INCREASING THE GROUPWISE DENSITY NUMBER BY C.C.C. FORCING

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ABSTRACT. We show that $\aleph_2 \leq \mathfrak{b} < \mathfrak{g}$ is consistent.

This work is dedicated to James Baumgartner on the occasion of his 60th birthday.

0. INTRODUCTION

We show that for every regular cardinal with a definition in the ground model, the statement $\kappa = \mathfrak{b} < \mathfrak{b}^+ = \mathfrak{g}$ is consistent. In particular this holds for $\kappa = \aleph_2$. This answers a question of Andreas Blass.

We recall the definitions of the three cardinal characteristics $\mathfrak{b}$, $\mathfrak{g}$, $\mathfrak{u}$. The set of functions from ω to ω is written as ${}^\omega\omega$. For $f, g \in {}^\omega\omega$, we say g dominates f and write $f \leq^* g$ iff for all but finitely many n, $f(n) \leq g(n)$. A family $B \subseteq {}^\omega\omega$ is unbounded iff for every $g \in {}^\omega\omega$ there is some $f \in B$ such that $f \not\leq^* g$. The bounding number $\mathfrak{b}$ is the smallest cardinal of an unbounded family $B \subseteq {}^\omega\omega$.

For $X, Y \in [\omega]^\omega$ we write $Y \subseteq^* X$ to denote that $Y \setminus X$ is finite. A subset $\mathscr{G}$ of $[\omega]^\omega$ is called groupwise dense if $(\forall X \in \mathscr{G})(\forall Y \subseteq^* X)(Y \in \mathscr{G})$ and for every partition $\{[\pi_i, \pi_{i+1}) : i < \omega\}$ of ω into finite intervals there is an infinite set A such that $\bigcup\{[\pi_i, \pi_{i+1}) : i \in A\} \in \mathscr{G}$. The groupwise density number, $\mathfrak{g}$, is the smallest number of groupwise dense families with empty intersection.

By an ultrafilter we mean a non-principal ultrafilter on ω. Such an ultrafilter is called a P-point if for any $A_i \in \mathscr{U}$, $i < \omega$, there is an $A \in \mathscr{U}$, such that $A \subseteq^* A_i$ for $i < \omega$. Such an A is called a pseudointersection of A_i, $i < \omega$. An ultrafilter is called a Q-point if, given a strictly increasing sequence π_i, $i < \omega$, of natural numbers, there is some $A \in \mathscr{U}$ that for all $i < \omega$, $|A \cap [\pi_i, \pi_{i+1})| \leq 1$. For an ultrafilter $\mathscr{U}$ the cardinal $\chi(\mathscr{U}) = \min\{|\mathscr{B}| : \mathscr{B} \subseteq \mathscr{U} \wedge (\forall X \in \mathscr{U})(\exists Y \in \mathscr{B})(Y \subseteq X)\}$ is called the

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character of $\mathscr{U}$. The cardinal $\mathfrak{u}$, the ultrafilter characteristic, is defined as the minimal $\chi(\mathscr{U})$ for all non-principal ultrafilters $\mathscr{U}$ on ω.

The bounding number $\mathfrak{b}$ and groupwise density number $\mathfrak{g}$ can be in either order. For a regular $\kappa > \aleph_1$, we get the constellation $\aleph_1 = \mathfrak{g} < \mathfrak{b} = \kappa$ for example after adding uncountably (— their number does not matter, the continuum can be larger than κ—) many random reals over a model of MA and $2^\omega = \kappa$ [4] or in a finite support iteration of Hechler forcings of length κ [13].

Also $\aleph_1 < \mathfrak{g} < \mathfrak{b}$ is consistent. We sketch a proof given by the referee. Let $\kappa < \lambda$ be regular uncountable and assume CH. We take a finite support iteration $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \lambda, \beta \leq \lambda \rangle$ of length λ adding Hechler generics in the odd steps and going through all c.c.c. partial orders of size $< \kappa$ in the even steps. Then $\mathfrak{b} = 2^\omega = \lambda$ and book-keeping gives $\mathrm{MA}_{<\kappa}$, so that $\mathfrak{g} \geq \kappa$. The proof of $\mathfrak{g} \leq \kappa$ is a standard modification of the argument for $\mathfrak{g} = \aleph_1$ in the Hechler model.

Recall the latter argument: if all iterands are Hechler forcing, then since Hechler forcing is Suslin, absoluteness gives us that $\mathbb{P}_A$ is completely embedded into $\mathbb{P}_\lambda$ for every $A \subseteq \lambda$, where $\mathbb{P}_A$ is defined as $\mathbb{P}_\lambda$ considering only coordinates from A and ignoring the others. Furthermore, when $\mathcal{A}$ is a directed family of subsets of λ such that for all countable subsets B of λ there is some $A \in \mathcal{A}$ with $B \subseteq A$, then $\mathbb{P}_\lambda$ is the direct limit of $\mathbb{P}_A$, $A \in \mathcal{A}$. This is so because the conditions in Hechler forcing are reals and hence arise in countable fragments of the iteration.

Now let $\mathcal{A}$ be a strictly increasing ω_1-chain of subsets of λ with $\bigcup \mathcal{A} = \lambda$. Then $V[G] \cap {}^\omega\omega = \bigcup_{A \in \mathcal{A}} V[G \cap \mathbb{P}_A] \cap {}^\omega\omega$, i.e., the reals arise in an ω_1-chain of intermediate models. By a standard argument, see [12, 4], this yields $\mathfrak{g} \leq \aleph_1$.

Now return to the above situation: Say $A \subseteq \lambda$ is closed if for all even $\alpha \in A$, $\mathrm{supp}(\underset{\sim}{\mathbb{Q}}_\alpha) \subseteq A$, where $\mathrm{supp}(\underset{\sim}{\mathbb{Q}}_\alpha)$ is the union of the supports of the conditions determining what the order $\underset{\sim}{\mathbb{Q}}_\alpha$ is. By the countable chain condition and since the supports of the conditions are finite, $|\mathrm{supp}(\underset{\sim}{\mathbb{Q}}_\alpha)| < \kappa$ for all even α. Then for each $B \subseteq \lambda$ of size $< \kappa$ there is some closed $A \supseteq B$ of size $< \kappa$. If A is closed then $\mathbb{P}_A$ is completely embedded into $\mathbb{P}_\lambda$. Furthermore, when $\mathcal{A}$ is a directed family of closed subsets of λ such that for all $B \subseteq \lambda$ of size $< \kappa$ there is some $A \in \mathcal{A}$ with $B \subseteq A$, then $\mathbb{P}_\lambda$ is the direct limit of the $\mathbb{P}_A$, $A \in \mathcal{A}$. Now there is a strictly increasing κ-chain $\mathcal{A}$ of closed subsets of λ with $\bigcup \mathcal{A} = \lambda$. Again we get $V[G] \cap {}^\omega\omega = \bigcup_{A \in \mathcal{A}} V[G \cap \mathbb{P}_A] \cap {}^\omega\omega$ and $\mathfrak{g} \leq \kappa$.

In all models so far known of the reverse inequality $\mathfrak{b} < \mathfrak{g}$ we have had $\aleph_1 = \mathfrak{b} < \mathfrak{g} = 2^\omega = \aleph_2$. The models given by a countable support iteration of Blass-Shelah, Miller or Matet forcing over a ground model satisfying CH fulfil even $\aleph_1 = \mathfrak{u} < \mathfrak{g} = 2^\omega = \aleph_2$. Since $\mathfrak{b} \leq \mathfrak{u}$ [11], the latter is stronger than $\mathfrak{b} < \mathfrak{g}$. For the constellation $\mathfrak{b} < \mathfrak{g} \leq \mathfrak{u}$ one can for example interweave random reals at the odd steps of a countable support iteration of Miller forcings, see [2, Model 7.5.5].

The main part of this work is to show that the inequality $\mathfrak{b} < \mathfrak{b}^+ = \mathfrak{g}$ can hold above $\aleph_2$. There is nothing special about $\aleph_2$; any regular cardinal that is definable without parameters can serve. Our construction yields $\aleph_2 = \mathfrak{b} < \mathfrak{g} = \mathfrak{u} = 2^\omega = \aleph_3$ and it is open how to keep $\mathfrak{u}$ small. Moreover, our construction does not allow to push $\mathfrak{g}$ strictly above $\mathfrak{b}^+$. In the last section of this work we show that $\mathfrak{g} \leq \mathfrak{d}_\mathfrak{b}$, and this is possibly a partial explanation for the obstacles in getting $\mathfrak{g} > \mathfrak{b}^+$.

The main part of this paper will be the proof of

Theorem 0.1. $\aleph_2 \leq \mathfrak{b} < \mathfrak{g}$ *is consistent relative to ZFC.*

Here is an outline: In section 1 we state and prove some properties of Matet forcing with stable ordered-union ultrafilters and prove a key lemma. In section 2 we finish the proof of Theorem 0.1. In section 3 we show $\mathfrak{g} \leq \mathfrak{d}_\mathfrak{b}$.

1. A variant of Matet forcing

We shall define a variant of Matet forcing. For this purpose, we first introduce some notation about ordered-union ultrafilters. Our nomenclature follows Blass [3] and Eisworth [8].

We let $\mathbb{F}$ be the collection of all finite subsets of ω. For $a, b \in \mathbb{F}$ we write $a < b$ if $(\forall n \in a)(\forall m \in b)(n < m)$. We shall work with filters on $\mathbb{F}$, i.e. subsets of $\mathscr{P}(\mathbb{F})$ that are closed under intersections and supersets. A sequence $\bar{a} = \langle a_n : n \in \omega \rangle$ of members of $\mathbb{F}$ is called unmeshed if for all n, $a_n < a_{n+1}$. The set $(\mathbb{F})^\omega$ denotes the collection of all infinite unmeshed sequences in $\mathbb{F}$. If X is a subset of $\mathbb{F}$, we write $\mathrm{FU}(X)$ for the set of all finite unions of members of X. We write $\mathrm{FU}(\bar{a})$ instead of $\mathrm{FU}(\{a_n : n \in \omega\})$. We let $\mathbb{P} \lessdot \mathbb{Q}$ denote that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$.

Definition 1.1. *Given $\bar{a}$ and $\bar{b}$ in $(\mathbb{F})^\omega$, we say that $\bar{b}$ is a condensation of $\bar{a}$ and we write $\bar{b} \sqsubseteq \bar{a}$ if $\bar{b} \subseteq \mathrm{FU}(\bar{a})$. We say $\bar{b}$ is almost a condensation of $\bar{a}$ and we write $\bar{b} \sqsubseteq^* \bar{a}$ iff there is an n such that $\langle b_t : t \geq n \rangle$ is a condensation of $\bar{a}$.*

Definition 1.2. *In the Matet forcing, $\mathbb{M}$, the conditions are pairs $(a, \bar{c})$ such that $a \in \mathbb{F}$ and $\bar{c} \in (\mathbb{F})^\omega$ and $a < c_0$. The forcing order is $(b, \bar{d}) \leq (a, \bar{c})$ (the stronger condition is the smaller one) iff $a \subseteq b$ and $b \setminus a$ is a union of finitely many of the c_n and $\bar{d}$ is a condensation of $\bar{c}$.*

Definition 1.3. *A filter* $\mathscr{F}$ *on* $\mathbb{F}$ *is said to be an ordered-union filter if it has a basis of sets of the form* $\mathrm{FU}(\bar{d})$ *for* $\bar{d} \in (\mathbb{F})^\omega$. *An ordered-union filter is said to be stable if, whenever it contains* $\mathrm{FU}(\bar{d}_n)$ *for* $\bar{d}_n \in (\mathbb{F})^\omega$, $n < \omega$, *then it also contains some* $\mathrm{FU}(\bar{e})$ *for some* $\bar{e}$ *that is almost a condensation of each* $\bar{d}_n$.

Ordered-union ultrafilters need not exist, as their existence implies the existence of Q-points [3] and there are models without Q-points [10]. Under MA(σ-centred) stable (even $< 2^\omega$-stable) ordered-union ultrafilters exist [3].

It is well known [9, 4] that the forcing $\mathbb{M}$ can be decomposed into two steps $\mathbb{P} * \mathbb{M}(\underset{\sim}{\mathscr{U}})$, such that $\mathbb{P}$ is ω_1-closed (that is, every descending sequence of conditions of countable length has a lower bound) and adds a stable ordered-union ultrafilter $\mathscr{U}$ on the set $\mathbb{F}$, and that $\mathbb{M}(\mathscr{U})$ is the Matet forcing with sequences from the ultrafilter (and hence it is σ-centred).

Definition 1.4. *Given a* $\sqsubseteq^*$*-descending sequence* $\bar{a}^\alpha$, $\alpha < \beta$, *the notion of forcing* $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ *consists of all pairs* $(s, \bar{a})$, *such that* $s \in \mathbb{F}$ *and* $\bar{a}$ *is an end segment of one of the* $\bar{a}^\alpha$*'s and* $s < \min(a_0)$. *The forcing order is the same as in the Matet forcing.*

We shall use $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ for $\sqsubseteq^*$-descending sequences of length 1, of length $< \kappa$ and of length κ. The forcing $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$ diagonalises ("shoots a real through") $\bigcup\{a^\alpha_n : n < \omega\}$, $\alpha < \beta$.

Note that for a $\sqsubseteq^*$-descending sequence with a last element, $\mathbb{M}(\bar{a}^\alpha : \alpha \leq \beta)$ is equivalent to $\mathbb{M}(\bar{a}^\beta)$ and this is in turn equivalent to Cohen forcing. However, $\mathbb{M}(\bar{a}^\gamma)$ is not a complete suborder of $\mathbb{M}(\bar{a}^\alpha : \alpha < \beta)$.

We shall show that given a set of κ groupwise dense families, there are $\bar{a}^\alpha$, $\alpha < \kappa$, such that $\mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ adds a real through all the families. This is similar to the fact shown by Blass [4], that the original Matet forcing $\mathbb{M}$ adds a real that lies in all groupwise dense families from the ground model. By unpublished results of Blass and Laflamme [4], Matet forcing preserves P-points and hence, by the iteration theorem for preserving P-points [7], it preserves $\mathfrak{u}$. However, our finite support iteration of iterands of the form $\mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ and other iterands will not preserve $\mathfrak{u}$, as the iteration adds Cohen reals in limit steps and also at some successor steps that force a part of $\mathrm{MA}_{<\kappa}$. We shall only keep $\mathfrak{b}$ small.

We write names for reals in c.c.c. forcings $\mathbb{P}$ in a standardised form $\underset{\sim}{g} = \mathrm{Name}(\bar{k}, \bar{p}) = \{\langle (n, k_{n,m}), p_{n,m}\rangle : n, m \in \omega\}$, such that $\{p_{n,m} : m \in \omega\}$ is predense in $\mathbb{P}$ and $p_{n,m} \Vdash_{\mathbb{P}} \underset{\sim}{g}(n) = k_{n,m}$ and such that $k_{n,m} = k_{n,m'}$ if $p_{n,m}$ and $p_{n,m'}$ are compatible.

Lemma 1.5. *Let* $\bar{a}^\alpha$, $\alpha < \delta$, *be a* $\sqsubseteq^*$*-descending sequence. Assume* $\mathbb{Q} = \mathbb{M}(\bar{a}^\alpha : \alpha < \delta)$ *and* $\mathrm{cf}(\delta) > \aleph_0$ *and* $\underset{\sim}{g}$ *is a* $\mathbb{Q}$*-name for a member of* ${}^\omega\omega$.

Then we can find an $\alpha_0 < \delta$ *such that for every* $\alpha \in [\alpha_0, \delta)$ *there are* $p_{n,m} \in \mathbb{M}(\bar{a}^\alpha)$ *and* $k_{n,m} \in \omega$ *such that* $\{p_{n,m} : m < \omega\}$ *is predense in* $\mathbb{Q}$ *and* $p_{n,m} \Vdash_{\mathbb{Q}} \underset{\sim}{g}(n) = k_{n,m}$.

Proof. We assume that $\underset{\sim}{g} = \{\langle (n, h_{n,m}), q_{n,m}\rangle : m, n < \omega\}$. Since $\mathrm{cf}(\delta) > \omega$, there is some $\alpha_0 < \kappa$ such that all $q_{n,m}$ are in $\mathbb{M}(\bar{a}^\beta : \beta \leq \alpha)$. Now, given $\alpha \in [\alpha_0, \delta)$, we take

$$I_n = \{q \in \mathbb{M}(\bar{a}^\alpha) : (\exists m)(q \leq_{\mathbb{Q}} q_{n,m})\}.$$

Then I_n is predense in $\mathbb{Q}$. Now let $p_{n,m}$, $m < \omega$, list I_n and choose $k_{n,m}$ such that $p_{n,m} \Vdash_{\mathbb{Q}} \underset{\sim}{g}(n) = k_{n,m}$. Then $\bar{k}$, $\bar{p}$ describe $\underset{\sim}{g}$ as desired. □

The following lemma will be used in those successor steps of our planned iterated forcing in which we want to add an infinite set that is in κ groupwise dense sets at the same time.

Lemma 1.6. *Assume that* κ *is a regular uncountable cardinal,* $2^\omega = \kappa$, $\mathrm{MA}_{<\kappa}$*(*σ*-centred),* $\{\mathscr{G}_\alpha : \alpha < \kappa\}$ *is a set of groupwise dense subsets and that* $\bar{f} = \langle f_\alpha : \alpha < \kappa\rangle$ *is a* $\leq^*$*-increasing and -unbounded sequence of functions in* ${}^\omega\omega$*. Then there is a* σ*-centred forcing notion* $\mathbb{Q}$ *of size* κ *such that*

$$\Vdash_{\mathbb{Q}} \text{“}\bar{f} \text{ is unbounded } \wedge \exists X \in [\omega]^\omega \bigwedge_{\alpha<\kappa} X \in \mathscr{G}_\alpha.\text{”}$$

Proof. We shall build $\mathbb{Q} = \mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ by choosing $\bar{a}^\alpha \in (\mathbb{F})^\omega$ by induction on $\alpha < \kappa$ such that $\bar{a}^\beta \sqsubseteq^* \bar{a}^\alpha$ for $\alpha < \beta$. Since $\mathrm{cf}(\kappa) > \omega$, each $\mathbb{Q}$-name for a real has an equivalent $\mathbb{M}(\bar{a}^\beta)$-name for all sufficiently large β. We shall show that we can choose $\mathbb{Q}$ carefully, with a sealing argument, such that in the end there will be no name for a new function dominating all the f_α, $\alpha < \kappa$.

Now we carry out the construction. Let $\langle \bar{b}^\alpha, \underset{\sim}{g}^\alpha : \alpha < \kappa\rangle$ list the pairs $(\bar{b}, \underset{\sim}{g})$ such that $\bar{b} \in (\mathbb{F})^\omega$ and $\underset{\sim}{g} = \{\langle (n, k_{n,m}), p_{n,m}\rangle : m, n \in \omega\}$ is an $\mathbb{M}(\bar{b})$-name for a function in ${}^\omega\omega$ such that each pair $(\bar{b}, \underset{\sim}{g})$ appears κ many times.

Now we shall choose by induction on $\alpha < \kappa$ some $\bar{a}^\alpha \in (\mathbb{F})^\omega$ with the following properties:

(a) If $\beta < \alpha$ then $\bar{a}^\alpha \sqsubseteq^* \bar{a}^\beta$.

(b) If $\alpha = 2\beta + 1$, then $\bigcup_{n<\omega} a_n^\alpha \in \mathscr{G}_\beta$.

(c) If $\alpha = 2\beta+2$ and for some $\gamma < 2\beta+2$, $\bar{b}^\beta = \bar{a}^\gamma$ and $\underset{\sim}{g}^\beta$ is a $\mathbb{M}(\bar{b}^\beta)$-name of a member of ${}^\omega\omega$ that can be construed as an $\mathbb{M}(\bar{a}^{2\beta+1})$-name, then $\bar{a}^\alpha$ guarantees that for some $\zeta_\alpha < \kappa$,

$$\Vdash_{\mathbb{Q}} \underset{\sim}{g}^\beta \not\geq^* f_{\zeta_\alpha}.$$

For $\alpha = 0$ we let $\bar{a}^0 = \langle \{n\} : n < \omega \rangle$.

Let $\alpha < \kappa$ be a limit ordinal. We apply $\mathrm{MA}_{<\kappa}$(σ-centred) to the σ-centred forcing notion $\{(\bar{a}, n, F) : \bar{a}$ is a finite unmeshed sequence of subsets of n and F is a finite subset of $\alpha\}$, ordered by $(\bar{b}, n', F') \le (\bar{a}, n, F)$ iff $n' \ge n$, $F' \supseteq F$, and $\bar{b} = \bar{a}\hat{\ }\bar{c}$ with $c_i \cap n = \emptyset$ and $(\forall \gamma \in F)(\forall k)(b_k \subseteq [n, n') \to b_k \in FU(\bar{a}^\gamma))$, and the dense sets $\mathscr{I}_{\beta,n} = \{(\bar{a}, m, F) : \bigcup \bar{a} \setminus n \ne \emptyset \wedge \beta \in F \wedge m \ge n\}$, $\beta < \alpha$, $n < \omega$, and thus we get a filter G intersecting all the $\mathscr{I}_{\beta,n}$ and set $\bar{a}^\alpha = \bigcup \{\bar{a} : (\exists n, F)((\bar{a}, n, F) \in G)\}$. Then $\bar{a}^\alpha$ is as desired.

Step $\alpha = 2\beta + 1$. We show that, given $\mathscr{G}_\beta$ and $\bar{a}^{2\beta}$, there is some condensation $\bar{a}^{2\beta+1} \sqsubseteq^* \bar{a}^{2\beta}$ such that $\bigcup_n a_n^{2\beta+1} \in \mathscr{G}_\beta$: We apply the definition of groupwise density to the partition $\{[\min(a_n^{2\beta}), \min(a_{n+1}^{2\beta})) : n < \omega\}$ and get an infinite set I such that $\bigcup \{[\min(a_i^{2\beta}), \min(a_{i+1}^{2\beta})) : i \in I\} \in \mathscr{G}_\beta$. Then also $\bigcup \{a_i^{2\beta} : i \in I\} \in \mathscr{G}_\beta$. Then we re-index the sequence $\langle a_i^{2\beta} : i \in I \rangle$ by the natural numbers, so $a_n^{2\beta+1} = a_{i_n}^{2\beta}$ for the increasing enumeration $\langle i_n : n < \omega \rangle$ of I.

Step $\alpha = 2\beta + 2$. We assume that for some $\gamma < 2\beta + 2$, $\bar{b}^\beta = \bar{a}^\gamma$ and $\underset{\sim}{g}^\beta$ is a $\mathbb{M}(\bar{b}^\beta)$-name of a member of ${}^\omega\omega$ that has an equivalent $\mathbb{M}(\bar{a}^{2\beta+1})$-name. Otherwise we can take $\bar{a}^{2\beta+2} = \bar{a}^{2\beta+1}$.

For each $n < \omega$ we choose a finite set $a_n^{\alpha+}$ such that $a_n^{2\beta+1}$ is an initial segment of $a_n^{\alpha+}$ and there is some $u_n \subseteq \{n, n+1, \dots, \ell_n - 1\}$ such that $n \in u_n$ and

$$a_n^{\alpha+} = \bigcup \{a_\ell^{2\beta+1} : \ell \in u_n\}$$

and such that for every $w \subseteq \{0, 1, \dots, \min(a_n^{2\beta+1}) - 1\}$ there is some $m_n^\beta(w)$ such that

$$p^\beta_{n, m_n^\beta(w)} \ge (w \cup a_n^{\alpha+}, \bar{a}^{2\beta+1} \restriction [\ell_n, \omega)).$$

Since there are only finitely many $w \subseteq \min a_n^{2\beta+1}$, there is such an $a_n^{\alpha+}$.

Now in order to be able to concatenate the $a_n^{\alpha+}$ and in order to ensure that $\underset{\sim}{g}^\beta$ will not be a dominating function we thin out: Let $k(w, n)$ be one $k^\beta_{n, m_n^\beta(w)}$ that is in $\underset{\sim}{g}^\beta$ together with $p^\beta_{n, m_n^\beta(w)} \ge (w \cup a_n^{\alpha+}, \bar{a}^{2\beta+1} \restriction [\ell_n, \omega))$. Now we take $h(n) = \max\{k(w, n) : w \subseteq \min(a_n^{2\beta+1})\}$. By our premise on $\bar{f}$ there is some $\zeta_\alpha < \kappa$ that that $X = \{n \in \omega : h(n) < f_{\zeta_\alpha}(n)\}$ is infinite. Now we choose an infinite $Y \subseteq X$ such that $(\forall n \in Y)(\ell_n < \min(Y \setminus (n+1)))$. Let n_i^β, $i \in \omega$, enumerate Y. Then we set $\bar{a}^\alpha = \langle a^{\alpha+}_{n_i^\beta} : i < \omega \rangle$.

For every $n \in Y$ and $w \subseteq \min(a_n^{2\beta+1})$ we have that
$(w \cup a_n^\alpha, \bar{a}^\alpha \restriction [n+1, \omega)) \le_{\mathbb{Q}} (w \cup a_n^\alpha, \bar{a}^{2\beta+1} \restriction [\ell_n, \omega))$.

Now we show that $\mathbb{Q} = \mathbb{M}(\bar{a}^\alpha : \alpha < \kappa)$ is as desired. It is σ-centred, because for every $w \in \mathbb{F}$, $\mathbb{Q}_w = \{(w, \bar{a}^\beta \restriction [\ell, \omega)) : \ell \in \omega, w < a^\beta_\ell, \beta \in \kappa\}$ is centred.

Then the generic $W = \bigcup\{w : \exists \bar{a}(w, \bar{a}) \in G\}$ is an infinite subset of ω and since every $(w, \bar{a}) \in \mathbb{Q}$ forces in $\mathbb{Q}$ that $w \subseteq W \subseteq w \cup \bigcup\{a_n : n < \omega\}$, we have by the choice of the $\bar{a}^\alpha$ in the odd steps, that the generic W is in each $\mathscr{G}_\alpha$, $\alpha < \kappa$.

Now we show that

$$\Vdash_{\mathbb{Q}} \bar{f} \text{ is unbounded.}$$

Assume towards a contradiction that there is a $\mathbb{Q}$-name $\underset{\sim}{g}$ for a real and there is $p \in \mathbb{Q}$ such that $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{g}$ dominates $\bar{f}$". By Lemma 1.5 there is some $\gamma < \kappa$ such that $\underset{\sim}{g}$ is an $\mathbb{M}(\bar{a}^\gamma)$-name. Then for some $\beta \geq \gamma$ we have $(\bar{b}^\beta, \underset{\sim}{g}^\beta) = (\bar{a}^\gamma, \underset{\sim}{g})$. So at stage $\alpha = 2\beta + 2$ in our construction we take care of $\underset{\sim}{g}$'s equivalent $\mathbb{M}(\bar{a}^{2\beta+1})$-name $\text{Name}(\bar{k}^\beta, \bar{p}^\beta)$. Let ζ_α and $\bar{a}^\alpha$ be as in this step. Assume that there are some $p \geq q$ and some $n(*)$ such that $q \Vdash_{\mathbb{Q}} (\forall n \geq n(*))(\underset{\sim}{g}(n) \geq^* f_{\zeta_\alpha}(n))$. By the form of $\mathbb{Q}$, $q = (s, \bar{a}^{\alpha(1)})$ for some $\alpha(1) \geq \alpha$ and some s, such that $\bar{a}^{\alpha(1)}$ is a condensation of $\bar{a}^\alpha$. So there is some $n^\beta_i \geq n(*)$ such that there are r_i, r_{i+1} and j such that $a^{\alpha(1)}_j \subseteq r_{i+1}$ and $a^{\alpha(1)}_j \cap [r_i, r_{i+1}) = a^{\alpha+}_{n^\beta_i} = a^\alpha_i$. Then we set $s' = s \cup (\bigcup \bar{a}^{\alpha(1)} \cap [0, r_i))$, and we set $q' = (s' \cup a^\alpha_i, a^{\alpha(1)}_{j+1}, \dots)$.

We set $m^\beta_{n^\beta_i}(s') = m$. Then q' witnesses that q and $p^\beta_{n^\beta_i, m}$ are compatible, because $q \geq q'$ and $p^\beta_{n^\beta_i, m} \geq q'$. However, our choice of m yields $p^\beta_{n^\beta_i, m} \Vdash_{\mathbb{Q}} \underset{\sim}{g}(n^\beta_i) = k^\beta_{n^\beta_i, m} < f_{\zeta_\alpha}(n^\beta_i)$. Contradiction. □

2. A finite support iteration

Now we describe a finite support iteration.

Theorem 2.1. *Let $\kappa = \text{cf}(\kappa) > \aleph_1$ and assume $\kappa^{<\kappa} = \kappa$ and assume that $\diamondsuit(S)$ holds for some stationary $S \subseteq \{\alpha < \kappa^+ : \text{cf}(\alpha) = \kappa\}$. There is some finite support iteration $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa^+, \beta \leq \kappa^+ \rangle$ such that*

$$\Vdash_{\mathbb{P}_{\kappa^+}} \text{MA}_{<\kappa} \wedge 2^\omega = \kappa^+ \wedge \mathfrak{g} = \kappa^+ \wedge \mathfrak{b} = \kappa.$$

Proof. By $\diamondsuit(S)$ there is $\bar{Y} = \langle Y_\delta : \delta \in S \rangle$, such that $Y_\delta \subseteq \delta$ and for all $Y \subseteq \kappa^+$ the set $\{\delta \in S : Y_\delta = Y \cap \delta\}$ is a stationary subset of κ.

As the ground model has $\kappa^{<\kappa} = \kappa$, we can fix an enumeration $\underset{\sim}{\mathbb{Q}}'_\beta$, $\beta \in \kappa^+ \setminus (S \cup \kappa)$ of all c.c.c. names of partial orders on all ordinals $< \kappa$, such that each name appears cofinally often before each $\alpha \in \kappa^+$ of cofinality κ.

We choose $\underset{\sim}{\mathbb{Q}}_\beta$ by induction on $\beta < \kappa^+$. In the first κ steps we add κ Hechler reals $\underset{\sim}{f}_\alpha$, $\alpha < \kappa$, and these will be the $\leq^*$-increasing unbounded sequence whose unboundedness will be preserved through the rest of the iteration.

In the following steps we distinguish two cases: First case: If $\beta \in S$ and $\Vdash_{\mathbb{P}_\beta}$ "Y_β is a code for a $\mathbb{P}_\beta$-name of a family $\{\underset{\sim}{\mathscr{G}}_\zeta : \zeta < \kappa\}$ of κ groupwise dense subsets of $[\omega]^\omega$". Then we take $\underset{\sim}{\mathbb{Q}}_\beta$ such that $\Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}_\beta$ is as in Lemma 1.6," and we get $\Vdash_{\mathbb{P}_\beta * \underset{\sim}{\mathbb{Q}}_\beta}$ "there is an infinite subset of ω that is in each $\underset{\sim}{\mathscr{G}}_\zeta$, $\zeta < \kappa$".

Second case: Not all the criteria from the first case are fulfilled. Then, as in the usual iteration for Martin's axiom, $\underset{\sim}{\mathbb{Q}}_\beta$ will be $\underset{\sim}{\mathbb{Q}}'_\beta$ with weights p, where we have $p \Vdash_{\mathbb{P}_\beta}$ "$\underset{\sim}{\mathbb{Q}}'_\beta$ is a c.c.c. forcing of cardinality less than κ", and $\underset{\sim}{\mathbb{Q}}_\beta$ will be the trivial partial order with orthogonal weight.

As $\kappa^{<\kappa} = \kappa$ also in the final model we have $\mathrm{MA}_{<\kappa}$, because if $\mathbb{P}$ is a c.c.c.-notion of forcing of cardinality $< \kappa$ in $\mathbf{V}^{\mathbb{P}_{\kappa^+}}$ and if $\gamma < \kappa$ and D_α, $\alpha < \gamma$, is a sequence of predense subsets of $\mathbb{P}$, then this holds in an initial segment $\mathbf{V}^{\mathbb{P}_\delta}$ for some $\delta \in \kappa^+ \setminus S$ and hence by what we did in successor steps for $\delta \notin S$, there is a directed $G \subseteq \mathbb{P}$ such that $\bigwedge_{\alpha<\gamma} G \cap D_\alpha \neq \emptyset$.

By Lemma 1.6, in each Matet step of the iteration the unbounded family f_α, $\alpha < \kappa$, is preserved. By [1, 2.1] also in each extension by $\mathbb{Q}$ of size $< \kappa$ the unbounded family is preserved. By the preservation theorem for finite support iterations from [2, 6.5.3], the unbounded well-ordered family f_α, $\alpha < \kappa$, is preserved in all limit steps of the iteration. Thus we have $\mathfrak{b} = \kappa$ in the extension.

Let $\mathscr{G}_\alpha$, $\alpha < \kappa$, be a family of groupwise dense sets in $V^{\mathbb{P}}$. As $\langle Y_\delta : \delta \in S \rangle$ is a diamond sequence and as being κ groupwise dense families reflects down into a κ-club set in κ^+ (a proof for the countable support iteration of proper forcings is given by [6], and a simpler form thereof works for finite support iteration of c.c.c. forcings), at stationarily many steps Y_δ guesses a name for $\mathscr{G}_\alpha \cap \mathbf{V}^{\mathbb{P}_\delta}$, $\alpha < \kappa$, and by the choice of $\mathbb{P}_{\delta+1}$ in the first case, the forcing adds a real that is in all the $\mathscr{G}_\alpha$. Hence $\mathfrak{g} = \kappa^+$. □

Corollary 2.2. $\aleph_2 \leq \mathfrak{b} < \mathfrak{g}$ *is consistent relative to* ZFC.

Proof. We take a ground model of GCH and then we force $\diamondsuit(S)$ for some stationary $S \subseteq \{\alpha < \aleph_3 : \mathrm{cf}(\alpha) = \aleph_2\}$. Then we apply the previous theorem with $\kappa = \aleph_2$. □

3. An upper bound on $\mathfrak{g}$

Definition 3.1. *Let κ be a regular cardinal. On ${}^\kappa\kappa$ we define the almost order $f \leq^* g$ iff there is some $\alpha < \kappa$ such that for all $\beta \geq \alpha$, $f(\beta) \leq g(\beta)$. A set $D \subseteq {}^\kappa\kappa$ is called dominating in $({}^\kappa\kappa, \leq^*)$ iff for every $f \in {}^\kappa\kappa$ there is some $g \in D$ such that $g \geq^* f$. So we have the dominating number $\mathfrak{d}_\kappa$ which is the smallest size of a dominating set.*

Theorem 3.2. $\mathfrak{g} \leq \mathfrak{d}_\mathfrak{b}$.

Proof. Let $D = \{f_\varepsilon : \varepsilon < \mathfrak{d}_\mathfrak{b}\}$ be a dominating family. We shall build groupwise dense families $\mathscr{G}_f$, $f \in D$, such that their intersection is empty. First we introduce some notation and present a characterisation of $\mathfrak{b}$ in terms of a slightly different ordering than $\leq^*$ on ${}^\omega\omega$.

Definition 3.3. *(1)* $\mathrm{Inc}(\omega) = \{\bar{n} : \bar{n} = \langle n_i : i < \omega\rangle$ *is increasing*$\}$.

(2) *(*[5, Def. 2.9]*)* $\bar{m} \leq^{**} \bar{n}$ *iff* $(\forall^\infty i)(|\{j : m_j \in [n_i, n_{i+1}]\}| \geq 2)$.

We thank Boaz Tsaban for telling us that the following lemma was originally proved by Blass. We nevertheless let our proof stand, since it is self-contained and in contrast to Blass' elegant proof, does not speak about morphisms and duality.

Lemma 3.4. *(*[5, Theorem 2.10]*)*

(1) *$\leq^{**}$ is a partial order.*

(2) $(\mathrm{Inc}(\omega), \leq^{**})$ *is $\mathfrak{b}$-directed.*

(3) *There is an $\leq^{**}$-increasing sequence of length $\mathfrak{b}$ with no upper bound.*

Proof. (1) is easy. (2) Let $\gamma < \mathfrak{b}$ and $\bar{n}_\alpha$, $\alpha < \gamma$, be given. We first need the two-fold iteration operation. For a strictly increasing function $f\colon \omega \to \omega$ we define $\tilde{f}$ by $\tilde{f}(0) = 0$, $\tilde{f}(n+1) = f(f(\tilde{f}(n)))$. We take $f \geq^* \bar{n}_\alpha$ for all $\alpha < \gamma$.

Now we have $(\forall \alpha < \gamma)(\forall^\infty i)(f(i) \geq n_\alpha(i))$. We show that $\tilde{f} \geq^{**} \bar{n}_\alpha$ for all $\alpha < \gamma$. We fix α and take i_0 so that $(\forall i \geq i_0)(f(i) \geq n_\alpha(i) \wedge f(\tilde{f}(i)) - \tilde{f}(i) \geq 2)$. Then for $i \geq i_0$ we get: $\tilde{f}(i+1) = f(f(\tilde{f}(i)))$ and $f(\tilde{f}(i)) \geq n_\alpha(\tilde{f}(i)) \geq \tilde{f}(i)$ and $f(f(\tilde{f}(i))) \geq n_\alpha(f(\tilde{f}(i)))$, so at least $n_\alpha(\tilde{f}(i))$, $n_\alpha(\tilde{f}(i)+1)$, $\dots$, $n_\alpha(f(\tilde{f}(i)))$ are in the interval $[\tilde{f}(i), \tilde{f}(i+1)]$, so at least 2 elements.

(3) Let f_α, $\alpha < \mathfrak{b}$, be an unbounded family of strictly increasing functions. We let $n_{\alpha,i} = f_\alpha(i)$. There is no $\bar{n} \geq^{**} \bar{n}_\alpha$ for all $\alpha < \mathfrak{b}$ as otherwise $\bar{n} \geq^* f_\alpha$ for all $\alpha < \mathfrak{b}$. Now we use (2) to choose by induction on $\alpha < \mathfrak{b}$ an $\leq^{**}$-increasing sequence $\langle \bar{m}_\alpha : \alpha < \mathfrak{b}\rangle$ by taking for each $\alpha < \mathfrak{b}$ some $\bar{m}_\alpha \geq^{**} \bar{n}_\alpha$ such that $\bar{m}_\alpha \geq^{**} \bar{m}_\beta$ for all $\beta < \alpha$. □

Definition 3.5. *Let* $\langle \bar{n}_\alpha \,:\, \alpha < \mathfrak{b}\rangle$ *be a* $\leq^{**}$*-increasing and -unbounded sequence in* $\mathrm{Inc}(\omega)$*.*

(1) Let $A \in [\omega]^\omega$ *and* $\bar{n} \in \mathrm{Inc}(\omega)$*. We let* $\mathrm{In}(A, \bar{n}) = \{i \,:\, A \cap [n_i, n_{i+1}) \neq \emptyset\}$*.*

(2)

$$\mathscr{G}(\langle \bar{n}_\alpha \,:\, \alpha < \mathfrak{b}\rangle) = \{A \in [\omega]^\omega \,:\, (\exists \alpha) \\ \langle n_{\alpha,i} \,:\, i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle \geq^{**} \bar{n}_{\alpha+1}\}$$

Remark: Since $\bar{n}_\alpha$, $\alpha < \mathfrak{b}$, is increasing and unbounded, there is some minimal $\beta \geq \alpha$ such that $\langle n_{\alpha,i} \,:\, i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle \not\geq^{**} \bar{n}_\beta$. The requirement for $\bar{n}_\beta$ in the definition of $\mathscr{G}(\langle \bar{n}_\alpha \,:\, \alpha < \mathfrak{b}\rangle)$ goes in the opposite direction: $\bar{n}_\alpha \leq^{**} \bar{n}_\beta \leq^{**} \langle n_{\alpha,i} \,:\, i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle$ and hence A has to be sufficiently small.

Lemma 3.6. *If* $\langle \bar{n}_\alpha \,:\, \alpha < \mathfrak{b}\rangle$ *is* $\leq^{**}$*-unbounded and* $\alpha_0 < \mathfrak{b}$*, then* $\mathscr{G}(\langle \bar{n}_\alpha \,:\, \alpha_0 < \alpha < \mathfrak{b}\rangle)$ *is groupwise dense.*

Proof. We have that $\mathrm{In}(B, \bar{n}_\alpha) \subseteq^* \mathrm{In}(A, \bar{n}_\alpha)$ if $B \subseteq^* A$ and thus $\mathscr{G}(\langle \bar{n}_\alpha \,:\, \alpha_0 < \alpha < \mathfrak{b}\rangle)$ is closed under infinite almost subsets. Now let a partition $\{[\pi_i, \pi_{i+1}) \,:\, i < \omega\}$ be given and set $\bar{\pi} = \langle \pi_{2i} \,:\, i < \omega\rangle$. Then take $\alpha \geq \alpha_0$ such that $\bar{n}_\alpha \not\leq^{**} \bar{\pi}$. So there are infinitely many i such that there is at most one element j such that $n_{\alpha,j} \in [\pi_{2i}, \pi_{2i+2}]$.

Now we inductively choose increasing sequences i_n, j_n, j'_n, $n \in \omega$ and $u_n \in 2$. We take i_0 such that there is at most one $n_{\alpha,j} \in [\pi_{2i_0}, \pi_{2i_0+2}]$ and such that there is some $n_{\alpha,j} \leq \pi_{2i_0+2}$. We name the largest j such that $n_{\alpha,j} \leq \pi_{2i_0+2}$ to be j_0. If $n_{\alpha,j_0} \leq \pi_{2i_0+1}$, then let $j'_0 = j_0$, otherwise let $j'_0 = j_0 - 1$.

Now let i_n and j_n be defined. Then we take $i_{n+1} > i_n$ such that there is at most one $n_{\alpha,j}$ in $[\pi_{2i_{n+1}}, \pi_{2i_{n+1}+2}]$ and again we let $j_{n+1} > j_n$ be so that $n_{\alpha,j_{n+1}}$ is the largest $n_{\alpha,j} \leq \pi_{2i_{n+1}+2}$. If $n_{\alpha,j_{n+1}} \leq \pi_{2i_{n+1}+1}$, then let $j'_{n+1} = j_{n+1}$, otherwise let $j'_{n+1} = j_{n+1} - 1$. In addition we take i_{n+1} so large such that $[n_{\alpha,j'_n}, n_{\alpha,j'_{n+1}}]$ contains at least two different $n_{\alpha+1,j}$. We let $u_n = 1 - (j_n - j'_n)$ and finally we let $A = \bigcup\{[\pi_{2i_n+u_n}, \pi_{2i_n+u_n+1}) \,:\, n \in \omega\}$. By the construction, $\mathrm{In}(A, \bar{n}_\alpha)$ is infinite and $\langle n_{\alpha,i} \,:\, i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle = \langle n_{\alpha,j'_n} \,:\, n \in \omega\rangle \geq^{**} \bar{n}_{\alpha+1}$. □

Proof of Theorem 3.2. Suppose that $\{f_\varepsilon \,:\, \varepsilon < \mathfrak{d}_\mathfrak{b}\}$ is a dominating family. We take some fixed $\leq^{**}$-increasing and -unbounded sequence $\langle \bar{n}_\gamma \,:\, \gamma < \mathfrak{b}\rangle$. For each $\varepsilon < \mathfrak{d}_\mathfrak{b}$ let

$$E_\varepsilon = \{\alpha < \mathfrak{b} \,:\, (\forall \beta < \alpha)(f_\varepsilon(\beta) < \alpha)\}.$$

This is a club in the regular cardinal $\mathfrak{b}$, and let $\langle \xi_{\varepsilon,\alpha} : \alpha < \mathfrak{b}\rangle$ be the increasing continuous enumeration of it. We show that

$$\bigcap_{\varepsilon \in \mathfrak{d}_{\mathfrak{b}}, \alpha_0 < \mathfrak{b}} \mathscr{G}(\langle \bar{n}_{\xi_{\varepsilon,\alpha}} : \alpha_0 < \alpha < \mathfrak{b}\rangle) = \emptyset.$$

Assume towards a contradiction that A is infinite and in this intersection. We define $f_A \colon \mathfrak{b} \to \mathfrak{b}$ by

$$f_A(\alpha) = \min\{\gamma : \gamma \geq \alpha \wedge \langle n_{\alpha,i} : i \in \mathrm{In}(A, \bar{n}_\alpha)\rangle \not\geq^{**} \bar{n}_\gamma\}.$$

Since f_ε, $\varepsilon < \mathfrak{d}_{\mathfrak{b}}$, is a dominating family, there is some ε and some α_0 such that for all $\alpha \geq \alpha_0$, $f_A(\alpha) \leq f_\varepsilon(\alpha)$. Since $A \in \mathscr{G}(\langle \bar{n}_{\xi_{\varepsilon,\beta}} : \alpha_0 < \beta < \kappa\rangle)$, there is some $\alpha_0 < \xi_{\varepsilon,\beta} \in E_\varepsilon$ such that $\langle n_{\xi_{\varepsilon,\beta},i} : i \in \mathrm{In}(A, \bar{n}_{\xi_{\varepsilon,\beta}})\rangle \geq^{**} \bar{n}_{\xi_{\varepsilon,\beta+1}}$.

Hence $\xi_{\varepsilon,\beta+1} < f_A(\xi_{\varepsilon,\beta})$. But $\xi_{\varepsilon,\beta+1} \in E_\varepsilon$, that means $f_\varepsilon(\xi_{\varepsilon,\beta}) < \xi_{\varepsilon,\beta+1} < f_A(\xi_{\varepsilon,\beta})$, which contradicts the choice of ε and α_0. $\square$

Remark: So Theorem 3.2 shows that c.c.c. forcing of any length over a model of GCH will give $\mathfrak{g} \leq \mathfrak{d}_{\mathfrak{b}} = \mathfrak{b}^+$, since c.c.c. forcing does not increase the value of $\mathfrak{d}_{\mathfrak{b}}$ if it preserves the value of $\mathfrak{b}$.

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