

# SACCHARINITY

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A . We present a method to iterate finitely splitting lim-sup tree forcings along non-wellfounded linear orders. As an application, we introduce a new method to force (weak) measurability of all definable sets with respect to a certain (non-ccc) ideal.

## I

1 **Non-wellfounded iterations.** We introduce a method to iterate lim-sup finitely splitting  
2 tree forcings along linear, non-wellfounded orders.

3 There is quite some literature about non-wellfounded iteration. E.g., Jech and Groszek [4]  
4 investigated the wellfounded but non-linear iteration of Sacks forcings. Building on this,  
5 Kanovei [7] and Groszek [5] develop non-wellfounded iterations of Sacks forcing. In spirit,  
6 their construction is close to the construction of this paper, but the implementation is quite  
7 different. Zapletal gives an illfounded iteration construction in the framework of “idealized  
8 forcing” [15], it seems that his results give some of the properties of our construction (e.g.,  
9  $\omega^\omega$ -bounding) for a more general class of forcings, cf. his Theorem 5.4.12.<sup>1</sup> Regarding fi-  
10 nite support, Brendle [1] developed finite-support non-wellfounded iteration constructions,  
11 based on the second author’s method of iterations along smooth templates [13]. Brendle’s  
12 paper also contains the important observation by Hjorth (answering a question of Hechler)  
13 that it is impossible to have an illfounded iteration of forcings that all add dominating reals.

14 **Measurability.** As an application of our method, we introduce a new way to force mea-  
15 surability of definable sets.

16 In the seminal paper [14] Solovay proved that in the Levy model (after collapsing an  
17 inaccessible) every definable set is measurable and has the Baire property.

18 In [12] the second author showed that the inaccessible is necessary for measurability,  
19 but the Baire property of every definable set can be obtained by a forcing  $P$  without the  
20 use of an inaccessible (i.e., in ZFC). This forcing  $P$  is constructed by amalgamation of  
21 universally meager forcings  $Q$ . So every  $Q$  adds a co-meager set of generics and has many  
22 automorphisms, and the forcing  $P$  has similar properties to the Levy collapse. The property  
23 of  $Q$  that implies that  $Q$  can be amalgamated is called “sweetness” (a strong version of  
24 ccc). One can ask about other ccc ideals than Lebesgue-null and meager (or their defining  
25 forcings, random and Cohen), and classify such ideals (respectively forcings) according to  
26 whether they behave like Cohen or like randoms see, e.g., Sweet & Sour [10].

27 For (non-ccc) ideals corresponding to tree forcings  $Q$ , forcing measurability can be  
28 much simpler, see Section 6 about the Cohen model. In this model, all definable set are  $Q$ -  
29 measurable (e.g., Marczewski measurable for  $Q =$  Sacks forcing). The proof is a simpler

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*Date:* November 2004.

1991 *Mathematics Subject Classification.* 03E35,03E40.

This paper is dedicated to the memory of Greg Hjorth. Both authors gratefully acknowledge partial support by the National Science Foundation Grant No. 0600940. The first author is partially supported by European Union FP7 grant PERG02-GA-2207-224747 and the FWF Austrian Science Fund project P21651-N13. The second author is supported by the United States-Israel Binational Science Foundation (Grant no. 2006108), publication 859.

<sup>1</sup>note that Zapletal’s basic construction can be applied to countable orders only, for longer orders additional work is required, see Section 5.5 there.

1 version of Solovay's: Cohen forcing is homogeneous and adds subtrees  $S \in Q \cap V[G]$  to  
 2 all  $T \in Q \cap V$  such that all branches of  $S$  are Cohen reals.

3 In this paper, we introduce a new construction that gives a variant of measurability  
 4 (weak measurability, as defined in 3.3) for all definable sets: Instead of iterating basic  
 5 forcings  $Q$  that have many automorphisms and add a measure 1 set of generics, we use a  $Q$   
 6 that adds only a null set of generics (a single one in our case, and this real remains the only  
 7 generic over  $V$  even in the final limit). So  $Q$  has to be very non-homogeneous. Instead of  
 8 having many automorphisms in  $Q$ , we assume that the skeleton of the iteration has many  
 9 automorphisms (so in particular a non-wellfounded iteration has to be used).

10 We use the word Saccharinity for this concept: a construction that achieves the same  
 11 effect as (an amalgamation of) sweet forcings, but using entirely different means.

12 **Acknowledgments.** We thank the referee for pointing out many typos and unclarities, and  
 13 for providing section 6.

14 **Annotated contents.**

15 Section 1, p. 2: We define a class of finitely splitting tree forcings with “lim-sup norm”:  
 16 The forcing conditions are subtrees of a basic finitely splitting tree that satisfy  
 17 “along every branch, many nodes have many successors”.

18 Section 2, p. 7: We introduce a general construction to iterate such lim-sup tree-forcings  
 19 along non-wellfounded total orders. It turns out that the limit is proper,  $\omega^\omega$ -  
 20 bounding and has other nice properties similar to the properties of the lim-sup  
 21 tree-forcings itself.

22 Section 3, p. 15: We define (with respect to a lim-sup tree-forcing  $Q$ ) the ideals  $\mathbb{I}$  and  $\mathbb{I}^c$   
 23 (the  $< 2^{\aleph_0}$ -closure of  $\mathbb{I}$ ). These ideals will generally not be ccc. We define what  
 24 we mean by “ $X$  is weakly measurable” and formulate our application: Assuming  
 25 CH and a Ramsey property for  $Q$  (see Section 5), we can force that all definable  
 26 sets are weakly measurable. (This section requires only Section 1.)

27 Section 4, p. 17: Assuming CH, we construct an order  $I$  which has many automor-  
 28 phisms and a cofinal sequence  $(j_\alpha)_{\alpha \in \omega_2}$ . We show that the non-wellfounded  
 29 iteration of  $Q$  along the order  $I$  forces that  $2^{\aleph_0} = \aleph_2$ , that  $\mathbb{I}^c$  is nontrivial, that for  
 30 every definable set  $X$  “locally” either all or none of the generic reals  $\eta_{j_\delta}$  are in  
 31  $X$  and that the set  $\{\eta_{j_\delta} : \delta \in \omega_2\}$  is of weak measure 1 in the set  $\{\eta_i : i \in I\}$ .

32 Section 5, p. 21: We assume a certain Ramsey property for  $Q$ . We show that  $\{\eta_i : i \in I\}$   
 33 is of weak measure 1. Together with the result of the previous section this proves  
 34 the application.

35 Section 6, p. 25: We give a brief comparison with the Cohen model. (This section re-  
 36 quires only Sections 1 and 3.)

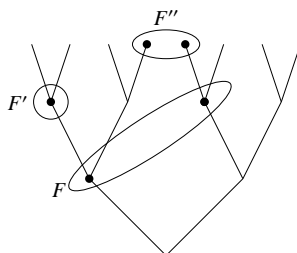
37 1. - -

38 We will define a class of finitely splitting tree forcings with “lim-sup norm”. The sim-  
 39 plest example is Sacks forcing. Such forcings (and generalizations) have been investigated  
 40 by many authors, e.g. in [9] under the name  $\mathbb{Q}_0^{\text{tree}}$  (see Definition 1.3.5 there).

41 1.1. **Basics.** Let us first introduce some notation:

42 **Definition 1.1.** Let  $T \subseteq \omega^{<\omega}$  be a tree (i.e.,  $T$  is closed under initial segments), let  $s, t \in$   
 43  $\omega^{<\omega}$ ,  $A \subseteq T$ .

- 44 • We write sequences as  $\langle a_1, \dots, a_n \rangle$  or as  $(a_1, \dots, a_n)$ . In particular,  $\langle \rangle$  denotes the  
 45 empty sequence.
- 46 •  $s \preceq t$  means that  $s$  is a restriction of  $t$  (or equivalently that  $s \subseteq t$ ).
- 47 •  $t$  is immediate successor of  $s$  if  $t \succeq s$  and  $\text{length}(t) = \text{length}(s) + 1$ .
- 48 •  $\text{succ}_T(t)$  is the set of immediate successors of  $t$  in a tree  $T$ . If the tree  $T$  is clear  
 49 from the context we will also write  $\text{succ}(t)$ .



F 1.  $F'$  is stronger than  $F$ ,  $F''$  is purely stronger than  $F$ .

- 1 • Nodes  $s$  and  $t$  are compatible ( $s \parallel t$ ), if they are comparable, i.e., if  $s \leq t$  or  $t \leq s$ .
- 2 Otherwise,  $s$  and  $t$  are incompatible ( $s \perp t$ ).
- 3 • The order in forcing notions is usually chosen such that  $q < p$  means that  $q$  is
- 4 stronger than  $p$ . We try to stick to Goldstern's alphabetic convention [3, 1.2]:
- 5 Whenever two conditions are comparable the notation is chosen so that the vari-
- 6 able used for the stronger condition comes "lexicographically" later.
- 7 • Two forcing conditions  $p$  and  $p'$  are compatible ( $p \parallel p'$ ), if there is a  $q$  stronger
- 8 than both  $p$  and  $p'$ . Otherwise,  $p$  and  $p'$  are incompatible ( $p \perp p'$ ).
- 9 •  $T^{[t]} := \{s \in T : s \parallel t\}$ . (So  $T^{[t]}$  is a tree.) If  $T$  is clear, we might also just write
- 10  $[t]$ .
- 11 •  $T \upharpoonright n := \{t \in T : \text{length}(t) < n\}$ .
- 12 •  $A \subseteq T$  is a chain if  $s \parallel t$  for all  $s, t \in A$ .
- 13 •  $b \subseteq T$  is a branch if it is a maximal chain.
- 14 If there exists a  $t \in b$  with length  $n$  then this  $t$  is unique and denoted by  $b \upharpoonright n$ .
- 15 •  $A \subseteq T$  is an antichain if  $s \perp t$  for all  $s \neq t \in A$ . Unless noted otherwise, we will
- 16 assume that antichains are nonempty.
- 17 •  $A \subseteq T$  is a front if it is an antichain and every branch  $b$  meets  $A$  (i.e.,  $|b \cap A| = 1$ ).
- 18 •  $t \leq A$  stands for: " $t \leq s$  for some  $s \in A$ ".
- 19 •  $T_{\text{cldn}}^A := \{t \in \omega^{<\omega} : t \leq A\}$ .
- 20 (We will use this downwards-closure only for finite sets  $A$ . Then  $T_{\text{cldn}}^A$  is a finite
- 21 tree.)
- 22 • If  $A$  and  $A'$  are antichains, then  $A'$  is stronger than  $A$  if for each  $t \in A'$  there is a
- 23  $s \in A$  such that  $s \leq t$  (cf. Figure 1).
- 24 • If  $A$  and  $A'$  are antichains then  $A'$  is purely stronger than  $A$  if it is stronger and for
- 25 each  $s \in A$  there is a  $t \in A'$  such that  $s \leq t$  (cf. Figure 1).
- 26 •  $\text{lim}(T)$  are the maximal branches of  $T$ . We use this notation only for  $T$  that are
- 27 "pruned", i.e., have no finite maximal branches; then  $\text{lim}(T) \subseteq \omega^\omega$  is the closed
- 28 set corresponding to  $T$ .

29 We are only interested in finitely splitting trees (i.e.,  $\text{succ}(t)$  is finite for all  $t \in T$ ). Then

30 all fronts are finite. Note that being a front is stronger than being a maximal antichain. For

31 example,  $\{0^n 1 : n \in \omega\}$  is a maximal antichain in  $2^{<\omega}$ , but not a front.

32 **Assumption.** Assume  $T_{\text{max}}$  and  $\mu$  satisfy the following:

- 33 •  $T_{\text{max}}$  is a finitely splitting tree.
- 34 •  $\mu$  assigns a non-negative real to every subset of  $\text{succ}_{T_{\text{max}}}(t)$  for every  $t \in T_{\text{max}}$ .
- 35 •  $\mu$  is monotone, i.e., if  $A \subseteq B$  then  $\mu(A) \leq \mu(B)$ .
- 36 • The measure of singletons is smaller than 1, i.e.,  $\mu(\{s\}) < 1$ .
- 37 • For all branches  $b$  in  $T_{\text{max}}$ ,  $\limsup_{n \rightarrow \infty} (\mu(\text{succ}(b \upharpoonright n))) = \infty$ .

38 Note that such a  $T_{\text{max}}$  has to be perfect.

39 **Definition 1.2.** (The tree forcing  $Q$ .)

- 1 • If  $T$  is a subtree of  $T_{\max}$  and  $t \in T$ , then  $\mu_T(t)$  is defined as the measure of the
- 2  $T$ -successors of  $t$ , i.e.,  $\mu_T(t) := \mu(\text{succ}_T(t))$ .
- 3 •  $Q$  consists of all subtrees  $T$  of  $T_{\max}$  (ordered by inclusion) such that along every
- 4 branch  $b$  of  $T$

$$\limsup(\mu_T(b \upharpoonright n)) = \infty.$$

6 So  $T_{\max}$  itself is the weakest element of  $Q$ .

7 For example, Sacks forcing can be defined in this way: Set  $T_{\max} := 2^{<\omega}$ , and for  $t \in T_{\max}$   
8 and  $A \subseteq \text{succ}(t)$  set

$$\mu(A) := \begin{cases} \text{length}(t) & \text{if } |A|=2, \\ 0 & \text{otherwise.} \end{cases}$$

10 Then a subtree  $T$  of  $2^{<\omega}$  is in  $Q$  iff  $T$  is a Sacks tree, i.e., iff along every branch there are  
11 infinitely many splitting nodes.<sup>2</sup>

12 **Definition 1.3.** A (finite or infinite) subtree  $T$  of  $T_{\max}$  is  $n$ -dense if there is a front  $F$  in  $T$   
13 such that  $\mu_T(t) > n$  for every  $t \in F$ .

14 **Lemma 1.4.** (1) A subtree  $T$  of  $T_{\max}$  is in  $Q$  iff  $T$  is  $n$ -dense for every  $n \in \mathbb{N}$ .

15 (2) “ $T \in Q$ ” and “ $T \leq_Q S$ ” are Borel statements, and “ $S \perp T$ ” is  $\Pi_1^1$  (in the real  
16 parameters  $T_{\max}$  and  $\mu$ ).

17 *Proof.* (1)  $\rightarrow$ : If  $D_n := \{s \in T : \mu_T(s) > n\}$  meets every branch, then

$$F_n := \{s \in D_n : (\forall s' \preceq s) s' \notin D_n\}$$

19 is a front.

20  $\leftarrow$ : If  $b$  is a branch, then  $b$  meets every  $F_n$ , i.e.,  $\mu_T(b \upharpoonright m) > n$  for some  $m$ . Since  $\mu_T(b \upharpoonright m)$   
21 is finite,  $\limsup(\mu_T(b \upharpoonright n))$  has to be infinite.

22 (2) Since  $T_{\max}$  is finitely splitting, “ $F$  is a front” is equivalent to “ $F$  is a finite maximal  
23 antichain”.  $\square$

24 A finite antichain  $A$  can be seen as an approximation to a tree: “ $A$  approximates  $T$ ”  
25 means that  $A$  is a front in  $T$ . If  $A'$  is purely stronger than  $A$ , then  $A'$  gives more information  
26 about the tree  $T$  that is approximated (i.e., every tree approximated by  $A'$  is also approx-  
27 imated by  $A$ ). And, informally, a stronger antichain approximates smaller (i.e., stronger)  
28 trees.

29 We will usually identify a finite antichain  $F$  and the corresponding finite tree  $T_{\text{cln}}^F$ .

30 **Definition 1.5.** • A finite antichain  $F$  is  $n$ -dense if  $T_{\text{cln}}^F$  is  $n$ -dense.

- 31 •  $\bar{F} = (F_n)_{n \in \omega}$  is a front-sequence, if  $F_{n+1}$  is  $n$ -dense and purely stronger than  $F_n$ .
- 32 • A front-sequence  $\bar{F}$  and a tree  $T \in Q$  correspond to each other if  $F_n$  is a front in  $T$   
33 for all  $n$ .

34 **Facts 1.6.** • If  $F$  is  $n$ -dense and  $F'$  is purely stronger than  $F$ , then  $F'$  is  $n$ -dense as  
35 well. (This is not true if  $F'$  is just stronger than  $F$ .)

- 36 • If  $T \in Q$  then there is a front-sequence corresponding to  $T$ .
- 37 • If  $\bar{F}$  is a front-sequence then there exists exactly one  $T \in Q$  corresponding to  $\bar{F}$ ,  
38 which we call  $\text{lim}(\bar{F})$ . It is the tree

$$\text{lim}(\bar{F}) := \{t \in T_{\max} : (\exists i \in \omega) t \leq F_i\},$$

40 or equivalently

$$\text{lim}(\bar{F}) := \{t \in T_{\max} : (\forall i \in \omega) (\exists s \in F_i) t \parallel s\}.$$

42 **Lemma 1.7.** Assume that  $Q$  is a finitely splitting lim-sup tree-forcing.

<sup>2</sup>This example is “atomic” in the following sense: For a node  $s \in T$  there is an  $A \subset \text{succ}(s)$  such that  $\mu(A)$  is large but  $\mu(B) < 1$  for every  $B \subsetneq A$ . In this paper, we will be interested in “finer” norms. In particular we will require the Ramsey property defined in 5.4.

- 1 (1) If  $T \in Q$  and  $t \in T$  then  $T^{[t]} \in Q$ . (Sometimes this fact is formulated as “ $Q$  is  
2 strongly arboreal”.)
- 3 (2) The finite union of elements of  $Q$  is in  $Q$ .<sup>3</sup>
- 4 (3) The generic filter on  $Q$  is determined by a real  $\eta$  defined by  $\mathbb{1}_Q \{ \eta \} = \bigcap_{T \in G_Q} \lim(T)$ ;  
5 or equivalently:  $\eta$  is the union of the stems of the trees in  $G_Q$ .  
6 It is forced that  $\eta \notin V$  and that  $T \in G_Q$  iff  $\eta \in \lim(T)$ .  
7 For every  $T \in Q$  and  $t \in T$ ,  $t < \eta$  is compatible with  $T$ . (In other words:  
8  $T \not\perp t \neq \eta$ .)
- 9 (4) (Fusion) If  $(T_i)_{i \in \omega}$  is a decreasing sequence in  $Q$  and  $\bar{F}$  is a front-sequence such  
10 that  $F_i$  is a front in  $T_i$  for all  $i$ , then  $\lim(\bar{F}) \leq_Q T_i$ .
- 11 (5) (Pure decision) If  $D \subseteq Q$  is dense,  $T \in Q$  and  $F$  is a front of  $T$ , then there is an  
12  $S \leq T$  such that  $F$  is a front of  $S$  and for every  $t \in F$ ,  $S^{[t]} \in D$ .
- 13 (6)  $Q$  is proper<sup>4</sup> and  $\omega^\omega$ -bounding.

14 *Sketch of proof.* (1) and (2) and (4) are clear. (1) and (2) imply (5).

15 (3): Let  $G$  be  $Q$ -generic over  $V$ , and define  $X := \bigcap_{T \in G} \lim(T)$ . Since  $\lim(T_{\max})$  is  
16 compact, it satisfies the finite intersection property. So  $X$  is nonempty. For every  $T \in G$   
17 and  $n \in \omega$  there is exactly one  $t \in T$  of length  $n$  such that  $T^{[t]} \in G$ . So  $X$  has at most one  
18 element.

19 If  $r \in V$ , then the set of trees  $S \in Q^V$  such that  $r \notin \lim(S)$  is dense: If  $r$  is a branch  
20 of  $T \in Q$  then pick an  $m$  such that  $\mu_T(r \upharpoonright m) > 2$ . Since singletons have measure less than  
21 1,  $r \upharpoonright m$  has at least two immediate successors in  $T$ , and one of them (we call it  $t$ ) is not an  
22 initial segment of  $r$ . So  $S := T^{[t]}$  forces that  $\eta \neq r$ .

23 Assume towards a contradiction that  $\eta \in \lim(T)$  for some  $T \in Q^V \setminus G$ . Then this is  
24 forced by some  $S \in G$ . In particular  $S$  can not be a subtree of  $T$ . So pick an  $s \in S \setminus T$ .  
25 Then  $S^{[s]} \leq S$  forces that  $\eta \notin \lim(T)$ , a contradiction.

26 If  $T \in Q$  and  $t \in T$  then  $T^{[t]}$  forces that  $t < \eta$ .

27 (4) and (5) imply that  $Q$  is  $\omega^\omega$ -bounding and satisfies a version of Axiom A (with fronts  
28 as indices instead of natural numbers).<sup>5</sup> So we get properness. (We will prove a more  
29 general case in 2.24.)  $\square$

30 So a front can be seen as a finite set of (pairwise incompatible) possibilities for initial  
31 segments of the generic real  $\eta$ . In the next section we will generalize this to finite sequences  
32 of generic reals instead of a single one.

### 33 1.2. Some additional facts needed later.

34 **Lemma 1.8.** *If  $S \in Q$  and the forcing  $R$  adds a new real  $\dot{r} \in 2^\omega$ , then  $R$  forces that there  
35 is a  $\bar{T} \leq_Q S$  such that  $\lim(\bar{T}) \cap V = \emptyset$ , and moreover  $\lim(\bar{T}) \cap V$  remains empty in every  
36 extension of the universe.*

37 *Proof.* Assume  $S$  corresponds to the front-sequence  $\bar{F}$ . Without loss of generality we can  
38 assume that along every branch in  $S$  there is exactly one split between  $F_{n-1}$  and  $F_n$  and  
39 this split has measure  $> n$ .

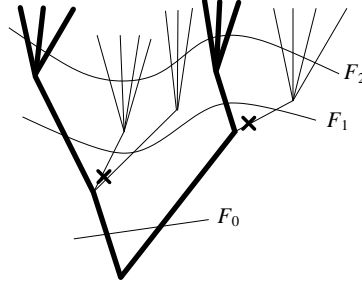
<sup>3</sup> $Q$  is generally not closed under countable unions.

<sup>4</sup>there even are generic conditions for arbitrary countable transitive ZFC models  $M$ , similarly to Suslin proper. Sometimes this is called “totally proper”.

<sup>5</sup>In the formulation of fusion and pure decision we could use the classical Axiom A version as well: Define  $F_n^T$  to be the minimal  $n$ -dense front, i.e.,

$$F_n^T := \{t \in T : \mu_T(t) > n \ \& \ (\forall s \preceq t) \mu_T(s) \leq n\},$$

and define  $T \leq_n S$  by  $T \leq S$  and  $F_n^T = F_n^S$ . It should be clear how to formulate fusion and pure decision for this notion, and that this proves Axiom A for  $Q$ . But in 1.7, we do not use this notion, instead we (implicitly) use the following one:  $T \leq_A S$  means that  $T \leq S$  and  $A$  that is a front in both  $T$  and  $S$ . The reason is that this is the notion that will be generalized for the non-wellfounded iteration.



F 2. An example for  $S$  and its subtree  $\underline{T}$  (bold) when  $\iota(0) = 0$ .

1 We define an  $R$ -name of a sequence of finite antichains  $(\underline{F}'_n)$  the following way (cf.  
 2 Figure 2): If  $n$  is even, we “take all splittings”, i.e.,  $\underline{F}'_n$  is the set of nodes in  $F_n$  that are  
 3 compatible with  $\underline{F}'_{n-1}$ . If  $n$  is odd, then we add no splittings at all: for every  $s \in \underline{F}'_{n-1}$   
 4 we put exactly one successor  $t \in F_n$  of  $s$  into  $\underline{F}'_n$ , namely the one continuing the  $\iota(\frac{n-1}{2})$ -th  
 5 successor of the (unique) splitting node over  $s$ . It is clear that the sequence  $(\underline{F}'_n)$  defines a  
 6 subtree  $\underline{T}$  of  $S$  that is in  $\mathcal{Q}$ .

7 Assume  $V'$  is an arbitrary extension of  $V$  containing an  $R$ -generic filter  $G$  over  $V$ . If  
 8  $\eta \in \lim(\underline{T}[G]) \cap V$ , then  $\iota[G]$  can be decoded in  $V$  using  $S$  and  $\eta$ . This is a contradiction  
 9 to  $\Vdash_R \iota \notin V$ .  $\square$

10 We will also need the following family of definable dense subsets of  $\mathcal{Q}$ :

11 **Definition 1.9.** Fix a recursive bijection  $\psi$  from  $\omega$  to  $2^{<\omega}$ . Assume that  $f : \omega \rightarrow \omega$  is  
 12 strictly increasing and that  $A \subseteq \omega$ .

- 13 • For  $g \in 2^\omega$ , define  $A_g^\psi := \{n \in \omega : \psi(n) < g\}$ .
- 14 •  $\mathcal{Q}_A^f$  is the set of all  $T \in \mathcal{Q}$  such that for all splitting nodes  $t \in T$ ,  $\text{length}(t)$  is in the  
 15 interval  $[f(n), f(n+1) - 1]$  for some  $n \in A$ .
- 16 •  $T \in \mathcal{Q}$  has full splitting with respect to  $f$  if for all  $n \in \omega$  and  $t \in T$  of length  
 17  $f(n+1)$  there is an  $s \leq t$  of length at least  $f(n)$  such that  $\mu_T(s) > n$ .
- 18 •  $D_f^{\text{spl}}$  is the set of all  $T \in \mathcal{Q}$  such that either  $T \in \mathcal{Q}_{A_g^\psi}^f$  for some  $g \in 2^\omega$  or  $T \perp_{\mathcal{Q}} S$   
 19 for all  $g \in 2^\omega$  and  $S \in \mathcal{Q}_{A_g^\psi}^f$ .

20 Of course the notions  $\mathcal{Q}_A^f$  and  $D_f^{\text{spl}}$  depend on the forcing  $\mathcal{Q}$  (i.e., on  $T_{\max}$  and  $\mu$ ), so  
 21 maybe it would be more exact to write  $\mathcal{Q}_A^f[T_{\max}, \mu]$  etc. However, we always assume that  
 22 the  $\mathcal{Q}$  is understood. 3.3).

23 **Lemma 1.10.** Assume that  $f : \omega \rightarrow \omega$  is strictly increasing and  $A, B \subseteq \omega$ .

- 24 (1) If  $g \neq g'$ , then  $A_g^\psi \cap A_{g'}^\psi$  is finite.
- 25 (2)  $\mathcal{Q}_\omega^f = \mathcal{Q}$ . If  $A$  is finite then  $\mathcal{Q}_A^f = \emptyset$ .
- 26 (3)  $\mathcal{Q}_A^f \cap \mathcal{Q}_B^f = \mathcal{Q}_{A \cap B}^f$ . If  $A \subseteq B$ , then  $\mathcal{Q}_A^f \subseteq \mathcal{Q}_B^f$ .
- 27 (4) If  $T \leq_{\mathcal{Q}} S$  and  $S \in \mathcal{Q}_A^f$  then  $T \in \mathcal{Q}_A^f$ .
- 28 (5) For every  $T \in \mathcal{Q}$  there is a strictly increasing  $f$  such that  $T$  has full splitting with  
 29 respect to  $f$ .
- 30 (6) If  $T \in \mathcal{Q}$  has full splitting with respect to  $f$  and  $|A| = \aleph_0$  then there is an  $S \leq_{\mathcal{Q}} T$   
 31 such that  $S \in \mathcal{Q}_A^f$ .

- 1 (7)  $D_f^{spl}$  is an (absolute definition of an) open dense subset of  $Q$  (using the parameters  
 2  $f, T_{max}$  and  $\mu$ ).<sup>6</sup>  
 3 (8) In any extension  $V'$  of  $V$  the following holds: If  $r \in 2^\omega \setminus V$  and  $S \in Q_{A_r}^f$ , then  
 4  $T \perp_Q S$  for all  $T \in V \cap D_f^{spl}$ .

5 *Proof.* (1)–(4) and (6) are clear.

6 (5): Let  $T$  be an element of  $Q$ . Assume we already constructed  $f(n)$ . Let  $N$  be the  
 7 maximum of  $\mu_T(t)$  for  $t \in T \setminus f(n)$ . There is an  $N + n + 1$ -dense front  $F$  in  $T$ . Let  $f(n + 1)$   
 8 be the maximum of  $\{\text{length}(t) : t \in F\}$ .

9 (7): “ $T$  is incompatible with all  $S \in Q_{A_g}^f$ ” is absolute, since it is equivalent to

$$10 (\forall g \in 2^\omega) (\forall S \subseteq T_{max}) \left[ S \notin Q_{A_g}^f \vee T \perp_Q S \right],$$

11 which is a  $\Pi_1^1$  statement.

12 (8): Let  $r \in 2^\omega \setminus V$  and  $T \in V \cap D_f^{spl}$ . If  $T \in Q_{A_g}^f$  for some  $g \in 2^\omega \cap V$ , then  $g \neq r$ , so  
 13  $A_g^\psi \cap A_r^\psi$  is finite and  $Q_{A_r}^f \cap Q_{A_g}^f$  is empty. If on the other hand  $T$  is incompatible with all  
 14  $S \in Q_{A_g}^f$  in  $V$  then this holds in  $V'$  as well.  $\square$

15 Assume  $f'(n) \geq f(n)$  for all  $n \in \omega$ . Define  $h(n)$  by induction:  $h(n+1) := f'(h(n)+1)$ . If  $T$   
 16 has full splitting with respect to  $f$ , then  $T$  has full splitting with respect to  $h$ :  $h(n) \leq f(h(n))$ ,  
 17 since  $f$  is strictly increasing.  $f(h(n)+1) \leq f'(h(n)+1) = h(n+1)$ , and there are  $h(n)$ -dense  
 18 splits between the levels  $f(h(n))$  and  $f(h(n)+1)$ . So there are  $n$ -dense splits between the  
 19 levels  $h(n)$  and  $h(n+1)$ . So we get:

20 **Lemma 1.11.** *If  $V'$  is an  $\omega^\omega$ -bounding extension of  $V$  and  $T \in Q^{V'}$ , then there is a strictly  
 21 increasing  $h \in V$  such that (in  $V'$ )  $T$  has full splitting with respect to  $h$ .*

## 22 2. A - I

23 In this section we introduce a general construction to iterate lim-sup tree-forcings  $Q_i$   
 24 (as defined in the last section) along non-wellfounded linear orders  $I$ . It turns out that the  
 25 limit  $P$  is proper,  $\omega^\omega$ -bounding and has other nice properties similar to the properties of  $Q_i$   
 26 itself. If  $I$  is wellfounded, then  $P$  is equivalent to the usual countable support iteration of  
 27 (the evaluations of the definitions)  $Q_i$ .

### 28 2.1. Conditions and approximations, the nw-iteration.

29 **Definition 2.1.** Let  $I$  be a linear order. For  $i \in I$  we set  $I_{<i} := \{j \in I : j < i\}$  and  
 30 analogously we define  $I_{\leq i}$  and  $I_{>i}$ . We also set  $I_{<\infty} := I$ .

<sup>6</sup> $X_0 := \{A_g^\psi : g \in 2^\omega\}$  is an almost disjoint family, but not maximal. So of course  $Q(X_0) := \bigcup_{A \in X_0} Q_A^f \subset Q$  is not dense. We add the incompatible conditions to get the dense set  $D_f^{spl}$ . One could ask whether  $Q(X)$  is dense for a m.a.d. family  $X$ . The following holds:

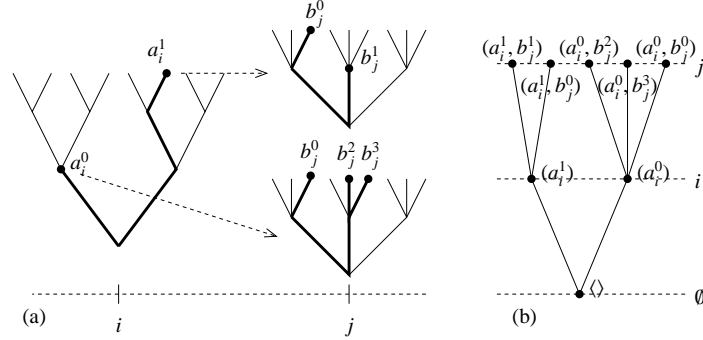
(a) For every  $f$  there is a m.a.d. family  $X$  such that  $Q(X)$  is not dense.

(b) (CH) For every  $f$  there is a m.a.d. family  $X$  such that  $Q(X)$  is dense.

*Proof:* Fix  $f$ . A node  $s \in T_{max}$  has level  $m$  if  $f(m) \leq \text{length}(s) < f(m+1)$ .  $S \in Q$  has unique splitting if  $S$  has at most one splitting point of level  $n$  for all  $n \in \omega$ . For every  $T \in Q$  there is an  $S \leq_Q T$  with unique splitting.

For (a), fix a  $T \in Q$  with unique splitting. Set  $Y := \{A \in [\omega]^{\aleph_0} : (\forall S \leq_Q T) S \notin Q_A^f\}$ .  $Y$  is open dense in  $([\omega]^{\aleph_0}, \subseteq)$ , therefore there is a m.a.d.  $X \subseteq Y$ .

For (b), list  $Q$  as  $(T_\alpha)_{\alpha \in \omega_1}$ , and build  $B_\alpha \in [\omega]^{\aleph_0}$  by induction on  $\alpha \in \omega_1$ : Find an  $S \leq_Q T_\alpha$  with unique splitting. If some  $S' \leq_Q S$  is in  $Q_{B_\beta}^f$  ( $\beta < \alpha$ ) (or equivalently in  $Q_{\bigcup_{i \in I} B_{\beta_i}}^f$  for some  $l \in \omega, \beta_0, \dots, \beta_{l-1} < \alpha$ ), then just pick any almost disjoint  $B_\alpha$ . Otherwise enumerate  $(B_\beta)_{\beta \in \alpha}$  as  $(C_n)_{n \in \omega}$ , and construct  $B_\alpha$  and  $S' \leq_Q S$  inductively: At stage  $n$ , add a split of  $S$  to  $S'$  whose level is not in  $\bigcup_{m \leq n} C_m$ , and use some bookkeeping to guarantee that  $S' \in Q$ . Let  $B_\alpha$  be the set of splitting-levels of  $S'$ .



F 3. An approximation  $g: u = \{i, j\}$ ,  $T_{\max}^i = 2^{<\omega}$ ,  $T_{\max}^j = 3^{<\omega}$ .  
 $\text{Pos}(g) = \text{Pos}_{\leq j}(g) = \{(a_i^1, b_j^0), (a_i^1, b_j^1), (a_i^0, b_j^2), (a_i^0, b_j^3), (a_i^0, b_j^0)\}$ .  
 (a): viewed as function:  $g(i)(\langle \rangle) = \{a_i^0, a_i^1\}$ ,  $g(j)(\langle a_i^1 \rangle) = \{b_j^0, b_j^1\}$  etc.  
 (b): viewed as tree, the heights labeled with  $\{\emptyset\} \cup u$ .

1 For every  $i \in I$  we fix a finitely splitting lim-sup tree-forcing  $Q_i$  (to be more exact, we  
 2 fix a pair  $T_{\max}^i, \mu^i$ ). In the application of this paper, each  $Q_i$  will be the same forcing  $Q$ .

3 **Definition 2.2.** (Pre-condition) We call  $p$  a pre-condition, if  $p$  is a function, the domain of  
 4  $p$  is a countable<sup>7</sup> subset of  $I$ , and for each  $i \in \text{dom}(p)$ ,  $p(i)$  consists of the following:

- 5 •  $\text{Dom}_i^p$ , a countable subset of  $\text{dom}(p) \cap I_{<i}$ , and  
 6 • a (definition of a) Borel function  $B_i^p: (\omega^\omega)^{\text{Dom}_i^p} \rightarrow Q_i$

7 **Remark 2.3.** The idea is that we calculate the condition  $B_i^p \in Q_i$  using countably many  
 8 generic reals  $(\eta_j)_{j \in \text{Dom}_i^p}$  that have already been produced at stage  $i$ . The forcing conditions  
 9  $p$  of the non-wellfounded iterations will be pre-conditions that satisfy additional properties,  
 10 in particular: all  $B_i^p$  are continuous (on a certain Borel set), i.e., if we want to know  $B_i^p$  up to  
 11 some finite height we only have to know  $(\eta_i \upharpoonright m)_{i \in u}$  for some finite  $u$  and  $m \in \omega$ . Moreover,  
 12 we will assume that we will have “wellfounded continuity parts”. This will be explained  
 13 in the following, here just an example: Assume that  $I = \omega^* = \{\dots, 3, 2, 1, 0\}$ , and each  
 14  $T_{\max}^i = 2^{<\omega}$ . Let  $p$  be the pre-condition with  $\text{Dom}_n^p = \{n+1\}$ , i.e.,  $B_n^p$  only depends on  
 15 the generic real  $\eta_{n+1}$ , and  $B_n^p(x) = [0]$  if  $x(0) = 0$  and  $B_n^p(x) = [1]$  otherwise. Then  $p$  is  
 16 continuous, but will not be a valid condition, since it is not well founded enough.

17 We now define finite “approximations” to conditions of the iteration; they will have  
 18 the same role for the iteration that finite antichains have for  $Q$  (see, e.g., Lemma 1.7).  
 19 The following definition looks rather unpleasant, but really is quite simple, as Figure 3  
 20 hopefully demonstrates. (We first define approximations as functions as in (a) of the figure;  
 21 sometimes it is more useful to think of them as trees as in (b), which will be described  
 22 in 2.6.)

23 **Definition 2.4.** (Approximation)

- 24 •  $g$  is an approximation, if  $g$  is a function with finite domain  $u \subseteq I$  of the following  
 25 form: Let  $i_0$  be the smallest element of  $u$ . We set  $\text{Pos}_{<i_0}(g) := \{\langle \rangle\}$ . By induction on  
 26  $i \in u$ , we assume that  $\text{Pos}_{<i}(g)$  is a set of sequences indexed by the set  $\{j \in u : j < i\}$ ,  
 27 and require the following:  $g(i)$  is a function from  $\text{Pos}_{<i}(g)$  to finite antichains in  
 28  $T_{\max}^i$ , and we set

$$\text{Pos}_{\leq i}(g) := \{\bar{a} \hat{\ } b : \bar{a} \in \text{Pos}_{<i}(g), b \in g(i)(\bar{a})\}.$$

29 If  $j$  is the successor of  $i$  in  $u$ , we set  $\text{Pos}_{<j}(g)$  to be  $\text{Pos}_{\leq i}(g)$ .

<sup>7</sup>this includes finite and empty.



- 1 • For any  $i \in I \cup \{\infty\}$ , we define  $\text{Pos}_{<i}(\mathfrak{g})$  as  $\text{Pos}_{\leq j}(\mathfrak{g})$ , where  $j = \max(\text{dom}(\mathfrak{g}) \cap I_{<i})$   
2 (or as  $\{\langle \rangle\}$ , if  $\text{dom}(\mathfrak{g}) \cap I_{<i}$  is empty). We set  $\text{Pos}(\mathfrak{g}) := \text{Pos}_{<\infty}(\mathfrak{g})$  and call it the set  
3 of possibilities of  $\mathfrak{g}$ .  
4 • If  $i \notin \text{dom}(\mathfrak{g})$  or  $\bar{a} \notin \text{Pos}_{<i}(\mathfrak{g})$  we set  $\mathfrak{g}(i)(\bar{a}) := \{\langle \rangle\}$  (i.e., the front in  $T_{\max}^i$  consisting  
5 only of the root. This corresponds to “no information”).  
6 • Let  $\mathfrak{g}$  be an approximation,  $J \subset I$ , and  $\bar{\eta} = (\eta_i)_{i \in J}$  a sequence of reals. Then  
7 “ $\bar{\eta}$  is compatible with  $\mathfrak{g}$ ”, if there is an  $\bar{a} \in \text{Pos}(\mathfrak{g})$  such that  $a_i < \eta_i$  for all  $i \in$   
8  $\text{dom}(\mathfrak{g}) \cap J$ . If in addition  $J \supseteq \text{dom}(\mathfrak{g})$ , then this  $\bar{a}$  is uniquely defined and called  
9  $\bar{\eta} \upharpoonright \mathfrak{g}$ . If  $J \supseteq \text{dom}(\mathfrak{g}) \cap I_{<i}$ , then  $\bar{a} \upharpoonright I_{<i}$  is uniquely defined, and therefore we can set  
10  $\mathfrak{g}(i)(\bar{\eta}) := \mathfrak{g}(i)(\bar{a} \upharpoonright I_{<i})$ .  
11 If  $\bar{b} = (b_i)_{i \in J}$  is a sequence of elements of  $\omega^{<\omega}$ , we define  $\bar{b}$  to be compatible  
12 with  $\mathfrak{g}$  if there is a sequence  $\bar{\eta}$  extending  $\bar{b}$  and compatible with  $\mathfrak{g}$ . If  $J \supseteq \text{dom}(\mathfrak{g})$   
13 and additionally each  $b_i$  is long enough, then such a  $\bar{b}$  defines a unique  $\bar{a} \in \text{Pos}(\mathfrak{g})$   
14 called  $\bar{b} \upharpoonright \mathfrak{g}$ ; if  $J \supseteq \text{dom}(\mathfrak{g}) \cap I_{<i}$  and additionally each  $b_i$  is long enough, then we  
15 can define  $\mathfrak{g}(i)(\bar{b})$  as above.  
16 • If  $\mathfrak{g}$  and  $\mathfrak{g}'$  are both approximations, then “ $\mathfrak{g}'$  is stronger than  $\mathfrak{g}$ ” if  $\text{dom}(\mathfrak{g}') \supseteq$   
17  $\text{dom}(\mathfrak{g})$  and for all  $\bar{b} \in \text{Pos}(\mathfrak{g}')$  there is an  $\bar{a} \in \text{Pos}(\mathfrak{g})$  such that  $\bar{b} \geq \bar{a}$  (i.e.,  $b_i \geq a_i$   
18 for all  $i \in \text{dom}(\mathfrak{g})$ ). In this case  $\bar{a}$  is  $\bar{b} \upharpoonright \mathfrak{g}$ .  
19 Equivalently,  $\mathfrak{g}'$  is stronger than  $\mathfrak{g}$  iff for all  $i \in \text{dom}(\mathfrak{g})$  and all  $\bar{b} \in \text{Pos}_{<i}(\mathfrak{g}')$   
20 there is a (unique)  $\bar{a} \in \text{Pos}_{<i}(\mathfrak{g})$  such that  $\bar{b} \geq \bar{a}$  and the antichain  $\mathfrak{g}'(i)(\bar{b})$  is stronger  
21 than  $\mathfrak{g}(i)(\bar{a})$ .  
22 •  $\mathfrak{g}'$  is purely stronger than  $\mathfrak{g}$  if  $\mathfrak{g}'$  is stronger than  $\mathfrak{g}$  and for all  $i \in \text{dom}(\mathfrak{g})$  and  
23  $\bar{b} \in \text{Pos}_{<i}(\mathfrak{g}')$  the front  $\mathfrak{g}'(i)(\bar{b})$  is purely stronger than  $\mathfrak{g}(i)(\bar{b} \upharpoonright \mathfrak{g})$ .  
24 • For  $u \subseteq \text{dom}(\mathfrak{g})$ ,  $\text{maxlength}_u(\mathfrak{g})$  is  $\max(\{\text{length}(a_i) : i \in u, \bar{a} \in \text{Pos}(\mathfrak{g})\})$ .  
25  $\text{maxlength}(\mathfrak{g})$  is  $\text{maxlength}_{\text{dom}(\mathfrak{g})}(\mathfrak{g})$ . Analogously we define  $\text{minlength}(\mathfrak{g})$ .  
26 •  $\mathfrak{g}$  is  $n$ -dense at  $i \in I$ , if  $i \in \text{dom}(\mathfrak{g})$  and for all  $\bar{a} \in \text{Pos}_{<i}(\mathfrak{g})$ ,  $\mathfrak{g}(i)(\bar{a})$  is  $n$ -dense for  
27  $Q_i$ . (See Definition 1.5.)  
28 • For all  $\bar{a} = (a_i)_{i \in u}$  such that  $a_i \in T_{\max}^i$  there is a (unique) approximation  $\mathfrak{g}$  such that  
29  $\text{Pos}(\mathfrak{g}) = \{\bar{a}\}$ . We will call this approximation  $\bar{a}$  as well.

30 **Facts 2.5.** • “stronger” is a partial order on the set of approximations; the same  
31 holds for “purely stronger”.  
32 • If  $\mathfrak{h}$  is stronger than  $\mathfrak{g}$ , then all  $\bar{\eta}$  compatible with  $\mathfrak{h}$  are compatible with  $\mathfrak{g}$ .

33 We could equivalently define approximations as trees, cf. Figure 3(b): Given an approx-  
34 imation  $\mathfrak{g}$ , we can define an approximation-tree with  $u = \text{dom}(\mathfrak{g})$  labeling the heights above  
35 the root, and the set of nodes at height  $i_n \in u$  is  $\text{Pos}_{\leq i_n}(\mathfrak{g})$ ; the tree order is just extension of  
36 sequences. Every such approximation-tree corresponds to an approximation:

37 **Fact 2.6.** Consider a finite tree where the heights above the root are labeled by the in-  
38 creasing sequence  $u = \{i_1, \dots, i_n\}$  in  $I$ . Assume that each node at height  $i_m$  is a sequence  
39  $(a_j)_{j=1, \dots, i_m}$  and that the tree order is the extension relation. Then this tree corresponds to  
40 an approximation, iff each branch has maximal height and the successors of each node at  
41 level  $i_{n-1}$  form an antichain in  $T_{\max}^{i_n}$ .

42 In particular, if we take a subset of the (maximal) branches in the approximation-tree  
43  $\mathfrak{g}$ , we get a “sub-approximation”  $\mathfrak{h}$ . A single branch  $\bar{a}$  is a special case of such a sub-  
44 approximation.

45 **Definition 2.7.** (Approximation to  $p$ ) Let  $p$  be a pre-condition.

- 46 •  $\mathfrak{g}$  approximates  $p$ , or:  $\mathfrak{g}$  is a  $p$ -approximation, if  $\text{dom}(\mathfrak{g}) \subseteq \text{dom}(p)$  and  $\mathfrak{g}$  is an  
47 approximation with the following property: If  $i \in \text{dom}(\mathfrak{g})$ ,  $\bar{a} \in \text{Pos}_{<i}(\mathfrak{g})$ , and  
48  $\bar{\eta} = (\eta_j)_{j \in \text{Dom}_i^p}$  is compatible with  $\bar{a}$ , then  $\mathfrak{g}(i)(\bar{a})$  is a front in  $B_i^p(\bar{\eta})$ .  
49 •  $\mathfrak{g}$  is an indirect approximation to  $p$  witnessed by  $\mathfrak{g}'$ , if  $\mathfrak{g}'$  approximates  $p$  and  $\mathfrak{g}'$  is  
50 purely stronger than  $\mathfrak{g}$ .

1 **Example 2.8.** The following trivial example should demonstrate the difference between  
 2 approximation and indirect approximation: Assume each  $T_{\max}^i$  is  $2^{<\omega}$ , and  $p$  is a condition  
 3 with  $\text{dom}(p) = \{i, j\}$  for some  $i < j$  in  $I$ . Accordingly  $\text{Dom}_i^p$  has to be empty, and  $B_i^p$  is  
 4 constant; we set it to have constant value  $[1]$ . We set  $\text{Dom}_j^p = \{i\}$  and  $B_j^p(x) = [x(0)]$ , i.e., if  
 5 the real  $x$  starts with 0 then  $B_j^p$  is  $[0]$  and otherwise it is  $[1]$ . We define the approximation  
 6  $\mathfrak{g}$  by  $\text{Pos}(\mathfrak{g}) = \{(\langle \cdot \rangle, 1)\}$  and  $\mathfrak{h}$  by  $\text{Pos}(\mathfrak{h}) = \{(1, 1)\}$ . Then  $\mathfrak{g}$  indirectly approximates  $p$ ,  
 7 witnessed by  $\mathfrak{h}$ .

8 Now we can define the forcing  $P$ , the non-wellfounded countable support limit along  $I$ :

9 **Definition 2.9.** (The nwf-iteration  $P = \text{nwf-lim}_I(Q_i)$ )

- 10 •  $p \in P$  means:  
 11  $p$  is a pre-condition, and for all finite  $u \subseteq \text{dom}(p)$ ,  $i \in u$  and  $n \in \omega$  there is a  $p$ -  
 12 approximation  $\mathfrak{g}$  such that  $\text{dom}(\mathfrak{g}) \supseteq u$ ,  $\mathfrak{g}$  is  $n$ -dense for  $i$ , and  $\text{minlength}_u(\mathfrak{g}) > n$ .
- 13 • For  $p, q \in P$ ,  $q \leq p$  means:  
 14 for all  $p$ -approximations  $\mathfrak{g}$  there is a  $q$ -approximation  $\mathfrak{h}$  which is stronger than  $\mathfrak{g}$   
 15 (so in particular,  $\text{dom}(q) \supseteq \text{dom}(p)$ ).
- 16 •  $q \leq_{\mathfrak{g}} p$  if  $q \leq p$  and  $\mathfrak{g}$  indirectly approximates  $p$  and  $q$ .

17 **Remark.** The definition of  $q \leq p$  is not equivalent to “for all  $i$  and  $\bar{\eta}$ ,  $B_i^q(\bar{\eta})$  is a subtree of  
 18  $B_i^p(\bar{\eta})$ .” (Informally speaking, we are only interested in “the generic  $\bar{\eta}$ , not in “all  $\bar{\eta}$ ”.) We  
 19 will see in Lemma 2.23(6) that  $q \leq p$  is equivalent to: for each  $i \in I$  it is forced by  $q \upharpoonright P_{<i}$   
 20 that  $B_i^q(\bar{\eta})$  is a subtree of  $B_i^p(\bar{\eta})$ , where  $\bar{\eta}$  is the generic sequence up to  $i$ .

21 **Facts 2.10.** •  $\leq$  is transitive, and for a fixed approximation  $\mathfrak{g}$  the relation  $\leq_{\mathfrak{g}}$  is transi-  
 22 sitive as well.  
 23 • If  $\mathfrak{h}$  is purely stronger than  $\mathfrak{g}$  then  $\leq_{\mathfrak{h}}$  implies  $\leq_{\mathfrak{g}}$ .  
 24 • For every  $p \in P$ , the approximations of  $p$  are directed: If  $\mathfrak{g}$  and  $\mathfrak{g}'$  both (indirectly)  
 25 approximate  $p$ , then there is a  $\mathfrak{h}$  approximating  $p$  that is (purely) stronger than  
 26 both  $\mathfrak{g}$  and  $\mathfrak{g}'$ . In fact, every  $p$ -approximation  $\mathfrak{h}$  has this property if  $\text{dom}(\mathfrak{h}) \supseteq$   
 27  $\text{dom}(\mathfrak{g}) \cup \text{dom}(\mathfrak{g}')$  and if  $\text{minlength}_{\text{dom}(\mathfrak{g}) \cup \text{dom}(\mathfrak{g}')}(\mathfrak{h})$  is large enough.

28 So in particular for every  $p \in P$  there is an approximating sequence:

29 **Definition 2.11.** An approximating sequence for  $p \in P$  is a sequence  $(\mathfrak{g}_n)_{n \in \omega}$  of approxima-  
 30 tions of  $p$  such that  $\mathfrak{g}_{n+1}$  is purely stronger than  $\mathfrak{g}_n$ , and  $\mathfrak{g}_{n+1}$  is  $n$ -dense for each  $i \in \text{dom}(\mathfrak{g}_n)$ ,  
 31 and  $\text{dom}(p) = \bigcup_{n \in \omega} \text{dom}(\mathfrak{g}_n)$ .

32 An approximating sequence contains all relevant information about  $p$ . In particular,  $\mathfrak{g}$  is  
 33 an indirect approximation to  $p$  iff there is an  $n$  such that  $\mathfrak{g}_n$  is purely stronger than  $\mathfrak{g}$ . So if  
 34  $p$  and  $q$  both have the approximating sequence  $(\mathfrak{g}_n)_{n \in \omega}$ , then  $p =^* q$  (i.e.,  $p \leq q$  and  $q \leq p$ ),  
 35 furthermore  $\mathfrak{g}$  indirectly approximates  $p$  iff it indirectly approximates  $q$ .

36 Approximating sequences provide an equivalent definition for  $P$ :

37 **Definition 2.12.** (Alternative definition of the nwf-iteration  $P$ ) Define the p.o.  $P'$  as fol-  
 38 lows:  $\bar{\mathfrak{g}} \in P'$  iff  $\bar{\mathfrak{g}}$  is a sequence of approximations  $(\mathfrak{g}_n)_{n \in \omega}$  such that  $\mathfrak{g}_{n+1}$  is purely stronger  
 39 than  $\mathfrak{g}_n$  and  $n$ -dense for every  $i \in \text{dom}(\mathfrak{g}_n)$ . We define  $\bar{\mathfrak{h}} < \bar{\mathfrak{g}}$  as: For every  $n$  there is an  $m$   
 40 such that  $\mathfrak{h}_m$  is stronger than  $\mathfrak{g}_n$ .

41 **Lemma 2.13.** There is a dense embedding<sup>8</sup>  $\phi : P' \rightarrow P$ .

42 *Proof.* Given a sequence  $\bar{\mathfrak{g}} \in P'$ , define  $p = \phi(\bar{\mathfrak{g}})$  the following way:  $\text{dom}(p) = \bigcup \text{dom}(\mathfrak{g}_n)$ .  
 43 For  $i \in \text{dom}(p)$ , set  $\text{Dom}_i^p := \text{dom}(p) \cap I_{<i}$ . Define  $T = B_i^p(\bar{\eta})$  as follows: If  $\bar{\eta}$  is compatible  
 44 with all  $\mathfrak{g}_n$ , then let  $T$  be  $\{t \in T_{\max}^i : (\exists n \in \omega) t \leq \mathfrak{g}_n(i)(\bar{\eta})\}$ . Otherwise, let  $n$  be maximal  
 45 such that  $\bar{\eta}$  is compatible with  $\mathfrak{g}_n$ , and let  $T$  be  $\{t \in T_{\max}^i : (\exists s \in \mathfrak{g}_n(i)(\bar{\eta})) t \parallel s\}$ . Clearly,

<sup>8</sup> $\phi$  is even an isomorphism modulo  $=^*$ , where  $p =^* q$  if  $q \leq p$  and  $p \leq q$ .

1  $B_i^p$  is a Borel function,  $B_i^p(\bar{\eta}) \in Q_i$  and each  $g_n$  approximates  $p$ . Therefore  $(g_n)_{n \in \omega}$  is an  
 2 approximating sequence for  $p \in P$ . It is clear that  $\phi$  preserves the order.

3 Let  $\psi$  map  $p \in P$  to any approximating sequence for  $p$ .  $\psi : P \rightarrow P'$  preserves order as  
 4 well and  $\phi(\psi(p)) =^* p$ . Therefore  $\phi$  is a dense embedding.  $\square$

5 **Notes 2.14.** (1) If  $g$  indirectly approximates  $p$ , then there is a  $q =^* p$  such that  $g$   
 6 approximates  $q$ . (Just let  $q$  correspond to an approximating sequence of  $p$  starting  
 7 with  $g_0 = g$ .)

8 (2) It doesn't matter whether the  $g_n$  in an approximating sequence are approximations  
 9 to  $p$  or just indirect approximations.

10 (3) It doesn't matter whether  $g_{n+1}$  proves  $n$ -density for every  $i \in \text{dom}(g_n)$  or for just  
 11 some  $i_n$ , provided that the sequence  $(i_n)_{n \in \omega}$  covers  $\bigcup \text{dom}(g_n)$  infinitely often.

12 (4) In Definition 2.2 of pre-condition, instead of requiring  $B_i^p$  to be a function into  
 13  $Q_i$ , we could have defined  $B_i^p$  to be a function to subtrees of  $T_{\max}^i$ . The additional  
 14 “ $n$ -dense” requirements on a condition guarantee  $B_i^p(\bar{\eta}) \in Q_i$  anyway (for generic  
 15 sequences  $\bar{\eta}$ ).

16 (5) Every approximation  $g$  can be interpreted as a condition in  $P$ , by

$$17 \quad B_i^g(\bar{\eta}) := \{t : t \parallel g(i)(\bar{\eta})\} \text{ for } i \in \text{dom}(g).$$

18 (Where we set  $g(i)(\bar{\eta}) := \{\langle \rangle\}$  if  $\bar{\eta}$  is incompatible with  $g$ .) Then  $g$  approximates  
 19 itself.

20 (6) For any approximation  $g$  and  $u \subseteq I$  finite we can assume  $u \subseteq \text{dom } g$ : Just set  $g(i)$   
 21 to be the constant function with value  $\{\langle \rangle\}$  for  $i \notin \text{dom } g$ . (Recall that  $\{\langle \rangle\}$  is the  
 22 “trivial front” corresponding to “no information”.)

23 (7) If  $g$  and  $h$  are approximations, we can assume without loss of generality that  
 24  $\text{dom}(g) = \text{dom}(h)$ .

25 (8) For any  $U \subseteq I$  countable and  $p \in P$  we can assume without loss of generality that  
 26  $\text{dom}(p) \supseteq U$ . This is clear if  $p$  is interpreted as a sequence of Borel-functions:  
 27 just set  $B_i^p$  to be (the constant function with value)  $T_{\max}^i$  for  $i \notin \text{dom}(p)$ . If  $p$   
 28 is interpreted as sequence  $(g_n)_{n \in \omega}$  of approximations, we have to set  $g_n(i)$  to be (the  
 29 constant function with value)  $T_{\max}^i \cap \omega^{k(n)}$  for some sufficiently large  $k(n)$ . (Using  
 30  $\{\langle \rangle\}$  does not work here, since it does not satisfy  $n$ -density.)

31 (9) So if  $q \leq p$  we can assume  $\text{dom}(q) = \text{dom}(p)$ , and if  $p$  is interpreted as sequence  
 32  $(g_n)_{n \in \omega}$  and  $q$  as  $(h_n)_{n \in \omega}$  then we can assume  $\text{dom}(g) = \text{dom}(h)$ .

33 **2.2. Fusion and pure decision.** We have seen: Every  $p \in P$  corresponds to a purely  
 34 increasing sequence  $(g_n)$  of approximations such that  $\bigcup \text{dom}(g_n) = \text{dom}(p)$  and  $g_{n+1}$  is  
 35  $n$ -dense for  $\text{dom}(g_n)$ . The approximating sequences immediately prove a version of fusion:

36 **Lemma 2.15.** (Fusion) Assume that  $(p_n)_{n \in \omega}$  is a sequence of conditions,  $(g_n)_{n \in \omega}$  a sequence  
 37 of approximations, and  $i_n \in \text{dom}(g_n)$  such that:

- 38 •  $p_{n+1} \leq_{g_n} p_n$ ,
- 39 •  $g_{n+1}$  is purely stronger than  $g_n$  and  $n$ -dense for  $i_n$ ,
- 40 •  $(i_n)_{n \in \omega}$  covers  $\bigcup \text{dom}(p_n)$  infinitely often.

41 Then there is a condition  $p_\omega$  such that  $p_\omega \leq_{g_n} p_n$  for all  $n$ .

42 *Proof.* We already know that the sequence  $(g_n)_{n \in \omega}$  of approximations defines a condition  
 43  $p_\omega$  such that each  $g_n$  approximates  $p_\omega$ . If  $h$  approximates  $p_n$ , then some  $g_m$  is stronger than  
 44  $h$ . Then  $g_m$  approximates  $p_\omega$ , so  $p_\omega \leq p_n$ .  $\square$

45 **Definition 2.16.**  $h$  is sub-approximation of  $g$  if  $\text{Pos}(h) \subseteq \text{Pos}(g)$ . (So in particular  $\text{dom}(g) =$   
 46  $\text{dom}(h)$ .)

47 Obviously any sub-approximation of  $g$  is stronger than  $g$ . In the interpretation of ap-  
 48 proximations as trees, a sub-approximation is just a nonempty subset of the (maximal)  
 49 branches, see Fact 2.6.

1 **Lemma 2.17.** (Sub-approximation) Assume that  $\mathfrak{g}$  indirectly approximates  $p$  and that  $\mathfrak{h}$  is  
 2 a sub-approximation of  $\mathfrak{g}$ . Then there is a weakest condition stronger than  $p$  and approxi-  
 3 mated by  $\mathfrak{h}$ , which we call  $p \upharpoonright \mathfrak{h}$ .

4 *Proof.* Without loss of generality, we can think of  $p$  as an approximation-sequence  $(\mathfrak{g}_n)_{n \in \omega}$   
 5 with  $\mathfrak{g} = \mathfrak{g}_0$ . We define approximations  $\mathfrak{h}_n$  as follows:  $\mathfrak{h}_n$  consists of those nodes in the  
 6 approximation-tree  $\mathfrak{g}_n$  that are compatible with an element of  $\mathfrak{h}$ . Then  $p \upharpoonright \mathfrak{h}$  is the sequence  
 7  $(\mathfrak{h}_n)_{n \in \omega}$ .  $\square$

8 A special case of a sub-approximation is a singleton:

9 **Definition 2.18.** Assume that  $\mathfrak{g}$  (indirectly) approximates  $p$  and  $\bar{a} \in \text{Pos}(\mathfrak{g})$ . We can in-  
 10 terpret  $\bar{a}$  as an approximation, a sub-approximation of  $\mathfrak{g}$ . Instead of  $p \upharpoonright \bar{a}$  we also write  
 11  $p^{[\bar{a}]}$ .

12 **Corollary 2.19.** (Specialization and pure decision) Assume that  $\mathfrak{g}$  indirectly approximates  
 13  $p$  and that  $\bar{a} \in \text{Pos}(\mathfrak{g})$ .

- 14 (1)  $p^{[\bar{a}]} \in P$ ,  $p^{[\bar{a}]} \leq p$  and  $\bar{a}$  indirectly approximates  $p^{[\bar{a}]}$ . If  $q \leq p$  and  $\bar{a}$  indirectly  
 15 approximates  $q$ , then  $q \leq p^{[\bar{a}]}$ .  
 16 (2) If  $q \leq_{\mathfrak{g}} p$ , then  $q^{[\bar{a}]} \leq p^{[\bar{a}]}$ .  
 17 (3) If  $q \leq p^{[\bar{a}]}$  then there is a  $r \leq_{\mathfrak{g}} p$  such that  $r^{[\bar{a}]} =^* q$ .  
 18 (4) The set  $\{p^{[\bar{a}]} : \bar{a} \in \text{Pos}(\mathfrak{g})\}$  is predense below  $p$ .  
 19 (5) Abusing notation, we denote with  $(i, a)$  the approximation  $\mathfrak{g}$  with domain  $\{i\}$  such  
 20 that  $\mathfrak{g}(i)(\langle \rangle) = \{a\}$ . For all  $i \in I$ ,  $n \in \omega$  the following set is dense:

$$21 \quad \{p \in P : (\exists a \in \omega^n)(i, a) \text{ approximates } p\}.$$

22 (Or, in the notation introduced later: We can densely determine the generic  $\eta_i$  up  
 23 to  $n$ .)

- 24 (6) (Pure decision) If  $D \subseteq P$  is open dense, and  $\mathfrak{g}$  indirectly approximates  $p$ , then  
 25 there is an  $r \leq_{\mathfrak{g}} p$  such that  $r^{[\bar{a}]} \in D$  for all  $\bar{a} \in \text{Pos}(\mathfrak{g})$ .

26 *Proof.* (1) and (2) follow easily from the definition.

27 (3) We set  $r$  to be  $q$  “below  $\bar{a}$ ” and  $p$  otherwise. Let  $p$  correspond to  $(\mathfrak{g}_n)_{n \in \omega}$  with  $\mathfrak{g}_0 = \mathfrak{g}$ ,  
 28 and  $q$  corresponds to  $(\mathfrak{h}_n)_{n \in \omega}$  with  $\mathfrak{h}_0 = \bar{a}$  such that each  $\mathfrak{h}_n$  is stronger than  $\mathfrak{g}_n$ . According  
 29 to Note 2.14(9), we can assume that  $\text{dom}(\mathfrak{h}_n) = \text{dom}(\mathfrak{g}_n) = u_n$ . We define by induction on  
 30  $n$  a sub-approximation  $\bar{f}_n$  of  $\mathfrak{g}_n$ : Let  $i_0$  be minimal in  $u_n$ . So  $\text{Pos}_{<i_0}(\bar{f}_n) = \{\langle \rangle\}$ . By induction  
 31 on  $i \in u_n$ , define for all  $\bar{b} \in \text{Pos}_{<i}(\bar{f}_n)$

$$32 \quad \bar{f}_n(i)(\bar{b}) := \begin{cases} \mathfrak{g}_n(i)(\bar{b}) & \text{if } \bar{b} \text{ is incompatible with } \mathfrak{h}_n, \\ \mathfrak{h}_n(i)(\bar{b}) \cup \{t \in \mathfrak{g}_n(i)(\bar{b}) : t \perp \mathfrak{h}_n(i)(\bar{b})\} & \text{otherwise.} \end{cases}$$

33 It is clear that the possibilities of  $\bar{f}_n$  follow  $\mathfrak{h}_n$  up to some  $i \in \text{dom } \mathfrak{g}_n$  and from then on  
 34 become incompatible with  $\mathfrak{h}_n$  and follow  $\mathfrak{g}_n$ . To be more exact:  $\bar{b} \in \text{Pos}(\bar{f}_n)$  iff  $\bar{b} \in \text{Pos}(\mathfrak{g}_n)$   
 35 and for some  $i \in \text{dom}(\mathfrak{g}_n) \cup \{\infty\}$ ,  $\bar{a} \upharpoonright I_{<i}$  is in  $\text{Pos}(\mathfrak{h}_n)$  and either  $i = \infty$  or  $a_i \perp \mathfrak{h}_n(i)(\bar{a})$ .  
 36 From this it follows that  $\bar{f}_n$  is purely stronger than  $\mathfrak{g}_n$ , and that the  $\bar{f}_n$  are an approximating  
 37 sequence (converging to some  $r \leq p$ ).

38 (4) If  $\mathfrak{g}$  indirectly approximates  $p$  and  $q \leq p$ , then there is a  $\mathfrak{h}$  stronger than  $\mathfrak{g}$  approxi-  
 39 mating  $q$ . Let  $\bar{b} \in \text{Pos}(\mathfrak{h})$  and  $\bar{a} = \bar{b} \upharpoonright \mathfrak{g} \in \text{Pos}(\mathfrak{g})$ . Then  $q^{[\bar{b}]} \leq q, p^{[\bar{a}]}$ .

40 (5) Let  $\mathfrak{h}$  approximate  $p$  such that  $\text{minlength}_{(i)}(\mathfrak{h}) > n$ . Let  $\bar{a} \in \text{Pos}(\mathfrak{h})$ . Then  $(a_i)$  indi-  
 41 rectly approximates  $p^{[\bar{a}]} \leq p$ . By 2.14(1) we can find a  $q =^* p$  such that  $(a_i)$  approximates  
 42  $q$ .

43 (6) Let  $\text{Pos}(\mathfrak{g}) = \{\bar{a}_0, \dots, \bar{a}_l\}$ . Pick  $q_0 \leq p^{[\bar{a}_0]}$  in  $D$ , and  $r_0 \leq_{\mathfrak{g}} p$  as in (3). So  $r_0^{[\bar{a}_0]} \in D$ .  
 44 Pick  $q_1 \leq r_0^{[\bar{a}_1]}$  in  $D$  and  $r_1 \leq_{\mathfrak{g}} r_0$  as above, etc. Then  $r_l$  has the required property.  $\square$

45 **Remark.** Similarly, we can define conjunctions of two approximations  $\mathfrak{g}, \mathfrak{g}'$ . More specifi-  
 46 cally: let us call  $\mathfrak{g}$  and  $\mathfrak{g}'$  compatible if there is an  $\mathfrak{h}$  stronger than both  $\mathfrak{g}$  and  $\mathfrak{g}'$ . Then

1 for every compatible pair  $g, g'$  there is a weakest approximation  $g \wedge g'$  stronger than  $g$  and  
 2  $g'$ . If  $p$  and  $q$  have incompatible approximations, then they are incompatible (in  $Q$ ). This  
 3 can be used to define the conjunction of an approximation and a condition (if the condition  
 4  $p$  corresponds to the sequence  $g_n$ , let  $p \wedge h$  correspond to the sequence  $g_n \wedge h$ ; it is the  
 5 weakest condition stronger than  $p$  that is approximated by  $h$ ). Similarly one can define the  
 6 conjunction of two conditions. However, all of this will not be needed in this paper.

7 **2.3. Restrictions.** We now list some trivial properties of  $P$  regarding restriction:

8 **Definition 2.20.** For  $i \in I \cup \{\infty\}$  we define  $P_{<i} := \{p \in P : \text{dom}(p) \subseteq I_{<i}\}$ . In particular,  
 9  $P = P_{<\infty}$ . Analogously we define  $P_{\leq i}$  for  $i \in I$ .

10 **Facts 2.21.** (*Restriction*) Assume  $p, q \in P$  and  $i, j \in I \cup \{\infty\}$ .

- 11 • If  $\text{dom}(q) \supseteq \text{dom}(p)$ ,  $q \upharpoonright \text{dom}(p) = p$  and  $g$  approximates  $p$ , then  $q \leq_g p$ .
- 12 •  $p \upharpoonright I_{<i} \in P_{<i}$  and  $p \leq p \upharpoonright I_{<i}$ .
- 13 • If  $p' \leq p$  then  $p' \upharpoonright I_{<i} \leq p \upharpoonright I_{<i}$ . If  $p \in P_{<i}$  then  $p \upharpoonright I_{<i} = p$ .
- 14 • Let  $q \in P_{<i}$ ,  $q \leq p \upharpoonright I_{<i}$ . Define  $q \wedge p := q \cup p \upharpoonright I_{\geq i}$ . Then  $q \wedge p \in P$  is the weakest  
 15 condition stronger than both  $q$  and  $p$ .
- 16 •  $p \upharpoonright I_{<i}$  is a reduction of  $p$  (i.e.,  $r' \in P_{<i}$  and  $r' \leq p \upharpoonright I_{<i}$  implies  $r' \parallel p$ ).
- 17 • In particular,  $P_{<i} < P_{<j}$  (i.e.,  $P_{<i}$  is a complete subforcing of  $P_{<j}$ ) for  $i < j$ .
- 18 • If  $p \upharpoonright I_{<i} \parallel q \upharpoonright I_{<i}$  and  $\text{dom}(p) \cap \text{dom}(q) \subseteq I_{<i}$ , then  $p \parallel q$ .
- 19 • Similar facts hold for  $P_{\leq i}$ . E.g., if  $i < j$ , then  $P_{\leq i} < P_{<j}$ .

20 **Definition 2.22.** Assume that  $j \in I \cup \{\infty\}$  and  $i < j$ , and that  $G_{<j}$  is a  $P_{<j}$ -generic filter  
 21 over  $V$ .

- 22 • Since  $P_{<i}$  is a complete subforcing of  $P_{<j}$ , the filter  $G_{<j} \cap P_{<i} =: G_{<i}$  is  $P_{<i}$ -generic  
 23 over  $V$ . We set  $V_{<i} := V[G_{<i}]$ . The canonical  $Q_i$ -generic filter over  $V_{<i}$  is called  
 24  $G(i)$ . Analogously we can define  $V_{\leq i}$  and  $G_{\leq i}$  (which turns out to be  $V_{<i}[G(i)]$  and  
 25  $G_{<i} * G(i)$ , respectively).
- 26 • In  $V_{<j}$  or  $V_{\leq i}$  we define  $\eta_i$  to be the union of all  $t \in \omega^{<\omega}$  such that  $(i, t)$  is an  
 27 approximation<sup>9</sup> of  $p$  for some  $p \in G_{<j}$  (or  $G_{\leq i}$ ).

28 **Lemma 2.23.** Let  $i, j, G_{<j}$  be as above,  $p \in G_{<j}$ , and set  $\bar{\eta} = (\eta_i)_{i < j}$ .

- 29 (1)  $\eta_i$  is a well-defined real. In particular we can calculate  $B_i^q(\bar{\eta} \upharpoonright \text{Dom}_i^q)$  for all  $q \in P$ ;  
 30 abusing notation, we will just write  $B_i^q(\bar{\eta})$ .
- 31 (2) If  $g$  indirectly approximates  $p$ , then  $\bar{\eta}$  is compatible with  $g$ .
- 32 (3)  $\{\eta_i\} = \bigcap \{\lim B_i^q(\bar{\eta}) : q \in G_{<j}, i \in \text{dom}(q)\}$ .
- 33 (4)  $q \in G_{<j}$  iff  $\eta_i \in \lim(B_i^q(\bar{\eta}))$  for all  $i \in \text{dom}(q)$ .
- 34 (5) (in  $V$ ):  $q \leq_p p$  iff  $\text{dom}(q) \supseteq \text{dom}(p)$  and  $q \Vdash \eta_i \in \lim(B_i^p(\bar{\eta}))$  for all  $i \in \text{dom}(p)$ .
- 35 (6) (in  $V$ ):  $q \leq_p p$  iff  $\text{dom}(q) \supseteq \text{dom}(p)$  and  $q \upharpoonright I_{<i} \Vdash B_i^q(\bar{\eta}) \subseteq B_i^p(\bar{\eta})$  for all  $i \in \text{dom}(p)$ .

36 *Proof.* (1) By 2.19(5), the set of conditions  $q$  such that for some  $t$  of length  $n$  the approx-  
 37 imation  $(i, t)$  approximates  $q$  is dense. Therefore  $\eta_i$  is infinite. Also, if  $s \perp t$ , if  $(i, t)$  is an  
 38 approximation of  $q$ , and if  $(i, s)$  is an approximation of  $q'$ , then  $q$  and  $q'$  are incompatible.  
 39 This shows that  $\eta_i$  is indeed a real.

40 (2) According to 2.19(4), the set  $\{p^{[\bar{a}]} : \bar{a} \in \text{Pos}(g)\}$  is predense below  $p$ . Let  $\bar{a}$  be such  
 41 that  $p^{[\bar{a}]} \in G$ . Any  $q \in G$  that is stronger than  $p^{[\bar{a}]}$  and decides  $\eta_i$  up to the length of  $a_i$   
 42 forces that  $\eta_i \supset a_i$ . So  $\bar{\eta}$  is compatible with  $\bar{a}$  and therefore with  $g$ .

43 (3) Let  $n \in \omega$ . We have to show that  $\eta_i \upharpoonright n \in B_i^q(\bar{\eta})$ . First pick an approximation  $g$  of  
 44  $q$  with  $\text{minlength}_{i(i)}(g) \geq n$ . We already know that  $\bar{\eta}$  is compatible with  $g$ , in particular  
 45  $\eta_i$  is compatible with  $g(i)(\bar{\eta})$ . And  $g(i)(\bar{\eta})$  is a front in  $B_i^q(\bar{\eta})$ , since  $g$  approximates  $q$ . It  
 46 remains to be seen that the intersection on the right-hand side is a singleton; this is clear  
 47 by genericity.

<sup>9</sup>as in Corollary 2.19(5)

1 (5) One direction follows immediately from the definition of the order in  $P$ : Assume  
 2 that  $q \leq p$  and that  $i \in \text{dom}(p)$ . Assume towards a contradiction that  $r \leq q$  forces that  
 3  $\eta_i \notin \text{lim}(B_i^p(\bar{\eta}))$ , more specifically that  $\eta_i \upharpoonright M \notin B_i^p(\bar{\eta})$  for some  $M$  (already determined by  
 4  $\bar{r}$ ). Pick a  $p$ -approximation  $g$  that has minimal height greater than  $M$  at position  $i$ ; and an  
 5  $r$ -approximation  $h$  stronger than  $g$ . Pick  $\bar{b} \in \text{Pos}(h)$  and let  $\bar{a} \in \text{Pos}(g)$  be the restriction.  
 6 Then  $r^{\bar{b}}$  forces that  $\eta_i \upharpoonright M < a_i < b_i$  for any  $\bar{b} \in \text{Pos}(h)$ , but  $a_i \in g(i)(\bar{a})$  which is a front in  
 7  $B_i^p(\bar{\eta})$ , a contradiction.

8 For the other direction, let  $g$  approximate  $p$  and  $h$  approximate  $q$  such that  $\text{dom}(h) \supseteq$   
 9  $\text{dom}(g)$  and the length of  $h$  is sufficiently large on  $\text{dom}(g)$ . Then  $h$  must be stronger than  $g$ ,  
 10 which shows that  $q \leq p$ .

11 (4) follows from (3) and (5); (6) follows from (5) (see also the proof of Lemma 2.25).  
 12 □

13 **2.4. Properness, bounding, continuous reading.** As immediate consequence of fusion  
 14 and pure decision we get:

15 **Theorem 2.24.** (1)  $P$  is  $\omega^\omega$ -bounding. For every  $p$  and  $P$ -name  $\tau$  for an  $\omega$ -sequence  
 16 of ordinals there is a  $q \leq p$  such that  $q$  reads  $\tau$  continuously.<sup>10</sup>

17 (2) Assume that the cofinality<sup>11</sup> of  $I$  is  $\geq \aleph_1$ , that  $G$  is  $P$ -generic over  $V$  and that  
 18  $r \in \mathbb{R}^{V[G]}$ . Then there is an  $i \in I$  such that  $r \in \mathbb{R}^{V_{<i}}$ .

19 (3)  $P$  is proper.<sup>12</sup>

20 (4)  $P$  forces that  $\eta_i$  is a  $Q_i$ -generic real over  $V_{<i}$ .

21 (5) If  $I = I_1 + I_2$ , then  $\text{nwf-lim}_I(Q_i) \cong \text{nwf-lim}_{I_1}(Q_i) * \text{nwf-lim}_{I_2}(Q_i)$ , the forcing-  
 22 composition of  $\text{nwf-lim}_{I_1}(Q_i)$  and (the  $\text{nwf-lim}_{I_1}(Q_i)$ -name for)  $\text{nwf-lim}_{I_2}(Q_i)$ .

23 (6) If  $I = \sum_{\beta \in \epsilon} J_\beta$  is the concatenation of the orders  $J_\beta$  along the ordinal  $\epsilon$ , then  
 24  $\text{nwf-lim}_I(Q_i)$  is equivalent to the countable support limit  $(P_\beta, Q'_\beta)_{\beta \in \epsilon}$ , where  $Q'_\beta$   
 25 is (the  $P_\beta$ -name for)  $\text{nwf-lim}_{J_\beta}(Q_i)$ .

26 (7) If  $I$  is well-founded, then  $\text{nwf-lim}_I(Q_i)$  is the countable support limit of the  $Q_i$ .

27 *Proof.* (1) Fix for every countable subset  $J$  of  $I$  an enumeration  $\{j_m : m \in \omega\}$ , and denote  
 28  $\{j_m : m \in n\}$  by  $\text{first}(n, J)$ .

29 Assume  $\tau$  is a name of a real and  $p \in P$ . We have to show that there is a  $p_\omega \leq p$   
 30 and an  $f \in \omega^\omega$  such that  $p_\omega \Vdash \tau(n) < f(n)$ . Let  $p_0 \leq p$ ,  $f(0) \in \omega$  be such that  $p_0 \Vdash$   
 31  $\tau(0) = f(0)$ , and let  $g_0$  approximate  $p_0$ . Assume that  $g_n$  and  $p_n$  are already defined. We  
 32 define  $p_{n+1} \leq_{g_n} p_n$ ,  $f(n)$  and  $g_{n+1}$  the following way: Let  $p_{n+1} \leq_{g_n} p_n$  be such that  $p_{n+1}^{\bar{a}}$   
 33 decides  $\tau(n)$  for every  $\bar{a} \in \text{Pos}(g_n)$ , see 2.19(6). Let  $f(n)$  be the maximum of the possible  
 34 values for  $\tau(n)$ . Let  $g_{n+1}$  be a  $p_{n+1}$ -approximation stronger than  $g_n$  which is  $n$ -dense at  
 35 every  $i \in \text{first}(n, \text{dom}(p_1)) \cup \dots \cup \text{first}(n, \text{dom}(p_n))$ . Then the sequence  $(p_n)_{n \in \omega}$  satisfies the  
 36 conditions for fusion 2.15 so there is a  $p_\omega \leq p_n$ . Clearly,  $p_\omega \Vdash \tau(n) \leq f(n)$ .

37 The same argument shows continuous reading of  $\omega$ -sequences: Now we do not require  
 38  $\tau(n)$  to be a natural number, and we do not care about the maximum possible value; the  
 39 rest is the same.

40 (2) The  $p_\omega$  above completely determines  $\tau$ , so if  $p_\omega \in P_{<i}$ , then  $p_\omega \Vdash_P \tau \in V_{<i}$ .

41 (3) is very similar to the above: Assume that  $N < H(\chi)$  and  $p_0 \in N$ . Let  $\{D_m : m \in \omega\}$   
 42 enumerate the dense sets in  $N$ . Assume  $p_n, g_n \in N$  are already defined. Find (in  $N$ )  
 43  $p_{n+1} \leq_{g_n} p_n$  such that  $p_{n+1}^{\bar{a}} \in D_n$  for all  $\bar{a} \in \text{Pos}(g_n)$ , and pick  $g_{n+1} \in N$  big enough. Then

<sup>10</sup>In more detail: Let  $(\tau(n))_{n \in \omega}$  be a sequence of  $P$ -names for ordinals and  $p \in P$ . Then there is a  $q \leq p$   
 corresponding to a sequence  $(g_n)_{n \in \omega}$  of approximations, and there are functions  $f_n$  from  $\text{Pos}(g_n)$  into the ordinals  
 such that  $q^{\bar{a}}$  forces  $\tau(n) = f_n(\bar{a})$  for all  $\bar{a} \in \text{Pos}(g_n)$ . If each  $\tau(n)$  is a natural number then this defines (in  $V$ ) a  
 continuous function  $F$  from  $(\omega^\omega)^{\text{dom}(q)}$  into  $\omega^\omega$  such that  $q$  forces that  $F(\bar{\eta} \upharpoonright \text{dom}(q)) = \bar{\tau}$ .

<sup>11</sup>We always mean the ‘‘upwards cofinality’’, i.e., the minimal size of an upwards cofinal subset.  $A \subset I$  is  
 upwards cofinal if for every  $i \in I$  there is an  $a \in A$  such that  $a \geq i$ .

<sup>12</sup> $P$  even is non-elementary-proper (nep), i.e., there are generic conditions for all (non-transitive, non-  
 elementary, but ord-transitive) countable ZFC models; cf. [11] or [8].

1 we can (in  $V$ ) fuse this sequence into a  $p_\omega \in P$ . Note that  $\text{dom}(p_\omega) \subseteq N \cap I$ . If  $G$  is  
 2  $P$ -generic over  $V$  and  $p_\omega \in G$ , then  $p_n \in G$  and  $\{p_n^{[\bar{a}]}\} : \bar{a} \in \text{Pos}(g_n)\}$  is predense below  $p_n$ ,  
 3 so some  $p_n^{[\bar{a}]} \in G$ , and  $p_n^{[\bar{a}]} \in D_n \cap N$ .

4 (4) is a special case of (5): Set  $I_1 := I_{<i}$  and  $I_2 := \{i\}$ . So  $\eta_i$  is  $V_{<i}$ -generic in  $V_{\leq i}$  and  
 5 therefore in  $V_{<\infty}$  as well.

6 (5) Set  $P := \text{nwf-lim}_I(Q_i)$ ,  $P_1 := \text{nwf-lim}_{I_1}(Q_i)$ , and  $P_2$  (the  $P_1$ -name of)  $\text{nwf-lim}_{I_2}(Q_i)$ .

7 There is a natural map  $\phi : p \mapsto (p_1, p_2)$  from  $P$  to  $P_1 * P_2$ :  $p_1 := p \upharpoonright I_1$ , and  $p_2$  is defined  
 8 by  $\text{dom}(p_2) := \text{dom}(p) \setminus I_1$  and  $B_i^{p_2}(\bar{\eta}) := B_i^p((\eta_i)_{i \in I_1} \widehat{\bar{\eta}})$ .

9 It is clear that  $\phi$  preserves  $\leq$ . We claim that it is dense and preserves  $\perp$ . Assume  
 10  $\phi(p) = (p_1, p_2)$ ,  $\phi(q) = (q_1, q_2)$ , and  $(r_1, r_2) \leq (p_1, p_2), (q_1, q_2)$ . We have to find a  $r' \leq_P p, q$   
 11 such that  $\phi(r') \leq (r_1, r_2)$ .

12  $r_1$  forces that  $p_2, q_2$  and  $r_2$  correspond to approximating sequences  $(g_n^p), (g_n^q)$  and  $(g_n^r)$ .  
 13 As in (1) we can find an  $r'_1 \leq r_1$  with an approximating sequence  $(h_n)$  such that  $h_n$  decides  
 14  $g_n^i$  (for  $i \in \{p, q, r\}$ ) in a way such that  $g_n^r$  is stronger than both  $g_n^p$  and  $g_n^q$ . Then we can  
 15 concatenate  $(h_n)$  with the  $(g_n^r)$  to an approximating sequence to some  $r' \in P$ . Then  $r' \leq p, q$   
 16 and  $\phi(r') \leq (r_1, r_2)$ .

17 (6) By induction on  $\epsilon$ . The successor step follows from (5). Let  $\text{cf}(\epsilon) > \omega$ . Then the  
 18 nwf-limit as well as the cs-limit are just the unions of the smaller limits, and therefore  
 19 equal by induction. If  $\text{cf}(\epsilon) = \omega$ , then the nwf-limit as well as the cs-limit are the full  
 20 inverse limits of the iteration system, and therefore again equal by induction.

21 (7) follows from (6). □

22 We will also use the following fact:

23 **Lemma 2.25.** *Assume that  $\mathcal{S}$  is a  $P_{<i}$ -name for an element of  $Q_i$ , that  $q \upharpoonright I_{<i}$  reads  $\mathcal{S}$  con-*  
 24 *tinuously and that  $q \Vdash \eta_i \in \mathcal{S}$ . Then  $q \upharpoonright I_{<i}$  forces that  $B_i^q(\bar{\eta}) \leq_{Q_i} \mathcal{S}$ .*

25 *Proof.* Assume otherwise. Then there is an approximation  $g$  of  $p := q \upharpoonright I_{<i}$ , an  $\bar{a} \in \text{Pos}(g)$   
 26 and a  $t \in T_{\max}^i$  such that  $p^{[\bar{a}]}$  forces  $t$  to be in  $B_i^q(\bar{\eta})$  but not in  $\mathcal{S}$ . Let  $\bar{a}^+$  be  $\bar{a} \widehat{t}$ . Then  $\bar{a}^+$  is a  
 27 possible value of some approximation of  $q$ , and  $q^{[\bar{a}^+]}$  forces that  $\eta_i \notin \mathcal{S}$ , a contradiction. □

28 **Remark.** The iteration technique defined here also works for larger classes of forcings,  
 29 e.g., for the tree forcings  $\mathbb{Q}_0^{\text{tree}}$  of [9] mentioned already. If we assume additional properties  
 30 such as bigness and halving, we could also use lim-inf forcings. It is also possible to extend  
 31 the construction to non-total orders, or to allow  $T_{\max}^i, \mu^i$  to be  $P_{<i}$ -names.

### 3. T ℐ<sup>c</sup>

33 To every tree forcing such as  $Q$  defined in Section 1 (and many other tree forcings as  
 34 well) there is an associated ideal  $\mathbb{I}$  and a notion of measurability. We will also use  $\mathbb{I}^c$ , the  
 35  $< 2^{\aleph_0}$ -closure of  $\mathbb{I}$ , and the associated notion of weak measurability. The application in  
 36 this paper of a nw-iteration will be: for certain  $Q$  we can force weak measurability for all  
 37 definable sets.

38 **Definition 3.1.** • The ideal  $\mathbb{I}$  on the reals is defined by:  $X \in \mathbb{I}$  if for all  $S \in Q$  there  
 39 is a  $T \leq S$  such that  $X \cap \text{lim}(T) = \emptyset$ .

- 40 •  $\mathbb{I}^c$  is the  $< 2^{\aleph_0}$ -closure of  $\mathbb{I}$ .
- 41 •  $X$  has weak measure 1 if  $\mathbb{R} \setminus X \in \mathbb{I}^c$ .  $X$  has strong measure 1, if  $\mathbb{R} \setminus X \in \mathbb{I}$ .

42 **Notes.** • Of course these notions depend on the forcing  $Q$ , so it might be more exact  
 43 to use notation such as  $\mathbb{I}_Q$  or  $\mathbb{I}_{(T_{\max}, \mu)}$  etc. In this paper this is not necessary, since  
 44 we will always use a fixed  $Q$ .

- 45 • We use the phrase “measure 1” although the ideals  $\mathbb{I}$  and  $\mathbb{I}^c$  are not related to a  
 46 measure (they are not even ccc).
- 47 • Of course, if CH holds, then  $\mathbb{I}^c = \mathbb{I}$ .
- 48 •  $\mathbb{I}$  is always nontrivial (i.e.,  $\text{lim}(T_{\max}) \notin \mathbb{I}$ ), but this is not clear for  $\mathbb{I}^c$ .

1  $F : Q \rightarrow Q$  is a witness for  $X \in \mathbb{I}$  if  $F(S) \leq S$  and  $X \cap \lim(F(S)) = \emptyset$  for all  $S \in Q$ .  
 2 So every  $X \in \mathbb{I}$  is contained in a set  $\bigcap \{\omega^\omega \setminus \lim(F(S)) : S \in Q\}$ .<sup>13</sup>

3 **Lemma 3.2.**  $\mathbb{I}$  is a non-trivial  $\sigma$ -ideal.

4 *Proof.* This follows from fusion: Assume  $X_i \in \mathbb{I}$  ( $i \in \omega$ ) and  $S = S_0 \in Q$ . Pick any front  
 5  $F_0 \in S_0$ , so  $S_0 = \bigcup_{t \in F_0} S_0^{[t]}$ . For each  $t \in F_0$  pick an  $S_{1,t} \leq S_0^{[t]}$  such that  $\lim(S_{1,t}) \cap X_1 = \emptyset$ .  
 6 Set  $S_1 := \bigcup_{t \in F_0} S_{1,t}$ . So  $S_1 \in Q$ , and  $F_0$  is a front in  $S_1$ . Pick a 1-dense front  $F_1$  in  $S_1$   
 7 (purely) stronger than  $F_0$ . Iterate the construction. Fusion produces a  $T < S$  such that  
 8  $\lim(T) \cap X_i = \emptyset$  for all  $i \in \mathbb{N}$ .  $\square$

9 For example, if  $Q$  is Sacks forcing, then  $\mathbb{I}$  is called Marczewski ideal.  $X \in \mathbb{I}$  iff in every  
 10 perfect set  $A$  there is a perfect subset  $A'$  of  $A$  such that  $A' \cap X = \emptyset$ . So if  $X$  is Borel (or if  $X$   
 11 has the perfect set property, e.g.,  $X$  is  $\Sigma_1^1$ ), then  $X \in \mathbb{I}$  iff  $X$  is countable.  $\mathbb{I}$  is not a ccc ideal:  
 12 For  $A \subseteq \omega$ , set

$$X_A := \{f \in 2^\omega : (\forall n \notin A) f(n) = 0\}.$$

14 Clearly  $X_A \cap X_B = X_{A \cap B}$ , and  $|X_A| = 2^{|A|}$ . So if  $\{A_i : i \in 2^{\aleph_0}\}$  is an almost disjoint family,  
 15 then  $\{X_{A_i}\}$  is a family of closed sets not in  $\mathbb{I}$  such that  $X_{A_i} \cap X_{A_j}$  is finite for  $i \neq j$ .

16 For a Borel ccc ideal  $I$ , “ $X \subseteq \mathbb{R}$  is measurable” can be defined by “there is a Borel set  
 17  $A$  such that  $A \Delta X \in I$ ”. (Usually the basis of the ideal is simpler, e.g., one can use open  
 18 sets instead of Borel sets for meager, or  $G_\delta$  sets for Lebesgue-null.) Equivalently,  $X$   
 19 is measurable iff for every  $I$ -positive Borel set  $A$  there is an  $I$ -positive Borel set  $B \subseteq A$  such  
 20 that either  $B \cap X \in I$  or  $B \setminus X \in I$ . For non-ccc ideals that do not live on the Borel sets, this  
 21 second notion is usually the one used to define measurability:

22 **Definition 3.3.**  $\bullet$   $X \subseteq \mathbb{R}$  is measurable if for every  $T \in Q$  there is an  $S \leq_Q T$  such  
 23 that either  $\lim(S) \cap X \in \mathbb{I}$  or  $\lim(S) \setminus X \in \mathbb{I}$ .  
 24  $\bullet$   $X \subseteq \mathbb{R}$  is weakly measurable if for every  $T \in Q$  there is an  $S \leq_Q T$  such that either  
 25  $\lim(S) \cap X \in \mathbb{I}^c$  or  $\lim(S) \setminus X \in \mathbb{I}^c$ .

26 Since  $\mathbb{I}^c$  is the bigger ideal, measurability implies weak measurability.

27 In the rest of the paper, we will construct a specific  $Q$  and a nwf-iteration  $P$  and show  
 28 that  $P$  forces all definable sets to be weakly measurable:

29 **Theorem 3.4.** Assume CH and that  $Q$  satisfies the Ramsey property 5.4. Then there is  
 30 a proper,  $\aleph_2$ -cc,  $\omega^\omega$ -bounding p.o.  $P$  forcing that every set of reals which is (first-order)  
 31 definable using a parameter in  $L(\mathbb{R})$  is weakly measurable.

32 We will see in Lemma 5.5 that there is such a  $Q$ , and the Theorem will be proven  
 33 by 4.8, 4.10 and 5.8.

34 **Remark 3.5.** It is natural to ask whether in our forcing extension every definable set is  
 35 measurable (and not just weakly measurable, as stated in the theorem). This seems un-  
 36 likely, but it is not clear how to prove it. It is not even clear how to prove that in our forcing  
 37 model  $\mathbb{I} \neq \mathbb{I}^c$  (i.e., that  $\text{add}(\mathbb{I}) < 2^{\aleph_0}$ ). (Of course,  $\mathbb{I} = \mathbb{I}^c$  would trivially imply that measur-  
 38 able sets and weakly measurable sets are the same, so in particular that all definable sets  
 39 are measurable.)

40 Let us first list some facts about (weak) measurability:

41 **Lemma 3.6.** Every Borel set is measurable. The family of measurable sets is closed under  
 42 complements and countable unions; the same holds for weakly measurable sets.

43 *Proof.* Closure under complement is trivial.

44 Every closed set is measurable: Let  $X = \lim(T')$  be closed and  $T \in Q$ . If there is a  
 45  $t \in T \setminus T'$  then  $S := T^{[t]}$  satisfies  $\lim(S) \cap X = \emptyset$ . Otherwise  $T \subseteq T'$  and  $S := T$  satisfies  
 $\lim(S) \setminus X = \emptyset$ .

<sup>13</sup>Note that this is not a countable intersection.



1  
2 Assume that  $(X_i)_{i \in \omega}$  is a sequence of weakly measurable sets and that  $T \in Q$ . If for some  
3  $i \in \omega$  there is an  $S \leq T$  such that  $\lim(S) \setminus X_i \in \mathbb{I}^c$  then the same obviously holds for  $\bigcup_{i \in \omega} X_i$ .  
4 So assume that for all  $i \in \omega$  and  $T' \leq T$  there is an  $S \leq T'$  such that  $\lim(S) \cap X_i \in \mathbb{I}^c$ . Now  
5 repeat the proof of 3.2.

6 The same proof also shows that the measurable sets are closed under countable unions.  
7 □

8  $\mathbb{I}^c$  could be trivial (i.e.,  $\text{cov}(\mathbb{I})$  could be less than  $2^{\aleph_0}$ ). If  $\mathbb{I}^c$  is “everywhere nontrivial”,  
9 then  $\mathbb{I}^c$  and  $\mathbb{I}$  are the same on measurable (in particular, Borel) sets:

10 **Lemma 3.7.** *Assume that  $\lim(S) \notin \mathbb{I}^c$  for all  $S \in Q$ . Then  $\mathbb{I}^c$  and  $\mathbb{I}$  agree on measurable*  
11 *sets. I.e., if  $X$  is measurable and  $X \in \mathbb{I}^c$ , then  $X \in \mathbb{I}$ .*

12 *Proof.* For every  $T \in Q$  there is an  $S \leq_Q T$  such that  $\lim(S) \cap X \in \mathbb{I}$ : Otherwise  $\lim(S) \setminus X \in$   
13  $\mathbb{I} \subseteq \mathbb{I}^c$ , a contradiction to  $X \in \mathbb{I}^c$  and  $\lim(S) \notin \mathbb{I}^c$ . So by the definition of  $\mathbb{I}$  there is a  
14  $S' \leq_Q S \leq_Q T$  such that  $\lim(S') \cap X = \emptyset$ . So  $X \in \mathbb{I}$ . □

15 Since any Borel set  $B$  is measurable,  $B \in \mathbb{I}$  iff  $(\forall S \in Q) \lim(S) \not\subseteq B$ , so we get:

16 **Fact 3.8.** *For a Borel code  $B$ , the statement “ $B \in \mathbb{I}$ ” is  $\prod_2^1$  and therefore invariant under*  
17 *forcing.*

18 On the other hand, since  $\mathbb{I}$  is not a Borel ideal (i.e., not every  $X \in \mathbb{I}$  is contained in  
19 a Borel set  $B \in \mathbb{I}$ ), there is no reason why  $X \in \mathbb{I}$  should be upwards absolute between  
20 universes.

21 For later reference, we will reformulate the definition of  $\mathbb{I}$ : If  $S \in Q$ ,  $X \subseteq Q$ ,  $T \in X$  and  
22  $T' \leq_Q S, T$ , then  $\lim(T') \cap (2^\omega \setminus \bigcup_{R \in X} \lim(R)) \subseteq \lim(T') \setminus \lim(T) = \emptyset$ . So we get:

23 **Lemma 3.9.** *If  $X \subseteq Q$  is predense then  $\bigcup_{T \in X} \lim(T)$  is of strong measure 1.*

#### 24 4. A

25 In this section we assume CH. We will construct an order  $I$  and define  $P$  to be the  
26 nwf-limit of  $Q$  along  $I$ .  $I$  is  $\omega_2$ -like,<sup>14</sup> has a cofinal sequence  $j_\alpha$  ( $\alpha \in \omega_2$ ) and many  
27 automorphisms. We show that these properties imply that  $P$  forces the following:

- 28 •  $2^{\aleph_0} = \aleph_2$ ,
- 29 •  $\mathbb{I}^c$  is nontrivial (and moreover  $\lim(S) \notin \mathbb{I}^c$  for all  $S \in Q$ ),
- 30 • for every definable set  $X$ , “locally” either all or no  $\eta_{j_\delta}$  are in  $X$  and
- 31 •  $\{\eta_{j_\delta} : \delta \in \omega_2\}$  is of weak measure 1 in  $\{\eta_i : i \in I\}$ .

32 In the next section it will be shown that the set  $\{\eta_i : i \in I\}$  is of weak measure 1, which  
33 will finish the proof Theorem 3.4

34 First note that for any  $I$  with uncountable cofinality,  $P$  makes the old reals null:

35 **Lemma 4.1.** *If  $I$  has cofinality  $\geq \aleph_1$  and  $i \in I$  then  $\Vdash_P V_{<i} \cap \lim(T_{\max}) \in \mathbb{I}$ .*

36 *Proof.* Let  $G_P$  be  $P$ -generic over  $V$ . If  $T \in V[G_P]$  then  $T \in V_{<j}$  for some  $i < j < \infty$   
37 because of 2.24(2). So in  $V_{\leq j}$  there is an  $S < T$  such that  $\lim(S) \cap V_{<i} = \emptyset$  (in  $V_{\leq j}$  and  
38  $V[G_P]$  as well, according to 1.8). □

39 **Lemma 4.2.** *Assume that CH holds and that  $I$  is  $\omega_2$ -like. Then*

- 40 (1)  $P$  has the  $\aleph_2$ -cc (and therefore preserves all cofinalities).
- 41 (2)  $P_{<i} \Vdash CH$  for each  $i \in I$  and  $P \Vdash 2^{\aleph_0} = \aleph_2$ .

<sup>14</sup> $I$  is  $\omega_2$ -like if  $|I_{<i}| < \aleph_2$  for all  $i \in I$  and  $|I| = \aleph_2$ .

1 *Proof.* (1) If  $|I_{<i}| \leq 2^{\aleph_0}$  then  $|P_{<i}| \leq 2^{\aleph_0}$ : There are at most  $|I_{<i}|^{\aleph_0} \leq 2^{\aleph_0}$  many countable subsets  
 2 of  $|I_{<i}|$ . For each  $p \in P_{<i}$  with a fixed domain and each  $j \in \text{dom}(p)$  there are  $2^{\aleph_0}$  many  
 3 possibilities for  $\text{Dom}_j^p$  and  $2^{\aleph_0}$  many possibilities for the Borel definition  $B_j^p$ .

4 If CH holds, then the usual delta system lemma applies: If  $A \subseteq P$  is a maximal antichain  
 5 of size  $\aleph_2$  then without loss of generality the domains of  $p \in A$  form a delta system (i.e.,  
 6 there is a countable  $x \subseteq I$  such that  $\text{dom}(p_1) \cap \text{dom}(p_2) = x$  for all  $p_1 \neq p_2 \in A$ ). Since  
 7  $I$  is  $\omega_2$ -like,  $x$  cannot be cofinal. Let  $i$  be an upper bound of  $x$ . Without loss of generality  
 8  $p_1 \upharpoonright I_{<i} = p_2 \upharpoonright I_{<i}$  for  $p_1 \neq p_2 \in A$  (since there are only  $\aleph_1$  many elements of  $P_{<i}$ ). But then  
 9  $p_1 \parallel p_2$  by Fact 2.21.

10 Proper and  $\aleph_2$ -cc imply preservation of all cofinalities and cardinalities.

11 (2) Let  $G$  be  $P$ -generic over  $V$ . Then the reals in  $V[G]$  are the union of the reals in  
 12  $V_{<i}$ . Every real in  $V_{<i}$  is read continuously from a condition  $p \in G_{<i}$ . There are only  
 13  $|P_{<i}| = (2^{\aleph_0})^V = \aleph_1$  many conditions, and given a condition there are only  $(2^{\aleph_0})^V = \aleph_1$   
 14 many possibilities to continuously read a real from the condition. So there are at most  $\aleph_1$   
 15 many reals in  $V_{<i}$ . And  $\eta_i \notin V_{<i}$ , so in particular  $\eta_{i_1} \neq \eta_{i_2}$  for  $i_1 \neq i_2$ .  $\square$

16 The following is well known:

17 **Lemma 4.3.** *If CH holds, then there is an  $\aleph_1$  saturated<sup>15</sup> linear order  $\tilde{I}$  of size  $\aleph_1$ , and all*  
 18 *such orders are isomorphic.*

19 *Proof.* Induction of length  $\omega_1$ : Assume at stage  $\alpha$  we have a linear order  $L_\alpha$  of size  $\omega_1 =$   
 20  $2^{\aleph_0}$ . List all the  $(\omega_1)$  many countable gaps and add points to fill these gaps. At limits, take  
 21 the union. Then at stage  $\omega_1$  we get a saturated order.

22 Uniqueness is proven by the standard back and forth argument.  $\square$

23 **Definition 4.4.** Let  $\mathfrak{S}$  be the set of  $0 < \alpha < \omega_2$  such that  $\text{cf}(\alpha) \in \{1, \omega_1\}$ . Note that  $\mathfrak{S} \subseteq \omega_2$   
 24 is stationary.

25 We will now define the order  $I$  along which we iterate. (We do this assuming CH.)

26 Given  $\tilde{I}$  as above, let  $I$  be the following order:

$$27 \quad \underbrace{\tilde{I}}_0 + \underbrace{\{j_1\}}_1 + \tilde{I} + \cdots + \underbrace{\tilde{I}}_\omega + \underbrace{\{j_{\omega+1}\}}_{\omega+1} + \tilde{I} + \cdots + \underbrace{\{j_{\omega_1}\}}_{\omega_1} + \tilde{I} + \cdots$$

28 So at stages  $\alpha \in \mathfrak{S}$ , we add an order of the type  $\{c\} + \tilde{I}$ , in other stages we add just  $\tilde{I}$ .

29 **Facts 4.5.**  $\bullet$   $I$  is  $\omega_2$ -like,

- 30  $\bullet$   $(j_\alpha)_{\alpha \in \mathfrak{S}}$  is an increasing (and therefore cofinal) continuous sequence in  $I$ , and  
 31  $\bullet$  every  $j_\alpha$  has cofinality  $\aleph_1$  in  $I$ .

32 Continuous means that  $j_\delta = \sup(j_\alpha : \alpha \in \mathfrak{S}, \alpha < \delta)$  whenever  $\delta = \sup(\mathfrak{S} \cap \delta) \in \mathfrak{S}$   
 33 (which is equivalent to  $\text{cf}(\delta) = \omega_1$ ).

34 **Note.** We could just as well define  $j_\alpha$  for  $\alpha$  with cofinality  $\omega_1$  only, or for all  $\alpha \in \omega_2$  (and  
 35 require continuity for points of cofinality  $\omega_1$  only). All these versions are equivalent by  
 36 simple relabeling, cf. the beginning of the proof of 4.8.

<sup>15</sup>A linear order  $\tilde{I}$  is  $\aleph_1$  saturated if “there are no countable gaps”, more exactly:

- $\bullet$   $I$  has neither a smallest or a largest element, i.e., no  $(-\infty, 1)$  and no  $(1, \infty)$  gaps.
- $\bullet$   $I$  does not have a cofinal sequence of order type  $\omega$  nor a coinital one of order type  $\omega^*$ , i.e., no  $(\omega, \infty)$  and no  $(-\infty, \omega^*)$  gaps.
- $\bullet$  If  $A \subset I$  has order type  $\omega$  and  $c > a$  for all  $a \in A$  ( $c > A$  in short) then there is a  $b < c$  such that  $b > A$ . I.e., there are no  $(\omega, 1)$  gaps.
- $\bullet$  Analogously for  $B$  of order type  $\omega^*$  and  $c < B$ . I.e., no  $(1, \omega^*)$  gaps.
- $\bullet$  If  $A$  has order type  $\omega$  and  $B$  has order type  $\omega^*$  and  $A < B$ , then there is an  $x \in I$  such that  $A < x < B$ . I.e., there are no  $(\omega, \omega^*)$  gaps.

1 **Definition 4.6.** We set  $Q_i = Q$  for all  $i \in I$  and let  $P$  be the nwf-iteration of  $Q_i$  along  $I$ .

2 We will use the notation  $I_\alpha, P_\alpha, V_\alpha$  and  $\eta_\alpha$  for  $I_{< j_\alpha}, P_{< j_\alpha}, V_{< j_\alpha}$  and  $\eta_{j_\alpha}$ . We set  $G_{\omega_2}$  to  
3 be (the name for) the  $P$ -generic (in previous notation,  $G_{< \infty}$ ) and  $V_{\omega_2}$  the generic extension  
4  $V[G_{\omega_2}]$  (in previous notation,  $V_{< \infty}$ ).

5 **Lemma 4.7.** (CH) Let  $S_0 \subseteq \mathfrak{S}$  be stationary.  $P$  forces the following:

- 6 (1)  $\{\eta_\delta : \delta \in S\} \notin \mathbb{I}^c$  for every stationary  $S \subseteq \mathfrak{S}$ , and  
7 (2)  $\{\eta_\delta : \delta \in S_0\} \cap \lim(T_0) \notin \mathbb{I}^c$  for every  $T_0 \in Q$ .

8 This lemma implies that in  $V_{\omega_2}$  the assumption of Lemma 3.7 is satisfied (i.e., that  $\mathbb{I}^c$  is  
9 “everywhere nontrivial”). This lemma holds for all  $I$  satisfying 4.5.

10 *Proof.* (1) Assume otherwise, i.e., there are  $P$ -names  $\underline{F}_\zeta$  ( $\zeta \in \omega_1$ ) for functions from  $Q$  to  
11  $Q$  and  $\underline{S}$  for a stationary set such that  $p_0 \in P$  forces

$$12 \quad \underline{F}_\zeta(T) \leq T \text{ and } (\forall \delta \in \underline{S}) (\exists \zeta \in \omega_1) (\forall T \in Q) \eta_\delta \notin \lim(\underline{F}_\zeta(T)).$$

13  $P$  forces that for each  $\alpha \in \mathfrak{S}$  there is a  $\beta \in \mathfrak{S}$  such that  $\underline{F}_\zeta(T) \in Q^{V_\beta}$  for all  $T \in Q^{V_\alpha}$  and  
14  $\zeta \in \omega_1$ . We need something slightly stronger: For every name  $\underline{T}$  for an element of  $Q^{V_\alpha}$  and  
15  $\zeta \in \omega_1$  there is a maximal antichain  $A \subset P$  such that for every  $q \in A$  there is a  $P$ -name  $\underline{T}'_q$   
16 such that  $q$  forces  $\underline{F}_\zeta(\underline{T}) = \underline{T}'_q$  and  $q$  continuously reads  $\underline{T}'_q$ . So if  $q \in G_{\omega_2}$  and  $\beta$  is bigger  
17 than  $\text{dom}(q)$ ,<sup>16</sup> then  $V_\beta$  not only contains  $\underline{T}'_q = F_\zeta(T)$ , but also knows that  $\underline{T}'_q$  will be  $F_\zeta(T)$   
18 in  $V_{\omega_2}$ .

19 Define  $f^-(\alpha)$  to be the smallest  $\beta$  which is bigger than  $\text{dom}(q)$  for every  $q \in A$ , where  
20  $A$  is an antichain for some  $\underline{T}$  and  $\zeta \in \omega_1$  as above.  $P$  is  $\aleph_2$ -cc, every  $q \in A$  has countable  
21 domain, and there are only  $\aleph_1$  many reals in  $V_\alpha$ . So  $f^-(\alpha) < \omega_2$ , and we can define  $f(\alpha)$  to  
22 be the smallest  $\beta \in \mathfrak{S}$  that is larger or equal to  $\max(\alpha, f^-(\alpha))$ .

23 If  $\text{cf}(\alpha) = \omega_1$ , then  $f(\alpha)$  is the supremum of  $\{f(\gamma) : \gamma \in \mathfrak{S} \cap \alpha\}$ , since the reals in  $V_\alpha$   
24 are the union of the reals in  $V_\gamma$ . So  $f$  is continuous.

25 Then  $P$  forces the following: Since  $\underline{S}$  is stationary, there is a  $\beta \in \underline{S}$  such that  $f(\beta) = \beta$ .  
26  $V_\beta$  can calculate every  $F_\zeta$ , and  $\underline{F}_\zeta'' Q$  is dense in  $Q$ . Since  $\eta_\beta$  is a  $Q$ -generic real over  $V_\beta$ ,  
27 there is (for every  $\zeta \in \omega_1$ ) a  $T \in Q^{V_\beta}$  such that  $\eta_\beta \in \lim(\underline{F}_\zeta(T))$ , a contradiction.

28 (2): We can assume that  $T_0 \in V$ . Again, choose names  $\underline{F}_\zeta$  as above, and assume that  
29  $p_0 \in P$  forces that

$$30 \quad \underline{F}_\zeta(T) \leq T \text{ and } (\forall \delta \in S_0) (\exists \zeta \in \omega_1) (\forall T \in Q) \eta_\delta \notin \lim(\underline{F}_\zeta(T)) \cap \lim(T_0).$$

31 Define  $f$  as above, so there is a  $\beta > \text{dom}(p)$  such that  $\beta \in S_0$  and  $f(\beta) = \beta$ . So the same  
32 argument proves that  $p_0$  forces that  $\eta_\beta \notin \lim(T_0)$ , a contradiction.  $\square$

33 We also get the following:

34 **Lemma 4.8.** (CH) For every  $C \subseteq \omega_2$  club,  $P$  forces the following:

$$35 \quad \{\eta_i : i \in I\} \setminus \{\eta_\alpha : \alpha \in \mathfrak{S} \cap C\} \in \mathbb{I}^c.$$

36 Again, this lemma applies to all  $I$  satisfying 4.5.

37 *Proof.* We can assume that  $C = \omega_2$ , since we can just relabel the sequence  $\{j_\alpha : \alpha \in \mathfrak{S} \cap C\}$ :  
38 Set  $j'_\alpha := j_\beta$ , where  $\beta$  is the  $\alpha$ -th element of  $C \cap \mathfrak{S}$ . Then  $(j'_\alpha)_{\alpha \in \mathfrak{S}}$  satisfies 4.5 as well.

39 Recall Definition 1.9 of  $Q_{A_\psi}^f$  and  $D_f^{\text{spl}}$  (for  $f : \omega \rightarrow \omega$  increasing and  $r \in 2^\omega$ ). Enumerate  
40 all increasing  $f : \omega \rightarrow \omega$  in  $V$  as  $f_\zeta$  ( $\zeta \in \omega_1$ ). (CH holds in  $V$ .)

41 **Claim:** In  $V$ , we can find  $P_\alpha$ -names  $\underline{T}_\alpha^\zeta$  ( $\zeta \in \omega_1, \alpha < \omega_2$  successor) for elements of  $Q$   
42 such that the following is forced by  $P_{\omega_2}$ :

- 43 (1) The set  $\{\underline{T}_\alpha^\zeta : \alpha < \omega_2 \text{ successor}\} \subseteq Q$  is dense for all  $\zeta \in \omega_1$ .  
44 (2)  $\underline{T}_\alpha^\zeta \in D_{f_\zeta}^{\text{spl}}$  (in  $V_\alpha$  or equivalently in  $V_{\omega_2}$ ).<sup>17</sup>

<sup>16</sup>More formally: if  $j_\beta > i$  for all  $i \in \text{dom}(q)$ .

<sup>17</sup>Recall 1.9 and 1.10.

1 (3) If  $\beta < \alpha$  is a successor, then  $T_\alpha^\zeta$  has no branch in  $V_\beta$ , and for all  $i < j_\alpha$  there is a  $\zeta_0$   
2 such that  $T_\alpha^\zeta$  has no branch in  $V_{<i}$  for all  $\zeta \geq \zeta_0$ .

3 Proof of the claim: Pick for all  $\alpha + 1$  a function  $\phi_{\alpha+1} : \omega_1 \rightarrow I_{<j_{\alpha+1}} \setminus I_{<j_\alpha}$  which is  
4 increasing and cofinal. Also pick an enumeration  $(\mathcal{S}_{\alpha+1})_{\alpha \in \omega_2}$  such that  $\mathcal{S}_\alpha$  is an  $P_\alpha$ -name  
5 and  $P$  forces that  $Q = \{\mathcal{S}_{\alpha+1} : \alpha \in \omega_2\}$ . (This is possible since  $P$  forces that  $Q^{V_{\omega_2}} = \bigcup Q^{V_\alpha}$ ,  
6 cf. 2.24(2).)

7 To find  $T_\alpha^\zeta$  ( $\alpha$  successor) note that  $P_\alpha$  forces that we can perform the following construc-  
8 tion in  $V_\alpha$ : First pick an  $S' \leq \mathcal{S}_\alpha$  such that  $S' \in D_{f_\zeta}^{\text{spl}}$  (cf. 1.10(7)).  $\text{cf}(j_\alpha) = \aleph_1$ , so  $S' \in V_{<i}$   
9 for some  $i < j_\alpha$ . Pick some  $i'$  bigger than  $\max(i, \phi_\alpha(\zeta))$  and smaller than  $j_\alpha$ . There is a real  
10  $r \in V_\alpha \setminus V_{<i'}$  (e.g.,  $\eta_{i'}$ ). Therefore there is a  $T_\alpha^\zeta < S'$  such that  $\lim(T_\alpha^\zeta) \cap V_{<i'} = \emptyset$  (in  $V_\alpha$  and  
11  $V_{\omega_2}$  as well, cf. 1.8). Let  $T_\alpha^\zeta$  be a  $P_\alpha$ -name for  $T_\alpha^\zeta$ .

12 The  $T_\alpha^\zeta$  constructed this way satisfy the claim: (1):  $T_\alpha^\zeta \leq \mathcal{S}_\alpha$ , (2):  $D_{f_\zeta}^{\text{spl}}$  is open dense and  
13 absolute, (3): pick  $\zeta_0$  such that  $\phi_\alpha(\zeta_0) > i$ . This ends the proof of the claim.

14 From now on assume  $G$  is  $P$ -generic over  $V$ . We work in  $V_{\omega_2}$  and set  $T_\alpha^\zeta := T_\alpha^\zeta[G]$ . So  
15 if  $i \in I$  then the sequence  $(T_{\alpha+1}^\zeta)_{j_{\alpha+1} < i, \zeta \in \omega_1}$  is in  $V_{<i}$ .

16 For all  $\zeta \in \omega_1$ ,  $X_\zeta := \bigcup_{\alpha+1 < \omega_2} \lim(T_{\alpha+1}^\zeta)$  is of strong measure 1 (cf. 3.9). So the set  
17  $Y := \bigcap_{\zeta \in \omega_1} X_\zeta$  is of weak measure 1. It is enough to show that

$$18 \quad (\{\eta_i : i \in I\} \setminus \{\eta_\alpha : \alpha \in \mathfrak{S}\}) \cap Y = \emptyset.$$

19 Assume towards a contradiction that some  $\eta_i$  is in  $Y$  and  $\eta_i \neq \eta_\alpha$  for all  $\alpha \in \mathfrak{S}$ .

20 Let  $\alpha \in \mathfrak{S}$  be minimal such that  $\eta_i \in V_\alpha$  (i.e.,  $i < j_\alpha$ ). So  $\alpha$  is a successor (but not  
21 necessarily a successor of a  $\beta \in \mathfrak{S}$ ), and  $i > j_\beta$  for all  $\beta \in \mathfrak{S} \cap \alpha$ . So according to (3) there  
22 is a  $\zeta_0$  such that  $\eta_i \notin \lim(T_{\gamma+1}^\zeta)$  for all  $\zeta > \zeta_0$  and all  $\gamma + 1 \geq \alpha$ .

23 So we know the following:  $\eta_i \in Y$ , i.e.,

$$24 \quad \eta_i \in \bigcup_{\gamma+1 < \omega_2} \lim(T_{\gamma+1}^\zeta) \quad \text{for all } \zeta \in \omega_1.$$

25 But

$$26 \quad \eta_i \notin \bigcup_{\alpha \leq \gamma+1 < \omega_2} \lim(T_{\gamma+1}^\zeta) \quad \text{for all } \zeta \geq \zeta_0.$$

27 Therefore

$$28 \quad \eta_i \in \bigcup_{\gamma+1 < \alpha} \lim(T_{\gamma+1}^\zeta) \quad \text{for all } \zeta \geq \zeta_0.$$

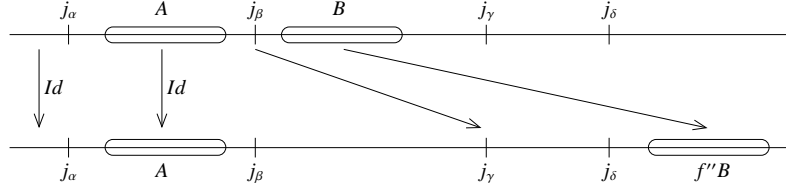
29 Recall that  $V_{<i}$  sees the sequence  $(T_{\gamma+1}^\zeta)_{\gamma+1 < \alpha, \zeta \in \omega_1}$ . So in  $V_{<i}$ , some  $T \in Q$  forces that for  
30 all  $\zeta > \zeta_0$  there is a successor  $\beta(\zeta) < \alpha$  such that  $\eta_i \in \lim(T_{\beta(\zeta)}^\zeta)$ . In  $V_{<i}$ ,  $T$  has full splitting  
31 for some  $f_\zeta \in V$ ,  $\zeta > \zeta_0$  (see 1.10(5), 1.11 and 2.24(1)).

32 Let  $r$  be a real in  $V_{<i} \setminus \bigcup_{\gamma+1 < \alpha} V_{\gamma+1}$ . Pick in  $V_{<i}$  a  $T' \leq T$  such that  $T' \in Q_{A_s^\psi}^{f_\zeta}$  (cf. 1.10(6))  
33 and  $T'$  decides  $\beta(\zeta)$ . Then  $T'$  forces that  $\eta_i \in \lim(T' \cap T_{\beta(\zeta)}^\zeta)$ , a contradiction to  $T' \perp$   
34  $T_{\beta(\zeta)}^\zeta \in V_{<i}$  (because of (2), either  $T_{\beta(\zeta)}^\zeta$  is in  $Q_{A_s^\psi}^{f_\zeta}$  for some old real  $s$ , or incompatible to all  
35  $Q_{A_s^\psi}^{f_\zeta}$ ).  $\square$

36 We call  $f$  an automorphism if it is a  $<$ -preserving bijection from  $I$  to  $I$ .

37 If  $f : I \rightarrow I$  is an automorphism, then  $f$  defines an automorphism of  $P$  in a natural way  
38 as well (provided of course that  $f(i) = j$  implies  $Q_i = Q_j$ , but in our case all the  $Q_i$  are the  
39 same). Also,  $f$  defines a map on all  $P$ -names, and we have:  $p \Vdash \varphi(\tau)$  iff  $f(p) \Vdash \varphi(f\tau)$ .

40 If  $\Vdash_P \dot{x} \in V_{<i}$ , then there is a  $V_{<i}$ -name  $\tau$  such that  $\Vdash_P \dot{x} = \tau$ . If  $f \upharpoonright I_{<i}$  is the identity, then  
41  $f(\tau) = \tau$ . So in this case  $p \Vdash \phi(\tau)$  iff  $f(p) \Vdash \phi(\tau)$ . Also, if  $f \upharpoonright \text{dom}(p) \cap I_{<i}$  is the identity  
42 then  $B_i^p(\bar{\eta}) = B_{f(i)}^{f(p)}(\bar{\eta})$ .

F 4. An automorphism  $f$ .

1 **Lemma 4.9.** *The following holds for  $I$  (see Figure 4): If  $\alpha < \beta < \gamma < \delta$  are in  $\mathfrak{S}$ , and*  
 2 *if  $A \subseteq I_\beta$  and  $B \subseteq I \setminus I_\beta$  are countable, then there is an automorphism  $f$  of  $I$  such that*  
 3  *$f \upharpoonright (I_\alpha \cup A)$  is the identity,  $f(j_\beta) = j_\gamma$  and  $f''B > j_\delta$ .*

4 *Proof.* For every  $i < j \in I$ ,  $I_{<i}$  and  $\{k : i < k < j\}$  are isomorphic and also isomorphic to  $\tilde{I}$   
 5 (since they are all  $\aleph_1$  saturated linear orders of size  $\aleph_1$ ). If  $A \subset I$  is countable, then there  
 6 are  $i < A < j$ , and for all such  $i, j$  the sets  $\{k : i < k < A\}$  and  $\{k : A < k < j\}$  are again  
 7 isomorphic to  $\tilde{I}$ . Also,  $I_{>i}$  is isomorphic to  $I$  (since  $\omega_2 \setminus \alpha$  is isomorphic to  $\omega_2$ ).

8 So assume  $\alpha < \beta < \gamma \in \mathfrak{S}$ ,  $A < i < j_\beta$  countable,  $i > j_\alpha$ . Then  $I_{<j_\beta} \setminus I_{\leq i} \cong I_{<j_\gamma} \setminus I_{\leq i} \cong \tilde{I}$ .  
 9 Also, if  $B \subset I$  is countable,  $\delta \in \mathfrak{S}$  and  $B > j_\beta$ , then there is an  $j_\beta < i < B$ , and  $I_{<i} \cong I_{<j_\delta} \cong \tilde{I}$ ,  
 10  $I \setminus I_{\leq i} \cong I \setminus I_{\leq j_\delta} \cong I$ . Now combine these automorphisms.  $\square$

11 **Lemma 4.10.** *For  $\beta \in \omega_2$  set  $Y_\beta := \{\eta_\gamma : \gamma \in \mathfrak{S}, \gamma > \beta\}$ .  $P$  forces the following: If  $X$  is*  
 12 *a set of reals defined with a parameter  $x \in \bigcup_{i \in I} V_{<i}$ , and if  $T \in \mathcal{Q}$ , then there is an  $S \leq T$*   
 13 *and a  $\beta \in \omega_2$  such that either  $\lim(S) \cap X \cap Y_\beta = \emptyset$  or  $(\lim(S) \setminus X) \cap Y_\beta = \emptyset$ .*

14 This lemma holds for all  $I$  satisfying 4.5 and 4.9.

15 Note that every real in  $V_{\omega_2}$  is in  $\bigcup_{i \in I} V_{<i}$ .

16 We will see in the next section that (using additional assumptions)  $Y_\beta$  is a weak measure  
 17 1 set. Then this lemma implies that  $X$  is weakly measurable, i.e., Theorem 3.4. Because of  
 18 4.8, it will be enough to show that the set  $\{\eta_i : i \in I\}$  is of weak measure 1.

19 *Proof.* Assume  $\tilde{X} = \{r : \varphi(r, \tilde{x})\}$  and fix some  $\tilde{T}$ . Some  $p_0$  forces that  $\tilde{x}$  and  $\tilde{T}$  are in  $V_\alpha$ , so  
 20 without loss of generality  $\tilde{x}, \tilde{T}$  are  $P_\alpha$  names and  $\text{dom}(p_0) \subset I_\alpha$ . Pick a  $p_1 \leq p_0$ ,  $p_1 \in P_\alpha$   
 21 such that  $p_1$  continuously reads  $\tilde{T}$ . Fix some  $\beta > \alpha$ . Then  $p_2 := p_1 \cup \{(j_\beta, \tilde{T})\}$  is an element  
 22 of  $P_{\leq j_\beta}$  (since  $\tilde{T}$  is read continuously).

23 Let  $p \leq p_2$  decide  $\varphi(\eta_\beta, \tilde{x})$ . Without loss of generality  $p \Vdash \varphi(\eta_\beta, \tilde{x})$ .  $p \upharpoonright I_\beta$  forces that  
 24  $\tilde{S} := B_{j_\beta}^p(\tilde{\eta}) \leq_Q \tilde{T}$  (since  $p \leq p_2$ ).

25 Assume towards a contradiction that for some  $q \leq p$ ,  $\gamma \in \mathfrak{S}$  and  $\gamma > \beta$

$$26 \quad q \Vdash \eta_\gamma \in \lim(\tilde{S}) \ \& \ \neg \varphi(\eta_\gamma, \tilde{x}).$$

27 Note that  $q \upharpoonright I_\gamma$  reads  $\tilde{S}$  continuously and forces that  $B_{j_\gamma}^q(\tilde{\eta}) \leq_Q \tilde{S}$  (cf. 2.25).

28 Set  $A := \text{dom}(p) \cap I_\beta$  and  $B := \text{dom}(p) \cap I_{>j_\beta}$ . Let  $j_\delta$  be bigger than  $\text{dom}(q)$ , and let  $f$   
 29 be an automorphism of  $I$  such that  $f \upharpoonright (I_\alpha \cup A)$  is the identity,  $f(j_\beta) = j_\gamma$  and  $f''B > \text{dom}(q)$   
 30 (cf. 4.9 or Figure 4).

31  $\text{dom}(f(p)) \cap \text{dom}(q) \subseteq A \cup \{j_\gamma\}$ .  $f(p) \upharpoonright A = p \upharpoonright A \geq q \upharpoonright A$ , and  $q \upharpoonright I_\gamma$  forces that

$$32 \quad B_{j_\gamma}^{f(p)}(\tilde{\eta}) = B_{j_\beta}^p(\tilde{\eta}) = \tilde{S} \geq_Q B_{j_\gamma}^q(\tilde{\eta}).$$

33 So  $f(p)$  and  $q$  are compatible, a contradiction to  $f(p) \Vdash \varphi(\eta_\gamma, \tilde{x})$ .  $\square$

## 34 5. A -

35 For the proof of Theorem 3.4 it remains to be shown that  $\{\eta_i : i \in I\}$  is of weak measure  
 36 1. For this we will need a certain Ramsey property for  $\mathcal{Q}$ .

1 **Definition 5.1.** A subtree  $T$  of  $T_{\max}$  is called  $(n, r)$ -meager if  $\mu_T(t) < r$  for all  $t \in T$  with  
 2 length at least  $n$ .

3 **Lemma 5.2.** *If  $T$  is meager for some  $(n, r)$ , then  $\lim(T) \in \mathbb{I}$ .*

4 *Proof.* For any  $S \in \mathcal{Q}$  there is an  $s \in S$  of length at least  $n$  such that  $\mu_S(s) > r$ . So there is  
 5 an immediate successor  $t$  of  $s$  in  $S$  such that  $t \notin T$ . Then  $\lim(S^{[t]}) \cap \lim(T) = \emptyset$ .  $\square$

6 **Definition 5.3.** Let  $M, N$  be natural numbers.  $N \rightarrow M$  means: If

- 7 •  $r_1, \dots, r_M \in T_{\max}$  such that  $\text{length}(r_i) > N$ ,
- 8 •  $t \in T_{\max}$  such that  $r_i \perp t$  for  $1 \leq i \leq M$ ,
- 9 •  $A \subseteq \text{succ}(t)$  such that  $\mu(A) > N$ ,
- 10 •  $f_i : A \rightarrow T_{\max}^{[r_i]}$  for  $1 \leq i \leq M$ ,

11 then there is a  $B \subseteq A$  such that

- 12 •  $\mu(B) > M$  and
- 13 •  $\{s \in T_{\max} : (\exists i \leq M) (\exists t \in B) s \leq f_i(t)\}$  is  $(N, 1/M)$ -meager.

14 **Definition 5.4.** A lim-sup tree-forcing  $\mathcal{Q}$  is strongly non-homogeneous if  $\mu$  is sub-additive<sup>18</sup>  
 15 and for all  $M$  there is an  $N$  such that  $N \rightarrow M$ .

16 There are many similar notions of bigness, see e.g., [9, 2.2].

17 **Lemma 5.5.** *There is a forcing  $\mathcal{Q}$  that is strongly non-homogeneous.*

18 *Proof.* First note that it is enough to show that for each  $M$  there is an  $N$  such that  $N \rightarrow^- M$ ,  
 19 where  $N \rightarrow^- M$  is defined as above but with just one  $r$  and  $f$  instead of  $M$  many. To see  
 20 this, just set  $K_0 := M^2$  and find  $K_i$  such that  $K_{i+1} \rightarrow^- K_i$ . Then  $K_M \rightarrow M$ . (Here we  
 21 use that  $\mu$  is sub-additive, since we need that the union of  $m$  many  $(n, x)$ -meager trees is  
 22  $(n, x \cdot m)$ -meager.)

23 We will construct  $T_{\max}$  and  $\mu$  by induction. We define  $s \triangleleft t$  by:  $\text{length}(s) < \text{length}(t)$  or  
 24  $\text{length}(s) = \text{length}(t)$  and  $s$  is lexicographically smaller than  $t$ .

25 Fix some  $t \in \omega^{<\omega}$ . Assume that we already decided which  $s \triangleleft t$  will be elements of  $T_{\max}$   
 26 and that we already defined the set of successors of all these  $s$  as well as the measure of  
 27 their subsets. Assume that we have decided to put  $t$  into  $T_{\max}$ . So we have to define  $\text{succ}(t)$   
 28 and the measure on it.

29 Let  $m_t$  be the number of nodes  $s \triangleleft t$  already defined, including the already defined  
 30 successors of  $s$  for  $s \triangleleft t$ . Set  $M_t := (2m_t)^{m_t}$ . Then we define  $\text{succ}(t)$  to be of size  $M_t^{m_t}$ .<sup>19</sup>  
 31 For  $A \subseteq \text{succ}(t)$  we set  $\mu(A) := \log_{M_t}(|A|/M_t + 1)$ .

32 Then  $0 \leq \mu(A) < m_t$ ,  $\mu(A) = 0$  iff  $A = \emptyset$ , and  $\mu$  is strictly monotonous and sub-additive.<sup>20</sup>  
 33 If  $A, B \subseteq \text{succ}(t)$  and  $|B| \geq |A|/M_t$ , then  $\mu(B) > \mu(A) - 1$ . If  $|B| \leq m_t$  then  $\mu(B) < 1/m_t$ . If  
 34  $\mu(\text{succ}(t)) > M$ , then  $m_t > M$ .

35 Now fix an arbitrary  $M \in \omega$ . There is an  $N_0$  such that  $\mu(A) < 1/M$  for all  $s$  with  
 36  $\text{length}(s) > N_0$  and all  $A \subseteq \text{succ}(s)$  with  $|A| < m_s$ . (Just note that  $m_s$  strictly increases with  
 37  $\text{length}(s)$ .) Let  $N$  be larger than  $M + 1$  and  $N_0$ .

38 So assume that  $r \perp t \in T_{\max}$ ,  $\text{length}(r) > N \geq N_0$ ,  $A \subseteq \text{succ}(t)$ ,  $\mu(A) > N \geq M + 1$  (in  
 39 particular  $m_t > M$ ), and  $f : A \rightarrow T_{\max}^{[r]}$ .

40 Set  $X := \{s' \geq r : s' \triangleleft t, \text{length}(s') \geq N\}$ . Enumerate  $X$  as  $\{s_0, \dots, s_{l-1}\}$  (for some  $l \geq 0$ ).  
 41 Set  $A_0 := A$ . Assume that  $A_n$  is already defined, and define

$$42 \quad S_n := \{s' \in T_{\max} : (\exists t' \in A_n) s' \leq f(t')\}.$$

43 If  $n > 0$  assume that  $|\text{succ}_{S_n}(s_{n-1})| \leq 1$  and that  $|A_n| > |A_{n-1}|/(2m_t)$ .

44 Then we define  $A_{n+1}$  as follows: Since  $s_n \in X$ ,  $|\text{succ}(s_n)| < m_t$ . By a simple pigeon-hole  
 45 argument, there is an  $A_{n+1} \subseteq A_n$  such that  $|A_{n+1}| > |A_n|/(2m_t)$  and  $|\text{succ}_{S_{n+1}}(s_n)| \leq 1$ . So in the

<sup>18</sup> $\mu(A \cup B) \leq \mu(A) + \mu(B)$ .

<sup>19</sup>We can e.g., set  $\text{succ}(t) := \{t \frown k : 0 \leq k < M_t^{m_t}\}$ .

<sup>20</sup>Since the function  $g(x) := \log_{M_t}(x + 1)$  is concave and satisfies  $g(0) = 0$ .

1 end we get a  $B := A_l$  with cardinality at least  $|A|/(2m_l)^{m_l} = |A|/M_l$ , i.e.,  $\mu(B) > \mu(A) - 1 \geq M$ .  
 2 Also,  $|\text{succ}_{S_l}(s')| \leq 1$  for every  $s' \in X$ , so  $\mu_{S_l}(s') \leq 1/M$  (since  $\text{length}(s')$  was sufficiently  
 3 large).

4 We claim that  $B$  is as required. We have to show that  $S_l$  is  $(N, 1/M)$ -meager. Pick an  
 5  $s' \in S_l$  of length  $\geq N$ . We already dealt with the case  $s' \in X$ . Otherwise  $s' \triangleright t$  (note that  
 6  $s' \neq t$  since  $s' \perp t$ ). In this case  $|\text{succ}_{S_l}(s')| \leq |\text{succ}_{T_{\max}}(t)| \leq m_s$ . So  $\mu(\text{succ}_{S_l}(s')) \leq 1/M$ ,  
 7 since  $\text{length}(s') > N_0$ .  $\square$

8 **Lemma 5.6.** *If  $Q$  is strongly non-homogeneous, then  $P$  forces the following: If  $r \in$   
 9  $\lim(T_{\max}) \setminus \{\eta_i : i \in I\}$  then there is a  $T \in V$  such that  $r \in \lim(T)$  and  $T$  is  $(1, 1)$ -meager.*

10 If additionally the assumptions of Lemma 4.2 hold, then there are only  $\aleph_1$  many  $T \in V$ ,  
 11 and  $\aleph_1 < (2^{\aleph_0})^{V_{\omega_2}}$ . This implies that the set  $\{\eta_i : i \in I\}$  is of weak measure 1:

12 If  $r \in \lim(T_{\max}) \setminus \{\eta_i : i \in I\}$ , then  $r \in \bigcup_{T \in V_{\text{meager}}} \lim(T) \in \mathbb{I}^c$ .

13 *Proof.* Fix a  $P$ -name  $\underline{r}$  for a real and a  $p \in P$  such that  $p \Vdash \underline{r} \notin \{\eta_i : i \in I\}$ . We will show  
 14 that there is a  $p_\omega \leq p$  and a  $(1, 1)$ -meager tree  $T$  such that  $p_\omega \Vdash \underline{r} \in \lim(T)$ .

15 We will by induction construct  $p_n \in P$ , approximations  $\mathfrak{g}_n$ ,  $k_n \in \omega$  and  $i_n \in u_n = \text{dom}(\mathfrak{g}_n)$   
 16 such that

- 17 (1)  $p_{n+1} \leq_{\mathfrak{g}_n} p_n$ ,  $\mathfrak{g}_{n+1}$  is purely stronger than  $\mathfrak{g}_n$ .
- 18 (2)  $\mathfrak{g}_n$  is  $n$ -dense at  $i_n$ .
- 19 (3) the sequence  $(i_n)_{n \in \omega}$  covers  $\bigcup \text{dom}(p_n)$  infinitely often.
- 20 (4)  $k_n \rightarrow \max(n+1, |\text{Pos}(\mathfrak{g}_n)|)$ .
- 21 (5) If  $n > 0$ , then for each  $\bar{a} \in \text{Pos}(\mathfrak{g}_n)$ ,  $p_n^{\bar{a}}$  forces a value to  $\underline{r} \upharpoonright k_n$ , and the tree  
 22  $\{r_n^{\bar{a}} : \bar{a} \in \text{Pos}(\mathfrak{g}_n)\} \subseteq T_{\max} \upharpoonright k_n$  is  $(k_{n-1}, 1)$ -meager.

23 (1)–(3) allow us to fuse the  $(p_n)_{n \in \omega}$  into a  $p_\omega \leq p$  (cf. 2.15), and (5) implies that the tree of  
 24 all initial segments of  $r$  compatible with  $p_\omega$  is meager.

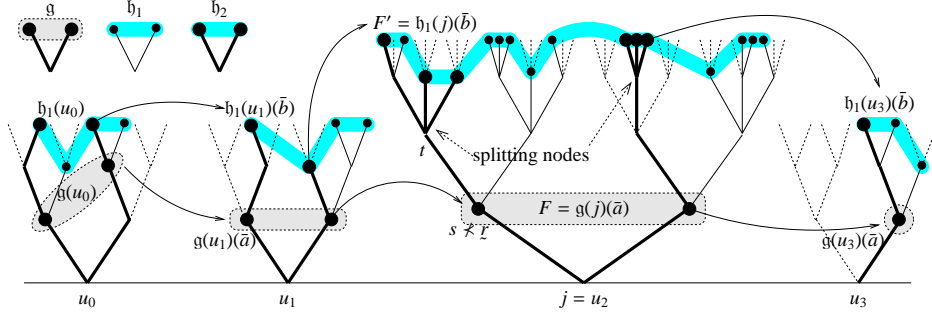
25 We start by picking any  $i_0 \in \text{dom}(p)$ , some  $p$ -approximation  $\mathfrak{g}_0$  that is 0-dense at  $i_0$ , a  
 26  $k_0$  satisfying (4). So assume by induction we have found  $p_n$ ,  $\mathfrak{g}_n$  and  $k_n$  satisfying (1,2,4).

- 27 (a) Set  $p := p_n$ ,  $\mathfrak{g} := \mathfrak{g}_n$ ,  $M := |\text{Pos}(\mathfrak{g}_n)|$  and  $N := k_n$ . So we have  $N \rightarrow M$ .
- 28 (b) Choose the position  $i_{n+1} \in \text{dom}(p_n)$  according to some simple bookkeeping. This  
 29 takes care of (3). Set  $j := i_{n+1}$ .
- 30 (c) Find a  $p_1 \leq_{\mathfrak{g}} p$  and  $m > N$  such that  $p_1 \Vdash (\eta_j \upharpoonright m \neq \underline{r} \upharpoonright m)$  and for all  $\bar{a} \in \text{Pos}(\mathfrak{g})$  the  
 31 condition  $p_1^{\bar{a}}$  determines  $\eta_j \upharpoonright m$  and  $\underline{r} \upharpoonright m$ .  
 32 (How to do this? First apply pure decision 2.19(6) to get a  $p' \leq_{\mathfrak{g}} p$  such that  
 33 for all  $\bar{a} \in \text{Pos}(\mathfrak{g})$  there is an  $m^{\bar{a}} > N$  and  $\eta^* \neq r^*$  such that  $p'^{\bar{a}} \Vdash (r^* = \underline{r} \upharpoonright m^{\bar{a}}, \eta^* =$   
 34  $\eta_j \upharpoonright m^{\bar{a}})$ . Then we apply pure decision again to get  $p_1 \leq_{\mathfrak{g}} p'$  determining  $\underline{r}$  and  $\eta_j$   
 35 up to  $\max\{m^{\bar{a}} : \bar{a} \in \text{Pos}(\mathfrak{g})\}$ .)
- 36 (d) Pick a  $p_1$ -approximation  $\mathfrak{h}_1$  which is  $\max(n, N)$ -dense at  $j$  and (purely) stronger  
 37 than  $\mathfrak{g}$ .
- 38 (e) Pick a  $k_{n+1} > m$  such that  $k_{n+1} \rightarrow \max(n+2, |\text{Pos}(\mathfrak{h}_1)|)$ .
- 39 (f) Pick a  $q \leq_{\mathfrak{h}_1} p_1$  such that  $q^{\bar{b}}$  determines  $\underline{r} \upharpoonright k_{n+1}$  up to  $k_{n+1}$  for all  $\bar{b} \in \text{Pos}(\mathfrak{h}_1)$ .

40 So far we have taken care of (1–4):  $q \leq_{\mathfrak{g}} p$ ,  $\mathfrak{h}_1$  approximates  $q$  and witnesses  $N$ -density (at  
 41  $j$ ). However, the tree of possible values for  $\underline{r}$  could be very thick in the levels between  $k_n$   
 42 and  $k_{n+1}$ . We will thin out the approximation  $\mathfrak{h}_1$  so that we still have  $(n+1)$ -density, and  
 43 the tree of possible values for  $\underline{r}$  gets sufficiently thin. We do this in two steps:

- 44 (g) Find a sub-approximation  $\mathfrak{h}_2$  of  $\mathfrak{h}_1$  that is still purely stronger than  $\mathfrak{g}$  and has only  
 45 as many splittings as  $\mathfrak{g}$ , apart from one additional split (for each possibility) that  
 46 witnesses  $N$ -density at  $j$  (see Figure 5).

47 In more detail: we construct  $\mathfrak{h}_2$  the following way: Given  $\bar{b} \in \text{Pos}_{< i}(\mathfrak{h}_2)$ , set  
 48  $\bar{a} = \bar{b} \upharpoonright \mathfrak{g}$ . We have to define  $\mathfrak{h}_2(i)(\bar{b})$ . If  $i \neq j$ , pick for each  $t \in \mathfrak{g}(i)(\bar{a})$  exactly  
 49 one successor  $s \in \mathfrak{h}_1(i)(\bar{b})$ . So  $\mathfrak{h}_2$  makes the branches of  $\mathfrak{g}$  longer, but does not  
 50 add any splittings. At  $j$ , we have the front  $F := \mathfrak{g}(j)(\bar{a})$  and the purely stronger



F 5.  $h_2$  (bold) is a subapprox. of  $h_1$  and still purely stronger than  $g$ . Here, we assume  $\text{dom}(h_1) = \{u_0, \dots, u_3\}$ ,  $j = u_2$ ,  $\bar{b} \in \text{Pos}(h_2)$  and  $\bar{a} = \bar{b} \upharpoonright g$ .

1  $n'$ -dense front  $F' := h_1(j)(\bar{b})$ . Recall that  $T := T_{\text{cln}}^{F'} = \{s : s \leq F'\}$  is the finite  
 2 tree corresponding to the front  $F'$ . We continue each  $t \in F$  in  $T$  uniquely (without  
 3 splits) until we reach a node  $t$  with many (i.e.,  $n'$ -dense) splittings. We call  $t$   
 4 “splitting node”. We take all the immediate successors of the splitting node and  
 5 continue them uniquely in  $T$  until we reach a leaf of  $T$ , i.e., an element of  $F'$ . This  
 6 process leads to a subset  $F''$  of  $F'$ . Set  $h_2(j)(\bar{b}) := F''$ .

7 (h) So we get: There are  $|\text{Pos}_{\leq j}(g)| \leq M$  many pairs  $(\bar{b}, t)$ , where  $\bar{b} \in \text{Pos}_{< j}(h_2)$  and  $t$  is  
 8 a splitting node.

9 Also, for  $\bar{b} \in \text{Pos}_{\leq j}(h_2)$ , there are at most  $M$  continuations of  $\bar{b}$  to some  $\bar{b}' \in$   
 10  $\text{Pos}(h_2)$ .

11 Such a  $\bar{b} \in \text{Pos}_{\leq j}(h_2)$  corresponds to a pair  $(\bar{a}, t)$  as above together with a choice  
 12 of an (immediate) successor of  $t$ .

13 (i) Now we are ready to apply the Ramsey property. First fix a  $\bar{b} \in \text{Pos}_{< j}(h_2)$  and a  
 14 splitting node  $t$ . (There are at most  $M$  many such pairs.)

15 This pair corresponds to a unique  $\bar{a} \in \text{Pos}_{\leq j}(g)$ . There are at most  $M$  many  
 16 continuations of  $\bar{a}$  to some  $\bar{c} \in \text{Pos}(g)$ . Fix an enumeration  $\bar{c}_1 \dots \bar{c}_M$  of these  
 17 possible continuations. According to (c), each  $\bar{c}_l$  forces a value  $r_l \upharpoonright m$ , call this  
 18 value  $r_l$ .

19 Back to  $h_2$ . Set  $A := \text{succ}(t)$  in the tree  $T_{\text{cln}}^{h_2(j)(\bar{b})}$  (or equivalently  $T_{\text{cln}}^{h_1(j)(\bar{b})}$ ). So  
 20  $\mu(A) \geq n' > N$ . For every  $s \in A$  there is a unique  $s' \geq s$  such that  $\bar{a} \cup \{(j, s')\} \in$   
 21  $\text{Pos}_{\leq j}(h_2)$ , and for every  $s \in A$ ,  $l \in M$  there is a unique  $\bar{d} \in \text{Pos}(h_2)$  continuing  
 22  $\bar{c}_l \in \text{Pos}(g)$  and  $\bar{a} \cup \{(j, s')\}$ . Each such  $\bar{d}$  decides  $r_l$  up to  $k_{n+1}$ . We call this value  
 23  $r^{s,l}$ . So  $r^{s,l} \upharpoonright m = r_l$ . According to (d) we know that  $\text{length}(t) > m$ , so in particular  
 24  $t \perp r_l$ , according to (c).

25 So for every  $l \in M$  we define a function  $f_l : A \rightarrow T_{\text{max}}^{[r_l]} \upharpoonright k_{n+1}$  by mapping  $s$  to  $r^{s,l}$ .  
 26 So we can apply the Ramsey property and get a  $B \subseteq A$  such that  $\mu(B) > M \geq n + 1$ ,  
 27 and the tree of possibilities for  $r_l$  induced by  $\bar{a}, B$  is  $(k_n, 1/M)$ -meager. We repeat  
 28 that for all pairs  $(\bar{a}, t)$  where  $\bar{a} \in \text{Pos}_{\leq j}(h_2)$  and  $t$  is a splitting node, and get a  
 29 subapproximation  $g_{n+1}$  of  $h_2$  such that the tree of possibilities for  $r_l$  induced by  
 30  $g_{n+1}$  is  $(k_n, 1)$ -meager (here we again use the sub-additivity of  $\mu$ ).

31 This results in a sub-approximation  $g_{n+1}$  of  $h_2$  (and therefore  $h_1$ ) which is still purely  
 32 stronger than  $g = g_n$ . Since  $g_{n+1}$  is a sub-approximation of  $h_1$ ,  $|\text{Pos}(g_{n+1})| \leq |\text{Pos}(h_1)|$ , and  
 33 therefore  $k_{n+1}, g_{n+1}$  satisfy (4).  $\square$

34 Note that we did not use the  $j_\alpha$  or automorphisms of  $I$ , the proof works for all  $I$ . In  
 35 particular, for  $I = \{i\}$  we get: If  $G$  is  $Q$ -generic over  $V$ , and if  $r \neq \eta$  in  $V[G]$ , then there is



1 a  $(1, 1)$ -meager  $T$  in  $V$  such that  $r \in \lim(T)$ . In particular, such an  $r$  cannot be  $Q$ -generic  
 2 over  $V$ . So we get:

3 **Corollary 5.7.** *If  $Q$  is strongly non-homogeneous then  $Q$  forces that  $\eta$  is the only  $Q$ -generic*  
 4 *real over  $V$  in  $V[G_Q]$ .*

5 **Remark.** A similar forcing  $Q^{\text{JeSh}}$  (finitely splitting, rapidly increasing number of succes-  
 6 sors) was used in [6] to construct a complete Boolean algebra without proper atomless  
 7 complete subalgebra.  $Q^{\text{JeSh}}$  can also be written as lim-sup forcing. However, the difference  
 8 is that the norm in  $Q^{\text{JeSh}}$  is “binary” (as e.g., Sacks): either  $s$  has a minimum number of  
 9 successors, then the norm is large, or the norm is 0. Such a norm cannot satisfy a Ramsey  
 10 property as the one above. For  $Q^{\text{JeSh}}$  we can only prove Corollary 5.7 for the “single step  
 11 iteration”, but not Lemma 5.6 for the iteration.

12 We have already mentioned another corollary:

13 **Corollary 5.8.** *If  $Q$  is strongly non-homogeneous, then  $P$  forces that  $\{\eta_i : i \in I\}$  is of weak*  
 14 *measure 1.*

15 This, together with 4.8 and 4.10 proves Theorem 3.4.

16 **Remark.** There are various ways to extend the constructions in this paper. As already  
 17 mentioned, we could use non-total orders  $I$  or allow  $Q_i$  to be a  $P_{<i}$ -name. A more difficult  
 18 change would be to use lim-inf trees instead of lim-sup trees. In this case we need addi-  
 19 tional assumptions such as bigness and halving. This could allow us to apply Saccharinity  
 20 to a ccc ideal  $\mathbb{I}$ , i.e., to force (without inaccessible or amalgamation) weak measurability  
 21 of all definable sets.

## 6. T C

23 We thank the referee for providing this section.

24 There is a well known and much simpler way to force that every definable set is even  
 25 measurable (not just weakly measurable) with respect to many tree forcings: Just add many  
 26 Cohen reals.

27 Let  $\mathbb{C}^\kappa$  be the forcing notion adding  $\kappa$  many Cohen reals (in a finite support product, or,  
 28 equivalently, a finite support iteration). Any  $\kappa$  with uncountable cofinality will work. We  
 29 call the forcing extension the “Cohen model”. If in the ground model  $\kappa^{\aleph_0} = \kappa$ , then the  
 30 continuum has size  $\kappa$  in the Cohen model.

31 **Lemma 6.1.** *In the Cohen model, every definable (e.g., projective) set is  $Q$ -measurable.*

32 This works for all  $Q$  as in Section 1, in particular for Sacks forcing, and also many  
 33 other tree forcings, such as Silver forcing (as was shown in [2]). So in particular, in the  
 34 Cohen model all definable sets are Marczewski measurable (corresponding to  $Q = \text{Sacks}$ )  
 35 and have the doughnut property (corresponding to  $Q = \text{Silver}$ ).

36 *Proof.* This is similar to, but simpler than, Solovay’s argument that all definable sets are  
 37 Lebesgue measurable in the Solovay model.

38 Assume that in the Cohen model the parameter  $p$  is in the union of the intermediate  
 39 extensions (i.e., already added by the first  $\alpha$  Cohen reals for some  $\alpha < \kappa$ ) and that

$$X = \{x : \varphi(x, p)\}.$$

41 for some first order formula  $\varphi$ . Pick  $T \in Q$ . We can assume without loss of generality (by  
 42 factoring  $\mathbb{C}^\kappa$ ) that  $p$  and  $T$  are in  $V$ .

43 Work in  $V$  and consider the (countable) forcing notion  $T$  (ordered by  $\leq_T$ , the standard  
 44 tree order). This forcing (which is obviously equivalent to a single Cohen forcing) adds a  
 45 real  $c$  that is Cohen over  $V$  in the natural topology of  $\lim(T)$  (we call such a real  $T$ -Cohen,  
 46 for short). In the same way as for “standard Cohen” forcing, one can see that  $c$  determines  
 the  $T$ -generic filter, and  $c^*$  is  $T$ -Cohen iff  $c^*$  is  $c[G]$  for some  $T$ -generic  $G$  over  $V$ .

1  
2 In particular, whenever  $R$  is some forcing notion,  $G_R$  is  $R$ -generic over  $V$  and  $c^* \in V[G_R]$   
3 is  $T$ -Cohen (over  $V$ ), then we can factor the extension by first adding the  $T$ -generic  $c^*$  and  
4 then forcing with some quotient forcing to extend  $V[c^*]$ . If  $R$  is  $\mathbb{C}^\kappa$ , then the quotient  
5 forcing is again equivalent to  $\mathbb{C}^\kappa$ .

6 Let  $c^*$  be  $T$ -Cohen (i.e.,  $T$ -generic) over  $V$ . In  $V[c^*]$  consider the forcing notion  $\mathbb{C}^\kappa$ .  
7 Since this forcing is homogeneous, either  $\Vdash_{\mathbb{C}^\kappa} \varphi(c^*, p)$  or  $\Vdash_{\mathbb{C}^\kappa} \neg\varphi(c^*, p)$ . Without loss of  
8 generality assume the former. So in  $V$  we can pick some condition  $t^* \in T$  such that

$$9 \quad t^* \Vdash_T \Vdash_{\mathbb{C}^\kappa} \varphi(\zeta, p).$$

10 Let  $c^*$  in a  $\mathbb{C}^\kappa$ -extension  $V'$  of  $V$  be any  $T$ -Cohen real extending  $t^*$ . As described above,  
11 we can get  $V'$  by first extending  $V$  with the  $T$ -generic  $c^*$  and then some  $\mathbb{C}^\kappa$ -extension of  
12  $V[c^*]$ . In particular,  $\varphi(c^*, p)$  holds in  $V'$ . To summarize:

13 (1) In the Cohen model  $V'$ , all  $T$ -Cohen reals  $c^*$  that extend  $t^*$  satisfy  $\varphi(c^*, p)$ .

14 Back in  $V$ , let  $T'$  be the tree  $T^{[t^*]}$ . So  $T' \in Q^V$ . Set

$$15 \quad P = \{(t, n) : n \geq \text{length}(t^*), t \text{ is a subtree of } T', \text{ each maximal branch has height } n\}$$

16 ordered by end-extension (more exactly:  $(t, n)$  is stronger than  $(s, m)$  iff  $n \geq m$  and  $t$  end-  
17 extends  $s$ ). Obviously  $P$  is equivalent to Cohen forcing as well, and  $P$  adds a generic  
18 subtree  $S$  of  $T'$  (and  $S$  determines the generic filter). By density, the lim-sup condition  
19 will be satisfied, so  $S$  is in  $Q^{V[S]}$ . In any forcing extension  $V'$  of  $V[S]$ , we get:

20 (2) Every branch  $\nu \in \text{lim}(S)$  is  $T$ -Cohen over  $V$  and extends  $t^*$ .

21 To see this, fix some nowhere dense set  $N$  in  $V$ . Without loss of generality  $N$  is closed,  
22 i.e., corresponds to a nowhere dense subtree  $N'$  of  $T$ . Then (by a simple density argument)  
23 there is some  $(t, n)$  in the  $P$ -generic such that each maximal branch of  $t$  is not in  $N'$ . So any  
24  $\nu \in \text{lim}(S)$  extends one of the maximal branches of  $t$ , and therefore is not in  $N$ .

25 Now we can finally fix a  $\mathbb{C}^\kappa$ -extension  $V'$  of  $V$ . We can use the equivalence of  $\mathbb{C}^\kappa$  and  
26  $P * \mathbb{C}^\kappa$  to get in  $V'$  some  $S \leq_Q T$  such that (2) holds. Then by (1) we get that each  
27  $c^* \in \text{lim}(S)$  satisfies  $\varphi(c^*, p)$ , i.e., that  $\text{lim}(S) \subseteq X$ .  $\square$

28 What is the difference between the Cohen model and the model obtained in the non-  
29 wellfounded iteration (let us call it nw-model, for short)? Note that in our nw-model,  
30 the continuum has size  $\aleph_2$  (of course we can get larger continuum as well). One obvious  
31 difference is that in the nw-model  $\mathbb{I}^\kappa$  (the  $< \aleph_2$ -closure of  $\mathbb{I}$ ) is non-trivial (or, in the language  
32 of cardinal characteristics,  $\text{cov}(\mathbb{I}) = \aleph_2$ ), which is not the case in the Cohen model for  
33  $\kappa \geq \aleph_2$ :

34 **Lemma 6.2.** *In the Cohen model,  $\text{cov}(\mathbb{I}) = \omega_1$ .*

35 *Proof.* The Cohen model is obtained by a finite support product of  $\kappa$  many Cohen reals.  
36 We can write  $\kappa$  as the strictly increasing union  $\bigcup_{\alpha \in \omega_1} A_\alpha$  (each  $A_\alpha$  of size  $\kappa$ ). Let  $\mathbb{C}_\alpha$  be the  
37 complete subforcing of  $\mathbb{C}^\kappa$  consisting of the conditions that only use coordinates in  $A_\alpha$ . Let  
38  $G$  be  $\mathbb{C}^\kappa$ -generic over  $V$ , and let  $G_\alpha$  be the induced  $\mathbb{C}_\alpha$ -generic filters over  $V$ . Then we get:

- 39 (1)  $V[G_\alpha] \cap \omega^\omega$  is a proper subset of  $V[G_{\alpha+1}] \cap \omega^\omega$ .  
40 (2)  $V[G] \cap \omega^\omega = \bigcup_{\alpha \in \omega_1} V[G_\alpha] \cap \omega^\omega$ .

41 From (1) and Lemma 1.8 we know that each  $V[G_\alpha] \cap \omega^\omega$  is  $Q$ -null in the final Cohen  
42 extension; so by (2)  $\omega^\omega$  is the union of  $\aleph_1$  many  $Q$ -null sets.  $\square$

43 (This argument works not only for the Cohen extension, but also for the random model  
44 and similarly for finite support iteration of Suslin ccc forcings of length  $\aleph_2$ ; also, it works  
45 for other ideals than the ones defined by lim-sup tree forcings.)

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