

THE SPECTRUM OF CHARACTERS
OF ULTRAFILTERS ON ω
SH846

SAHARON SHELAH

The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Edmond J. Safra Campus, Givat Ram
Jerusalem 91904, Israel

Department of Mathematics
Hill Center-Busch Campus
Rutgers, The State University of New Jersey
110 Frelinghuysen Road
Piscataway, NJ 08854-8019 USA

ABSTRACT. We show the consistency of statement: “the set of regular cardinals which are the character of some ultrafilter on ω is not convex”. We also deal with the set of π -characters of ultrafilters on ω .

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§0 INTRODUCTION

Some cardinal invariants of the continuum are actually the minimum of a natural set of cardinals $\leq 2^{\aleph_0}$ which can be called the spectrum of the invariant. Such a case is Sp_χ , the set of characters $\chi(D)$ of non-principal ultrafilters on ω (the minimal number of generators). On the history see [BnSh 642]; there this spectrum and others were investigated and it was asked if Sp_χ can be non-convex (formally 0.2(2) below).

The main result here is 1.1, it solves the problem (starting with a measurable). This was presented in a conference in honor of Juhasz, quite fitting as he had started the investigation of consistency on $\chi(D)$. In §2 we note what we can say on the strict π -character of ultrafilters.

The investigation is continued in [Sh:915] trying to get more “disorderly” behaviours in smaller cardinals and in particular answering negatively the original question, 0.2(2).

Recall

- 0.1 Definition.** 1) $\text{Sp}_\chi = \text{Sp}(\chi)$ is the set of cardinals θ such that: $\theta = \chi(D)$ for some non-principal ultrafilter D on ω where
 2) For D an ultrafilter on ω let $\theta = \chi(D)$ be the minimal cardinality θ such that D is generated by some family of θ members, i.e. $\text{Min}\{|\mathcal{A}| : \mathcal{A} \subseteq D \text{ and } (\forall B \in D)(\exists A \in \mathcal{A})[A \subseteq^* B]\}$, it does not matter if we use “ $A \subseteq B$ ”.

Now, Brendle and Shelah [BnSh 642, Problem 5], asked the question formulated in 0.2(2) below, but it seems to me, at least now that the question is really 0.2(1)+(3).

- 0.2 Problem** 1) Can $\text{Sp}(\chi) \cap \text{Reg}$ have gaps, i.e., can it be that $\theta < \mu < \lambda$ are regular, $\theta \in \text{Sp}(\chi)$, $\mu \notin \text{Sp}(\chi)$, $\lambda \in \text{Sp}(\chi)$?
 2) In particular does $\aleph_1, \aleph_3 \in \text{Sp}(\chi)$ imply $\aleph_2 \in \text{Sp}(\chi)$?
 3) Are there any restrictions on $\text{Sp}(\chi) \cap \text{Reg}$?

We thank the referee for helpful comments and in particular 2.5(1).

Discussion: This rely on [Sh 700, §4], there is no point to repeat it but we try to give a description. Let $\aleph_0 < \kappa < \mu < \lambda$ be regular cardinals, κ measurable.

Let $S = \{\alpha < \lambda : \text{cf}(\alpha) \neq \kappa\}$ or any unbounded subset of it. We define ([Sh 700, 4.3]) the class $\mathfrak{K} = \mathfrak{K}_{\lambda, S}$ of objects \mathfrak{t} approximating our final forcing. Each $\mathfrak{t} \in K$ consists mainly of a finite support iteration $\langle \mathbb{P}_i^{\mathfrak{t}}, \mathbb{Q}_i^{\mathfrak{t}} : i < \mu \rangle$ of c.c.c. forcing of cardinality $\leq \lambda$ with limit $\mathbb{P}_{\mathfrak{t}}^* = \mathbb{P}^{\mathfrak{t}} = \mathbb{P}_{\mu}^{\mathfrak{t}}$, but also $Q_i^{\mathfrak{t}}$ -names $\tau_i^{\mathfrak{t}}$ ($i < \mu$) so formally

\mathbb{P}_{i+1}^t -names, satisfying a strong version of the c.c.c. and for $i \in S$, also \underline{D}_i^t , a \mathbb{P}_i^t -name of a non-principal ultrafilter on ω from which \mathbb{Q}_i^t is nicely defined and \underline{A}_i^t , a \mathbb{Q}_i^t -name (so \mathbb{P}_{i+1}^t -name) of a pseudo-intersection (and $\mathbb{Q}_i, i \in S$, nicely defined) of \underline{D}_i^t such that $i < j \in S \Rightarrow \underline{A}_i^t \in \underline{D}_j^t$. So $\{\underline{A}_i : i \in S\}$ witness $\mathfrak{u} \leq \mu$ in $\mathbf{V}^{\mathbb{P}^t}$; not necessarily we have to use nicely defined \mathbb{Q}_i , though for $i \in S$ we do.

The order $\leq_{\mathfrak{R}}$ is natural order, we prove the existence of the so-called canonical limit.

Now a major point of [Sh 700] is: for $\mathfrak{s} \in \mathfrak{R}$ letting \mathcal{D} be a uniform κ -complete ultrafilter on κ , (or just κ_1 -complete $\aleph_0 < \theta < \kappa$), we can consider $\mathfrak{t} = \mathfrak{s}^\kappa / \mathcal{D}$; by Los theorem, more exactly by Hanf's Ph.D. Thesis, (the parallel of) Los theorem for $\mathbb{L}_{\kappa, \kappa}$ apply, it gives that $\mathfrak{t} \in \mathfrak{R}$, well if $\lambda = \lambda^\kappa / \mathcal{D}$; and moreover $\mathfrak{s} \leq_{\mathfrak{R}} \mathfrak{t}$ under the canonical embedding.

The effect is that, e.g. being "a linear order having cofinality $\theta \neq \kappa$ " is preserved, even by the same witness whereas having cardinality $\theta < \lambda$ is not necessarily preserved, and sets of cardinality $\geq \kappa$ are increased. As \mathfrak{d} is the cofinality (not of a linear order but) of a partial order there are complications, anyhow as \mathfrak{d} is defined by cofinality whereas \mathfrak{a} by cardinality of sets this helps in [Sh 700], noting that as we deal with c.c.c. forcing, names reals are represented by ω -sequences of conditions, the relevant thing are preserved. So we use a $\leq_{\mathfrak{R}}$ -increasing sequence $\langle \mathfrak{t}_\alpha : \alpha \leq \lambda \rangle$ such that for unboundedly many $\alpha < \lambda$, $\mathfrak{t}_{\alpha+1}$ is essentially $(\mathfrak{t}_\alpha^\alpha)^\kappa / \mathcal{D}$.

What does "nice" $\mathbb{Q} = \mathbb{Q}(D)$, for D a non-principal ultrafilter over ω mean? We need that

- (α) \mathbb{Q} satisfies a strong version of the c.c.c.
- (β) the definition commute with the ultra-power used
- (γ) if \mathbb{P} is a forcing notion then we can extend D to an ultrafilter \underline{D}^+ for every (or at least some) \mathbb{P} -name of an ultrafilter \underline{D} extending D we have $\mathbb{Q}(D) \triangleleft \mathbb{P} * \mathbb{Q}(\underline{D}^+)$ (used for the existence of canonical limit).

Such a forcing is combining Laver forcing and Mathias forcing for an ultrafilter D on ω , that is: if $p \in D$ iff p is a subtree of ω with trunk $\text{tr}(p) \in p$ such that for $\eta \in p$ we have $\ell g(\eta) < \ell g(\text{tr}(p)) \rightarrow (\exists! n)(\eta \hat{\ } \langle n \rangle \in p)$ and $\ell g(\eta) \geq \ell g(\text{tr}(p)) \Rightarrow \{n : \eta \hat{\ } \langle n \rangle \in p\} \in D$.

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§1 USING MEASURABLES AND FS ITERATIONS WITH NON-TRANSITIVE MEMORY

We use [Sh 700] in 1.1 heavily. We use measurables (we could have used extenders to get more). The question on $\aleph_1, \aleph_2, \aleph_3$, i.e. Problem 0.2(2) remains open.

1.1 Theorem. *There is a c.c.c. forcing notion \mathbb{P} of cardinality λ such that in $\mathbf{V}^{\mathbb{P}}$ we have $\mathfrak{a} = \lambda, \mathfrak{b} = \mathfrak{d} = \mu, \mathfrak{u} = \mu, \{\mu, \lambda\} \subseteq \text{Sp}_\chi$ but $\kappa_2 \notin \text{Sp}(\chi)$ if*

⊗ κ_1, κ_2 are measurable and $\kappa_1 < \mu = \text{cf}(\mu) < \kappa_2 < \lambda = \lambda^\mu = \lambda^{\kappa_2} = \text{cf}(\lambda)$.

Proof. Let \mathcal{D}_ℓ be a normal ultrafilter on κ_ℓ for $\ell = 1, 2$. Repeat [Sh 700, §4] with (κ_1, μ, λ) here standing for (κ, μ, λ) there, getting $\mathfrak{t}_\alpha \in \mathfrak{K}$ for $\alpha \leq \lambda$ which is $\leq_{\mathfrak{K}}$ -increasing and letting $\mathbb{P}_i^\alpha = \mathbb{P}_i^{\mathfrak{t}_\alpha}$ we have $\mathbb{Q}^\alpha = \langle \mathbb{P}_\varepsilon^\alpha : \varepsilon < \mu \rangle$ is a \leftarrow -increasing continuous sequence of c.c.c. forcing notions, $\mathbb{P}_\mu^\alpha = \mathbb{P}^\alpha = \mathbb{P}_{\mathfrak{t}_\alpha} := \text{Lim}(\mathbb{Q}^\alpha) = \cup\{\mathbb{P}_\varepsilon^\alpha : \varepsilon < \mu\}$; in fact $\langle \mathbb{P}_\varepsilon^\alpha, \mathbb{Q}_\varepsilon^\alpha : \varepsilon < \mu \rangle$ is an FS iterations, etc., but add the demand that for unboundedly many $\alpha < \lambda$

⊗ $_1^\alpha$ $\mathbb{P}^{\alpha+1}$ is isomorphic to the ultrapower $(\mathbb{P}^\alpha)^{\kappa_2}/\mathcal{D}_2$, by an isomorphism extending the canonical embedding.

More explicitly we choose \mathfrak{t}_α by induction on $\alpha \leq \lambda$ such that

- ⊗ $_1$ (a) $\mathfrak{t}_\alpha \in \mathfrak{K}$, see Definition [Sh 700, 4.3] so the forcing notion $\mathbb{P}_i^{\mathfrak{t}_\alpha}$ for $i \leq \mu$ is well defined and is \leftarrow -increasing with i
- (b) $\langle \mathfrak{t}_\beta : \beta \leq \alpha \rangle$ is $\leq_{\mathfrak{K}}$ -increasing continuous which means that:
 - (α) $\gamma \leq \beta \leq \alpha \Rightarrow \mathfrak{t}_\gamma \leq_{\mathfrak{K}} \mathfrak{t}_\beta$, see Definition [Sh 700, 4.6](1) so $\mathbb{P}_i^{\mathfrak{t}_\gamma} \leftarrow \mathbb{P}_i^{\mathfrak{t}_\beta}$ for $i \leq \mu$
 - (β) if α is a limit ordinal then \mathfrak{t}_α is a canonical $\leq_{\mathfrak{K}}$ -u.b. of $\langle \mathfrak{t}_\beta : \beta < \alpha \rangle$, see Definition [Sh 700, 4.6](2)
- (c) if $\alpha = \beta + 1$ and $\text{cf}(\beta) \neq \kappa_2$ then \mathfrak{t}_α is essentially $\mathfrak{t}_\beta^{\kappa_1}/\mathcal{D}_1$ (i.e. we have to identify $\mathbb{P}_\varepsilon^{\mathfrak{t}_\beta}$ with its image under the canonical embedding of it into $(\mathbb{P}_\varepsilon^{\mathfrak{t}_\beta})^{\kappa_1}/\mathcal{D}_1$, in particular this holds for $\varepsilon = \mu$, see Subclaim [Sh 700, 4.9])
- (d) if $\alpha = \beta + 1$ and $\text{cf}(\beta) = \kappa_2$ then \mathfrak{t}_α is essentially $\mathfrak{t}_\beta^{\kappa_2}/\mathcal{D}_2$.

So we need

- ⊗ $_2$ Subclaim [Sh 700, 4.9] applies also to the ultrapower $\mathfrak{t}_\beta^{\kappa_2}/D$. [Why? The same proof applies as $\mu^{\kappa_2}/\mathcal{D}_2 = \mu$, i.e., the canonical embedding of μ into $\mu^{\kappa_2}/\mathcal{D}_2$ is one-to-one and onto (and $\lambda^{\kappa_1}/\mathcal{D}_1 = \lambda^{\kappa_2}/\mathcal{D}_2 = \lambda$, of course).]

Let $\mathbb{P}_\varepsilon^\alpha = \mathbb{P}_\varepsilon^{\mathfrak{t}_\alpha}$ for $\varepsilon \leq \mu$ so $\mathbb{P}^\alpha = \cup\{\mathbb{P}_\varepsilon^\alpha : \varepsilon < \mu\}$ and $\mathbb{P} = \mathbb{P}^\lambda$. It is proved in [Sh 700, 4.10] that in $\mathbf{V}^{\mathbb{P}}$, by the construction, $\mu \in \text{Sp}(\chi)$, $\mathfrak{a} \leq \lambda$ and $\mathfrak{u} = \mu$, $2^{\aleph_0} = \lambda$. By [Sh 700, 4.11] we have $\mathfrak{a} \geq \lambda$ hence $\mathfrak{a} = \lambda$, and always $2^{\aleph_0} \in \text{Sp}(\chi)$ hence $\lambda = 2^{\aleph_0} \in \text{Sp}(\chi)$. So what is left to be proved is $\kappa_2 \notin \text{Sp}(\chi)$. Assume toward contradiction that $p^* \Vdash \underline{D}$ is a non-principal ultrafilter on ω and $\chi(\underline{D}) = \kappa_2$ and let it be exemplified by $\langle \underline{A}_\varepsilon : \varepsilon < \kappa_2 \rangle$.

Without loss of generality $p^* \Vdash_{\mathbb{P}} \underline{A}_\varepsilon \in \underline{D}$ does not belong to the filter on ω generated by $\{\underline{A}_\zeta : \zeta < \varepsilon\} \cup \{\omega \setminus n : n < \omega\}$, for each $\varepsilon < \kappa_2$ and trivially also $\omega \setminus \underline{A}_\varepsilon$ does not belong to this filter.

As λ is regular $> \kappa_2$ and the forcing notion \mathbb{P}^λ satisfies the c.c.c., clearly for some $\alpha < \lambda$ we have $p^* \in \mathbb{P}^\alpha$ and $\varepsilon < \kappa_2 \Rightarrow \underline{A}_\varepsilon$ is equivalent to a \mathbb{P}^α -name.

So for every $\beta \in [\alpha, \lambda)$ we have

\boxtimes_β^2 $p^* \Vdash_{\mathbb{P}^\beta}$ “for each $i < \kappa_2$ the set $\underline{A}_i \in [\omega]^{\aleph_0}$ is not in the filter on ω which $\{\underline{A}_j : j < i\} \cup \{\omega \setminus n : n < \omega\}$ generates, and also the complement of \underline{A}_i is not in this filter (as \underline{D} exemplifies this)”.

But for some such β , the statement \boxtimes_β^1 holds, i.e. $\otimes_1(d)$ apply, so in $\mathbb{P}^{\beta+1}$ which essentially is a $(\mathbb{P}^\beta)^{\kappa_2}/\mathcal{D}_2$ we get a contradiction. That is, let \mathbf{j}_β be an isomorphism from $\mathbb{P}^{\beta+1}$ onto $(\mathbb{P}^\beta)^{\kappa_2}/\mathcal{D}_2$ which extends the canonical embedding of \mathbb{P}^β into $(\mathbb{P}^\beta)^{\kappa_2}/\mathcal{D}_2$. Now \mathbf{j}_β induces a map $\hat{\mathbf{j}}_\beta$ from the set of $\mathbb{P}^{\beta+1}$ -names of subsets of ω into the set of $(\mathbb{P}^\beta)^{\kappa_2}/\mathcal{D}_2$ -names of subsets of ω , and let $\underline{A}^* = \hat{\mathbf{j}}_\beta^{-1}(\langle \underline{A}_i : i < \kappa_2 \rangle/\mathcal{D}_2)$ so $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ “ $\underline{A}^* \in [\omega]^{\aleph_0}$ and the sets $\underline{A}^*, \omega \setminus \underline{A}^*$ do not include any finite intersection of some members of $\{\underline{A}_\varepsilon : \varepsilon < \kappa_2\} \cup \{\omega \setminus n : n < \omega\}$ ”. So $p^* \Vdash_{\mathbb{P}^{\beta+1}}$ “ $\{\underline{A}_\varepsilon : \varepsilon < \kappa_2\}$ does not generate an ultrafilter on ω ” but $\mathbb{P}^{\beta+1} \triangleleft \mathbb{P}$, contradiction. $\square_{1.1}$

1.2 Remark. 1) As the referee pointed out we can in 1.1, if we waive “ $\mathfrak{u} < \mathfrak{a}$ ” we can forget κ_1 (and \mathcal{D}_1) so not taking ultra-powers by \mathcal{D}_1 , so $\mu = \aleph_0$ is allowed, but we have to start with \mathfrak{t}_0 such that $\mathbb{P}_0^{\mathfrak{t}_0}$ is adding κ_2 -Cohen.

2) Moreover, in this case we can demand that $\mathbb{Q}_\alpha^{\mathfrak{t}} = \mathbb{Q}(\underline{D}_\alpha^{\mathfrak{t}})$ and so we do not need the $\tau_\alpha^{\mathfrak{t}}$. Still this way was taken in [Sh:915, §1]. But this gain in simplicity has a price in lack of flexibility in choosing the \mathfrak{t} . We use this mildly in §2; mildly as only for \mathbb{P}_1 . See more in [Sh:915, §2, §3].

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§2 REMARKS ON π -BASES

2.1 Definition. 1) \mathcal{A} is a π -base if:

- (a) $\mathcal{A} \subseteq [\omega]^{\aleph_0}$
- (b) for some ultrafilter D on ω , \mathcal{A} is a π -base of D , see below, note that D is necessarily non-principal

1A) We say \mathcal{A} is a π -base of D if $(\forall B \in D)(\exists A \in \mathcal{A})(A \subseteq^* B)$.

1B) $\pi\chi(D) = \text{Min}\{|\mathcal{A}| : \mathcal{A} \text{ is a } \pi\text{-base of } D\}$.

2) \mathcal{A} is a strict π -base if:

- (a) \mathcal{A} is a π -base of some D
- (b) no subset of \mathcal{A} of cardinality $< |\mathcal{A}|$ is a π -base.

3) D has a strict π -base when D has a π -base \mathcal{A} which is a strict π -base.

4) $\text{Sp}_{\pi\chi}^* = \{|\mathcal{A}| : \text{there is a non-principal ultrafilter } D \text{ on } \omega \text{ such that } \mathcal{A} \text{ is a strict } \pi\text{-base of } D\}$.

2.2 Definition. For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ let $\text{Id}_{\mathcal{A}} = \{B \subseteq \omega : \text{for some } n < \omega \text{ and partition } \langle B_\ell : \ell < n \rangle \text{ of } B \text{ for no } A \in \mathcal{A} \text{ and } \ell < n \text{ do we have } A \subseteq^* B_\ell\}$.

2.3 Observation. For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ we have:

- (a) $\text{Id}_{\mathcal{A}}$ is an ideal on $\mathcal{P}(\omega)$ including the finite sets, though may be equal to $\mathcal{P}(\omega)$
- (b) if $B \subseteq \omega$ then: $B \in [\omega]^{\aleph_0} \setminus \text{Id}_{\mathcal{A}}$ iff there is a (non-principal) ultrafilter D on ω to which B belongs and \mathcal{A} is a π -base of D
- (c) \mathcal{A} is a π -base iff $\omega \notin \text{Id}_{\mathcal{A}}$.

Proof.

Clause (a): Obvious.

Clause (b):

The “if” direction: Let D be a non-principal ultrafilter on ω such that $B \in D$ and \mathcal{A} is a π -base of D . Now for any $n < \omega$ and partition $\langle B_\ell : \ell < n \rangle$ of B as $B \in D$ and D is an ultrafilter clearly there is $\ell < n$ such that $B_\ell \in D$ hence by Definition 2.1(1A) there is $A \in \mathcal{A}$ such that $A \subseteq^* B_\ell$. By the definition of $\text{Id}_{\mathcal{A}}$ it follows that $B \notin \text{Id}_{\mathcal{A}}$ but $[\omega]^{<\aleph_0} \subseteq \text{Id}_{\mathcal{A}}$ so we are done.

The “only if” direction: So we are assuming $B \notin \text{Id}_{\mathcal{A}}$ so as $\text{Id}_{\mathcal{A}}$ is an ideal of $\mathcal{P}(\omega)$ there is an ultrafilter D on ω disjoint to $\text{Id}_{\mathcal{A}}$ such that $B \in D$. So if $B' \in D$ then $B' \subseteq \omega \wedge B' \notin \text{Id}_{\mathcal{A}}$ hence by the definition of $\text{Id}_{\mathcal{A}}$ it follows that $(\exists A \in \mathcal{A})(A \subseteq^* B')$. By Definition 2.1(1A) this means that \mathcal{A} is a π -base of D .

Clause (c): Follows from clause (b). □_{2.4}

2.4 Observation. 1) If D is an ultrafilter on ω then D has a π -base of cardinality $\pi\chi(D)$.

2) \mathcal{A} is a π -base iff for every $n \in [1, \omega)$ and partition $\langle B_\ell : \ell < n \rangle$ of ω to finitely many sets, for some $A \in \mathcal{A}$ and $\ell < n$ we have $A \subseteq^* B_\ell$.

3) $\text{Min}\{\pi\chi(D) : D \text{ a non-principal ultrafilter on } \omega\} = \text{Min}\{|\mathcal{A}| : \mathcal{A} \text{ is a } \pi\text{-base}\} = \text{Min}\{|\mathcal{A}| : \mathcal{A} \text{ is a strict } \pi\text{-base}\}.$

Proof. 1) By the definition.

2) For the “only if” direction, assume \mathcal{A} is a π -base of D then $\text{Id}_{\mathcal{A}} \subseteq \mathcal{P}(\omega) \setminus D$ (see the proof of 2.2) so $\omega \notin \text{Id}_{\mathcal{A}}$ and we are done.

For the “if” direction, use 2.2.

3) Easy. □_{2.4}

2.5 Theorem. *In $\mathbf{V}^{\mathbb{P}}$ as in 1.1, we have $\{\mu, \lambda\} \subseteq \text{Sp}_{\pi\chi}^*$ and $\kappa_2 \notin \text{Sp}_{\pi\chi}^*$.*

Proof. Similar to the proof of 1.1 but with some additions. Defining \mathfrak{K} in [Sh 700, 4.1] we allow $\mathbb{Q}_0 = \mathbb{Q}_0^t = \mathbb{P}_1^t$ to be any c.c.c. forcing notion of cardinality $\leq \lambda$ (this makes no change). The main change is in the proof of $\Vdash_{\mathbb{P}} \text{“}\lambda \in \text{Sp}_{\pi\chi}\text{”}$. The main addition is that choosing t_α by induction on α we also define \mathcal{A}_α such that

- $\otimes'_1 (a), (b)$ as in \otimes_1 in the proof of ?
 $\text{scite}\{1.1\}$ undefined
- (c) as in $\otimes_1(c)$ but only if $\alpha \neq 2 \bmod \omega$ (and $\alpha = \beta + 1$)
 - (d) \underline{A}_α is a $\mathbb{P}_1^{t_\alpha}$ -name of an infinite subset of ω
 - (e) if $\alpha \neq 2 \bmod \omega$ then $\Vdash_{\mathbb{P}^{t_\alpha}} \underline{A}_\alpha = \omega$ (or do not define \underline{A}_α)
 - (f) if $\alpha < \beta$ are $\equiv 2 \bmod \omega$ then $\Vdash_{\mathbb{P}_\mu^{t_\beta}} \underline{A}_\beta \subseteq^* \underline{A}_\alpha$
 - (g) if $\beta = \alpha + 1$ and $\beta \equiv 2 \bmod \omega$ and \underline{B} is a $\mathbb{P}_\mu^{t_\alpha}$ -name of an infinite subset of ω then $\Vdash_{\mathbb{P}_\mu^{t_\beta}} \underline{B} \not\subseteq^* \underline{A}_\alpha$.

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This addition requires that we also prove

- ⊗₃ if $\mathfrak{s} \in \mathfrak{K}$ and \underline{D} is a $\mathbb{P}_1^{\mathfrak{s}}$ -name of a filter on ω including all co-finite subsets of ω (such that $\emptyset \notin \underline{D}$) then for some $(\mathfrak{t}, \underline{A})$ we have
- (a) $\mathfrak{s} \leq_{\mathfrak{K}} \mathfrak{t}$
 - (b) $\Vdash_{\mathbb{P}_1^{\mathfrak{t}}} \text{“}\underline{A} \text{ is an infinite subset of } \omega\text{”}$
 - (c) if \underline{B} is a $\mathbb{P}^{\mathfrak{s}}$ -name of an infinite subset of ω then $\Vdash_{\mathbb{P}^{\mathfrak{t}}} \text{“}\underline{B} \not\subseteq^* \underline{A}\text{”}$.

[Why ⊗₃ holds? Without loss of generality $\Vdash_{\mathbb{P}_1^{\mathfrak{s}}} \text{“}\underline{D} \text{ is an ultrafilter on } \omega\text{”}$. We can find a pair $(\mathbb{P}', \underline{A}')$

- (α) \mathbb{P}' is a c.c.c. forcing notion
- (β) $\mathbb{P}_1^{\mathfrak{s}} \triangleleft \mathbb{P}'$ moreover $\mathbb{P}' = \mathbb{P}_1^{\mathfrak{s}} * \mathbb{Q}(\underline{D})$
- (γ) $|\mathbb{P}'| \leq \lambda$
- (δ) $\Vdash_{\mathbb{P}'} \text{“}\underline{A} \text{ is an almost intersection of } \underline{D} \text{ (i.e. } \underline{A} \in [\omega]^{\aleph_0} \text{ and } (\forall B \in \underline{D})(A \subseteq^* B)\text{)”}$
- (ε) $\eta' \in {}^\omega \omega$ is the generic of $\mathbb{Q}[\underline{D}]$ and $\underline{A}' = \text{Rang}(\eta')$ so both are \mathbb{P}' -names.

Now we define $\mathfrak{t}' : \mathfrak{t} \leq_{\mathfrak{K}} \mathfrak{t}'$ and $\mathbb{P}_1^{\mathfrak{t}'} = \mathbb{P}'$, we do it by defining $\mathbb{Q}_i^{\mathfrak{t}'}$ by induction on i as in the proof of [Sh 700, 4.8] and we choose $\tau_i^{\mathfrak{t}'}$ naturally. Let $\langle \underline{n}_\rho : \rho \in {}^\omega 2 \rangle$ be a $\mathbb{P}_0^{\mathfrak{t}'}$ -name listing the members of \underline{A} .

Now we choose \mathfrak{t} such that $\mathfrak{t}' \leq_{\mathfrak{K}} \mathfrak{t}$ and for some $\mathbb{P}_0^{\mathfrak{t}}$ -name $\underline{\rho}$ of a member of ${}^\omega 2$ we have $\Vdash_{\mathbb{P}^{\mathfrak{t}}} \text{“}\underline{\rho} \neq \underline{\nu}\text{”}$ for any $\mathbb{P}_{\mathfrak{t}'}$ -name (clearly exists, e.g. when $(\mathfrak{t}, \mathfrak{t}')$ is like $(\mathfrak{t}', \mathfrak{s})$ above, e.g. do as above with \mathbb{P}' , adding λ^+ such raelis, and reflect). Now $\underline{A} := \{\underline{n}_{\underline{\rho} \upharpoonright k} : k < \omega\}$ is forced to be an infinite subset of \underline{A}' , and if it includes a member of $\mathcal{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{s}}]}$ or even $\mathcal{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}}]}$ we get that $\underline{\rho}$ is from $({}^\omega 2)^{\mathbf{V}[\mathbb{P}^{\mathfrak{t}'}]}$, contradiction.]

- (*)₁ $\mu \in \text{Sp}_{\pi\chi}^*$, in $\mathbf{V}^{\mathbb{P}}$, of course.

[Why? As there is a \subseteq^* -decreasing sequence $\langle B_\alpha : \alpha < \mu \rangle$ of sets which generates a (non-principle ultrafilter). We can use B_α as the generic of $\mathbb{Q}_\alpha^{\mathfrak{t}'} = \mathbb{P}_{\alpha+1}^{\mathfrak{t}'} / \mathbb{P}_\alpha^{\mathfrak{t}'}$.]

- (*)₂ $\kappa_2 \notin \text{Sp}_{\pi\chi}^*$.

[Why? Toward contradiction assume $p^* \in \mathbb{P}$ and $p^* \Vdash_{\mathbb{P}} \text{“}\underline{D} \text{ is a non-principal ultrafilter on } \omega \text{ and } \{\mathcal{U}_\varepsilon : \varepsilon < \kappa_2\} \text{ is a sequence of infinite subsets of } \omega \text{ which is a strict } \pi\text{-base of } \underline{D}\text{”}$; so $p^* \Vdash_{\mathbb{P}} \text{“}\{\mathcal{U}_\varepsilon : \varepsilon < \zeta\} \text{ is not a } \pi\text{-base of any ultrafilter on } \omega\text{”}$ for every $\zeta < \kappa_2$, hence for some $\langle \underline{B}_{\zeta, \ell} : \ell < \underline{n}_\zeta \rangle$ we have $p^* \Vdash \text{“}\underline{n}_\ell < \omega \text{ and } \langle \underline{B}_{\zeta, \ell} : \ell < \underline{n}_\ell \rangle \text{ is a partition of } \omega \text{ and } \varepsilon < \zeta \wedge \ell < \underline{n}_\zeta \Rightarrow \mathcal{U}_\varepsilon \not\subseteq^* \underline{B}_{\zeta, \ell}\text{”}$. We now as in the proof of 1.1, choose suitable $\beta < \lambda$ and consider $\langle \underline{B}_\ell^* : \ell < \underline{n} \rangle = \hat{\mathbf{j}}_\beta^{-1}(\langle \underline{B}_{\zeta, \ell} : \ell < \underline{n}_\zeta \rangle : \zeta < \kappa_2) / \mathcal{D}_2$ so $p^* \Vdash_{\mathbb{P}^{\beta+1}} \text{“}\langle \underline{B}_\ell^* : \ell < \underline{n} \rangle \text{ is a partition of } \omega \text{ to finitely many sets and } \varepsilon < \kappa_2 \wedge \ell < \underline{n} \Rightarrow \mathcal{U}_\varepsilon \not\subseteq^* \underline{B}_\ell^*\text{”}$. But this contradicts $p^* \Vdash_{\mathbb{P}} \text{“}\{\mathcal{U}_\varepsilon : \varepsilon < \kappa_2\} \text{ is a } \pi\text{-base.}$]

(*)₃ $\lambda \in \text{Sp}_\pi^*$.

[Why? Clearly it is forced (i.e. $\Vdash_{\mathbb{P}^\lambda}$) that $\langle \underline{A}_{\omega\alpha+2} : \alpha < \lambda \rangle$ is a \subseteq^* -decreasing sequence of infinite subsets of ω , hence there is an ultrafilter of \underline{D} on ω including it. Now $\underline{A}_{\omega\alpha+2}$ witness that $\mathcal{P}(\omega)^{\mathbf{V}[\mathbb{P}^{\omega\alpha+2}]}$ is not a π -base of \underline{D} (recalling clause (g) of \otimes'_1). As λ is regular we are done.] □_{2.5}

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