

THE DEPTH OF ULTRAPRODUCTS  
OF BOOLEAN ALGEBRAS  
SH853

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ABSTRACT. We show that if  $\mu$  is a compact cardinal then the depth of ultraproducts of less than  $\mu$  many Boolean Algebras is at most  $\mu$  plus the ultraproduct of the depths of those Boolean Algebras.

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## §0 INTRODUCTION

Monk has looked systematically at cardinal invariants of Boolean Algebras. In particular, he has looked at the relations between  $\text{inv}(\prod_{i<\kappa} \mathbf{B}_i/D)$  and  $\prod_{i<\kappa} \text{inv}(\mathbf{B}_i)/D$ , i.e., the invariant of the ultraproducts of a sequence of Boolean Algebras vis the ultraproducts of the sequence of the invariants of those Boolean Algebras for various cardinal invariants  $\text{inv}$  of Boolean Algebras. That is: is it always true that  $\text{inv}(\prod_{i<\kappa} \mathbf{B}_i/D) \leq \prod_{i<\kappa} (\text{inv}(\mathbf{B}_i)/D)$ ? is it consistently always true? Is it always true that  $\prod_{i<\kappa} \text{inv}(\mathbf{B}_i)/D \leq \text{inv}(\prod_{i<\kappa} \mathbf{B}_i/D)$ ? is it consistently always true? See more on this in Monk [Mo96]. Roslanowski Shelah [RoSh 534] deals with specific  $\text{inv}$  and with more on kinds of cardinal invariants and their relationship with ultraproducts. Monk [Mo90a], [Mo96], in his list of open problems raises the question for the central cardinal invariants, most of them have been solved by now; see Magidor Shelah [MgSh 433], Peterson [Pe97], Shelah [Sh 345], [Sh 462], [Sh 479], [Sh 589, §4], [Sh 620], [Sh 641], [Sh 703], Shelah and Spinas [ShSi 677].

We here throw some light on problem 12 of [Mo96], pg.287 and will be continued in [Sh:F683].

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**0.1 Definition.** For a Boolean Algebra  $\mathbf{B}$  let

- (a)  $\text{Depth}(\mathbf{B}) = \sup\{\theta: \text{in } \mathbf{B} \text{ there is an increasing sequence of length } \theta\}$
- (b)  $\text{Depth}^+(\mathbf{B}) = \sup\{\theta^+: \text{in } \mathbf{B} \text{ there is an increasing sequence of length } \theta\}$ .

*0.2 Remark.* So  $\text{Depth}^+(\mathbf{B}) = \lambda^+ \Rightarrow \text{Depth}(\mathbf{B}) = \lambda$  and if  $\text{Depth}^+(\mathbf{B})$  is a limit cardinal then  $\text{Depth}^+(\mathbf{B}) = \text{Depth}(\mathbf{B})$ .

## §1 ABOVE A COMPACT CARDINAL

The following claim gives severe restrictions on any try to build a ZFC example for  $\text{Depth}(\prod_{\varepsilon < \kappa} \mathbf{B}_\varepsilon)/D > \prod_{\varepsilon < \kappa} \text{Depth}(\mathbf{B}_\varepsilon)/D$  if  $\mathbf{V}$  is near  $\mathbf{L}$ , see [Sh 652] for complementary to §1.

**1.1 Claim.** 1) Assume

- (a)  $\kappa < \mu \leq \lambda$
- (b)  $\mu$  is a compact cardinal
- (c)  $D$  is an ultrafilter on  $\kappa$
- (d)  $\lambda = \text{cf}(\lambda)$  such that  $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$
- (e)  $\mathbf{B}_i$  ( $i < \kappa$ ) is a Boolean Algebra with  $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$
- (f)  $\mathbf{B} = \prod_{i < \kappa} \mathbf{B}_i/D$ .

Then  $\text{Depth}^+(\mathbf{B}) \leq \lambda$ .

2) Instead  $(\forall \alpha < \lambda)|\alpha|^\kappa < \lambda$  it suffices that  $(\forall \alpha < \lambda)(|\alpha^\kappa/D| < \lambda = \text{cf}(\lambda))$ .

3) We can weaken clause (e) (for parts (1) and (2)) to

- (g)  $\{i < \kappa : \mathbf{B}_i \text{ is a Boolean Algebra with } \text{Depth}^+(\mathbf{B}_i) \leq \lambda\} \in D$ .

*Proof.* 1) Toward contradiction assume that  $\langle a_\alpha : \alpha < \lambda \rangle$  is an increasing sequence in  $\mathbf{B}$ . So let  $a_\alpha = \langle a_i^\alpha : i < \kappa \rangle/D$ , so for  $\alpha < \beta$ ,  $A_{\alpha,\beta} =: \{i < \kappa : \mathbf{B}_i \models a_i^\alpha < a_i^\beta\} \in D$ .

Let  $E$  be a  $\mu$ -complete uniform ultrafilter on  $\lambda$ .

For each  $\alpha < \lambda$  let  $A_\alpha$  be such that the set  $\{\beta : \alpha < \beta < \lambda \text{ and } A_{\alpha,\beta} = A_\alpha\}$  is a member of  $E$  so an unbounded subset of  $\lambda$  (exist as  $\lambda = \text{cf}(\lambda) \geq \mu > 2^\kappa$ ).

We choose  $C$  as follows

$$C =: \{\delta < \lambda : \delta \text{ is a limit ordinal and if } u \subseteq \delta \\ \text{is bounded of cardinality } \leq \kappa \text{ then } \delta = \sup(S_u \cap \delta)\}$$

where

$$S_u =: \{\beta < \lambda : \beta > \sup(u) \text{ and } (\forall \alpha \in u)(A_{\alpha,\beta} = A_\alpha)\}.$$

As  $\lambda = \text{cf}(\lambda) > 2^\kappa = |D|$ , for some  $A_* \in D$  the set  $S =: \{\alpha < \lambda : \text{cf}(\alpha) > \kappa \text{ and } A_\alpha = A_*\}$  is a stationary subset of  $\lambda$ .

As we have assumed  $\lambda = \text{cf}(\lambda)$  and  $(\forall \alpha < \lambda)(|\alpha|^\kappa < \lambda)$ , clearly  $C$  is a club of  $\lambda$ . Let  $\{\delta_\varepsilon : \varepsilon < \lambda\} \subseteq C$ ,  $\delta_\varepsilon$  increases continuous with  $\varepsilon$  and  $\delta_{\varepsilon+1} \in S$ . For each  $\varepsilon < \lambda$  the family  $\mathfrak{A}_\varepsilon = \{S_u \cap \delta_{\varepsilon+1} \setminus \delta_\varepsilon : u \in [\delta_{\varepsilon+1}]^{\leq \kappa}\}$  is a downward  $\kappa^+$ -directed family of non-empty subsets of  $[\delta_\varepsilon, \delta_{\varepsilon+1})$  hence there is a  $\kappa^+$ -complete filter  $E_\varepsilon$  on  $[\delta_\varepsilon, \delta_{\varepsilon+1})$  extending  $\mathfrak{A}_\varepsilon$ .

For  $\varepsilon < \lambda$  and  $i < \kappa$  let  $W_{\varepsilon,i} =: \{\beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon+1} \text{ and } i \in A_{\beta, \delta_{\varepsilon+1}}\}$  and let  $B_\varepsilon =: \{i < \kappa : W_{\varepsilon,i} \in E_\varepsilon^+\}$ . As  $E_\varepsilon$  is  $\kappa^+$ -complete clearly  $W_\varepsilon =: \cap\{[\delta_\varepsilon, \delta_{\varepsilon+1}) \setminus W_{\varepsilon,i} : i \in \kappa \setminus B_\varepsilon\} \in E_\varepsilon$  hence there is  $\beta \in W_\varepsilon$ ; if  $i \in A_{\beta, \delta_{\varepsilon+1}}$  then  $\{\gamma : \delta_\varepsilon \leq \gamma < \delta_{\varepsilon+1} \text{ and } i \in A_{\gamma, \delta_{\varepsilon+1}}\} \in E_\varepsilon^+$ , so  $A_{\beta, \delta_{\varepsilon+1}}$  is a subset of  $B_\varepsilon$  and belongs to  $D$  hence  $B_\varepsilon \in D$ . So  $B_\varepsilon \cap A_* \in D$  is non-empty.

So for each  $\varepsilon$  for some  $i_{\delta_{\varepsilon+1}} \in A_*$  we have

$$\{\beta : \delta_\varepsilon \leq \beta < \delta_{\varepsilon+1} \text{ and } i_{\delta_{\varepsilon+1}} \in A_{\beta, \delta_{\varepsilon+1}}\} \in E_\varepsilon^+.$$

We can find  $i_* \in A_*$  such that

$$Y = \{\varepsilon < \lambda : \varepsilon \text{ is an even ordinal and } i_{\delta_{\varepsilon+1}} = i_*\}$$

has cardinality  $\lambda$ , and let  $Z = \{\delta_{\varepsilon+1} : \varepsilon \in Y\}$  so  $Z \in [\lambda]^\lambda$ . Now

- (\*)<sub>0</sub>  $\varepsilon \in Y \Rightarrow A_{\delta_{\varepsilon+1}} = A_*$   
[why? as  $\delta_{\varepsilon+1} \in S$ ]
- (\*)<sub>1</sub>  $i_* \in A_* \in D$   
[trivial; note if  $\forall \alpha < \lambda, |\alpha|^{2^\kappa} < \lambda$  we can have  $E_\varepsilon$  is  $(2^\kappa)^+$ -complete filter so we have  $B_{\delta_{\varepsilon+1}}$  instead of  $i_{\delta_\varepsilon}$  so we can weaken “ $D$  ultrafilter” to:  $D \subseteq \mathcal{P}(\kappa)$  upward closed and the intersection of any two non-empty]
- (\*)<sub>2</sub> if  $\alpha < \beta$  are from  $Z$  then  $i_* \in A_{\alpha, \beta}$   
[why? let  $\alpha = \delta_{\varepsilon+1}, \beta = \delta_{\zeta+1}$  so  $\varepsilon < \zeta$ ; let

$$\mathcal{U}_1 := \{\gamma : \delta_\zeta < \gamma < \delta_{\zeta+1}, A_{\alpha, \gamma} = A_\alpha (= A_{\delta_{\varepsilon+1}})\}$$

so

$$\mathcal{U}_1 = S_{\{\delta_{\varepsilon+1}\}} \cap [\delta_\zeta, \delta_{\zeta+1}) \in \mathfrak{A}_\zeta \subseteq E_\zeta$$

and let

$$\mathcal{U}_2 := \{\gamma : \delta_\zeta \leq \gamma < \delta_{\zeta+1}, i_* \in A_{\gamma, \delta_{\zeta+1}}\} \in E_\zeta^+.$$

[Why? As this is how  $i_{\delta_{\zeta+1}}$  is defined.]

So for any  $\alpha < \beta$  from  $Z$  as  $\mathcal{U}_1 \in E_\zeta$  and  $\mathcal{U}_2 \in E_\zeta^+$  clearly there is  $\gamma \in \mathcal{U}_1 \cap \mathcal{U}_2$  hence  $(\alpha = \delta_{\varepsilon+1} < \delta_\zeta \leq \gamma < \delta_{\zeta+1} = \beta$  and) for  $i = i_*$  we have  $\mathbf{B}_i \models a_i^{\delta_{\varepsilon+1}} < a_i^\gamma$  (because  $\gamma \in \mathcal{U}_1$ ) and  $\mathbf{B}_i \models a_i^\gamma < a_i^{\delta_{\zeta+1}}$  (because  $\gamma \in \mathcal{U}_2$ ) so together  $\mathbf{B}_i \models a_i^{\delta_{\varepsilon+1}} < a_i^{\delta_{\zeta+1}}$  but  $\alpha = \delta_{\varepsilon+1}, \beta = \delta_{\zeta+1}$  so we have gotten  $\mathbf{B}_i \models a_i^\alpha < a_i^\beta$  so we are done.

2) We change the choice of the club  $C$ . By the assumption, for each  $\alpha < \lambda$  let  $\langle f_\gamma^\alpha/D : \gamma < \gamma_\alpha \rangle$  be a list of the members of  $\alpha^\kappa/D$  without repetitions, so  $\gamma_\alpha < \lambda$ . Let

$$\begin{aligned} C = \{ & \delta : (i) \quad \delta < \lambda \text{ is a limit ordinal} \\ & (ii) \quad \text{if } \alpha < \delta \text{ then } \gamma_\alpha < \delta \\ & (iii) \quad \text{if } \alpha < \delta \text{ and } \gamma < \gamma_\alpha \text{ and} \\ & \quad \bar{A} = \langle A_i : i < \kappa \rangle \in {}^\kappa D \text{ and there is } \xi \in [\delta, \lambda) \text{ such that} \\ & \quad i < \kappa \Rightarrow A_{f_\gamma^\alpha(i), \xi} = A_i \text{ then there is} \\ & \quad \xi \in (\alpha, \delta) \text{ such that } i < \kappa \Rightarrow A_{f_\gamma^\alpha(i), \xi} = A_i \}. \end{aligned}$$

Clearly  $C$  is a club of  $\lambda$ . The only additional point in the proof is

(\*) if  $\delta_1 < \delta_2$  are from  $C$  and  $A_{\delta_2} = A_*$  then there is  $i_* \in A_*$  such that: for every  $\alpha \in S \cap \delta_1$  there is  $\beta \in [\delta_1, \delta_2)$  satisfying  $A_{\alpha, \beta} = A_* \wedge i_* \in A_{\beta, \delta_2}$ .

[Why (\*) holds? If not, then for every  $i \in A_*$  there is  $\alpha_i \in S \cap \delta_1$  satisfying  $\beta \in [\delta_1, \delta_2) \wedge A_{\alpha_i, \beta} = A_* \Rightarrow i \notin A_{\beta, \delta_2}$ . Let  $f \in {}^\kappa \alpha$  be defined by:  $f(i) = \alpha_i$ , if  $i \in A_*$ ,  $f(i) = 0$  otherwise, so for some  $\gamma < \gamma_{\delta_1}$  we have  $f = f_\gamma^{\delta_1} \text{ mod } D$  hence  $A =: \{i \in A_* : f(i) = f_\gamma^{\delta_1}(i)\} \in D$ . As  $\kappa < \mu$  and  $D$  is  $\mu$ -complete there is  $\xi_1 \in (\delta_2, \lambda)$  such that  $i < \kappa \Rightarrow A_{f_\gamma^{\delta_1}(i), \xi_1} = A_{f_\gamma^{\delta_1}(i)}$  hence by the choice of  $C$  there is  $\xi_2 \in (\delta_1, \delta_2)$  such that  $i < \kappa \Rightarrow A_{f_\gamma^{\delta_1}(i), \xi_2} = A_{f_\gamma^{\delta_1}(i), \xi_1} = A_{f_\gamma^{\delta_1}(i)}$ . But  $i \in A \Rightarrow f_\gamma^{\delta_1}(i) = f(i) = \alpha_i \in S \Rightarrow A_{\alpha_i, \xi_2} = A_{f_\gamma^{\delta_1}(i), \xi_2} = A_{f_\gamma^{\delta_1}(i)} = A_*$  so  $i \in A \Rightarrow A_{\alpha_i, \xi_2} = A_*$ . Now  $A_{\xi_2, \delta_2} \in D$  hence there is  $i_* \in A_* \cap A_{\xi_2, \delta_2}$  and for it we get contradiction.]

Of course, the set of such  $i_*$ 's belongs to  $D$ .

3) Obvious. □<sub>1.1</sub>

1.2 Conclusion: Let  $\mu$  be a compact cardinal. If  $\kappa < \mu$  and  $D$  is an ultrafilter on  $\kappa$ ,  $\mathbf{B}_i$  is a Boolean Algebra for  $i < \kappa$  then

- (\*) (a) if  $D$  is a regular ultrafilter then  $\text{Depth}(\prod_{i < \kappa} \mathbf{B}_i/D) \leq \mu + \prod_{i < \kappa} \text{Depth}(\mathbf{B}_i)/D$   
 (b) this holds if  $\kappa = \aleph_0$ .

*Proof.* If this fails, let  $\lambda = (\mu + \prod_{i < \kappa} \text{Depth}(\mathbf{B}_i)/D)^+$ , so  $\lambda$  is a regular cardinal  $> \mu$  and  $(\forall \alpha < \lambda)[|\alpha^\kappa/D| < \lambda]$  - see below and  $\lambda \leq \text{Depth}(\prod_{i < \kappa} \mathbf{B}_i/D)$ , so by 1.1 we get a contradiction.  $\square_{1.2}$

*1.3 Remark.* 1) Actually we prove that if  $\mu$  is a compact cardinal,  $\kappa < \mu \leq \lambda = \text{cf}(\lambda)$  and  $\mathbf{c} : [\lambda]^2 \rightarrow \kappa$  then we can find an increasing sequence  $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$  of ordinals  $< \lambda$  and  $i, j < \kappa$  such that for every  $\varepsilon < \zeta < \lambda$  for some  $\gamma$  satisfying  $\alpha_\varepsilon < \gamma < \alpha_\zeta$  we have  $\mathbf{c}\{\alpha_\varepsilon, \gamma\} = i, \mathbf{c}\{\gamma, \alpha_\zeta\} = j$  (the result follows using  $\mathbf{c} : [\lambda]^2 \rightarrow D$ ).

2) We use  $i_*$  rather than some  $B \in D$  in order to help clarify what we need.

3) Note that if  $D$  is a normal ultrafilter on  $\kappa > \aleph_0$  and  $\langle \lambda_i : i < \kappa \rangle$  is increasing continuous with limit  $\lambda, i < \kappa \Rightarrow \prod_{j \leq i} \lambda_j < \lambda_{i+1}$  then  $\lambda = \prod_{i < \kappa} \lambda_i/D$  but  $\lambda^\kappa/D >$

$\lambda$ . This is essentially the only reason for the undesirable extra assumption “ $D$  is regular” in 1.2.

Note

**1.4 Claim.** 1) In 1.1 instead “ $\mu \in (\kappa, \lambda]$  is a compact cardinal” it suffices to demand:  $\otimes_{\kappa^+, 2^\kappa, \lambda}$  where

$\otimes_{\sigma, \theta, \lambda}$  if  $\mathbf{c} : [\lambda]^2 \rightarrow \theta$  then we can find a stationary  $S \subseteq \lambda$  and  $\gamma < \theta$  such that for every  $u \in [S]^{< \sigma}$  the set  $S_u = \{\beta < \lambda : (\forall \alpha \in u)[\mathbf{c}\{\alpha, \beta\} = \gamma]\}$  is unbounded in  $\lambda$ .

2) If  $\mu$  is supercompact  $\sigma < \theta = \text{cf}(\theta) < \mu < \lambda = \text{cf}(\lambda)$  and  $\mathbb{Q} =$  adding  $\mu$  Cohen subsets of  $\theta$  then in  $\mathbf{V}, \otimes_{\sigma, \mu, \lambda}$  holds (even  $\otimes_{\sigma, \mu_1, \lambda}$  if  $\mu_1^{< \sigma} < \lambda$  in  $\mathbf{V}$ ).

In 1.4 we cannot get such results for  $\kappa > \mu$  because for  $\mu$  supercompact Laver indestructible and regular  $\lambda > \kappa > \mu$  we can force  $\{\delta < \lambda : \text{cf}(\delta) > \mu\}$  to have a square preserving the supercompactness.

**1.5 Claim.** Assume  $\lambda = \text{cf}(\lambda) > \kappa^+$  and  $\kappa = \text{cf}(\kappa)$ , and there is a square on  $S = \{\delta < \lambda : \text{cf}(\delta) \geq \kappa\}$  (see 1.6 below). Then

(a) there is a sequence  $\langle \mathbf{B}_i : i < \kappa \rangle$  of Boolean Algebras such that

( $\alpha$ )  $\text{Depth}^+(\mathbf{B}_i) \leq \lambda$

( $\beta$ ) for any uniform ultrafilter  $D$  on  $\kappa$ ,  $\text{Depth}^+(\prod_{i < \kappa} \mathbf{B}_i/D) > \lambda$

(b) the proof of [Sh 652, 5.1] can be carried.

Where

**1.6 Definition.** For  $\lambda = \text{cf}(\lambda) > \aleph_0$ ,  $S \subseteq \lambda = \text{sup}(S)$  we say that  $S$  has a square when we can find  $S^+$  and  $\langle C_\alpha : \alpha \in S^+ \rangle$  such that

- (a)  $S \setminus S^+$  is not a stationary subset of  $\lambda$
- (b)  $C_\alpha$  is a closed subset of  $\alpha$
- (c)  $\beta \in C_\alpha \Rightarrow \beta \in S \cap C_\beta = C_\alpha \cap \beta$
- (d) we stipulate  $C_\alpha = \{\emptyset\}$  for  $\alpha \notin S^+$ .

*Proof of 1.5.* As in [Sh 652, 5.1] using  $\bar{C} = \langle C_\alpha : \alpha \in S^+ \rangle$  from 1.6 instead  $\langle \text{acc}(C_\alpha) : \alpha < \lambda^+ \rangle$ . The only change being that in the proof of [Sh 652, Fact 5.3] in case 3 we have just  $\text{cf}(\alpha) \leq \kappa$  and let  $\langle \beta_\zeta : \zeta < \text{cf}(\alpha) \rangle$  be increasing continuous with limit  $\alpha$ . If  $\text{cf}(\alpha) < \kappa$  we can find  $\varepsilon(*) < \kappa$  such that  $\zeta_1 < \zeta_2 < \kappa \Rightarrow \beta_{\zeta_1} \in A_{\beta_{\zeta_2}, \varepsilon(*)}$  and let  $A_{\alpha, \varepsilon} = \emptyset$  if  $\varepsilon < \varepsilon(*)$  and  $A_{\alpha, \varepsilon} = \cup \{A_{\beta_\zeta, \varepsilon} : \zeta < \text{cf}(\alpha)\}$  if  $\varepsilon \in [\varepsilon(*), \kappa)$ .  $\square_{1.6}$

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