REASONABLY COMPLETE FORCING NOTIONS

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Abstract. We introduce more properties of forcing notions which imply that their \( \lambda \)-support iterations are \( \lambda \)-proper, where \( \lambda \) is an inaccessible cardinal. This paper is a direct continuation of Roslanowski and Shelah [5, §A.2]. As an application of our iteration result we show that it is consistent that dominating numbers associated with two normal filters on \( \lambda \) are distinct.

0. Introduction

There are serious ZFC obstacles to easy generalizations of properness to the case of iterations with uncountable supports (see, e.g., Shelah [8, Appendix 3.6(2)]). This paper belongs to the series of works aiming at localizing “good properness conditions” for such iterations and including Shelah [9], [10], Roslanowski and Shelah [4], [5] and Eisworth [2]. Our results continue Roslanowski and Shelah [5, §A.2], but no familiarity with the previous paper is assumed and the current work is fully self-contained.

In Section 2 we introduce 3 bounding–type properties (A, B, C) and we essentially show that the first two are almost preserved in \( \lambda \)-support iterations (Theorems 2.5, 2.8). “Almost” as the limit of the iteration occurs to have a somewhat weaker property, but equally applicable. In the following section we show that reasonably \( A \)-bounding forcing notions are exactly the ones introduced in [5, §A.2], thus showing that Theorem 2.8 improves [5, Thm A.2.4]. In the fourth section of the paper, we give an example of an interesting reasonably \( B \)-bounding forcing notion and we use it to show that it is consistent that dominating numbers associated with two normal filters on \( \lambda \) are distinct (Corollary 4.13). Finally, in the last section we present two forcing notions that are not yet covered by existing iteration theorems. We hope that the further development of the theory will include also them.

Like in [5], we assume here that our cardinal \( \lambda \) is inaccessible. We do not know at the moment if any parallel work can be done for a successor cardinal, though some progress will be presented in a subsequent paper [6].

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Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [3]). In forcing we keep the older convention that a stronger condition is the larger one.

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(1) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet (α, β, γ, δ . . .) and also by i, j (with possible sub- and superscripts).

Cardinal numbers will be called κ, λ, µ; λ will be always assumed to be inaccessible (we may forget to mention it).

By χ we will denote a sufficiently large regular cardinal; \( H(\chi) \) is the family of all sets hereditarily of size less than χ. Moreover, we fix a well ordering \( <_\chi \) of \( H(\chi) \).

(2) For two sequences η, ν we write \( \nu \triangleright \eta \) whenever \( \nu \) is a proper initial segment of η, and \( \nu \trianglerighteq \eta \) when either \( \nu \triangleright \eta \) or \( \nu = \eta \). The length of a sequence \( \eta \) is denoted by \( \text{lh}(\eta) \).

(3) We will consider several games of two players. One player will be called Generic or Complete or just COM, and we will refer to this player as “she”. Her opponent will be called Antigeneric or Incomplete or just INC and will be referred to as “he”.

(4) For a forcing notion \( P \), \( \Gamma_\alpha P \) stands for the canonical \( P \– name for the generic filter in \( P \). With this one exception, all \( P \– names for objects in the extension via \( P \) will be denoted with a tilde below (e.g., \( \tau, X \)). The weakest element of \( P \) will be denoted by \( \emptyset_P \) (and we will always assume that there is one, and that there is no other condition equivalent to it). We will also assume that all forcing notions under consideration are atomless.

By “λ–support iterations” we mean iterations in which domains of conditions are of size \( \leq \lambda \). However, we will pretend that conditions in a λ–support iteration \( \bar{Q} = \langle P_\xi, Q_\xi : \xi < \zeta^* \rangle \) are total functions on \( \zeta^* \) and for a condition \( p \) in the limit \( \lim(\bar{Q}) \) of the iteration \( \bar{Q} \) and \( \alpha \in \zeta^* \setminus \text{Dom}(p) \) we will let \( p(\alpha) = \emptyset_{\zeta^*} \).

(5) For a filter \( D \) on \( \lambda \), the family of all \( D \– positive subsets of \lambda \) is called \( D^+ \).

So \( A \in D^+ \) if and only if \( A \subseteq \lambda \) and \( A \cap B \neq \emptyset \) for all \( B \in D. \)

In this paper we assume the following.

Context 0.1. (a) \( \lambda \) is a strongly inaccessible cardinal,
(b) \( \bar{\mu} = (\mu_\alpha : \alpha < \lambda) \), each \( \mu_\alpha \) is a regular cardinal satisfying (for \( \alpha < \lambda \))
\[ \aleph_0 \leq \mu_\alpha \leq \lambda \quad \text{and} \quad (\forall f \in ^\alpha \mu_\alpha)(\prod_{\xi < \alpha} |f(\xi)| < \mu_\alpha), \]
(c) \( U \) is a normal filter on \( \lambda \).

1. Preliminaries on λ–support iterations

Definition 1.1. Let \( P \) be a forcing notion.

(1) For a condition \( r \in P \) let \( \mathcal{G}_0 P(r) \) be the following game of two players, Complete and Incomplete:

the game lasts at most \( \lambda \) moves and during a play the players construct a sequence \( \langle (p_i, q_i) : i < \lambda \rangle \) of pairs of conditions from \( P \) in such a way that \( (\forall j < i < \lambda)(r \leq p_j \leq q_j \leq p_i) \) and at the stage \( i < \lambda \) of the game, first Incomplete chooses \( p_i \) and then Complete chooses \( q_i \).

Complete wins if and only if for every \( i < \lambda \) there are legal moves for both players.
(2) We say that the forcing notion $P$ is \textit{strategically $〈<\lambda〉$–complete} if Complete has a winning strategy in the game $\mathcal{G}^0_P(r, r)$ for each condition $r \in P$.

(3) Let $N \prec (\mathcal{H}(\lambda), <, \mathcal{L})$ be a model such that $\langle <\lambda \rangle \subseteq N$, $|N| = \lambda$ and $\mathcal{P} \in N$. We say that a condition $p \in P$ is $(N, P)$–generic in the standard sense (or just: $(N, P)$–generic) if for every $P$–name $\tau \in N$ for an ordinal we have $p \Vdash \tau \in N$.

(4) $P$ is a $\lambda$–proper in the standard sense (or just: $\lambda$–proper) if there is $x \in \mathcal{H}(\lambda)$ such that for every model $N \prec (\mathcal{H}(\lambda), <, \mathcal{L})$ satisfying $\langle <\lambda \rangle \subseteq N$, $|N| = \lambda$ and $\mathcal{P} \in N$, and every condition $q \in N \cap \mathcal{P}$ there is an $(N, P)$–generic condition $p \in P$ stronger than $q$.

Proposition 1.2 ([5, Prop. A.1.4]). Suppose that $P$ is a ($<\lambda$)–strategically complete (atomless) forcing notion, $\alpha^* < \lambda$ and $p_\alpha \in P$ (for $\alpha < \alpha^*$). Then there are conditions $q_\alpha \in P$ (for $\alpha < \alpha^*$) such that $p_\alpha \leq q_\alpha$ and for distinct $\alpha, \alpha' < \alpha^*$ the conditions $q_\alpha, q_{\alpha'}$ are incompatible.

Proposition 1.3 ([5, Prop. A.1.6]). Suppose $\bar{Q} = \langle \mathbb{P}_i, \mathcal{Q}_i : i < \gamma \rangle$ is a $\lambda$–support iteration and, for each $i < \gamma$,

\[
\forall \mathbb{P}_i, " \mathcal{Q}_i \ is \ strategically \ (<\lambda)–complete \." \]

Then, for each $i < \gamma$ and $r \in \mathbb{P}_i$, there is a winning strategy $\mathcal{S}(\varepsilon, r)$ of Complete in the game $\mathcal{G}^0_\mathbb{P}(\varepsilon, r)$ such that, whenever $\varepsilon_0 < \varepsilon_1 \leq \gamma$ and $r \in \mathbb{P}_{\epsilon_1}$, we have:

(i) if $\langle (p_i, q_i) : i < \lambda \rangle$ is a play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_0, r|\varepsilon_0)$ in which Complete follows the strategy $\mathcal{S}(\varepsilon_1, r)$, then $\langle (p_i|\varepsilon_0, q_i|\varepsilon_0) : i < \lambda \rangle$ is a play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_1, r)$ in which Complete uses $\mathcal{S}(\varepsilon_1, r)$;

(ii) if $\langle (p_i, q_i) : i < \lambda \rangle$ is a play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_1, r)$ in which Complete plays according to the strategy $\mathcal{S}(\varepsilon_1, r)$, then $\langle (p_i|\varepsilon_0, q_i|\varepsilon_0) : i < \lambda \rangle$ is a play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_1, r)$ in which Complete uses $\mathcal{S}(\varepsilon_0, r|\varepsilon_0)$;

(iii) if $\varepsilon_1$ is limit and a sequence $\langle (p_i, q_i) : i < \lambda \rangle \subseteq \mathbb{P}_{\varepsilon_1}$ is such that for each $\xi < \varepsilon_1$, $\langle (p_i|\varepsilon_0, q_i|\varepsilon_0) : i < \lambda \rangle$ is a play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_0, r|\varepsilon_0)$ in which Complete plays according to the strategy $\mathcal{S}(\varepsilon_1, \xi)$, then $\langle (p_i, q_i) : i < \lambda \rangle$ is a play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_1, r)$ in which Complete plays according to $\mathcal{S}(\varepsilon_1, r)$;

(iv) if $\langle (p_i, q_i) : i < \lambda' \rangle$ is a partial play of $\mathcal{G}^0_\mathbb{P}(\varepsilon_1, r)$ in which Complete uses $\mathcal{S}(\varepsilon_1, r)$ and $p' \in \mathbb{P}_{\varepsilon_1}$ is stronger than all $p_i|\varepsilon_0$ (for $i < \lambda'$), then there is $p^* \in \mathbb{P}_{\varepsilon_1}$ such that $p' = p^*|\varepsilon_0$ and $p^* \geq p_i$ for $i < \lambda'$.

Definition 1.4 (Compare [5, Def. A.1.7], see also [9, A.3.3, A.3.2]).

(1) Let $\gamma$ be an ordinal, $\emptyset \neq w \subseteq \gamma$. A standard $(w, 1)^\gamma$–tree is a pair $T = (T, \text{rk})$ such that

- $\text{rk} : T \rightarrow w \cup \{\gamma\}$,
- if $t \in T$ and $\text{rk}(t) = \varepsilon$, then $t$ is a sequence $\langle (t)|\zeta : \zeta \in w \cap \varepsilon \rangle$,
- $(T, <)$ is a tree with root $\emptyset$ and such that every chain in $T$ has a $<\varepsilon$–upper bound it $T$,
- if $t \in T$, then there is $t' \in T$ such that $t \leq t'$ and $\text{rk}(t') = \gamma$.

We will keep the convention that $T''_y = (T'_y \text{rk}_y)$. \( T''_y \text{rk}_y \)

(2) Let $\bar{Q} = \langle \mathbb{P}_i, \mathcal{Q}_i : i < \gamma \rangle$ be a $\lambda$–support iteration. A standard tree of conditions in $\bar{Q}$ is a system $\bar{p} = \langle p_t : t \in T \rangle$ such that

- $(T, \text{rk})$ is a standard $(w, 1)^\gamma$–tree for some $w \subseteq \gamma$, \( (T, \text{rk}) \)}
\begin{itemize}
  \item \( p_t \in \mathbb{P}_{\text{rk}(t)} \) for \( t \in T \), and
  \item if \( s, t \in T \), \( s < t \), then \( p_s = p_t | \text{rk}(s) \).
\end{itemize}

(3) Let \( \bar{\rho}^0, \bar{\rho}^1 \) be standard trees of conditions in \( \bar{Q} \), \( \bar{\rho}^0 = \{ p_t^i : t \in T \} \). We write \( \bar{\rho}^0 \leq \bar{\rho}^1 \) whenever for each \( t \in T \) we have \( p_t^0 \leq p_t^1 \).

Note that our standard trees and trees of conditions are a special case of \( (w, \alpha)^\gamma \)-trees introduced in \([5, \text{Def. A.1.7}]\) (for \( \alpha = 1 \)). Our notation preserves the redundant “1” to keep the compatibility with the established terminology. For the same reason we use \( (t, \zeta) \) instead of \( t(\zeta) \).

**Proposition 1.5.** Assume that \( \bar{Q} = \langle \mathbb{P}_i, \bar{Q}_i : i < \gamma \rangle \) is a \( \lambda \)-support iteration such that for all \( i < \gamma \) we have 
\[ \forall \bar{P}_i \text{ " } \bar{Q}_i \text{ is strategically } (\lambda, \gamma, \text{complete}. " \]

(1) \([5, \text{Prop. A.1.9}]\) Suppose that \( \tilde{p} = \{ p_t : t \in T \} \) is a standard tree of conditions in \( \bar{Q} \), \( |T| < \lambda \), and \( T \subseteq \mathbb{P}_\gamma \) is open dense. Then there is a standard tree of conditions \( \bar{q} = \{ q_t : t \in T \} \) such that \( \tilde{p} \leq \bar{q} \) and \( (\forall t \in T) |\text{rk}(t) = \gamma \Rightarrow q_t \in T \).

(2) If \( \tilde{p} = \{ p_t : t \in T \} \) is a standard tree of conditions in \( \bar{Q} \) and \( |T| < \lambda \), then there is a standard tree of conditions \( \bar{q} = \{ q_t : t \in T \} \) such that \( \tilde{p} \leq \bar{q} \) and
\[ \text{if } t_0, t_1 \in T, \text{ rk}(t_0) = \text{rk}(t_1), \xi \in \text{Dom}(t_0) \text{ and } (t_0)_{\xi} \neq (t_1)_{\xi}, t_0|\xi = t_1|\xi, \text{ then} \]
\[ q_{t_0}|\xi \vDash \exists \bar{\kappa}_\xi \text{ " the conditions } q_{t_0}(\xi), q_{t_1}(\xi) \text{ are incompatible in } \bar{Q}_\xi. " \]

(3) Suppose that
\[ \text{ if } (w, 1)^\gamma \text{-tree } T = (T, \text{rk}) \text{ is a standard } (w, 1)^\gamma \text{-tree,} \]
\[ \tilde{p} = \{ p_t : t \in T \} \text{ is a standard tree of conditions in } \bar{Q}, \]
\[ \text{for } \xi \in w, \xi_\zeta \text{ is a } \mathbb{P}_\xi \text{-name for a non-zero ordinal below } \mu_\xi. \]

Then there are a standard \( (w, 1)^\gamma \)-tree \( T' = (T', \text{rk}') \) and a tree of conditions \( \bar{q} = \{ q_t : t \in T' \} \) such that
\[ \text{ if } t \in T', \text{ rk}(t) = \xi \in w, \text{ then } \]
\[ \{ \alpha < \mu_\xi : t \cup \{ (\xi, \alpha) \} \in T' \} = \varepsilon'_\xi. \]

**Proof.** (2) Straightforward application of 1.2.

(3) Note that we cannot apply the first part directly, as the tree \( T \) may be of size \( \lambda \). So we will proceed inductively constructing initial levels of \( T' \) of size \( < \lambda \) and applying (1) to them.

For \( \varepsilon \leq \gamma \) and \( r \in \mathbb{P}_\varepsilon \) let \( \text{st}(\varepsilon, r) \) be the winning strategy of Complete in \( D^\lambda_0(\mathbb{P}_\varepsilon, r) \) given by 1.3 (so these strategies have the coherence properties listed there). Let \( (\xi_\beta : \beta \leq \beta^\ast) \) be the increasing enumeration of \( w \cup \{ \gamma \} \), \( \beta^\ast < \lambda \). By induction on \( \beta \leq \beta^\ast \) we will pick \( \tau_\beta, q_\beta^3, r_\beta^3 \) and \( \varepsilon_\beta^3 \) such that
\[ \text{ if } t \in \tau_\beta, \text{ rk}(t) = (T_\beta, \text{rk}_\beta) \text{ is a standard } (w \cap \xi_\beta, 1)^\gamma \text{-tree, } T_\beta \subseteq T, |T_\beta| < \lambda, \text{ and} \]
\[ q_\beta^3 = \{ q_t^3 : t \in T_\beta \}, r_\beta^3 = \{ r_t^3 : t \in T_\beta \} \text{ are trees of conditions, } q_\beta^3 \leq r_\beta^3 \text{ and} \]
\[ r_t^3 \in \mathbb{P}_{\text{rk}(t)} \text{ for } t \in T_\beta \text{ (note: rk}(t), \text{ not rk}_\beta(t)). \]
(b) if $\beta_0 < \beta_1 \leq \beta^*$, then $T_{\beta_0} = \{ t | \xi_{\beta_0} : t \in T_{\beta_1} \}$ and $r^{\beta_0}_{t | \xi_{\beta_0}} \leq q^\beta_{\xi_t \xi_{\beta_0}}$ for $t \in T_{\beta_1}$.

c) if $\beta < \beta^*$, $t \in T_{\beta}$ and $r_{\beta}(t) = \gamma$ (so $r_k(t) = \xi_\beta$), then

$$\langle \{ r^{\alpha}_{t | \xi_\alpha} : t \in T_{\beta} \} : \alpha < \beta \rangle \sim \langle \{ q^\beta_{\xi_t \xi_\beta} \} : \beta \leq \alpha < \beta^* \rangle$$

is a partial play of $\mathcal{O}_0(\mathcal{P}_{\xi_\beta}, p_t)$ in which Complete uses her winning strategy $st(\xi_\beta, p_t)$:

(d) $\xi^\beta = \langle \tilde{\xi}_t^\beta \rangle : t \in T_{\beta}$, where $r_{\beta}(t) = \gamma$ (so $r_k(t) = \xi_\beta$), then $p_t \leq q^\beta_{\xi_t \xi_\beta} \in \mathcal{P}_{\xi_\beta}$ and

$$q^\beta_{\xi_t \xi_\beta} \ni \mathcal{P}_{\xi_\beta} \xi_\beta = \xi^\beta_t,$$

where $\xi_t = \xi^\beta_t$.

We let $T_0 = \{ \langle \rangle \}$ and choose $q_0^\beta \in \mathcal{P}_{\xi_\beta}$ and $\tilde{\xi}_t^\beta$ so that $p_0 \leq q_0^\beta$ and $\tilde{\xi}_0^\beta = \xi^\beta_0$. Then we let $\tilde{\xi}_t^\beta$ be the answer given to Complete by $st(\xi_\beta, p_t)$ in $\mathcal{O}_0(\mathcal{P}_{\xi_\beta}, p_0)$ to $q_0^\beta$. Now suppose that we have defined $T_\alpha, q^\alpha_t, r^\alpha_t$ and $\xi^\alpha_t$ for $\alpha < \beta \leq \beta^*$.

If $\beta$ is a limit ordinal then the demands (a) and (b) uniquely define the standard tree $T_\beta$. Note that $|T_\beta| \leq \lambda$ as $\lambda$ is inaccessible; remember also clause (f). It follows from the choice of $st(\xi, r)$ (see clause 1.3(iii)) and demand (c) at previous stages that

$$\langle \langle q^\alpha_{t | \xi_\alpha} : t \in T_{\beta} \rangle : r^\alpha_{t | \xi_\alpha} \ni \mathcal{P}_{\xi_\alpha} \xi_\alpha \rangle$$

$$\langle \rangle : t \in T_{\beta} \rangle$$

is a partial play of $\mathcal{O}_0(\mathcal{P}_{\xi_\beta}, p_t)$ in which Complete uses her winning strategy $st(\xi_\beta, p_t)$.

For $t \in T_{\beta}$ we define a condition $q_t \in \mathcal{P}_{\xi_\beta}$ as follows:

- $\text{Dom}(q_t) = \bigcup_{\alpha < \beta} \text{Dom}(r^\alpha_{t | \xi_\alpha}) \cup \text{Dom}(p_t) \subseteq \text{rk}(t)$,

- if $\zeta \in \text{Dom}(q_t)$, then $q_t(\zeta)$ is the $<^\zeta$-first of $\mathcal{P}_{\zeta}$-name for an element of $\mathcal{Q}_{\zeta}$ such that

$$q_t(\xi_\alpha) \ni \mathcal{P}_{\xi_\alpha} \xi_\alpha \$$

if the set $\{ r_{t | \xi_\alpha}(\zeta) : \xi_\alpha < \xi_\beta \} \cup \{ p_t(\zeta) \}$ has an upper bound, then $q_t(\xi_\beta)$ is such an upper bound.

It follows from (3) (and 1.3(iv)) that $p_t \leq q_t$ and $r^\alpha_{t | \xi_\alpha} \leq r^{\alpha+1}_{t | \xi_\alpha}$ for $\alpha < \beta$. Now, by the $\langle \langle \rangle \rangle$-first condition, clearly $\bar{q} = \langle q_t : t \in T_{\beta} \rangle$ is a tree of conditions. Applying 1.5(1) we may choose a tree of conditions $\bar{q}^\beta = \langle q_t^\beta : t \in T_{\beta} \rangle$ such that $\bar{q} \leq \bar{q}^\beta$ and

- if $\beta < \beta^*$, $t \in T_{\beta}$ and $r_{\beta}(t) = \gamma$, then the condition $q^\beta_t$ decides the value of $\xi_\beta$ (and let $q^\beta_t \ni \xi_\beta = \xi^\beta_t$) and $q^\beta_t \in \mathcal{P}_{\xi_\beta}$.

Then, for $t \in T_{\beta}$, we let $r^\beta_t$ be the answer given to Complete by $st(\bar{q}_{\beta}, p_t)$ in the appropriate partial play of $\mathcal{O}_0(\mathcal{P}_{\bar{q}_{\beta}}, p_t)$, where at stage $\beta$ Incomplete put $q^\beta_t$ (see (c), (3)). It follows from 1.3(ii) that $\bar{q}^\beta = \langle r^\beta_t : t \in T_{\beta} \rangle$ is a tree of conditions. Plainly, $T_{\beta}, \bar{q}^\beta, r^\beta$ and $\epsilon^\beta$ satisfy all relevant (restrictions of the) demands (a)–(f).

Now suppose that $\beta$ is a successor ordinal, say $\beta = \beta_0 + 1$. Let

$$T_{\beta} = T_{\beta_0} \cup \{ t \cup \{ \xi_{\beta_0}, \epsilon \} : t \in T_{\beta_0} \land r_{\beta_0}(t) = \gamma \land \epsilon \leq \epsilon^\beta_t \}$$

and for $t \in T_{\beta}$ define $q_t$ as follows:

- if $t \in T_{\beta_0}$, then $q_t = r^\beta_{t | \xi_{\beta_0}}$.
• if $t \in T_\beta \setminus T_{\beta_0}$, then $q_t = r^\beta_{t \lor \xi_\beta} \Vdash p_t [\xi_{\beta_0}, \xi_\beta]$.

Then $q = \langle q_t : t \in T_\beta \rangle$ is a tree of conditions, $r^\beta_t \leq q_t$ for $t \in T_{\beta_0}$. It follows from 1.5(1) that we may choose a tree of conditions $q^\beta = \langle q^\beta_t : t \in T_\beta \rangle$ such that $q \leq q^\beta$ and

- if $\beta < \beta^*$, $t \in T_\beta$ and $\ rk(\beta) = \gamma$, then the condition $q^\beta_t$ decides $\xi_{\xi_\beta}$ and, say, $q^\beta_t \Vdash \xi_{\xi_\beta} = \epsilon_{\xi_t}$.

Next, like in the limit case, $r^\beta = \langle r^\beta_t : t \in T_\beta \rangle$ is obtained by applying the strategies $st(\rho(t), p_t)$ suitably. Easily, $T_\beta$, $q^\beta$, $r^\beta$ and $\epsilon^\beta$ satisfy the demands (a)–(f).

After the inductive construction is carried out look at $T_{\beta^*}$, $q^{\beta^*}$ and $\langle \epsilon^{\beta^*} : \beta < \beta^* \rangle$.

2. ABC of reasonable completeness

Remark 2.1. Note that if a forcing notion $Q$ is strategically $(<\lambda)$–complete and $U$ is a normal filter on $\lambda$, then the normal filter generated by $U$ in $V^Q$ is proper. Abusing notation, we may denote the normal filter generated by $U$ in $V^Q$ also by $U$ or by $U^Q$. Thus if $A$ is a $Q$–name for a subset of $\lambda$, then $p \Vdash Q A \in U^Q$ if and only if for some $Q$–names $A_\alpha$ for elements of $U^Q$ we have that $p \Vdash Q \bigcap_{\alpha<\lambda} A_\alpha \subseteq A$

(where $\bigcap$ denotes the operation of diagonal intersection).

Let us note that many of the arguments in this section would be much simpler if we restricted ourselves to $(<\lambda)$–complete forcing notions. Unfortunately, the forcing notions that we would like to cover tend to be only strategically $(<\lambda)$–complete, see [5, §B.6].

Definition 2.2. Let $Q$ be a strategically $(<\lambda)$–complete forcing notion.

(1) For a condition $p \in Q$ we define a game $^1 \mathcal{G}^\mathcal{C}_{\mathcal{P}}(p, Q)$ between two players,Generic and Antigeneric, as follows. A play of $^1 \mathcal{G}^\mathcal{C}_{\mathcal{P}}(p, Q)$ lasts $\lambda$ steps and during a play a sequence

$$\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \rangle$$

is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now,

$(\forall)_\alpha$ first Generic chooses a non-empty set $I_\alpha$ of cardinality $< \mu_\alpha$ and a system $\langle p^\alpha_t : t \in I_\alpha \rangle$ of conditions from $Q$.

$(\exists)_\alpha$ then Antigeneric answers by picking a system $\langle q^\alpha_t : t \in I_\alpha \rangle$ of conditions from $Q$ such that $(\forall t \in I_\alpha) (p^\alpha_t \leq q^\alpha_t)$.

At the end, Generic wins the play

$$\langle I_\alpha, \langle p^\alpha_t, q^\alpha_t : t \in I_\alpha \rangle : \alpha < \lambda \rangle$$

of $^1 \mathcal{G}^\mathcal{C}_{\mathcal{P}}(p, Q)$ if and only if

$(\oplus)^{\mathcal{C}_{\mathcal{P}}} \mathcal{P}$ there is a condition $p^* \in Q$ stronger than $p$ and such that$^2$

$$p^* \Vdash Q \alpha = \{ \alpha < \lambda : (\exists t \in I_\alpha) (q^\alpha_t \in \Gamma_\alpha) \} = \lambda^\mu.$$

(2) Games $^1 \mathcal{G}^\mathcal{B}_{\mathcal{P}}(p, Q), ^1 \mathcal{G}^\mathcal{C}_{\mathcal{P}}(p, Q)$ are defined similarly, except that the winning criterion $(\oplus)^{\mathcal{C}_{\mathcal{P}}} \mathcal{P}$ is replaced by

$^1 rc$ stands for reasonable completeness

$^2$equivalently, for every $\alpha < \lambda$ the set $\{ q^\alpha_t : t \in I_\alpha \}$ is pre-dense above $p^*$
(a)\textsuperscript{bc}$ there is a condition $p^* \in Q$ stronger than $p$ and such that
\[ p^* \Vdash Q \models \{ \alpha < \lambda : (\exists t \in I_\alpha) (q^\alpha_t \in \Gamma_q) \} \in U^Q \]
(b)\textsuperscript{bc}$ there is a condition $p^* \in Q$ stronger than $p$ and such that
\[ p^* \Vdash Q \models \{ \alpha < \lambda : (\exists t \in I_\alpha) (q^\alpha_t \in \Gamma_q) \} \in (U^Q)^+ \]
respectively.
(3) For a condition $p \in Q$ we define a game $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$ between Generic and Antigeneric as follows. A play of $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$ lasts $\lambda$ steps and during a play a sequence
\[ \langle \zeta_\alpha, (p^\xi_\alpha, q^\xi_\alpha) : \xi < \zeta_\alpha : \alpha < \lambda \rangle \]
is constructed. Suppose that the players have arrived to a stage $\alpha < \lambda$ of the game. Now, Generic chooses a non-zero ordinal $\zeta_\alpha < \mu_\alpha$ and then the two players play a subgame of length $\zeta_\alpha$ alternately choosing successive terms of a sequence $\langle p^\xi_\alpha, q^\xi_\alpha + \xi < \zeta_\alpha \rangle$. At a stage $\xi < \zeta_\alpha$ of the subgame, first Generic picks a condition $p^\xi_\alpha \in Q$ and then Antigeneric answers with a condition $q^\xi_\alpha$ stronger than $p^\xi_\alpha$.
At the end, Generic wins the play
\[ \langle \zeta_\alpha, (p^\xi_\alpha, q^\xi_\alpha) : \xi < \zeta_\alpha : \alpha < \lambda \rangle \]
of $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$ if and only if
(\textsuperscript{bc})\textsuperscript{bc}$ there is a condition $p^* \in Q$ stronger than $p$ and such that
\[ p^* \Vdash Q \models \{ \alpha < \lambda : (\exists t \in I_\alpha) (q^\alpha_t \in \Gamma_q) \} \in U^Q \]
(4) Games $\mathcal{G}_{\bar{\mu}}^\text{rc}(p, Q)$ and $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$ are defined similarly except that the winning criterion (a)\textsuperscript{bc} is changed so that $\models U^Q$ is replaced by $\models U^Q$ or $\models (U^Q)^+$, respectively.
(5) We say that a forcing notion $Q$ is reasonably $A$–bounding over $\bar{\mu}$ if
(a) $Q$ is strategically ($<\lambda$)–complete, and
(b) for any $p \in Q$, Generic has a winning strategy in the game $\mathcal{G}_{\bar{\mu}}^\text{rc}(p, Q)$.
In an analogous manner we define when the forcing notion $Q$ is reasonably $X$–bounding over $\text{U}, \bar{\mu}$ (for $X \in \{ B, C, a, b, c \}$) — just using the game $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$ appropriately.
If $\mu_\alpha = \lambda$ for each $\alpha < \lambda$, then we may omit $\bar{\mu}$ and say reasonably $B$–bounding over $\text{U}$ etc. If $\text{U}$ is the filter generated by club subsets of $\lambda$, we may omit it as well.
(6) Let $\text{st}$ be a strategy for Generic in the game $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$. We will say that a sequence $\langle I_\alpha, (p^\alpha_t, q^\alpha_t) : t \in I_\alpha : \delta < \alpha < \lambda \rangle$ is a $\delta$–delayed play according to $\text{st}$ if it has an extension $\langle I_\alpha, (p^\alpha_t, q^\alpha_t) : t \in I_\alpha : \alpha < \lambda \rangle$ which is a play agreeing with $\text{st}$ and such that $p^\alpha_t = q^\alpha_t$ for $\alpha \leq \delta, t \in I_\alpha$.

Remark 2.3. If $\text{st}$ is a winning strategy for Generic in the game $\mathcal{G}_{\text{U},\bar{\mu}}^\text{rc}(p, Q)$, and $\bar{\sigma} = \langle I_\alpha, (p^\alpha_t, q^\alpha_t) : t \in I_\alpha : \delta \leq \alpha < \lambda \rangle$ is a $\delta$–delayed play according to $\text{st}$, then $\bar{\sigma}$ satisfies the condition (a)\textsuperscript{bc}.

Observation 2.4. For $\text{U}, \bar{\mu}$ as in 0.1, $X \in \{ A, B, C, a, b, c \}$ and a forcing notion $Q$, let $\Phi(Q, X, \text{U}, \bar{\mu})$ be the statement
Then the following implications hold

\[ \Phi(Q, A, \bar{\mu}) \Rightarrow \Phi(Q, B, \bar{U}, \bar{\mu}) \Rightarrow \Phi(Q, C, \bar{U}, \bar{\mu}) \]

\[ \Phi(Q, a, \bar{\mu}) \Rightarrow \Phi(Q, b, \bar{U}, \bar{\mu}) \Rightarrow \Phi(Q, c, \bar{U}, \bar{\mu}) \Rightarrow Q \text{ is \(\lambda\)-proper.} \]

**Theorem 2.5.** Assume that \(\lambda, U, \bar{\mu}\) are as in 0.1 and \(Q = (P_\xi, \bar{Q}_\xi : \xi < \gamma)\) is a \(\lambda\)-support iteration such that for every \(\xi < \gamma\),

\[ \models_{P_\xi} " Q_\xi \text{ is reasonably } B \text{-bounding over } U, \bar{\mu} " \]

Then \(P_\gamma = \lim(\bar{Q})\) is reasonably \(b\)-bounding over \(U, \bar{\mu}\) (and so also \(\lambda\)-proper).

**Proof.** For each \(\xi < \gamma\) pick a \(P_\xi\)-name \(\bar{\mathfrak{q}}^0_\xi\) such that

\[ \models_{P_\xi} " \bar{s}_\xi^0 \text{ is a winning strategy for Complete in } \mathcal{D}_\delta^\lambda(Q_\xi, \bar{Q}_\xi) \text{ such that if } \text{Incomplete plays } \bar{\emptyset}_\xi \text{ then Complete answers with } \bar{\emptyset}_\xi \text{ as well}. " \]

Also, for \(\xi \leq \gamma\) and \(r \in P_\xi\), let \(\mathfrak{s}_\xi(r)\) be a winning strategy of Complete in \(\mathcal{D}_\delta^\lambda(P_\xi, r)\) with the coherence properties given in 1.3.

We are going to describe a strategy \(\bar{s}_\xi\) for Generic in the game \(\mathcal{D}_\delta^\lambda(p, P_\gamma)\). In the course of the play, at a stage \(\delta \leq \lambda\), Generic will be instructed to construct aside

\[ (\otimes)_\delta \quad T_b, p^\delta_*, q^\delta_*, r^\delta_*, s: w, (\xi, \bar{s}, \bar{q}: \xi < w) \text{, and } \bar{s}^\delta_\xi \text{ for } \xi \in w_{\delta+1} \setminus w_\delta. \]

These objects will be chosen so that if

\[ \langle \xi_\delta, \langle p^\delta_*, q^\delta_* : \zeta \leq \xi_\delta : \delta < \lambda \rangle \]

is a play of \(\mathcal{D}_\delta^\lambda(p, P_\gamma)\) in which Generic follows \(\bar{s}_\xi\), and the side objects constructed at stage \(\delta < \lambda\) are listed in \((\otimes)_\delta\), then the following conditions are satisfied (for each \(\delta < \lambda\))

\[ (*)_1 \quad r^\delta_\delta, s \in \mathcal{P}_\gamma, r_0(0) = p(0), w_\delta \subseteq \gamma, |w_\delta| = |\delta| + 1, \bigcup_{\alpha < \lambda} \text{Dom}(r_\alpha) = \bigcup_{\alpha < \lambda} w_\alpha, w_0 = \{0\}, w_\delta \subseteq w_{\delta+1} \text{ and if } \delta \text{ is limit then } w_\delta = \bigcup_{\alpha < \delta} w_\alpha. \]

\[ (*)_2 \quad \text{For each } \alpha \leq \delta < \lambda \text{ we have } (\forall \xi \in w_{\alpha+1})(r_\alpha(\xi) = r_\delta(\xi)) \text{ and } p \leq r^\alpha_\delta \leq r_\delta, r_\alpha \leq r^\delta_\alpha \leq r_\delta. \]

\[ (*)_3 \quad \text{If } \xi \in \gamma \setminus w_\delta, \text{ then} \]

\[ r_\delta(\xi) \models " \text{the sequence } \langle r^\alpha_\delta(\xi), r_\alpha(\xi) : \alpha \leq \delta \rangle \text{ is a legal partial play of } \]

\[ \mathcal{D}_\delta^\lambda(Q_\xi, \bar{Q}_\xi) \text{ in which Complete follows } \bar{s}_\xi^\delta " \]

and if \(\xi \in w_{\delta+1} \setminus w_\delta\), then \(\bar{s}_\delta\) is a \(P_\xi\)-name for a winning strategy of Generic in \(\mathcal{D}_\delta^\lambda(r_\delta(\xi), Q_\xi)\) such that if \(\langle p^\delta_* : t \in I_\alpha \rangle\) is given by that strategy to Generic at stage \(\alpha\), then \(I_\alpha\) is an ordinal below \(\mu_\alpha\). (And \(\bar{s}_0^\delta\) is a suitable winning strategy of Generic in \(\mathcal{D}_{\delta, \mu}(p(0)), Q_0)\).)

\[ (*)_4 \quad T_b = (T_b, r_\delta) \text{ is a standard } (w_\delta, 1)^\gamma \text{-tree, } |T_b| < \mu_\delta. \]

\[ (*)_5 \quad \bar{p}^\delta_\alpha = \langle p^\delta_* : t \in T_b \rangle \text{ and } \bar{q}^\delta_\alpha = \langle q^\delta_* : t \in T_b \rangle \text{ are standard trees of conditions, } \bar{p}^\delta_* \leq \bar{q}^\delta_* \]
$(*)_6$ For $t \in T_\delta$ we have $(\text{Dom}(p) \cup \bigcup_{\alpha < \delta} \text{Dom}(r_\alpha) \cup w_\delta) \cap \text{rk}_3(t) \subseteq \text{Dom}(p^*_s,t)$, and for each $\xi \in \text{Dom}(p^*_s,t) \setminus w_\delta$:

$$p^*_s,t|_\xi \vDash p^*_\xi \quad \text{if the set } \{r_\alpha(\xi) : \alpha < \delta\} \cup \{p(\xi)\} \text{ has an upper bound in } Q_\xi,$$

then $p^*_s,t(\xi)$ is such an upper bound $\pi$.

$(*)_7$ $\zeta_\delta = \{t \in T_3 : \text{rk}_3(t) = \gamma\}$ and for some enumeration $\{t \in T_3 : \text{rk}_3(t) = \gamma\} = \{t_\zeta : \zeta < \zeta_\delta\}$, for each $\zeta < \zeta_\delta$ we have

$$p^*_s,t_\zeta \leq p^*_\zeta \leq q^*_{s,t_\zeta} \leq q^*_{s,t}.$$

$(*)_8$ If $\xi \in w_\delta$, then $\varepsilon_{\xi,\delta}$ is a $P_\xi$-name for an ordinal below $\mu_\delta$, $\bar{p}_{\delta,\xi}, \bar{q}_{\delta,\xi}$ are $P_\xi$-names for sequences of conditions in $Q_\xi$ of length $\varepsilon_{\xi,\delta}$.

$(*)_9$ If $\xi \in w_{\beta+1} \setminus w_\delta$, $\beta < \lambda$, then

$$\vDash p^*_\xi \quad \langle \varepsilon_{\alpha,\xi}, \bar{p}_{\alpha,\xi}, \bar{q}_{\alpha,\xi} : \beta < \alpha < \lambda \rangle \text{ is a delayed play of } \overrightarrow{\text{rb}}_{\uparrow,\mu}(r_\beta(\xi), Q_\xi) \text{ in which Generic uses } \text{st}_{\xi} \text{.}$$

$(*)_{10}$ If $t \in T_\delta$, $\text{rk}_3(t) = \xi < \gamma$, then the condition $p^*_s,t$ decides the value of $\varepsilon_{\delta,\xi}$, say $p^*_s,t \vDash \varepsilon_{\delta,\xi} = \varepsilon_{t\xi}^\nu$, and $\{(s) : t < s \in T_\delta = \varepsilon_{\delta,\xi}^\nu\}$ and

$$q^*_s,t \vDash p_{\delta,\xi}(s) \leq p^*_s,t(\xi) \quad \text{for } s < \varepsilon_{\delta,\xi}^\nu \text{ and } \bar{q}_{\delta,\xi} = \langle q^*_s,t(\xi) : t < s \in T_\delta \rangle \text{.}$$

$(*)_{11}$ If $t_0, t_1 \in T_3$, $\text{rk}_3(t_0) = \text{rk}_3(t_1)$ and $\xi \in w_3 \cap \text{rk}_3(t_0)$, $t_0|\xi = t_1|\xi$ but $(t_0)_\xi \neq (t_1)_\xi$, then

$$q^*_s,t_0,\xi \vDash p^*_\xi \quad \text{the conditions } q^*_s,t_0(\xi), q^*_s,t_1(\xi) \text{ are incompatible } \pi.$$}

$(*)_{12}$ $\text{Dom}(r_\delta) = \bigcup_{t \in T_3} \text{Dom}(q^*_s,t) \cup \text{Dom}(p)$ and if $t \in T_3$, $\xi \in \text{Dom}(r_\delta) \cap \text{rk}_3(t) \setminus w_\delta$, and $q^*_s,t|_\xi \leq q \in P_\xi$, $r_\delta|_\xi \leq q$, then

$$q \vDash p^*_\xi \quad \text{if the set } \{r_\alpha(\xi) : \alpha < \delta\} \cup \{q^*_s,t(\xi), p(\xi)\} \text{ has an upper bound in } Q_\xi,$$

then $r_\delta(\xi)$ is such an upper bound $\pi$.

To describe the instructions given by $\text{st}$ at stage $\delta < \lambda$ of a play of $\overrightarrow{\text{rb}}_{\uparrow,\mu}(p, P_\gamma)$ let us assume that

$$\langle \zeta_\alpha, \langle p^*_\alpha, q^*_\alpha : \zeta < \zeta_\alpha \rangle : \alpha < \delta \rangle$$

is the result of the play so far and that Generic constructed objects listed in $(\otimes)_\alpha$ (for $\alpha < \delta$) with properties $(\ast)_1 -(\ast)_{12}$.

First, Generic uses her favourite bookkeeping device to determine $w_\delta$ such that the demands in $(\ast)_1$ are satisfied (and that at the end we will have $\bigcup_{\alpha < \lambda} \text{Dom}(r_\alpha) = \bigcup_{\alpha < \lambda} w_\alpha$). Now Generic lets $T_\delta$ be a standard $(w_\delta, 1)^7$-tree such that for each $\xi \in w_3 \cup \{\gamma\}$ we have $\{t \in T_3 : \text{rk}_3(t) = \xi\} = \bigoplus_{s \in w_\delta \cap \xi} \mu_\delta$. Then for $\xi \in w_3$ she chooses $P_\xi$-names $\varepsilon_{\xi,\delta}, \bar{p}_{\delta,\xi}$ such that $\varepsilon_{\xi,\delta}$ is a name for an ordinal below $\mu_\delta$ and $\bar{p}_{\delta,\xi}$ is a name for a sequence of conditions in $Q_\xi$ of length $\varepsilon_{\xi,\delta}$ and

$$\vDash p^*_\xi \quad \text{is the answer to the delayed play } \langle \varepsilon_{\alpha,\xi}, \bar{p}_{\alpha,\xi}, \bar{q}_{\alpha,\xi} : \xi \in w_\alpha \text{ & } \alpha < \delta \rangle \text{ given to Generic by } \text{st}_{\xi} \text{.}$$

She lets $p^*_{s,0} = (p^*_{s,t} : t \in T_3)$ be a tree of conditions defined so that $\text{Dom}(p^*_{s,0}) = (\text{Dom}(p) \cup \bigcup_{\alpha < \delta} \text{Dom}(r_\alpha) \cup w_\delta) \cap \text{rk}_3(t)$ and for each $\xi \in \text{Dom}(p^*_{s,0})$
ζ

(H) Suppose that the two players have arrived at a stage ζ < ζ
Thus Generic has written aside

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Now Generic uses 1.5(3) and then 1.5(2) to choose a standard tree (w_3, 1)^\gamma-tree
T_δ = (T_δ, r_k) and a tree of conditions p_\delta = \{p_{\delta,t} : t ∈ T_δ\} such that

(\ast)_i^{(1)} T_δ ⊆ T_δ and for every t ∈ T_δ such that r_k(t) = ξ ∈ w_δ the condition p_{\delta,t}
decides the value of ξ_δ, ξ_δ = \varepsilon_δ, and

(\ast)_i^{(2)} if t ∈ T_δ, r_k(t) = ξ ∈ w_δ, then \{α < λ : t ∪ \{(ξ, α)\} ∈ T_δ\} = \varepsilon_δ, and

(\ast)_i^{(3)} p_{\delta,i} ≤ p_{\delta,t} for all t ∈ T_δ, and if t_0, t_1 ∈ T_δ, r_k(t_0) = r_k(t_1), ξ ∈ Dom(t_0),
and t_0|ξ = t_1|ξ but (t_0)ξ \neq (t_1)ξ, then

p_{\delta,t_0,ξ} ⊨ p_\delta \text{ "the conditions } p_{\delta,t_0,ξ}, p_{\delta,t_1,ξ} \text{ are incompatible in } Q_\xi " ,

Thus Generic has written aside T_δ, p_\delta, w_3 and (ξ_δ, ξ_δ) : t ∈ w_3). (It should be clear that they satisfy the relevant demands in (\ast)_i^{(1)} - (\ast)_i^{(3)}, (\ast)_i^{(4)} and (\ast)_i^{(5)})
Now she turns to the play of G_{[\mu, \mu]}(p, P_\gamma) and she puts

ζ_δ = |\{t ∈ T_δ : r_k(t) = \gamma\}|

and she also picks an enumeration \{ζ_i : ζ < ζ_δ\} of \{t ∈ T_δ : r_k(t) = \gamma\}. The two players start playing the subgame of level δ of length ζ_δ. During the subgame
Generic constructs partial plays \langle (r_i^{1,1}, s_i^{1,1}) : i ≤ ζ_δ\rangle of \check{\name}{\{\gamma, p_{\delta,\gamma,\xi}\}} (for ζ < ζ_δ) in
which Complete uses the strategy st(\gamma, p_{\gamma,\delta,\gamma,\xi}) and such that

(\ast)_i^{(6)} if ζ_i, ξ < ζ_δ, t < T_δ, t < t_ξ, t < ζ_δ, then r_i^{1,1}[r_k(t) = r_i^{1,1}[r_k(s_i^{1,1})
and s_i^{1,1}[r_k(t) = s_i^{1,1}[r_k(s_i^{1,1})];

(\ast)_i^{(7)} if p_i^{\delta,\gamma} = q_i^{\delta,\gamma} are the conditions played at stage ζ_i of the subgame, then p_i^{\delta,\gamma,\xi,\gamma,\delta,\gamma} ≤
r_i^{1,1} ≤ p_i^{\delta,\gamma} ≤ q_i^{\delta,\gamma} = r_i^{1,1} for all i < ζ_δ.

So suppose that the two players have arrived at a stage ζ < ζ_δ of the subgame and
\langle (r_i^{1,1}, s_i^{1,1}) : i < ζ_δ \rangle has been defined. Generic looks at \langle (r_i^{1,1}, s_i^{1,1}) : i < ζ_δ \rangle
- it is a play of \check{\name}{\{\gamma, p_{\gamma,\delta,\gamma,\xi}\}} in which Complete uses st(\gamma, p_{\gamma,\delta,\gamma,\xi}), so we may find
a condition p_i^{\delta,\gamma} ⊆ P_\gamma stronger than all r_i^{1,1}, s_i^{1,1} for i < ζ (and p_i^{\delta,\gamma} ≥ p_i^{\delta,\gamma,\xi}). She plays
this condition as her move at stage ζ of the subgame and Antigeneric answers with
q_i^{\delta,\gamma} ≥ p_i^{\delta,\gamma}. Generic lets r_i^{\delta,\gamma} = q_i^{\delta,\gamma} and she defines r_i^{\delta,\gamma} for ζ < ζ_δ, ζ \neq ζ_δ, as follows. Let
i, t ∈ T_δ be such that t < t_ξ, t < t_ξ and r_k(t) is the largest possible. Generic declares that

\text{Dom}(r_i^{\delta,\gamma}) = \text{Dom}(r_i^{\delta,\gamma}) \cap r_k(t) \cup \bigcup_{i < ζ_δ} \text{Dom}(s_i^{1,1}) \cup \text{Dom}(p_{\delta,t_ξ})

and r_i^{\delta,\gamma}[r_k(t) = r_i^{\delta,\gamma}[r_k(s_i^{1,1})], and for ε ∈ \text{Dom}(r_i^{\delta,\gamma}) \cap r_k(t) she lets r_i^{\delta,\gamma}(ε) be the <\gamma-first
\text{P}_\gamma-name for a member of Q_ε such that

r_i^{\delta,\gamma}(ε) \in \text{P}_\gamma \text{ " } r_i^{\delta,\gamma}(ε) \text{ is an upper bound to } \{p_{\delta,t_ξ}(ε) \cup \{s_i^{1,1}(ε) : i < ζ_δ \} "

}
(remember 1.3(iv)). Finally, $s^\xi_\alpha$ (for $\xi < \zeta$) is defined as the condition given to Complete by $\mathcal{ST}(\gamma, \nu^\alpha_{\xi, \tau})$ in answer to $\langle (r^\xi_\alpha, s^\xi_\alpha) : i < \zeta \rangle \langle r^\xi_\alpha \rangle$. It follows from 1.3(ii) that $(*)_n^{\alpha_0}$ is still satisfied for the $s^\xi_\alpha$.

After the subgame is completed and both $p^\xi_{\zeta, 0}, q^\xi_{\zeta, 0}$ and $\langle (r^\xi_\alpha, s^\xi_\alpha) : i < \zeta \rangle : \xi < \zeta\rangle$ have been determined, Generic chooses $r^0_{\zeta, \delta}$ as any upper bound to $\langle s^0_i : i < \zeta \rangle$ and then defines $r^\xi_{\zeta, \delta}$ for $\xi \in \zeta_\delta \setminus \zeta$ like $r^\xi_{\zeta, 0}$ for $\xi \notin \zeta$ above. Also $s^\xi_{\zeta, \delta}$ (for $\xi < \zeta_\delta$) are chosen like earlier (as results of applying $\mathcal{ST}(\gamma, \nu^\alpha_{\zeta, \tau})$). Finally, Generic picks a standard tree of conditions $q^\delta_{\zeta, t} \in \langle q^\delta_{\zeta, t} : t \in T_\delta \rangle$ such that $(\forall \zeta < \zeta_\delta)(q^\delta_{\zeta, t} = s^\xi_{\zeta, \delta})$.

(Note that $(*)_5$, $(*)_7$ hold.)

Now Generic defines $r^\delta_{\zeta, \delta}, r^\delta_{\zeta} \in \mathcal{P}_\gamma$ so that

$$\text{Dom}(r^\delta_{\zeta}) = \text{Dom}(r^\delta_{\zeta}) = \bigcup_{t \in T_\delta} \text{Dom}(q^\delta_{\zeta, t}) \cup \text{Dom}(p)$$

and

$(*)_0^{\alpha_0}$ if $\xi \in \text{Dom}(r^\delta_{\zeta}) \setminus w_\delta$, then:

$$r^\delta_{\zeta, \xi}(\xi) = \langle \mathcal{P}_\zeta \text{-name for an element of } Q_{\xi} \rangle$$

$r^\delta_{\zeta, \xi}(\xi)$ is an upper bound of $\{r_{\alpha}(\xi) : \alpha < \delta\} \cup \{\nu(\xi)\}$ and if $t \in T_\delta$, $\nu(t) > \xi$, and $q^\delta_{\zeta, t}\xi \in \Gamma_{\nu(t)}$ and the set

$\{r_{\alpha}(\xi) : \alpha < \delta\} \cup \{q^\delta_{\zeta, t}\xi, p(\xi)\}$ has an upper bound in $Q_{\xi}$,

then $r^\delta_{\zeta, \xi}(\xi)$ is such an upper bound

and $r^\delta_{\zeta, \delta}(\xi)$ is the $<^\delta_{\alpha_0}$-first $\mathcal{P}_\delta$-name for an element of $Q_{\xi}$ such that

$r^\delta_{\zeta, \delta}(\xi)$ is given to Complete by $\mathcal{ST}^{\alpha_1}_\delta$ as the answer to $\langle (r^\alpha_{\alpha}(\xi), r_{\alpha}(\xi) : \alpha < \delta) \rangle$.

$(*)_0^{\alpha_0}$ if $\xi \in w_{\alpha+1}, \alpha < \delta$, then $r^\delta_{\zeta, \delta}(\xi) = r^\delta_{\zeta, \delta}(\xi)$. (Note that by straightforward induction on $\xi \in \text{Dom}(r^\delta_{\zeta, \delta})$ one easily applies $(*)_3$ from previous stages to show that $r^\delta_{\zeta, \delta}, r^\delta_{\zeta, \delta}$ are well defined and $r^\delta_{\zeta, \delta}, r^\delta_{\zeta, \delta} \geq r_{\alpha}(p)$ for $\alpha < \delta$. Remember also $(*)_1^{\alpha_1}$ and/or $(*)_1^{\alpha_1}$.) If $\delta = 0$ we also stipulate $r^\delta_{0}(0) = r^0_{0}(0) = p(0)$.

Finally, for each $\xi \in w_\delta$, Generic chooses a $\mathcal{P}_\delta$-name $\check{q}^\xi_{\delta, \zeta}$ for a sequence of conditions in $Q_{\xi}$ of length $\check{e}_{\delta, \zeta}$ such that

$$\langle \forall e < \check{e}_{\delta, \zeta} \rangle \langle \check{p}^\xi_{\delta, e}(e) \leq \check{q}^\xi_{\delta, \zeta}(e) \rangle$$

$$\langle \forall t \in T_\delta, \text{rk}_t(t) > \xi, \text{ and } q^\delta_{\zeta, t}\xi \in \Gamma_{\text{rk}_t(t)} \rangle \text{ then } \check{q}^\xi_{\delta, \zeta}(t)_{\nu(t)} = q^\delta_{\zeta, t}\xi(\nu(t)) \text{.}$$

Generic also picks $w_{\delta+1}$ by the bookkeeping device mentioned at the beginning and for $\xi \in w_{\delta+1} \setminus w_\delta$ she fixes $e_{\xi}$ as in $(*)_3$.

This completes the description of the side objects constructed by Generic at stage $\delta$. Verification that they satisfy our demands $(*)_1^{\delta} - (*)_1^{\delta_2}$ is straightforward, and thus the description of the strategy $\mathcal{ST}$ is complete.

We are going to argue now that $\mathcal{ST}$ is a winning strategy for Generic. To this end suppose that $\langle \zeta_{\delta}, \langle p^\delta_{\xi}, q^\delta_{\xi} : \xi < \zeta_{\delta} \rangle : \delta < \lambda \rangle$ is the result of a play of $\mathcal{O}_{\alpha_1, \alpha_1}(p, \mathcal{P}_{\gamma})$ in which Generic followed $\mathcal{ST}$ and constructed side objects listed in $(\odot)_\delta$ (for $\delta < \lambda$) so that $(*)_1^{\delta} - (*)_1^{\delta_2}$ hold.
We define a condition \( r \in \mathbb{P}_\gamma \) as follows. Let \( \text{Dom}(r) = \bigcup_{\delta < \lambda} \text{Dom}(r_\delta) \) and for \( \xi \in \text{Dom}(r) \) let \( r(\xi) \) be a \( \mathbb{P}_\xi \)-name for a condition in \( Q_\xi \) such that if \( \xi \in w_{\alpha+1} \setminus w_\alpha \), \( \alpha < \lambda \) (or \( \xi = 0 = \alpha \)), then
\[
\not\vdash_{\mathbb{P}_\xi} " r(\xi) \geq r_\alpha(\xi) \text{ and } r(\xi) \not\in Q_\xi \{ \delta < \lambda : (\exists \varepsilon \leq \xi) (\tilde{q}_{\delta,\varepsilon}(\varepsilon) \in \Gamma_{Q_\xi}) \} \}
\]
Clearly \( r \) is well defined (remember \((*)_9\)) and \((\forall \delta < \lambda)(r_\delta \leq r) \) and \( r \geq p \). For each \( \xi \in \text{Dom}(r) \) choose a sequence \( \langle A^\xi_i : i < \lambda \rangle \) of \( \mathbb{P}_{\xi+1} \)-names for elements of \( U \cap V \) such that
\[
(*)_{17} \quad r(\xi+1) \not\vdash_{\mathbb{P}_{\xi+1}} " (\forall \delta \in \Delta) (\exists e < \varepsilon_\delta, \xi) (\tilde{q}_{\delta,\varepsilon}(\varepsilon) \in \Gamma_{Q_\xi}) \".
\]

Claim 2.5.1. For each limit ordinal \( \delta < \lambda \),
\[
r \not\vdash_{\mathbb{P}_\gamma} " (\forall \xi \in w_\delta) (\exists i < \lambda : A^\xi_i) \} \implies (\exists t \in T_\delta) (\forall k \in \Gamma_{P_\gamma} r_k(t) = \gamma' \text{ and } q^\xi_i(t) \in \Gamma_{P_\gamma} ) \".
\]

Proof of the Claim. Suppose that \( r' \geq r \) and a limit ordinal \( \delta < \lambda \) are such that
\[
(*)_{18} \quad r' \not\vdash_{\mathbb{P}_\gamma} " (\forall \xi \in w_\delta) (\exists i < \lambda : A^\xi_i) \} \}. \]

We are going to show that there is \( t \in T_\delta \) such that \( r_k(t) = \gamma \) and the conditions \( q^\xi_i \) and \( r' \) are compatible (and then the claim will follow). To this end let \( \langle \varepsilon_\alpha : \alpha \leq \alpha^* \rangle = w_\delta \cup \{ \gamma \} \) be the increasing enumeration. By induction on \( \alpha \leq \alpha^* \) we will choose conditions \( r^\alpha_\gamma, r^\alpha^* \in \mathbb{P}_{\varepsilon_\alpha} \) and \( t = (t)_{\varepsilon_\alpha} : \alpha < \alpha^* \in T_\delta \) such that letting \( t^\alpha_\delta = ((t)_{\varepsilon_\alpha} : \beta < \alpha) \in T_\delta \) we have
\[
(*)_{19} \quad q^\xi_i \xi^\delta \leq r^\alpha_\gamma \text{ and } r' \not\vdash_{\mathbb{P}_\gamma} " r_{\varepsilon_\alpha} \leq r^\alpha_\gamma \".
\]
\[
(*)_{20} \quad r''_{\varepsilon_\alpha} \not\vdash_{\mathbb{P}_\gamma} " r_{\varepsilon_\alpha} \leq r^\alpha_\gamma \".
\]
\[
(*)_{21} \quad (r''_{\varepsilon_\alpha} \not\vdash_{\mathbb{P}_\gamma} " r_{\varepsilon_\alpha} \leq r^\alpha_\gamma \") \text{ is a partial legal play of } \Gamma_{P_\gamma}, \gamma', r' \text{ in which Complete uses her winning strategy } st(\gamma, r') \text{.}
\]

Suppose that \( \alpha \leq \alpha^* \) is a limit ordinal and we have already defined \( t^\alpha_\delta = ((t)_{\varepsilon_\alpha} : \beta < \alpha) \) and \( (r^\alpha_{\varepsilon_\alpha}, r^\alpha^* : \beta < \alpha) \). Let \( \xi = \sup(\varepsilon_{\beta} : \beta < \alpha) \). It follows from \((*)_{20}\) (for \( \beta < \alpha \)) that we may find a condition \( s \in \mathbb{P}_\xi \) stronger than all \( r^\alpha_{\varepsilon_\alpha} \) (for \( \beta < \alpha \)). Let \( r^\alpha_\gamma \in \mathbb{P}_{\varepsilon_\alpha} \) be such that \( r^\alpha_\gamma | \xi \leq s \) and \( \text{ for } \varepsilon_{\gamma} : (r^\alpha_{\varepsilon_\gamma} : \varepsilon_{\gamma} < \xi) = r' | [\xi, \varepsilon_{\alpha}] \). It follows from \((*)_{19}\) that \( q^\xi_i | \xi \leq s \) and \( r' | \xi \leq s \) and \( r^\alpha_\gamma | \xi \). Note also that \( (\forall \beta < \alpha)(r^\alpha_{\varepsilon_\gamma} \leq s | e_{\beta} = r^\alpha_{\varepsilon_\gamma} | e_{\beta}, \xi) \). Now by induction on \( \xi < \varepsilon_{\alpha} \) we show that \( q^\xi_i | \xi \leq r^\alpha_\gamma | \xi \) and \( r' | \xi \leq r^\alpha_\gamma | \xi \). For \( \xi \leq \xi \) we are already done, so assume that \( \xi \notin [\xi, \varepsilon_{\alpha}] \) and we have shown \( q^\xi_i | \xi \leq r^\alpha_\gamma | \xi \) and \( r' | \xi \leq r^\alpha_\gamma | \xi \). It follows from \((*)_{10} + (*)_{13}\) that \( r^\alpha_\gamma | \xi \vdash_{\mathbb{P}_\gamma} " (\forall i < \delta)(r_i(\varepsilon) \leq p^\delta_{\varepsilon_\alpha} (\varepsilon)) \) and therefore we may use \((*)_{12}\) to conclude that
\[
r^\alpha_\gamma | \xi \vdash_{\mathbb{P}_\gamma} " q^\xi_i | \xi \leq r(\varepsilon) | \xi \leq r'(\xi) = r^\alpha_\gamma | \xi \). \]

The limit stages are trivial and we see that \((*)_{10}\) and (a part of) \((*)_{20}\) hold. Finally we let \( r^\alpha_\beta \in \mathbb{P}_{\varepsilon_\alpha} \) be the condition given to Complete by \( st(\gamma, r') \) as the response to \( (r''_{\varepsilon_\alpha} | \xi, \gamma), r'_{\varepsilon_\alpha} | \xi, \gamma) : \beta < \alpha \) \( r^\alpha_\beta \). Now suppose that \( \alpha = \beta + 1 \leq \alpha^* \) and we have already defined \( r^\alpha_\beta, r^\beta_{\varepsilon_\alpha}, \beta = \alpha \), \( r^\delta_{\varepsilon_\alpha} | \xi : (\exists e < \varepsilon_{\delta, \varepsilon_\alpha}) (q^\delta_{\varepsilon_\alpha} | \xi) \in \Gamma_{Q_\xi}) \). Then
\[
r^\beta_{\varepsilon_\alpha} \not\vdash_{\mathbb{P}_\gamma} " (\exists \varepsilon_{\beta} : (\exists e < \varepsilon_{\delta, \varepsilon_\alpha}) (\tilde{q}_{\delta, \varepsilon}(\varepsilon) \in \Gamma_{Q_\xi}) \".
\]
Therefore we may choose \( \varepsilon = (t)_{\gamma_3} < \varepsilon_3^{\beta_3} \) (thus defining \( t_0^\mu \)) and a condition 
\( s \in \mathbb{F}_{\varepsilon_\beta + 1} \) such that 
\( s | \varepsilon_\beta \geq r^\beta_{\varepsilon_\beta} \geq q^\delta_\gamma \) and 
\( s | \varepsilon_\beta \models " s | \varepsilon_\beta \geq r^\beta_{\varepsilon_\beta} \geq q^\delta_\gamma (\varepsilon_\beta) "." 

We let \( r^*_\alpha \in \mathbb{F}_{\varepsilon_\alpha} \) be such that \( r^*_\alpha | (\varepsilon_\beta + 1) = s \) and \( r^*_\alpha | (\varepsilon_\alpha) = r^\gamma | (\varepsilon_\alpha) \). Exactly like in the limit case we argue that \( (\ast)^{\gamma_3}_\gamma \) and (a part of) \( (\ast)^{\gamma_3}_\delta \) hold and then in the same manner as there we define \( r^*_\alpha \). 

Finally note that \( t \in T_{\delta} \), \( \text{rk}_{\delta}(t) = \gamma \), and the condition \( r^*_\alpha \) witnesses that \( r^\gamma \) and \( q^\delta_\gamma \) are compatible. \( \square \)

Now note that 
\[ \mathcal{P}_\gamma \models \{ \delta < \lambda : (\forall \xi \in w_3)(\delta \in \triangle \mathcal{A}_\xi) \} \in \mathcal{U}^\gamma \text{ } \]
and hence by 2.5.1 we have 
\[ r \models \mathcal{P}_\gamma \models \{ \delta < \lambda : (\exists t \in T_\delta)(\text{rk}_{\delta}(t) = \gamma \& q^\delta_\gamma \in \Gamma_{\mathcal{P}_\gamma}) \} \in \mathcal{U}^\gamma \text{ } \]
Therefore, by \( (\ast)_\gamma \), 
\[ r \models \mathcal{P}_\gamma \models \{ \delta < \lambda : (\exists \zeta < \zeta_\delta)(q^\delta_\zeta \in \Gamma_{\mathcal{P}_\gamma}) \} \in \mathcal{U}^\gamma \text{ } \]
and the proof of the theorem is complete. \( \square \)

**Remark 2.6.** The reason for the weaker “\( \mathbb{b} \)-bounding” in the conclusion of 2.5 (and not “\( \mathbb{B} \)-bounding”) is that in our description of the strategy \( st \), we would have to make sure that the conditions played by Antigeneric form a tree of conditions. Playing a subgame and keeping the demands of \( (\ast)_1 \) are a convenient way to deal with this issue.

Similar work and arguments may be carried out for \( \mathbb{A}/\mathbb{a} \)-bounding. However, in a subsequent paper [7] we find out that getting reasonably \( \mathbb{a} \)-bounding for the limit of the iteration is not sufficient for the applications there. With these applications in mind we introduce a stronger property which more precisely captures what can be claimed on iterations of reasonably \( \mathbb{A} \)-bounding forcing notions.

**Definition 2.7.** Let \( \mathcal{Q} = (\mathbb{P}_\zeta, \mathbb{Q}_\xi : \xi < \gamma) \) be a \( \lambda \)-support iteration.

1. For a condition \( p \in \mathbb{P}_\gamma = \text{lim}(\mathcal{Q}) \) we define a game \( \mathcal{G}^{\text{tree}}_{\mu}(p, \mathcal{Q}) \) between two players, Generic and Antigeneric, as follows. A play of \( \mathcal{G}^{\text{tree}}_{\mu}(p, \mathcal{Q}) \) lasts \( \lambda \) steps and in the course of the play a sequence \( \langle T_\alpha, p, \tilde{q}^\alpha : \alpha < \lambda \rangle \) is constructed. Suppose that the players have arrived to a stage \( \alpha < \lambda \) of the game. Now, 
   \( (\mathbb{N})_\alpha \) first Generic chooses a standard \( (w, 1)^\gamma \)-tree \( T_\alpha \) such that \( |T_\alpha| < \mu_\alpha \) and a tree of conditions \( \bar{p}^\alpha = \{ p^\alpha_t : t \in T_\alpha \} \subseteq \mathbb{P}_\gamma \), 
   \( (\mathbb{Z})_\alpha \) then Antigeneric answers by picking a tree of conditions \( \bar{q}^\alpha = \{ q^\alpha_t : t \in T_\alpha \} \subseteq \mathbb{P}_\gamma \) such that \( \bar{p}^\alpha \leq \bar{q}^\alpha \). At the end, Generic wins the play \( \langle T_\alpha, \bar{p}^\alpha, \bar{q}^\alpha : \alpha < \lambda \rangle \) of \( \mathcal{G}^{\text{tree}}_{\mu}(p, \mathcal{Q}) \) if and only if 
   \( (\ast)_\mathbb{A} \) there is a condition \( p^* \in \mathbb{P}_\gamma \) stronger than \( p \) and such that 
   \( p^* \models \mathcal{P}_\gamma \models (\forall \alpha < \lambda)((\exists t \in T_\alpha)(\text{rk}_{\alpha}(t) = \gamma \& q^\alpha_t \in \Gamma_{\mathcal{P}_\gamma}) \) " 

2. We say that \( \mathbb{P}_\gamma = \text{lim}(\mathcal{Q}) \) is reasonably* \( \mathbb{A}(\mathcal{Q}) \)-bounding over \( \mu \) if Generic has a winning strategy in the game \( \mathcal{G}^{\text{tree}}_{\mu}(p, \mathcal{Q}) \) for every \( p \in \mathbb{P}_\gamma \).
Theorem 2.8. Assume that $\lambda, \mu$ are as in 0.1 and $\mathcal{Q} = \{\mathcal{P}_\xi, \mathcal{Q}_\xi : \xi < \gamma\}$ is a $\lambda$–support iteration such that for every $\xi < \gamma$,

$$\models_{\mathcal{P}_\xi} \ " \mathcal{Q}_\xi \ is \ reasonably \ A$-$bounding \ over \ \mu \ " \). Then $\mathbb{P}_\gamma = \text{lim}(\mathcal{Q})$ is reasonably* $A(\mathcal{Q})$–bounding over $\mu$.

Proof. This is a variation on the proof of Theorem 2.5, but let us sketch the proof of our present version. For each $\xi < \gamma$ pick a $\mathcal{P}_\xi$–name $\mathbf{st}_\xi^0$ such that

$$\models_{\mathcal{P}_\xi} \ " \mathbf{st}_\xi^0 \ is \ a \ winning \ strategy \ for \ Complete \ in \ \mathcal{O}_\mu^{\mathcal{Q}_\xi}(\mathcal{P}_\xi, \mathcal{Q}_\xi) \ such \ that \ if \ Incomplete \ plays \ \emptyset \mathcal{Q}_\xi \ then \ Complete \ answers \ with \ \emptyset \mathbb{Q}_\xi \ as \ well \ " \). Let $p \in \mathbb{P}_\gamma$. We are going to describe a strategy $\mathbf{st}$ for Generic in the game $\mathcal{O}_\mu^{\text{tree}}(p, \mathcal{Q})$. In the course of the play, at a stage $\delta < \lambda$, Generic is instructed to construct aside

$$(\circ)_{\delta} \ r_\delta^-, r_\delta, w_\delta, \langle \xi_{\delta, \xi}, \bar{p}_{\delta, \xi}, \bar{g}_{\delta, \xi} : \xi \in w_\delta\rangle, \ and \ \mathbf{st}_\xi \ for \ \xi \in w_{\delta+1} \ \setminus \ w_\delta.$$ These objects are to be chosen so that if $\langle T_\delta, \bar{p}_\delta, \bar{q}_\delta : \delta < \lambda\rangle$ is a play of $\mathcal{O}_\mu^{\text{tree}}(p, \mathcal{Q})$ in which Generic follows $\mathbf{st}$, and the additional objects constructed at stage $\delta < \lambda$ are listed in $(\circ)_{\delta}$, then the following conditions are satisfied (for each $\delta < \lambda$).

\begin{enumerate}
\item For each $\alpha < \delta < \lambda$ we have $(\forall \xi \in w_{\alpha+1})(r_\alpha(\xi) = r_\delta(\xi))$ and $p \leq r_\alpha^- \leq r_\delta^- \leq r_\delta$.
\item For each $\xi \in \gamma \ \setminus \ w_\delta$, then $r_\delta^- | \xi \models \ " \ the \ sequence \ \langle r_\delta^-(\xi), r_\alpha(\xi) : \alpha \leq \delta \rangle \ is \ a \ legal \ partial \ play \ of \ \mathcal{O}_\mu^{\mathcal{Q}_\xi}(\mathcal{P}_\xi, \emptyset \mathcal{Q}_\xi) \ in \ which \ Complete \ follows \ \mathbf{st}_\xi^0 \ " \) and if $\xi \in w_{\delta+1} \ \setminus \ w_\delta$, then $\mathbf{st}_\xi$ is a $\mathcal{P}_\xi$–name for a winning strategy of Generic in $\mathcal{O}_\mu^{\text{tree}}(r_\delta(\xi), \mathcal{Q}_\xi)$ such that if $\langle p_\delta^\alpha : t \in I_\alpha\rangle$ is given by that strategy to Generic at stage $\alpha$, then $I_\alpha$ is an ordinal below $\mu_\alpha$. Also $\mathbf{st}_0$ is a winning strategy of Generic in $\mathcal{O}_\mu^{\text{tree}}(p(0), \mathcal{Q}_0)$.
\item $T_\delta = \langle T_\delta, r_\delta \rangle$ is a standard $(w_\delta, 1)^\gamma$–tree, $|T_\delta| < \mu_\delta$.
\item $\bar{p}_\delta = \langle p_\delta^\alpha : t \in T_\delta\rangle$ and $\bar{q}_\delta = \langle q_\delta^\alpha : t \in T_\delta\rangle$ are standard trees of conditions in $\mathcal{Q}$, $p_\delta^- \leq q_\delta^\alpha$ and $T_\delta, \bar{p}_\delta$ and $\bar{q}_\delta^\alpha$ are the innings of the two players at stage $\delta$.
\item For each $t \in T_\delta$ we have $(\text{Dom}(p) \cup \bigcup_{\alpha < \delta} \text{Dom}(r_\alpha(\cup w_\delta)) \cap \text{rk}_\delta(t) \subseteq \text{Dom}(p_\delta^t))$ and
\begin{itemize}
\item for each $\xi \in \text{Dom}(p_\delta^t) \ \setminus \ w_\delta$:
\item $p_\delta^t | \xi \models_{\mathcal{P}_\xi} \ " \ the \ set \ \{r_\alpha(\xi) : \alpha < \delta\} \cup \{p(\xi)\} \ has \ an \ upper \ bound \ in \ \mathcal{Q}_\xi, \ then \ p_\delta^t(\xi) \ is \ such \ an \ upper \ bound \ " \).\end{itemize}
\item If $\xi \in w_\delta$, then $\xi_{\delta, \xi}$ is a $\mathcal{P}_\xi$–name for an ordinal below $\mu_\delta$, $\bar{p}_{\delta, \xi}, \bar{g}_{\delta, \xi}$ are $\mathcal{P}_\xi$–names for $\xi_{\delta, \xi}$–sequences of conditions in $\mathcal{Q}_\xi$.
\item If $\xi \in w_{\beta+1} \ \setminus \ w_\delta$, $\beta < \lambda$, then
\begin{itemize}
\item $\models_{\mathcal{P}_\xi} \ " \ (\xi_{\beta, \xi}, \bar{p}_{\alpha, \xi}, \bar{g}_{\alpha, \xi} : \beta < \alpha < \lambda) \ is \ a \ delayed \ play \ of \ \mathcal{O}_\mu^{\text{tree}}(r_\delta(\xi), \mathcal{Q}_\xi) \ in \ which \ Generic \ uses \ \mathbf{st}_\xi \ " \).\end{itemize}
\item If $t \in T_\delta$, $\text{rk}_\delta(t) = \xi < \gamma$, then the condition $p_\delta^t$ decides the value of $\xi_{\delta, \xi}$, say $p_\delta^t | " \xi_{\delta, \xi} = e_{\delta, \xi}^\alpha "$ and $\{ (\alpha) : t < s \in T_\delta \} = e_{\delta, \xi}^t$ and
\begin{itemize}
\item $p_\delta^\alpha | " \bar{p}_{\delta, \xi}(e) \leq p_\delta^{\alpha-\xi}(\xi)$ and $\bar{g}_{\delta, \xi}(e) = q_\delta^{\alpha-\xi}(\xi)$ for $e < e_{\delta, \xi}^t$ \).\end{itemize}
Problem 2.9.

being reasonably stronger than listed in (⊗) Definition 3.1 why, we should recall the following definition. The detailed description of the strategy \( \text{st} \) closely follows the description of the strategy \( \text{st} \) in the proof of 2.5 (after the formulation of (++)_12 there). To argue that \( \text{st} \) is a winning strategy for Generic, suppose that \( \langle T_\delta, p^\delta, q^\delta : \delta < \lambda \rangle \) is the result of a play of \( \bigcirc_{\mu, \text{aA}}^\mu(p, Q) \) in which Generic followed \( \text{st} \) and constructed objects listed in \( (\otimes)_3 \) (for \( \delta < \lambda \)) so that (++)_1(++)_1 hold. Define a condition \( r \in P_\xi \) as follows. Let \( \text{Dom}(r) = \bigcup_{t \in T_\delta} \text{Dom}(q^t) \cup \text{Dom}(p) \) and if \( t \in T_\delta, \xi \in \text{Dom}(r_\xi) \cap \text{rk}(t) \setminus w_\delta, \) then

\[
p^t_\delta \upharpoonright p_\xi \models q^t_\xi \text{ " the conditions } p^t_\delta(\xi), p^t_\xi(\xi) \text{ are incompatible "}.
\]

\[
(*)_{11} \text{ Dom}(r_\xi) = \bigcup_{t \in T_\delta} \text{Dom}(q^t_\xi) \cup \text{Dom}(p_\xi) \text{ and if } t \in T_\delta, \xi \in \text{Dom}(r_\xi) \cap \text{rk}(t) \setminus w_\delta,
\]

and \( q^t_\xi \upharpoonright \xi \leq q \in P_\xi, r_\xi \xi \leq q, \) then

\[
q \upharpoonright P_\xi \text{ " if the set } \{r_\alpha(\xi) : \alpha < \delta \} \cup \{q^t_\xi(\xi), p_\xi \} \text{ has an upper bound in } Q_\xi, \text{ then } r^t_\xi(\xi) \text{ is such an upper bound "}.
\]

The detailed description of the strategy \( st \) closely follows the description of the strategy \( st \) in the proof of 2.5 (after the formulation of (++)_1(++)_12 there). To argue that \( st \) is a winning strategy for Generic, suppose that \( \langle T_\delta, p^\delta, q^\delta : \delta < \lambda \rangle \) is the result of a play of \( \bigcirc_{\mu, \text{aA}}^\mu(p, Q) \) in which Generic followed \( st \) and constructed objects listed in (⊗)_3 (for \( \delta < \lambda \)) so that (++)_1(++)_1 hold. Define a condition \( r \in P_\xi \) as follows. Let \( \text{Dom}(r) = \bigcup_{t \in T_\delta} \text{Dom}(r_\xi) \) and for \( \xi \in \text{Dom}(r) \) let \( r(\xi) \) be a \( P_\xi \)-name for a condition in \( Q_\xi \) such that if \( \xi \in w_{\alpha+1} \setminus w_\alpha, \alpha < \lambda \) (or \( \xi = 0 = \alpha \)), then

\[
\models r(\xi) \geq r_\alpha(\xi) \text{ and } r(\xi) \upharpoonright Q_\xi \text{ (} \forall \delta < \lambda (\exists \varepsilon < \varepsilon_s, \xi) (q^\delta_\varepsilon \varepsilon_s, \xi(e) \in \Gamma_\varepsilon) \text{ "}.}
\]

Clearly \( r \) is well defined (remember (++)_9) and (\( \forall \delta < \lambda(r_\delta \leq r) \) and \( r \geq p \). An argument following the lines of the proof of Claim 2.5.1 shows that for each \( \delta < \lambda \) the family \( \{q^\delta_\varepsilon : \varepsilon \in T, \varepsilon \leq \varepsilon_\delta \} \) is pre-dense above \( r \). \( \square \)

We do not know if iterations of reasonably \( x \)-bounding forcing notions are reasonably \( x \)-bounding or even \( \lambda \)-proper (for \( x \in \{a, b\} \)). In a subsequent paper [7] we introduce a property called nice double \( x \)-bounding and we show that it is preserved in \( \lambda \)-support iterations (see [7, 2.9, 2.10]). This property is in some cases stronger than being reasonably \( x \)-bounding, but it puts some restrictions on \( \mu \). In this context the following problem is very natural.

**Problem 2.9.**

1. Do we have a result parallel to 2.5 for reasonably \( C \)-bounding forcings?
2. Let \( x \in \{a, b\} \). Are \( \lambda \)-support iterations of reasonably \( x \)-bounding forcing notions still reasonably \( x \)-bounding? At least \( \lambda \)-proper?

3. Consequences of reasonable ABC

Let us note that Theorem 2.8 improves [5, Theorem A.2.4]. Before we explain why, we should recall the following definition.

**Definition 3.1** ([5, Def. A.2.1]). Let \( P \) be a forcing notion.

1. A complete \( \lambda \)-tree of height \( \alpha < \lambda \) is a set of sequences \( s \subseteq \leq^\alpha \lambda \) such that
   - \( s \) has the \( \prec \)-smallest element denoted \( \text{root}(s) \),
   - \( s \) is closed under initial segments longer than \( \text{lh} \text{root}(s) \), and
   - the union of any \( \prec \)-increasing sequence of members of \( s \) is in \( s \), and
   - \( (\forall \eta \in s) (\exists \nu \in s) (\eta < \nu \& \text{lh}(\nu) = \alpha) \).
2. For a condition \( p \in P \) and an ordinal \( i_0 < \lambda \) we define a game \( \bigcirc_{\mu, \text{aA}}^\mu(i_0, p, P) \) of two players, Generic and Antigeneric. A play lasts at most \( \lambda \) moves indexed by ordinals from the interval \( [i_0, \lambda] \), and during it the players construct a sequence \( \langle (s_i, p^i, q^i) : i_0 \leq i < \lambda \rangle \) as follows. At stage \( i \) of the
play (where \( i_0 \leq i < \lambda \)), first Generic chooses \( s_i \subseteq \leq \lambda \) and a system \( p^i = (p^i_\eta : \eta \in s_i \cap \lambda) \) such that

(\( \alpha \)) \( s_i \) is a complete \( \lambda \)-tree of height \( i + 1 \) and \( \text{lh}(\text{root}(s_i)) = i_0 \),

(\( \beta \)) for all \( j \) such that \( i_0 \leq j < i \) we have \( s_j = s_i \cap \lambda \),

(\( \gamma \)) \( p^i_\eta \in \mathbb{P} \) for all \( \eta \in s_i \cap \lambda \), and

(\( \delta \)) if \( i_0 \leq j < i \), \( \nu < \lambda \) and \( \nu < \eta < \lambda \), then \( q^i_\nu \leq p^i_\eta \) and

\[ p \leq p^i_\eta. \]

(\( \varepsilon \)) \( |s_i \cap \lambda| < \mu_i \).

Then Antigeneric answers choosing a system \( q^i = (q^i_\eta : \eta \in s_i \cap \lambda) \) of conditions in \( \mathbb{P} \) such that \( p^i_\eta \leq q^i_\eta \) for each \( \eta \in s_i \cap \lambda \).

Generic wins a play if she always has legal moves (so the play really lasts \( \lambda \) steps) and there are a condition \( q \geq p \) and a \( \mathbb{P} \)-name \( \rho \) such that

(\( \star \)) \( q \Vdash \rho \in \lambda \) \& \( \forall \eta \in [i_0, \lambda) (p^i(i+1) \in s_i \land q^i_{\rho^i(i+1)} \in \Gamma_{\rho^i}) \).

(3) We say that \( \mathbb{P} \) has the strong \( \mu \)-Sacks property whenever

(a) \( \mathbb{P} \) is strategically \((\leq \lambda)\)-complete, and

(b) Generic has a winning strategy in the game \( \mathcal{D}_\mu^{\text{Sacks}}(i_0, p, \mathbb{P}) \) for any \( i_0 < \lambda \) and \( p \in \mathbb{P} \).

The following proposition explains why 2.8 is stronger than [5, Theorem A.2.4].

**Proposition 3.2.** Assume that \( \lambda, \mu \) are as in Context 0.1 and that additionally \( (\forall i < j < \lambda)(\mu_i \leq \mu_j) \). Let \( \mathcal{Q} \) be a forcing notion. Then \( \mathcal{Q} \) is reasonably \( \lambda \)-bounding over \( \bar{\mu} \) if and only if

\[ \mathcal{Q} \text{ has the strong } \mu \text{-Sacks property.} \]

**Proof.** Suppose that \( \mathcal{Q} \) is reasonably \( \lambda \)-bounding over \( \bar{\mu} \). Since the sequence \( \bar{\mu} \) is non-decreasing, it is enough to show that Generic has a winning strategy in \( \mathcal{D}_\mu^{\text{Sacks}}(0, p, \mathcal{Q}) \) for each \( p \in \mathcal{Q} \) (as then almost the same strategy will be good in \( \mathcal{D}_\mu^{\text{Sacks}}(i, p, \mathcal{Q}) \) for any \( i < \lambda \)).

Let \( p \in \mathcal{Q} \). We are going to define a strategy \( \text{st} \) for Generic in the game \( \mathcal{D}_\mu^{\text{Sacks}}(0, p, \mathcal{Q}) \). To describe it, let us fix a winning strategy \( \text{st}_0 \) of Complete in \( \mathcal{D}_\mu^{\text{rcA}}(0, p, \mathcal{Q}) \) and a winning strategy \( \text{st}_1 \) of Generic in \( \mathcal{D}_\mu^{\text{rcA}}(p, \mathcal{Q}) \). Now, at a stage \( \delta < \lambda \) of the play the strategy \( \text{st} \) will tell Generic to write aside

(\( \Xi \)) \( I_\delta \) and \( \langle r^0_\delta, r^1_\delta : t \in I_\delta \rangle \) and \( \langle r^\delta_\eta : \eta \in s_\delta \cap \lambda \rangle \)

so that if \( \langle s_\delta, p^\delta, q^\delta : \delta < \lambda \rangle \) is a play of \( \mathcal{D}_\mu^{\text{Sacks}}(0, p, \mathcal{Q}) \) in which Generic follows \( \text{st} \), then the following conditions (\( \odot \))\(_1\)–(\( \odot \))\(_4\) are satisfied (for each \( \delta < \lambda \)).

(\( \odot \))\(_1\) \( I_\alpha \) \( r^0_\alpha, r^1_\alpha : t \in I_\alpha \) : \( \alpha \leq \delta \) is a partial legal play of \( \mathcal{D}_\mu^{\text{rcA}}(p, \mathcal{Q}) \) in which Generic uses \( \text{st}_1 \).

(\( \odot \))\(_2\) For each \( \eta \in s_\delta \cap \lambda \) the sequence \( \langle q^\eta_\alpha^{\eta_\alpha+1}, r^\alpha_\eta^{\eta_\alpha+1} : \alpha \leq \delta \rangle \) is a partial legal play of \( \mathcal{D}_\mu^{\text{rcA}}(p, \mathcal{Q}) \) in which Complete uses \( \text{st}_0 \).

(\( \odot \))\(_3\) If \( t \in I_\delta \), \( \alpha < \delta, \nu \in s_\alpha \cap \lambda \) such that \( r^\alpha_\nu \leq r^\delta_\eta \), and \( r^\alpha_\nu \) are incompatible then either \( r^\alpha_\nu, r^\delta_\eta \) are incompatible or \( r^\alpha_\nu \leq r^\delta_\eta \).

(\( \odot \))\(_4\) \( \langle p^\delta_\nu : \nu \in s_\delta \cap \lambda \rangle \) is an antichain in \( \mathcal{Q} \).

So suppose that the two players arrived to a stage \( \delta < \lambda \) of the game \( \mathcal{D}_\mu^{\text{Sacks}}(0, p, \mathcal{Q}) \) and the objects listed in (\( \Xi \))\(_\alpha\) (for \( \alpha < \delta \)) as well as \( \langle s_\delta, p^\delta, q^\delta : \alpha < \delta \rangle \) have been constructed. First Generic uses \( \text{st}_1 \) to pick the answer \( \langle I_\delta, \langle r^0_\delta, r^1_\delta : t \in I_\delta \rangle \rangle \)
to \( \langle I_\alpha, \langle r^{0,\alpha}_{t^1}, r^{1,\alpha}_{t^1} : t \in I_\alpha \rangle : \alpha < \delta \rangle \) in \( \mathcal{O}^{\text{IC}}_{\mu}(p, Q) \). Then she uses the strategic completeness of \( Q \) and 1.2 to choose a system \( \langle r^*_t : t \in I_\delta \rangle \) of conditions in \( Q \) such that

\[(\circ)_5 \text{ if } t \in I_\delta, \text{ then } r^{0,\delta}_t \leq r^*_t \text{ and for every } \alpha < \delta \text{ and } \nu \in s_\alpha \cap \alpha^{+1}\lambda, \text{ either } r^{r}_{\nu}, r^{s}_{\nu} \text{ are incompatible or } r^{r}_{\nu} \leq r^*_t, \text{ and also either } p, r^*_t \text{ are incompatible or } p \leq r^*_t,\]

\[(\circ)_6 \text{ if } t_0, t_1 \in I_\delta, \text{ } t_0 \neq t_1, \text{ then the conditions } r^{r}_{\nu}, r^{s}_{\nu} \text{ are incompatible in } Q.\]

Now she lets \( s^* = \{ \eta \in \delta^\lambda : (\forall \alpha < \delta)(\eta|(\alpha + 1) \in s_\alpha) \} \) and

\[s_* = \{ \eta \in s^* : (\exists t \in I_\delta)(\forall \alpha < \delta)(r^{r}_{\eta|(\alpha + 1)} \leq r^*_t \text{ and } p \leq r^*_t) \},\]

and for each \( \eta \in s^- \) she fixes an enumeration \( \{ t^\eta_\zeta : \zeta < \xi_\eta \} \) of the set

\[\{ t \in I_\delta : (\forall \alpha < \delta)\{ r^{r}_{\eta|(\alpha + 1)} \leq r^*_t \text{ and } p \leq r^*_t \} \}.\]

Now Generic defines

\[s^*_\nu = \{ \nu \in \delta^{1+1}: (\nu|\delta \in s^\ast \setminus s^- \text{ and } \nu(\delta) = 0) \} \cup \{ p \} \text{ (remember } (\circ)_2),\]

and she lets \( s_\delta \) be a \( \lambda \)-tree of height \( \delta + 1 \) such that \( s_\delta \cap \delta^{+1} = s^*_\nu \). For \( \nu \in s^*_\nu \)

she also chooses \( p^\delta_\nu \) so that

- if \( \nu|\delta \notin s^- \), then \( p^\delta_\nu \in Q \) is an upper bound to \( \{ r^{r}_{\nu|(\alpha + 1)} : \alpha < \delta \} \cup \{ p \} \) (remember \( (\circ)_2 \)),
- if \( \nu|\delta \in s^- \), then \( p^\delta_\nu = r^{r}_{\nu|\delta} \).

And now, in the play of \( \mathcal{O}^{\text{Sacks}}_{\mu}(0, p, Q) \), Generic puts

\[s_\delta \quad \text{and} \quad \{ p^\delta_\nu : \nu \in s^*_\nu \}\]

and Antigeneric answers with \( \{ q^\delta_\nu : \nu \in s^*_\nu \} \) (so that \( q^\delta_\nu \geq p^\delta_\nu \)). Conditions \( r^\delta_\nu \) (for \( \nu \in s^*_\nu \)) are determined using \( st_0 \) (so that the demand in \( (\circ)_2 \) is satisfied). Finally, Generic defines also \( r^{1,\delta}_t \) for \( t \in I_\delta \) so that

- if \( t = t^\eta_\zeta \) for some \( \eta \in s^- \) and \( \zeta < \xi_\eta \), then \( r^{1,\delta}_t = r^{\delta}_{\eta|\zeta} \),
- otherwise \( r^{1,\delta}_t = r^*_t \).

This completes the description of what Generic plays and what she writes aside — it should be clear that the requirements of \( (\circ)_1-(\circ)_4 \) are satisfied. Now, why is \( st \) a winning strategy? So suppose that \( \langle (s_\delta, p^\delta, q^\delta) : \delta < \lambda \rangle \) is a play of \( \mathcal{O}^{\text{Sacks}}_{\mu}(0, p, Q) \) in which Generic follows \( st_1 \), and \( I_\delta, \langle r^{0,\delta}, r^{1,\delta}_t : t \in I_\delta \rangle \) and \( \langle r^\delta_\eta : \eta \in s_\delta \cap \delta^{+1}\lambda \rangle \) (for \( \delta < \lambda \)) are the objects constructed by Generic anyway, so they satisfy \( (\circ)_1-(\circ)_4 \). It follows from \( (\circ)_1 \) and the choice of \( st_1 \) that there is a condition \( p^* \geq p \) such that

\[(\circ)_7 \text{ for every } \delta < \lambda \text{ the set } \{ r^{1,\delta}_t : t \in I_\delta \} \text{ is pre-dense above } p^*.\]

We claim that then also

\[(\circ)_8 \text{ for every } \delta < \lambda \text{ the set } \{ r^\delta_\eta : \eta \in s_\delta \cap \delta^{+1}\lambda \} \text{ is pre-dense above } p^* \]

(and this clearly implies that Generic won the play, remember \( (\circ)_4 \)). Assume towards contradiction that \( (\circ)_8 \) fails and let \( \delta < \lambda \) be the smallest ordinal for which we may find a condition \( q \geq p^* \) such that \( q \) is incompatible with every \( r^\delta_\eta \) for \( \eta \in s_\delta \cap \delta^{+1}\lambda \). It follows from \( (\circ)_7 \) that we may pick \( t \in I_\delta \) such that the conditions \( r^{1,\delta}_t, q \) are compatible. By the previous sentence and by the definition of \( r^{1,\delta}_t \) we get that \( t \neq t^\eta_\zeta \) for all \( \xi < \xi_\eta, \eta \in s^- \) and thus \( r^{1,\delta}_t = r^*_t \). Look at the condition \( r^*_t \) (satisfying \( (\circ)_9 + (\circ)_8 \)) — it must be stronger than \( p \) and by the minimality of \( \delta \)
we have that \((\forall \alpha < \delta)(\exists v \in s_\alpha \cap \alpha+1) (r^\alpha_\eta \leq r^\delta_\eta)\). It follows from \((\land)_4\) from stages \(\alpha < \delta\) that there is \(\eta \in s^*\) such that \((\forall \alpha < \delta)(r^\alpha_{\eta(\alpha+1)} \leq r^\delta_\eta)\). Then \(t \in s^-\) and hence \(t = r^\delta_\eta\) for some \(\xi < \xi^*_\eta\), contradicting what we already got.

The converse implication should be clear. \(\square\)

The following easy proposition explains why the names of the properties defined in 2.2 include the adjective “bounding”.

**Proposition 3.3.** Let \(\lambda, \mathcal{U}\) and \(\bar{\mu}\) be as in 0.1. Assume that \(\mathcal{Q}\) is a forcing notion, \(p \in \mathcal{Q}\) and \(\tau\) is a \(\mathcal{Q}\)-name for an element of \(\lambda^\lambda\).

1. If \(\mathcal{Q}\) is reasonably \(a\)-bounding over \(\bar{\mu}\), then there are a condition \(q \geq p\) and a sequence \(\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle\) such that
   - \(a_\alpha \subseteq \lambda, |a_\alpha| < \mu_\alpha\) for all \(\alpha < \lambda\),
   - \(q \Vdash_{\mathcal{Q}} (\forall \alpha < \lambda) (\tau(\alpha) \in a_\alpha)\) ".

2. If \(\mathcal{Q}\) is reasonably \(b\)-bounding over \(\mathcal{U}, \bar{\mu}\), then there is a condition \(q \geq p\) and a sequence \(\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle\) such that
   - \(a_\alpha \subseteq \lambda, |a_\alpha| < \mu_\alpha\) for all \(\alpha < \lambda\),
   - \(q \Vdash_{\mathcal{Q}} \{ \alpha < \lambda : \tau(\alpha) \in a_\alpha \} \in \mathcal{U}^\lambda\) ".

3. If \(\mathcal{Q}\) is reasonably \(c\)-bounding over \(\mathcal{U}, \bar{\mu}\), then there are a condition \(q \geq p\) and a sequence \(\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle\) such that
   - \(a_\alpha \subseteq \lambda, |a_\alpha| < \mu_\alpha\) for all \(\alpha < \lambda\),
   - \(q \Vdash_{\mathcal{Q}} \{ \alpha < \lambda : \tau(\alpha) \in a_\alpha \} \in (\mathcal{U}^\lambda)^+".

4. A model

In this section, in addition to the assumptions stated in 0.1 we will also assume that

**Context 4.1.**

1. \(S \subseteq \lambda\) is stationary and co-stationary, \(S \subseteq \mathcal{U}\)
2. \(V\) is a normal filter on \(\lambda, \lambda \setminus S \in V\).

**Definition 4.2.**

1. Let \(\alpha < \beta < \lambda\). An \((\alpha, \beta)\)-extending function is a mapping \(c : \mathcal{P}(\alpha) \to \mathcal{P}(\beta) \setminus \mathcal{P}(\alpha)\) such that \(c(u) \cap \alpha = u\) for all \(u \in \mathcal{P}(\alpha)\).

2. Let \(C\) be an unbounded subset of \(\lambda\). A \(C\)-extending sequence is a sequence \(\varepsilon = \langle c_\alpha : \alpha \in C \rangle\) such that each \(c_\alpha\) is an \((\alpha, \min(C \setminus (\alpha + 1)))\)-extending function.

3. Let \(C \subseteq \lambda, |C| = \lambda, \beta \in C, w \subseteq \beta\) and let \(\varepsilon = \langle c_\alpha : \alpha \in C \rangle\) be a \(C\)-extending sequence. We define \(\text{pos}^+(w, \varepsilon, \beta)\) as the family of all subsets \(u \subseteq \alpha\) of \(\beta\) such that
   - (i) if \(\alpha_0 = \min(\{ \alpha \in C : (\forall \xi \in w)(\xi \prec \alpha)\})\), then \(u \cap \alpha_0 = w\) (so if \(c_\alpha = \beta\) then \(u = w\)), and
   - (ii) if \(\alpha_0, \alpha_1 \in C, w \subseteq \alpha_0 < \alpha_1 = \min(C \setminus (\alpha_0 + 1)) \leq \beta\), then either \(c_{\alpha_0}(u \cap \alpha_0) = u \cup \alpha_1\) or \(u \cup \alpha_0 = u \cap \alpha_1\),
   - (iii) if \(\sup(w) < \alpha_0 = \sup(C \cap \alpha_0) \notin C, \alpha_1 = \min(C \setminus (\alpha_0 + 1)) \leq \beta\), then \(u \cap \alpha_1 = u \cup \alpha_0\).

For \(\alpha_0 \in \beta \cap C\) such that \(w \subseteq \alpha_0\), the family \(\text{pos}^+(w, \varepsilon, \alpha_0, \beta)\) consists of all elements \(u\) of \(\text{pos}^+(w, \varepsilon, \beta)\) which satisfy also the following condition:

4. A \(C\)-extending sequence \(\varepsilon = \langle c_\alpha : \alpha \in C \rangle\) is \(S\)-closed provided that
   - (i) \(C\) is a club of \(\lambda\), and
Observation 4.3.  

(1) Assume that \( c \) is a \( \mathcal{C} \)-extending sequence, \( \alpha_0, \alpha_1, \beta \in C \), \( \alpha_0 < \alpha_1 < \beta \) and \( w \subseteq \alpha_0 \).

(a) If \( u \in \text{pos}^+(w, c, \alpha_0, \alpha_1) \) and \( v \in \text{pos}^+(u, c, \beta) \), then \( v \in \text{pos}^+(w, c, \alpha_0, \beta) \).

(b) If \( v \in \text{pos}^+(w, c, \alpha_0, \beta) \), then \( v \cap \alpha_1 \in \text{pos}(w, c, \alpha_0, \alpha_1) \) and \( v \in \text{pos}^+(v \cap \alpha_1, c, \beta) \).

(c) Similarly for \( \text{pos}^+ \).

(2) Assume that \( c \) is an \( \mathcal{S} \)-closed \( \mathcal{C} \)-extending sequence, \( \alpha_0, \alpha_1, \beta \in C \), \( \alpha_0 < \alpha_1 \leq \beta \), \( w \subseteq \alpha_0 \) and \( w \cup \{\alpha_0\} \) is \( \mathcal{S} \)-closed.

(a) If \( u \in \text{pos}(w, c, \alpha_0, \alpha_1) \) and \( v \in \text{pos}^+(u, c, \beta) \), then \( v \in \text{pos}(w, c, \alpha_0, \beta) \).

(b) If \( v \in \text{pos}(w, c, \alpha_0, \beta) \), then \( v \cap \alpha_1 \in \text{pos}(w, c, \alpha_0, \alpha_1) \) and \( v \in \text{pos}^+(v \cap \alpha_1, c, \beta) \).

(c) Similarly for \( \text{pos}^+ \).

(d) \( \emptyset \neq \text{pos}(w, c, \alpha_0, \beta) = \{u \in \text{pos}^+(w, c, \beta) : u \cap \alpha_0 = w \& \alpha \in u\} \).

Definition 4.4. We define a forcing notion \( Q^1_S \) as follows.

A condition in \( Q^1_S \) is a triple \( p = (w^p, C^p, \alpha^p) \) such that

(i) \( C^p \subseteq \lambda \) is a club of \( \lambda \) and \( w^p \subseteq \min(C^p) \) is such that the set \( w^p \cup \{ \min(C^p) \} \) is \( \mathcal{S} \)-closed,

(ii) \( \alpha^p = \langle c^p_\alpha : \alpha \in C^p \rangle \) is an \( \mathcal{S} \)-closed \( \mathcal{C} \)-extending sequence.

The order \( \leq_{Q^1_S} \leq \) of \( Q^1_S \) is given by

\[ p \leq_{Q^1_S} q \quad \text{if and only if} \]

(a) \( C^q \subseteq C^p \) and \( w^q \in \text{pos}^+_S(w^p, C^p, \min(C^q)) \) and

(b) if \( \alpha_0 < \alpha_1 \) are two successive members of \( C^q \), \( u \in \text{pos}^+_S(w^q, C^q, \alpha_0) \), then \( c^q_{\alpha_0}(u) \in \text{pos}(w^p, C^p, \alpha_0, \alpha_1) \).

For \( p \in Q^1_S \), \( \alpha \in C^p \) and \( u \in \text{pos}^+_S(w^p, C^p, \alpha) \) we let \( p \upharpoonright u \overset{\text{def}}{=} (u, C^p \setminus \alpha, \alpha') \) (\( C^p \setminus \alpha \)).

Remark 4.5. Note that in 4.4(b) we may replace \( \text{pos}(u, c^p, \alpha_0, \alpha_1) \) by \( \text{pos}^+(u, c^p, \alpha_1) \) (remember 4.2(4)(ii) and 4.3(2)(d)).

Proposition 4.6. (1) \( Q^1_S \) is a \( (< \lambda) \)–complete forcing notion of cardinality \( 2^\lambda \).

(2) If \( p \in Q^1_S \) and \( \alpha \in C^p \), then

- for each \( u \in \text{pos}^+_S(w^p, C^p, \alpha) \), \( p \upharpoonright u \in Q^1_S \) is a condition stronger than \( p \), and
- the family \( \{p \upharpoonright u : u \in \text{pos}^+_S(w^p, C^p, \alpha)\} \) is pre-dense above \( p \).

(3) Let \( p \in Q^1_S \) and \( \alpha < \beta \) be two successive members of \( C^p \). Suppose that for each \( u \in \text{pos}^+_S(w^p, C^p, \alpha) \) we are given a condition \( q_u \in Q^1_S \) such that \( p \upharpoonright q_u \overset{\text{def}}{=} (u, C^p \setminus \alpha, \alpha') \) (remember 4.2(4)(ii) and 4.3(2)(d)).

Then there is a condition \( q \in Q^1_S \) such that letting \( \alpha' = \min(C^p \setminus \beta) \) we have...
(a) \( p \leq q, w^q = w^p, C^\gamma \cap \beta = C^\gamma \cap \beta \) and \( c^q_\delta = c^p_\delta \) for \( \delta \in C^\gamma \cap \alpha, \) and
(b) \( \bigcup \{ w^{\alpha_0} : u \in \pos^+(w^p, c^p, \alpha) \} \subseteq \alpha', \)
(c) \( q_\alpha \leq q^{\alpha_0} c^{\alpha_0}_\delta(u) \) for every \( u \in \pos^+(w^p, c^p, \alpha). \)

(4) Assume that \( p \in \pos^+(\omega^c, \alpha) \) and \( \tau \) is a \( \pos^+(\omega^c) \)-name such that \( p \forces \langle \tau \subseteq \nu \rangle. \)

Then there is a condition \( q \in \pos^+(\omega^c) \) stronger than \( p \) and such that
(a) \( w^q = w^p, \alpha \in C^\gamma \) and \( C^\gamma \cap \alpha \subseteq \alpha', \)
(b) if \( u \in \pos^+(w^q, c^q, \alpha) \) and \( \gamma = \min(C^\gamma \setminus (\alpha + 1)) \), then the condition \( q^{\alpha_0} c^{\alpha_0}_\delta(u) \) forces a value to \( \tau. \)

Proof. (1) It should be clear that \( \pos^+(\omega^c) \) is a forcing notion of size \( 2^\lambda. \) To show that it is \( (\langle \lambda \rangle, \alpha) \)-complete suppose that \( \gamma < \lambda \) is a limit ordinal and \( \bar{p} = \langle p_\xi : \xi < \gamma \rangle \subseteq \pos^+(\omega^c) \)
is \( \leq \pos^+(\omega^c) \)-increasing. We put \( w^u = \bigcup_{\xi < \gamma} w^{p_\xi}, C^\gamma = \bigcap_{\xi < \gamma} C^{p_\xi} \) and for \( \delta \in C^\gamma \) we define
\( c^q_\delta : \mathcal{P}(\delta) \rightarrow \mathcal{P}(\min(C^\gamma \setminus (\alpha + 1))) \) so that
- if \( u \in \bigcap_{\xi < \gamma} \pos^+(w^{p_\xi}, c^{p_\xi}, \delta) \), then \( c^{p_\xi}_\delta(u) = \bigcup_{\xi < \gamma} c^{p_\xi}_\delta(u), \)
- if \( u \subseteq \delta \) but it is not in \( \bigcap_{\xi < \gamma} \pos^+(w^{p_\xi}, c^{p_\xi}, \delta) \), then \( c^{p_\xi}_\delta(u) = u \cup \{ \delta \}. \)

Finally we put \( c^q = (c^q_\delta : \delta \in C^\gamma) \) and \( q = (w^q, C^\gamma, c^q). \) One easily checks that \( q \in \pos^+(\omega^c) \) is a condition stronger than all \( p_\xi \)'s.

(2) Straightforward (remember 4.3(2)).

(3) Note that if \( u \in \pos^+(w^p, c^p, \alpha), \alpha_u = \min(C^{u_\alpha}) \), then \( w^{u_\alpha} \in \pos(u, \alpha, c^p, \alpha_u) \).
We let \( w^p = w^q \) and \( C^\gamma = (C^{p_\beta} \cup \gamma) \cup \bigcap \{ C^{p_\alpha} : u \in \pos^+(w^p, c^p, \alpha) \} \) (plainly, \( C^\gamma \) is a club of \( \lambda \).) Let \( \alpha' = \min(C^\gamma \setminus (\alpha + 1)) = \min(C^\gamma \setminus \beta). \) For \( \delta \in C^\gamma \cap \alpha \), \( C^{p_\delta} \cap \alpha \) put \( c^q_\delta = c^{p_\delta}_\alpha. \) Next, choose an \( (\alpha, \alpha') \)-extending function \( c^q_\delta : \mathcal{P}(\delta) \rightarrow \mathcal{P}(\alpha') \) such that \( (\forall u \in \pos^+(w^p, c^p, \alpha)) (c^q_\delta(u) \in \pos^+(w^p, c^p, \alpha')) \) and \( (c^q_\delta(u) \cap \{ \alpha' \}) \) is \( S \)-closed for each \( u \subseteq \alpha. \) (Remember 4.3(2d); note that, by the definition of \( C^\gamma, w^{u_\alpha} \subseteq \alpha' \) for each \( u \in \pos^+(w^p, c^p, \alpha). \)) Finally, if \( \delta_0 < \delta_1 \) are two successive members of \( C^\gamma \setminus \alpha' \), then choose a \( (\delta_0, \delta_1) \)-extending function \( c^q_{\delta_0} : \mathcal{P}(\delta_0) \rightarrow \mathcal{P}(\delta_1) \) so that
(i) \( \text{if } v \subseteq \delta_0, \text{ } u = v \cap \alpha \in \pos^+(\omega^c, c^p, \alpha) \) and \( v \in \pos^+(\omega^{u_\alpha}, c^{u_\alpha}, \delta_0), \) then \( c^q_{\delta_0}(v) \in \pos^+(v, c^p, \delta_0, \delta_1); \)
(ii) \( \text{if } v \subseteq \delta_0, \text{ } u \in \pos^+(\omega^c, c^p, \delta_0) \) but we are not in a case covered by (i), then \( c^q_{\delta_0}(v) \in \pos^+(v, c^p, \delta_0, \delta_1); \)
(iii) in all other cases we let \( c^q_{\delta_0}(v) = v \cup \{ \delta_0 \}. \)

Let \( c^q = (c^q_\delta : \delta \in C^\gamma) \) and \( q = (w^q, C^\gamma, c^q). \) It should be clear that \( q \in \pos^+(\omega^c) \) is a condition as required.

(4) Easily follows from (3). \( \square \)

Definition 4.7. Suppose that \( \gamma < \lambda \) is a limit ordinal and \( \bar{p} = \langle p_\xi : \xi < \gamma \rangle \subseteq \pos^+(\omega^c) \)
is \( \leq \pos^+(\omega^c) \)-increasing. The condition \( q \) constructed as in the proof of 4.6(1) for \( p \) will be called the natural limit of \( \bar{p}. \)

Proposition 4.8. (1) Suppose \( \bar{p} = \langle p_\xi : \xi < \lambda \rangle \) is a \( \leq \pos^+(\omega^c) \)-increasing sequence of conditions from \( \pos^+(\omega^c) \) such that
(a) \( w^{p_\xi} = w^{p_\lambda} \) for all \( \xi < \lambda, \)
(b) if \( \gamma < \lambda \) is limit, then \( p_\gamma \) is the natural limit of \( \bar{p}|\gamma, \)
(c) for each \( \xi < \lambda, \) if \( \delta \in C^{p_\xi}, \text{otp}(C^{p_\xi} \cap \delta) = \xi, \) then \( C^{p_{\xi+1}} \cap (\delta + 1) = C^{p_\xi} \cap (\delta + 1) \) and for every \( \alpha \in C^{p_{\xi+1}} \cap \delta \) we have \( c^{p_{\xi+1}}_\alpha = c^{p_\xi}_\alpha. \)
Then the sequence \( \bar{p} \) has an upper bound in \( Q_1^4 \).

(2) Suppose that \( p \in Q_1^4 \) and \( b \) is a \( Q_1^4 \)-name such that \( p \Vdash "b : \lambda \rightarrow V". \)

Then there is a condition \( q \in Q_1^4 \) stronger than \( p \) and such that

(\( \ast \)) if \( \delta = \delta' \) are two successive points of \( C^\kappa \), \( u \in pos^+_\delta(w^\kappa, c^\kappa, \delta) \), then the

condition \( q \Vdash c^\kappa(u) \) decides the value of \( b[(\delta + 1)] \).

Proof. (1) First let us note that if \( \delta \in \triangle C^\kappa \) is a limit ordinal, then \( \delta \in \bigcap_{\xi < \lambda} C^\kappa \)

and \( c^{\kappa+1}_\delta = c^\kappa_\delta \) for all \( \xi \geq \delta + 2 \) (by assumptions (b) and (c)). Now, we put \( w^\kappa = w^{p_0} \) and \( C^\kappa = \{ \delta \in \triangle C^\kappa : \delta \text{ is limit } \} \), and for \( \delta \in C^\kappa \) we let \( c^\kappa_\delta = c^{\kappa+1}_\delta \)

(thus defining \( c^\kappa = \langle c^\kappa_\delta : \delta \in C^\kappa \rangle \)). It should be clear that \( q = \langle w^\kappa, C^\kappa, c^\kappa \rangle \in Q_1^4 \) is

an upper bound to \( \bar{p} \).

(2) Follows from (1) above and 4.6(4). \( \Box \)

**Definition 4.9.** We let \( W \) and \( \eta, \nu \) be \( Q_1^1 \)-names such that

\[ \Vdash_{Q_1^1} W = \bigcup \{ w^p : p \in \Gamma_{Q_1^1} \} \]

and

\[ \Vdash_{Q_1^1} "\eta, \nu \in \lambda \text{ and if } (\delta_\xi : \xi < \lambda) \text{ is the increasing enumeration of } cl(W), \]

and \( \delta_\xi \leq \alpha < \delta_{\xi+1}, \xi < \lambda \), then \( \eta(\alpha) = \xi \) and \( \nu(\alpha) = \delta_\xi \). \]

**Proposition 4.10.** (1) \( \Vdash_{Q_1^1} "W \text{ is an unbounded } S \cdot \text{closed subset of } \lambda \". Consequently \( \Vdash_{Q_1^4} "W \in U^{Q_1^4} \". \)

(2) \( \Vdash_{Q_1^1} "W, \lambda \setminus W \in (U^{Q_1^1})^+ \". \)

(3) \( \Vdash_{Q_1^1} \langle \forall f \in \lambda \cdot \nu(V) \rangle (\forall A \in U^{Q_1^1}) \langle \exists \alpha \in A \rangle (f(\alpha) < \nu(\alpha)) \).

Proof. (2) Suppose that \( p \in Q_1^4 \) and \( A_i \) (for \( i < \lambda \)) are \( Q_1^1 \)-names for elements of \( V \cap V \). Build inductively sequences \( \langle p_i : i < \lambda \rangle \subseteq Q_4^1 \) and \( \langle A_i : i < \lambda \rangle \subseteq V \) such that

(a) \( \langle \forall i < j < \lambda \rangle (p_i \leq p_i \leq p_j) \),

(b) \( p_{i+1} \Vdash_{Q_1^1} A_i = A_i \) and \( i \leq \text{sup}(w^p_i) \) for all \( i < \lambda \),

(c) if \( \gamma < \lambda \) is limit, then \( p_\gamma = p_\gamma \) is the natural limit of \( \langle p_i : i < \gamma \rangle \).

Pick a limit ordinal \( \delta \in \triangle A_i \setminus S \) such that \( \delta = \text{sup} \bigcup_{i < \lambda} (w^p_i) \) \( \subseteq C^p_\delta \) (possible by the normality of \( V \); remember (b,c) above). Then \( p_\delta \Vdash \delta \in \triangle A_i \). Put \( \beta = \text{min}(C^p_\delta \setminus (\delta + 1)) \).

Let \( w = c^\kappa_{\delta} \langle w^p_i \rangle \) and \( p^* = p_\delta \mid \beta w \). Then \( p^* \geq p_\delta \) and \( p^* \Vdash \delta \in W \).

On the other hand, since \( \delta = \text{sup}(w^p_i) \notin S \), we have \( w^p_\delta \in pos^+_\delta(w^\kappa, c^\kappa, \beta) \) so we may let \( p^{**} = p_\delta \mid \beta w^p_\delta \). Then \( p^{**} \geq p_\delta \) and \( p^{**} \Vdash \delta \notin \bigcup_{i < \lambda} W \).

(3) Suppose that \( p \in Q_1^4 \), \( f \in \lambda \cdot \nu \) and \( \langle A_\alpha : \alpha < \lambda \rangle \) is a sequence of \( Q_1^1 \)-names for members of \( V \cap V \). By induction on \( \alpha < \lambda \) construct a sequence \( \langle p_\alpha, A_\alpha : \alpha < \lambda \rangle \)

such that for each \( \alpha \):

(i) \( p_\alpha \in Q_1^1 \), \( A_\alpha \subseteq V, p_0 = p, p_\alpha \leq_{Q_1^1} p_{\alpha+1} \), \( \text{min}(C^p_{\alpha+1}) > \alpha \)

(ii) if \( \alpha \) is a limit ordinal, then \( p_\alpha \) is the natural limit of \( \langle p_\beta : \beta < \alpha \rangle \), and

(iii) \( p_{\alpha+1} \Vdash_{Q_1^1} A_\alpha \cap (\lambda \setminus S) = A_\alpha \).
Next pick a limit ordinal $\delta \in \triangle A_\alpha \cap (\lambda \setminus S)$ such that $(\forall \alpha < \delta) (\min(C^{P^{\alpha}}) < \delta)$. Then $p^*_\delta \Vdash \alpha < \delta \in A_\alpha$ and $\delta = \min(C^{P^{\delta}})$ and $w^{P^{\delta}} \subseteq \delta$ is $S$-closed, so we may let $w^\alpha = w^{P^{\delta}}$, $C^\alpha = C^{P^{\delta}} \setminus (f(\delta) + 1)$ and $c^\alpha = c^{P^{\delta}} | C^\alpha$ to get a condition $q^* \in Q^1_S$ stronger than $p$ and such that

$$q^* \Vdash_{Q^1_S} \delta \in \triangle A_\alpha \text{ and } f(\delta) < \nu(\delta).$$

\[\square\]

**Proposition 4.11.** The forcing notion $Q^1_S$ is reasonably $B$-bounding over $U$.

**Proof.** By 4.6(1), $Q^1_S$ is $(<\lambda)$-complete, so we have to verify 2.2(5b) only. Let $p \in Q^1_S$ and let $\mu^* = (\mu^*_\alpha : \alpha < \lambda)$, $\mu^*_\alpha = \lambda$ for each $\alpha < \lambda$. We are going to describe a strategy $st$ for Generic in $\mathord{\square}^{\mathcal{U}_{\mathcal{U}'}^{\mathcal{R}}}_{p, Q^1_S}$.

In the course of a play the strategy $st$ instructs Generic to build aside an increasing sequence of conditions $p^* = (p^*_\alpha : \alpha < \lambda) \subseteq Q^1_S$ such that

(a) $p^*_0 = p$ and $w^{P^\alpha} = w^\alpha$ for all $\alpha < \lambda$, and

(b) if $\gamma < \lambda$ is limit, then $p^*_\gamma$ is the natural limit of $p^*_\delta | \gamma$, and

(c) for each $\alpha < \lambda$, if $\delta \in C^{P^{\alpha}}$, $\otp(C^{P^\gamma} \cap \delta) = \alpha$, then $C^{P^{\alpha+1}} \cap (\delta + 1) = C^{P^\gamma} \cap (\delta + 1)$ and for every $\xi \in C^{P^{\alpha+1}} \cap \delta$ we have $c^\xi_\alpha = c^\xi_{\alpha+1}$, and

(d) after stage $\alpha < \lambda$ of the play of $\mathord{\square}^{\mathcal{U}_{\mathcal{U}'}}_{p, Q^1_S}$, the condition $p^*_\alpha+1$ is determined (conditions $p^*_\alpha$ for non-successor $\alpha < \lambda$ are determined by (a), (b) above).

So suppose that the players arrived to a stage $\alpha < \lambda$ of $\mathord{\square}^{\mathcal{U}_{\mathcal{U}'}}_{p, Q^1_S}$, and Generic (playing according to $st$ so far) has constructed aside an increasing sequence $\langle p^*_\xi : \xi < \alpha \rangle$ of conditions (satisfying (a)–(d)). Let $\delta \in C^{P^\alpha}$ be such that $\otp(C^{P^\gamma} \cap \delta) = \alpha$ and let $\gamma = \min(C^{P^{\alpha+1}} \setminus (\delta + 1))$. Now Generic makes her move in $\mathord{\square}^{\mathcal{U}_{\mathcal{U}'}}_{p, Q^1_S}$:

- $I_\alpha = \text{pos}^+_S(w^{P^{\alpha}}, c^{P^\alpha}, \delta)$, and
- $p^*_\alpha = p^*_\alpha | \gamma^*_{\alpha+1}(u)$ for $u \in I_\alpha$.

Let $\langle q^*_u : u \in I_\alpha \rangle \subseteq Q^1_S$ be the answer of Antigeneric, so $p^*_{\alpha+1} | \gamma^*_{\alpha+1}(u) \leq q^*_u$ for each $u \in \text{pos}^+(w^{P^{\alpha+1}}, c^{P^{\alpha+1}}, \delta)$. Now Generic uses 4.6(3) (with $\delta, \gamma, p^*_\alpha, q^*_u$ here standing for $\alpha, \beta, p, q_u$ there) to pick a condition $p^*_{\alpha+1}$ such that, letting $\alpha' = \min(C^{P^{\alpha+1}} \setminus \gamma)$, we have

- $p^*_{\alpha} \leq p^*_{\alpha+1}$, $w^{P^{\alpha+1}} = w^{P^{\alpha}}$, $C^{P^{\alpha+1}} \cap \gamma = C^{P^{\alpha}} \cap \gamma$ and $c^\xi_{\alpha+1} = c^\xi_\alpha$ for $\xi \in C^{P^{\alpha+1}} \cap \delta$, and
- $\bigcup \{w^{c^\xi} : u \in I_\alpha \} \subseteq \alpha'$, and
- $q^*_u \leq p^*_{\alpha+1} | \gamma^*_{\alpha+1}(u)$ for every $u \in I_\alpha$.

We claim that $st$ is a winning strategy for Generic in $\mathord{\square}^{\mathcal{U}_{\mathcal{U}'}}_{p, Q^1_S}$. So suppose that

$$\langle I_\alpha, (p^*_\alpha, q^*_u : u \in I_\alpha) : \alpha < \lambda \rangle$$

is a play of $\mathord{\square}^{\mathcal{U}_{\mathcal{U}'}}_{p, Q^1_S}$ in which Generic uses $st$, and let $\bar{p}^* = (p^*_\alpha : \alpha < \lambda) \subseteq Q^1_S$ be the sequence constructed aside by Generic, so it satisfies (a)–(c) above, and thus also the assumptions of 4.8(1). Let $p^*$ be an upper bound to $\bar{p}$ (which exists by 4.8(1)). Now note that

$$p^* \Vdash_{Q^1_S} \text{ if } \alpha \in C^{P^\gamma} \cap W \text{ and } w = W \cap \alpha, \text{ then } q^*_u \in \Gamma_{Q^1_S} \text{ "}$$
and therefore
\[ p^* \Vdash_{Q^1_2} \left( \forall \alpha \in CP \cap W \right) \left( \exists u \in I \lambda \right) \left( q^u_{\alpha} \in \Gamma_{Q^1_2} \right). \]

Since \( p^* \Vdash CP \cap W \in U^{Q^1_1} \) (by 4.10) we may conclude that the condition \( p^* \) witnesses that Generic won the play. \( \square \)

**Definition 4.12.** Let \( F \) be a filter on \( \lambda \) including all co-bounded subsets of \( \lambda \), \( \emptyset \notin F \).

1. We say that a family \( F \subseteq \lambda \lambda \) is \( F \)-dominating whenever
\[ (\forall q \in \lambda \lambda) \left( \exists f \in F \right) \left( \{ \alpha : g(\alpha) < f(\alpha) \} \in F \right). \]
2. The \( F \)-dominating number \( \delta_F \) is the minimal size of an \( F \)-dominating family in \( \lambda \lambda \).
3. If \( F \) is the filter of co-bounded subsets of \( \lambda \), then the corresponding dominating number is also denoted by \( \delta_\lambda \). If \( F \) is the filter generated by club subsets of \( \lambda \), then the corresponding dominating number is called \( \delta_c \).

It was shown in Cummings and Shelah [1] that \( \delta_\lambda = \delta_c \) (whenever \( \lambda > \beth_\omega \) is regular). The following corollary is an interesting addition to that result.

**Corollary 4.13.** It is consistent that \( \lambda \) is an inaccessible cardinal and there are two normal filters \( U', U'' \) on \( \lambda \) such that \( \delta_{U'} \neq \delta_{U''} \).

**Proof.** Start with the universe where \( \lambda, U, V, S \) are as in 0.1 + 4.1 and \( 2^\lambda = \lambda^+ \). Let \( \bar{Q} = (F_\xi, G_\xi : \xi < \lambda^{++}) \) be a \( \lambda \)-support iteration such that for every \( \xi < \lambda^{++} \), \( \| P_\xi \| = Q_\xi \subseteq \bar{Q}^\lambda \).

It follows from 2.5 that \( P_{\lambda^{++}} \) is reasonably \( b \)-bounding over \( U \), and hence also \( \lambda \)-proper. Therefore using 4.6(1) and [5, Theorem A.1.10] (see also Eisworth [2, §3]) one can easily argue that the limit \( P_{\lambda^{++}} \) of the iteration satisfies the \( \lambda^{++} \)-cc, \( \| P_{\lambda^{++}} \| = 2^\lambda = \lambda^{++} \). \( P_{\lambda^{++}} \) is strategically \( (\prec \lambda) \)-complete and \( \lambda \)-proper. Thus, the forcing with \( P_{\lambda^{++}} \) does not collapse cardinals and it follows from 3.3 that
\[ \| P_{\lambda^{++}} \| < \lambda \lambda \cap V \text{ is } \langle U \rangle^{P_{\lambda^{++}}} \text{-dominating in } \lambda \lambda \]
and it follows from 4.10(3) that for each \( \xi < \lambda^{++} \)
\[ \| P_{\lambda^{++}} \| < \lambda \lambda \cap V^{\bar{Q}_\xi} \text{ is not } \langle V \rangle^{P_{\lambda^{++}}} \text{-dominating in } \lambda \lambda \]

Therefore we may easily conclude that
\[ \| P_{\lambda^{++}} \| < \text{ if } U' = \langle U \rangle^{P_{\lambda^{++}}}, U'' = \langle V \rangle^{P_{\lambda^{++}}} \text{ then } \delta_{U'} = \delta_{U''} = \lambda^+ < 2^\lambda = \lambda^{++} = \delta_{U''} = \delta_c = \delta_\lambda. \]

\( \square \)

5. Two bad examples of forcing notions

In this section we give two more examples of forcing notions that have some of the properties studied in the paper - but not strong enough to allow us to quote results obtained earlier. They are test cases for our future research.

**Definition 5.1.** We define a forcing notion \( P^\mu \) as follows.

**A condition in** \( P^\mu \) is a pair \( p = (f^P, C^P) \) such that
\[ C^P \subseteq \lambda \text{ is a club of } \lambda \text{ and } f^P \in \prod \{ \mu_\xi : \xi \in \lambda \setminus C^P \}. \]
The order $\leq_{P\hat{\mu}} \leq$ of $P\hat{\mu}$ is given by:

\[ p \leq_{P\hat{\mu}} q \text{ if and only if } C^q \subset C^p \text{ and } f^p \subset f^q. \]

**Proposition 5.2.** Assume in addition to 0.1 that the sequence $\hat{\mu}$ is increasing unbounded in $\lambda$. Let $D_\lambda$ be the club filter on $\lambda$. Then the forcing notion $P\hat{\mu}$ is reasonably $B$–bounding over $D_\lambda$ but it is not reasonably $B$–bounding over $D_\lambda$.

**Proof.** It should be clear that $P\hat{\mu}$ is a $(<\lambda)$–complete forcing notion.

Let $\mu \in P\hat{\mu}$. We are going to describe a winning strategy $st$ for Generic in the game $D_{\text{cb}}(\mu, P\hat{\mu})$. The strategy $st$ instructs Generic to construct aside (in addition to the innings in the play)

1. a closed increasing sequence $\langle \delta_\alpha : \alpha < \lambda \rangle \subseteq \lambda$ and an increasing sequence $\langle r_\alpha : \alpha < \lambda \rangle$ of conditions in $P\hat{\mu}$, so that
2. $r_0 = p$, $\delta_0 = \min(C^p)$ and $\langle \delta_\beta : \beta \leq \alpha \rangle = C^\alpha \cap (\delta_\alpha + 1)$ for $\alpha < \lambda$,
3. $\delta_{\alpha+1}$ and $r_{\alpha+1}$ are known right after the stage $\alpha$ of the play.

Suppose that the players have arrived at a stage $\alpha < \lambda$ of the play, and Generic playing according to $st$ constructed aside $\langle \delta_{\beta+1}, r_{\beta+1} : \beta < \alpha \rangle$. If $\alpha$ is limit, then $\langle \delta_\alpha \rangle$ determines $\alpha$ and Generic lets $r_\alpha \in P\hat{\mu}$ be such that $C^{\alpha+1} = \{ \beta < \lambda : C^{\beta} \subseteq C^{\alpha+1} \}$ and $f^{\alpha+1} = \bigcup_{\beta < \alpha} f^\beta$. (Clearly, relevant parts of $(\circ)_1$ and $(\circ)_2$ are satisfied.) Now Generic picks an enumeration $\langle \eta_\xi : \xi < \zeta_\alpha \rangle$ of $\prod \{ \mu_\xi : \xi \in \{ \delta_\beta : \beta \leq \alpha \} \}$ (for some limit $\zeta_\alpha < \lambda$) and puts $\zeta_\alpha$ as her inning in $D_{\text{cb}}(p, P\hat{\mu})$. Now the two players start a subgame of length $\zeta_\alpha$. The innings of Generic in the subgame are essentially determined by the following demands:

4. $p_\alpha^\mu = (p_0 \cup f^{\alpha}, C^{\alpha+1} \setminus \{ \delta_\beta : \beta \leq \alpha \})$,

and for $\xi < \alpha < \zeta_\alpha$

5. $\eta_\xi \subseteq f^\alpha$ and $(f^\alpha \setminus (\lambda \setminus \{ \delta_\beta : \beta \leq \alpha \}), C^\xi \cup \{ \delta_\beta : \beta \leq \alpha \}) \leq_{P\hat{\mu}} (f^\xi \setminus (\lambda \setminus \{ \delta_\beta : \beta \leq \alpha \}), C^\xi \cup \{ \delta_\beta : \beta \leq \alpha \})$.

After the subgame is over, Generic lets $r_{\alpha+1} \in P\hat{\mu}$ be such that

\[ f^{\alpha+1} = \bigcup \{ f^\xi \setminus (\lambda \setminus \{ \delta_\beta : \beta \leq \alpha \}) : \xi < \zeta_\alpha \} \text{ and } C^{\alpha+1} = \bigcap_{\zeta < \zeta_\alpha} C^\zeta \cup \{ \delta_\beta : \beta \leq \alpha \} \]

and she takes $\delta_{\alpha+1} = \min (C^{\alpha+1} \setminus (\delta_\alpha + 1))$.

This completes the description of the strategy $st$. Suppose that

\[ \langle \zeta_\alpha, (p_\alpha^\zeta, q_\alpha^\zeta : \xi < \zeta_\alpha) : \alpha < \lambda \rangle \]

is a play of $D_{\text{cb}}(p, P\hat{\mu})$ in which Generic followed $st$ and she constructed aside $\langle \delta_\alpha, r_\alpha : \alpha < \lambda \rangle$. Then $\bigcup_{\alpha < \lambda} f^{\alpha} \cap C^{\alpha+1} \in P\hat{\mu}$ is a condition witnessing that $(\spadesuit)^\zeta_{\alpha}$ of 2.2.3 holds, so Generic wins this play.

To show that $P\hat{\mu}$ is not reasonably $B$–bounding over $D_\lambda$, we will describe a winning strategy $st^*$ for Antigeneric in $D_{\text{cb}}(p, P\hat{\mu})$. In the course of the play Antigeneric will construct (in addition to his innings) an increasing sequence $\langle \xi_\delta : \delta < \lambda \rangle$ of ordinals below $\lambda$. Suppose that the two players have arrived at the stage $\delta < \lambda$ of the play and Generic has chosen a set $I_\delta$ of cardinality less than $\lambda$ and a system $\langle p_\delta^t : t \in I_\delta \rangle$ of conditions in $P\hat{\mu}$. Now, Antigeneric sets $C = \bigcap \{ C^\eta : t \in I_\delta \}$ (it is a club of $\lambda$) and then he picks $\xi_\delta \in C$ such that

$(\Box)_1$ $\mu_{\xi_\delta} \supset |I_\delta|$ and $\xi_\delta > \sup(\xi_\alpha : \alpha < \delta) + \delta$. 

Next for every $t \in I_\delta$, Antigeneric chooses a condition $q_\delta^t \in \mathbb{P}^\delta$ such that

\[(\Box)_2 C_\delta^\delta = C \setminus (\xi_\delta + 1), p_\delta^\delta \leq q_\delta^t \text{ and} \]

\[(\Box)_3 \text{ if } t, s \in I_\delta \text{ are distinct, then } f^{q_\delta^t}(\xi_\delta) \neq f^{q_\delta^s}(\xi_\delta).\]

This completes the description of $st^\ast$.

Suppose that $\langle I_\delta, (p_\delta^\delta, q_\delta^t : t \in I_\delta) : \delta < \lambda \rangle$ is a play of $\omega^{\text{rcB}}(\theta_p, \mathbb{P}^\delta)$ in which Antigeneric plays according to $st^\ast$ and $\langle \xi_\delta : \delta < \lambda \rangle$ is the sequence constructed aside. Let $p^* \in \mathbb{P}^\delta$. We claim that for every $\delta \in C_{p^*}^\ast$ the family $\{q_\delta^t : t \in I_\delta\}$ is not predense above $p^*$. So suppose that $\delta \in C_{p^*}^\ast$. It follows from $(\Box)_1 + (\Box)_2$ that $\delta, \xi_\delta \in \bigcap \{\lambda \setminus C_\delta^\delta : t \in I_\delta\}$.

**Case 1** $\xi_\delta \not\in C_{p^*}^\ast$.

Then we may find a condition $r \geq p^*$ such that $f^r(\xi_\delta) \not\in \{f^{q_\delta^t}(\xi_\delta) : t \in I_\delta\}$ (remember $(\Box)_1$). The condition $r$ is incompatible with all $q_\delta^t$ (for $t \in I_\delta$).

**Case 2** $\xi_\delta \in C_{p^*}^\ast$.

If $f^{p^*}(\xi_\delta) \not\in \{f^{q_\delta^t}(\xi_\delta) : t \in I_\delta\}$, then $p^*$ is incompatible with all $q_\delta^t$ for $t \in I_\delta$. Otherwise there is a unique $s \in I_\delta$ such that $f^{p^*}(\xi_\delta) = f^{q_\delta^s}(\xi_\delta)$ and the condition $p^*$ is incompatible with all $q_\delta^t$ for $t \in I_\delta \setminus \{s\}$. Since $\delta \in C_{p^*} \setminus C_\delta^\delta$, we may pick a condition $r \geq p^*$ such that $\delta \not\in C^r$ and $f^r(\delta) \neq f^{q_\delta^s}(\delta)$. Then $r$ is incompatible with all $q_\delta^t$ for $t \in I_\delta$.

Now we may easily argue that Generic lost the play. \(\Box\)

An iterable property which will be introduced in the subsequent paper [7] will capture also $\mathbb{P}^\delta$. The second example is a very close relative of the forcing notion $Q^\delta_2$ from the previous section. Yet at the moment we do not know if we can iterate it.

**Definition 5.3.** We define a forcing notion $Q^\delta_2$ as follows.

A condition in $Q^\delta_2$ is a triple $p = (w^p, C^p, \epsilon^p)$ such that

(i) $C^p \in \mathcal{U}$, $w^p \leq \min(C^p)$,

(ii) $\epsilon^p = \langle c_{\alpha}^p : \alpha \in C^p \rangle$ is a $C^p$-extending sequence.

The order $\leq_{Q^\delta_2}$ of $Q^\delta_2$ is given by

$p \leq_{Q^\delta_2} q$ if and only if

(a) $C^p \subseteq C^q$ and $w^q \in \text{pos}^+(w^p, \epsilon^p, \min(C^q))$ and

(b) if $\alpha_0, \alpha_1 \in C^q$, $\alpha_0 < \alpha_1 = \min(C^q \setminus \{\alpha_0 + 1\})$ and $u \in \text{pos}^+(w^q, \epsilon^p, \alpha_0)$, then

$\epsilon_{\alpha_0}^p(u) \in \text{pos}(u, \epsilon^p, \alpha_0, \alpha_1).$

For $p \in Q^\delta_2$, $\alpha \in C^p$ and $u \in \text{pos}^+(w^p, C^p, \alpha)$ we let $p\vert_\alpha \overset{\text{def}}{=} (u, C^p \setminus \alpha, \epsilon^p \upharpoonright (C^p \setminus \alpha))$.

**Proposition 5.4.**

1. $Q^\delta_2$ is a $(< \lambda)$-complete forcing notion of cardinality $2^\lambda$.

2. If $p \in Q^\delta_2$ and $\alpha \in C^p$, then

(a) for each $u \in \text{pos}^+(w^p, C^p, \alpha)$, $p\vert_\alpha u \in Q^\delta_2$ is a condition stronger than $p$; and

(b) the family $\{p\vert_\alpha u : u \in \text{pos}^+(w^p, C^p, \alpha)\}$ is pre-dense above $p$.

3. Let $p \in Q^\delta_2$ and $\alpha < \beta$ be two successive members of $C^p$. Suppose that for each $u \in \text{pos}^+(w^p, C^p, \alpha)$ we are given a condition $q_u \in Q^\delta_2$ such that $p\vert_\alpha q_u^p(u) \leq q_u$. Then there is a condition $q \in Q^\delta_2$ such that letting $\alpha' = \min(C^\beta \setminus \beta)$ we have

(a) $p \leq q$, $w^q = w^p$, $C^q \cap \beta = C^p \cap \beta$ and $c_{\alpha}^q = c_{\alpha}^p$ for $\delta \in C^\beta \cap \alpha$, and
(b) $\bigcup \{ w^\beta : u \in \text{pos}^+(w^\beta, c^\beta, \alpha) \} \subseteq \alpha'$, and
(c) $q_\alpha \leq q_{1_\alpha}^{c^\beta}(u)$ for every $u \in \text{pos}^+(w^\beta, c^\beta, \alpha)$.

(4) Assume that $p \in Q^\alpha_{U^\alpha}$, $\alpha \in C^\gamma$ and $\tau$ is a $Q^\gamma_{U^\alpha}$-name such that $p \Vdash \tau \in V$.

Then there is a condition $q \in Q^\alpha_{U^\alpha}$ stronger than $p$ and such that
(a) $w^\delta = w^\alpha$, $\alpha \in C^\gamma$ and $C^\gamma \cap \alpha = C^\delta \cap \alpha$, and
(b) if $u \in \text{pos}^+(w^\alpha, c^\alpha, \alpha)$ and $\gamma = \min(C^\gamma \setminus (\alpha + 1))$, then the condition
$q|_\gamma$, $c^\alpha(u)$ forces a value to $\tau$.

Proof. Fully parallel to 4.6. \hfill \Box

Definition 5.5. The natural limit of an $\leq_{Q^\gamma_{U^\alpha}}$-increasing sequence $\bar{p} = (p_\xi : \xi < \gamma) \subseteq Q^\gamma_{U^\alpha}$ where $\gamma < \lambda$ is a limit ordinal) is the condition $q = (w^\alpha, C^\gamma, c^\gamma)$ defined as follows:

- $w^\alpha = \bigcup_{\xi < \gamma} w^{p_\xi}$, $C^\gamma = \bigcap_{\xi < \gamma} C^{p_\xi}$ and
- $c^\gamma = \langle c^\delta : \delta \in C^\gamma \rangle$ is such that for $\delta \in C^\gamma$ and $u \subseteq \delta$ we have $c^\delta(u) = \bigcup_{\xi < \gamma} c^{p_\xi}(u)$.

Proposition 5.6. (1) Suppose $\bar{p} = (p_\xi : \xi < \lambda)$ is a $\leq_{Q^\gamma_{U^\alpha}}$-increasing sequence of conditions from $Q^\alpha_{U^\alpha}$ such that
(a) $w^\alpha = w^{p_\alpha}$ for all $\xi < \lambda$, and
(b) if $\gamma < \lambda$ is limit, then $p_\gamma$ is the natural limit of $\bar{p}|\gamma$, and
(c) for each $\xi < \lambda$, if $\delta \in C^{p_\xi}$, $\otp(C^{p_\xi} \cap \delta) = \xi$, then $C^{p_{\xi+1}} \cap (\delta + 1) = C^{p_\xi} \cap (\delta + 1)$ and for every $\alpha \in C^{p_{\xi+1}} \cap \delta$ we have $c^{p_{\xi+1}}(\alpha) = c^{p_\xi}(\alpha)$.

Then the sequence $\bar{p}$ has an upper bound in $Q^\gamma_{U^\alpha}$.

(2) Suppose that $p \in Q^\alpha_{U^\alpha}$ and $b$ is a $Q^\gamma_{U^\alpha}$-name such that $p \Vdash \check{b} : \lambda \rightarrow V$.

Then there is a condition $q \in Q^\alpha_{U^\alpha}$ stronger than $p$ and such that
(\bigwedge) if $\delta < \delta'$ are two successive points of $C^\alpha$, $u \in \text{pos}(w^\alpha, c^\alpha, \delta)$, then the condition $q|_\delta$, $c^\delta(u)$ decides the value of $b|\delta(\delta + 1)$.

Proof. Fully parallel to 4.8. \hfill \Box

Definition 5.7. We let $W$ and $\eta, \nu$ be $Q^\gamma_{U^\alpha}$-names such that
\[ \forces_{Q^\gamma_{U^\alpha}} W = \bigcup \{ w^\alpha : p \in \Gamma_{Q^\gamma_{U^\alpha}} \} \]
and
\[ \forces_{Q^\gamma_{U^\alpha}} " \eta, \nu \in \lambda \wedge \text{if } (\delta_\xi : \xi < \lambda) \text{ is the increasing enumeration of } \text{cl}(W), \]
\[ \text{and } \delta_\xi \leq \alpha < \delta_{\xi+1}, \xi < \lambda, \text{ then } \eta(\alpha) = \xi \text{ and } \nu(\alpha) = \delta_{\xi+4} " \].

Note that if $p \in Q^\alpha_{U^\gamma}$, then
\[ p \forces_{Q^\gamma_{U^\alpha}} " W \subseteq \bigcup \{ (\alpha_0, \alpha_1) : \alpha_0, \alpha_1 \in C^\alpha \& \alpha_1 = \min(C^\alpha \setminus (\alpha_0 + 1)) \} " \]
and
\[ p \forces_{Q^\gamma_{U^\alpha}} " \{ \alpha \in C^\alpha : \min(C^\alpha \setminus (\alpha + 1)) \} \cap W \neq \emptyset \}
\[ \{ \alpha \in C^\alpha : \min(C^\alpha \setminus (\alpha + 1)) \} \cap W = \emptyset \} \in (U^\alpha)^{Q^\gamma_{U^\alpha}} " \].

Proposition 5.8. $\forces_{Q^\gamma_{U^\alpha}} (\forall f \in \lambda \wedge V) (\forall A \in U^\alpha) (\exists \alpha \in A) (f(\alpha) < \nu(\alpha))$.

Proof. Fully parallel to 4.10. \hfill \Box

Proposition 5.9. The forcing notion $Q^\alpha_{U^\alpha}$ is reasonably $C$-bounding over $U$. 

\[ \text{modified: 2006-09-13} \]
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Proof. Fully parallel to 4.11. □

The following problem is a particular case of 2.9(1).

Problem 5.10. Are $\lambda$–support iterations of $Q^2_U$ $\lambda$–proper?

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