \in I} n_i / U$ is infinite. Then for every $k \in \omega$ there is a k–dimensional choice function on A such that the family of closed sets in the induced ultratopology consists of the finite subsets of ${}^k A$ and ${}^k A$.*

Proof. Let c be an arbitrary k–dimensional choice function on ${}^k A$. By modifying c, we will construct another choice function $\hat{}$ which induces the required ultratopology. Let E be the set of all triples $\langle s, i, m \rangle$ where $s \in \Pi_{l \in I} \mathcal{P}({}^k n_l)$ such that $\Pi_{l \in I} s(l)/U$ is infinite, and $i \in I$, $m \in {}^k n_i - s(i)$. It is easy to see that $|\Pi_{l \in I} \mathcal{P}({}^k n_l)| \leq |{}^I \omega| = 2^{|I|}$. Therefore $|E| \leq 2^{|I|} \times |I| \times \aleph_0 = 2^{|I|}$. Let $\{\langle s_\alpha, i_\alpha, m_\alpha \rangle :\ \alpha < 2^{|I|}\}$ be an enumeration of E.

By transfinite recursion we construct an injective function $f : E \to {}^k A$ such that for every $\langle s, i, m \rangle \in E$ we have $f(\langle s, i, m \rangle) \in \Pi_{l \in I} s(l)/U$. Suppose that f_β has already been defined on $\{\langle s_\alpha, i_\alpha, m_\alpha \rangle :\ \alpha < \beta\}$ for all $\beta < \gamma \leq 2^{|I|}$ such that

(i) if $\beta_1 < \beta_2 < \gamma$ then $f_{\beta_1} \subseteq f_{\beta_2}$ and

(ii) $|f_\beta| \leq |\beta|$.

If γ is a limit ordinal then let $f_\gamma = \cup_{\beta < \gamma} f_\beta$. Now suppose that γ is a successor ordinal, say $\gamma = \delta + 1$. Since U is a good ultrafilter, Theorem VI, 2.13 of [4] implies that the cardinality of $B = \Pi_{l \in I} s(l)/U$ is $2^{|I|}$. Therefore there is an element $b \in B$ which is not in the range of f_δ. Let f_γ be $f_\delta \cup \{\langle \langle s_\delta, i_\delta, m_\delta \rangle, b \rangle\}$. Clearly, $f = f_{2^{|I|}}$ is the required function.

Now we construct a k–dimensional choice function $\hat{}$ as follows. If a does not belong to the range of f then let $\hat{a} = c(a)$. Otherwise there is a unique $e = \langle s, i, m \rangle \in E$ such that $f(e) = a$. Let

$$\hat{a}(j) = \begin{cases} c(a)(j) & \text{if } i \neq j, \\ m & \text{otherwise.} \end{cases}$$

This way we really defined a k–dimensional choice function on A. We claim that the closed sets of the induced ultratopology are exactly the finite subsets of ${}^k A$ and ${}^k A$.

By Theorem 2.5 of [2], every ultratoplogy is T_1. Therefore every finite subset of ${}^kA$ is closed. Let F be an infinite closed subset of ${}^kA$ and suppose, seeking a contradiction, that there is an element

$$(*) \qquad b \in {}^kA - F.$$

By Corollary 2.2 of [2], F is a decomposable relation, say $F = \Pi_{l\in I} s(l)/U$. Therefore, there is a $J \in U$ such that for every $i \in J$ we have $\hat{b}(i) \notin s(i)$. Hence, for every $i \in J$ we have $e_i := \langle s, i, \hat{b}(i)\rangle \in E$. By construction, for every $i \in J$ we have $f(e_i) \in F$ and $f(e_i)\hat{}(i) = \hat{b}(i)$. This means that

$$\{i \in I : \ (\exists a \in F)\hat{a}(i) = \hat{b}(i)\} \supseteq J \in U.$$

That is, $T(F, b) \in U$ (where T is understood according to the new choice function $\hat{}$). Since we assumed that F is closed, this implies $b \in F$ which contradicts to $(*)$. ∎

Corollary 2.2 *There are ultrafilters U_1, U_2 (over I and J, respectively) and sequences $s = \langle n_i,\ i \in I\rangle$ and $z = \langle m_j,\ j \in J\rangle$ of natural numbers such that*

(i) n_i and m_j are different for all $i \in I, j \in J$ and

(ii) for every $k \in \omega$, there are k–dimensional ultratopologies $\mathcal{C}_k$ in $\Pi_{i\in I} n_i/U_1$ and $\mathcal{D}_k$ in $\Pi_{j\in J} m_j/U_2$ such that $\mathcal{C}_k$ and $\mathcal{D}_k$ are homeomorphic.

Proof. Let U_1, U_2 be good ultrafilters, and let s and z be arbitrary sequences of natural numbers satisfying the requirements of the corollary such that $|I| = |J|$ and both $A = \Pi_{i\in I} s_i/U_1$ and $B = \Pi_{j\in J} z_j/U_2$ are infinite. Let $k \in \omega$ be arbitrary. By Theorem 2.1 there are ultratopologies $\mathcal{C}_k$ and $\mathcal{D}_k$, respectively on A and B, such that

the closed sets of $\mathcal{C}_k$ are exactly the finite subsets of ${}^kA$ and ${}^kA$, and

the closed sets of $\mathcal{D}_k$ are exactly the finite subsets of ${}^kB$ and ${}^kB$.

By Theorem VI, 2.13 of [4], $|A| = |B| = 2^{|I|}$. Let $f : A \to B$ be any bijection. Then $f_k : {}^kA \to {}^kB$, $f_k(\langle a_0, ..., a_{k-1}\rangle) = \langle f(a_0), ..., f(a_{k-1})\rangle$ is clearly a bijection from ${}^kA$ onto ${}^kB$ mapping finite subsets of ${}^kA$ to finite subsets of ${}^kB$. Thus, f is the required homeomorphism. ∎

3 Cardinalities of dense sets

Problem 5.3 (A) of [2] asks whether it is possible to choose a sequence s of finite sets and an ultrafilter U such that there is an utratopology on $A = \Pi_{i\in I} n_i/U$ in which every dense set has cardinality $|A|$. In this section we will show that this is impossible if A is infinite. We start with a simple observation: every k-dimensional ultratopology is homeomorphic to an appropriate 1–dimensional ultratopology.

Theorem 3.1 *Suppose $\mathcal{C}_k$ is a k–dimensional ultratopology on A. Then there is a 1–dimensional ultratopology $\mathcal{D}$ which is homeomorphic to $\mathcal{C}$.*

Proof. The idea is to identify k–tuples of sequences with sequences of k–tuples. By a slight abuse of notation we will use this identification freely. Let $A = \Pi_{i\in I} A_i/U$ (here the A_i's are not necessarily finite) and suppose that $\hat{}$ is a k–dimensional choice function inducing $\mathcal{C}_k$. Let $\mathcal{B} = \Pi_{i\in I}{}^k A_i/U$. We define a 1–dimensional choice function c in B as follows. If $s = \langle s_i : i \in I\rangle/U \in B$ then for each $j \in k$ let $s^j = \langle s_i(j) : i \in I\rangle/U$. Define $c(s) = \langle s^0, ..., s^{k-1}\rangle\hat{}$ and $\varphi : {}^kA \to B$, $\varphi(\langle s^0/U, ..., s^{k-1}/U\rangle) = \langle\langle s^0(i), ..., s^{k-1}(i)\rangle : i \in I\rangle/U$. Then clearly, c is a 1–dimensional choice function which induces an ultratopology $\mathcal{D}$ on B. Then for any $a \in {}^kA$ and $i \in I$ we have $\hat{a}(i) = c(\varphi(a))(i)$. Now it is straightforward to check that φ is a homeomorphism between $\mathcal{C}$ and $\mathcal{D}$. ∎

Let $\mathcal{C}$ be an ultratopology on an ultraproduct $A = \Pi_{i\in I} n_i/U$ of finite sets. Suppose $\mathcal{C}$ can be induced by a choice function $\hat{}$. Let

$$G = \{\langle i, m\rangle : i \in I, m \in n_i \text{ and } (\exists a \in A)(\hat{a}(i) = m)\},$$

and for each $\langle i, m\rangle \in G$, let $a_{i,m} \in A$ be such that $\hat{a}_{i,m}(i) = m$. Clearly, if I is infinite then $|G| \leq |I|$. We claim that there is a dense subset R of A such that $|R| \leq |G|$. In fact, R can be chosen to be $R = \{a_{i,m} : \langle i, m\rangle \in G\}$. To see this, suppose $a \in A$. Then for every $i \in I$ we have $\langle i, \hat{a}(i)\rangle \in G$ and therefore $T(R, a) = I \in U$. Hence $cl(R) = A$, as desired.

Now we are able to give a negative answer to Problem 5.3 (A) of [2]. Recall, that if X is a topological space then the *density* $d(X)$ of X is defined as follows:

$$d(X) = min\{|S| : S \subseteq X \text{ and } S \text{ is dense in } X\}.$$

Theorem 3.2 *Suppose $\mathcal{C}$ is a 1–dimensional ultratopology on an infinite ultraproduct $A = \Pi_{i\in I} n_i/U$ where each n_i is a finite set. Then $d(\mathcal{C}) < |A|$.*

Proof. Suppose, seeking a contradiction, that $\mathcal{C}$ is an ultratopology on A such that every dense subset of $\mathcal{C}$ has cardinality $\kappa := |A|$. Using the notation introduced in the remark before the theorem, R is a dense subset of A, and therefore $|R| = |A|$. Let $<^A$ be a well-ordering of A (having order type κ). By transfinite recursion we define a sequence $\langle b_i \in A : i < \kappa\rangle$ as follows. Assume $j < \kappa$ and b_l has already been defined for every $l < j$. Let $W_j = \{b_l : l < j\}$ and let $V_j = \{a \in A : T(W_j, a) \in U\}$. Since $j < \kappa$, $V_j \subseteq cl(W_j) \neq A$. If j is an odd ordinal, then let b_j be the $<^A$–first element of $R - W_j$. Otherwise let b_j be the $<^A$–first element in $A - V_j$. Clearly, the following conditions are satisfied:

(i) for every $\langle i, m\rangle \in G$ there is a $j < \kappa$ such that $\hat{b}_j(i) = m$, in fact, $R \subseteq \{b_l : l < \kappa\} = W_\kappa$;

(ii) for every $a \in A$ there is a $j < \kappa$ such that $a \in V_j$ (the smallest such j will be denoted by j_a);

(iii) for every $j < \kappa$ there is an ordinal $j < s(j)$ such that $s(j) < \kappa$ and $b_{s(j)} \notin cl(W_j)$. (This is true because otherwise, by (i), one would have $cl(W_j) \supseteq cl(R) = A$ which is impossible since $|W_j| < \kappa$.)

Now let $H = \{i \in I : n_i = \{\hat{b}_j(i) : j < \kappa\}\}$. We show that $H \in U$.

Again, seeking a contradiction, assume $I - H \in U$. For every $i \in I - H$ let $e_i \in n_i - \{\hat{b}_j(i) : j < \kappa\}$ be arbitrary, let $e = \langle e_i : i \in I - H\rangle/U$, and let $O = \{i \in I : \hat{e}(i) = e_i\}$. Clearly, $O \in U$. In addition, if $i \in O \cap (I - H)$ then, by (i), there is a $j < \kappa$ such that $\hat{b}_j(i) = \hat{e}(i) = e_i$ – contradicting to the selection of e_i's.

For every $i \in H$ we introduce a binary relation $\prec^i$ as follows. If $n, m \in n_i$ then $n \prec^i m$ means that there is a $j \in \kappa$ such that $\hat{b}_j(i) = n$ but for every $l \leq j$ $\hat{b}_l(i) \neq m$. Clearly, the relation $\prec^i$ is irreflexive, transitive, and trichotome for every $i \in H$. Since n_i is finite, for every $i \in H$ there is an $\prec^i$–maximal element $y_i \in n_i$. Let $y = \langle y_i : i \in H\rangle/U$ and let $K = \{i \in I : \hat{y}(i) = y_i\}$. Now from ($ii$) and ($iii$) it follows that $y \in V_{j_y}$ and $b_{s(j_y)} \notin cl(W_{j_y}) = cl(V_{j_y})$. Thus, for every $i \in K \cap T(W_{j_y}, y)$ we have

$$(*) \qquad \hat{y}(i) = y_i \in \{\hat{b}_l(i) : l < j_y\}.$$

Since $cl(V_{j_y})$ is closed, there is an $L \in U$ such that for all $i \in L$ we have $\hat{b}_{s(j_y)}(i) \notin \{\hat{b}_l(i) : l < j_y\}$ and thus $\hat{b}_l(i) \prec^i \hat{b}_{s(j_y)}(i)$ for every $i \in L \cap H$ and for every $l < j_y$. Particularly, $(*)$ implies that if $i \in L \cap H \cap K \cap T(W_{j_y}, y)$ then $\hat{y}(i) = y_i \prec^i \hat{b}_{s(j_y)}(i)$ which is impossible since, by construction, y_i is the $\prec^i$–maximal element in n_i. This contradiction completes the proof. ∎

Using Theorem 3.1 the above results can be generalized to higher dimensional ultratopologies as well.

Corollary 3.3 *Let $k \in \omega$ be arbitrary and suppose $\mathcal{C}$ is a k–dimensional ultratopology on an infinite ultraproduct $A = \Pi_{i\in I} n_i/U$ where each n_i is a finite set. Then $d(\mathcal{C}) < |A|$.*

Proof. Assume, seeking a contradiction, that $\mathcal{C}$ is a k–dimensional ultratopology on A whose density is $|A|$. Then, by Theorem 3.1, there is a 1–dimensional ultratopology with the above property, contradicting to Theorem 3.2 ∎

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