POWER SET MODULO SMALL, THE SINGULAR OF UNCOUNTABLE COFINALITY

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Abstract. Let μ be singular of uncountable cofinality. If $\mu > 2^{\mathrm{cf}(\mu)}$, we prove that in $\mathbb{P} = ([\mu]^\mu, \supseteq)$ as a forcing notion we have a natural complete embedding of $\mathrm{Levy}(\aleph_0, \mu^+)$ (so $\mathbb{P}$ collapses μ^+ to $\aleph_0$) and even $\mathrm{Levy}(\aleph_0, \mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\mu))$. The "natural" means that the forcing $(\{p \in [\mu]^\mu : p \text{ closed}\}, \supseteq)$ is naturally embedded and is equivalent to the Levy algebra. Moreover we prove more than conjectured: if $\mathbb{P}$ fails the χ-c.c. then it collapses χ to $\aleph_0$. We even prove the parallel results for the case $\mu > \aleph_0$ is regular or of countable cofinality. We also prove: for regular uncountable κ, there is a family $\mathbf{P}$ of $\mathfrak{b}_\kappa$ partitions $\bar{A} = \langle A_\alpha : \alpha < \kappa \rangle$ of κ such that for any $A \in [\kappa]^\kappa$ for some $\langle A_\alpha : \alpha < \kappa \rangle \in \mathbf{P}$ we have $\alpha < \kappa \Rightarrow |A_\alpha \cap A| = \kappa$.

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§0 Introduction

This work on the one hand continue the celebrated work of the Czech school on the completion of the Boolean algebras $\mathscr{P}(\lambda)/[\lambda]^{<\lambda}$ solving some of their questions and on the other hand tries to confirm the "pcf is effective" thesis.

We may consider the completions of the Boolean Algebras $\mathscr{P}(\mu)/\{u \subseteq \mu : |u| < \mu\} = \mathscr{P}(\mu)/[\mu]^{<\mu}$. This is equivalent to considering the partial orders $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$, viewing them as forcing notions, so actually looking at their completion $\hat{\mathbb{P}}_\mu$, which are complete Boolean Algebras. Recall that forcing notions $\mathbb{P}^1, \mathbb{P}^2$ are equivalent iff their completions are isomorphic Boolean Algebras. The Czech school has investigated them, in particular, (letting $\ell(\mu)$ be 0 if $\mathrm{cf}(\mu) > \aleph_0$ and 1 if $\mu > \mathrm{cf}(\mu) = \aleph_0$, (and $\aleph_{\ell(\mu)} = \mathfrak{h}$ if $\mu = \aleph_0$) consider the questions:

$\otimes_1$ (a) is $\hat{\mathbb{P}}_\mu$ isomorphic to the completion of the Levy collapse $\mathrm{Levy}(\aleph_{\ell(\mu)}, 2^\mu)$?

(b) which cardinals χ the forcing notion $\mathbb{P}_\mu$ collapse to $\aleph_{\ell(\mu)}$ in particular is μ^+ collapsed

(c) is $\mathbb{P}_\mu$ (θ, χ)-nowhere distributive for $\theta = \aleph_{\ell(\mu)}$? This can be phrased as: for some $\mathbb{P}_\mu$-name $\underset{\sim}{f}$ of a function from $\aleph_{\ell(\mu)}$ to χ, for every $p \in \mathbb{P}_\mu$ for some $i < \theta$ the set $\{\alpha < \chi : p \nVdash \underset{\sim}{f}(i) \neq \alpha\}$ has cardinality χ.

The first, (a) is a full answer, the second, (b)seems central for set theories and essentially give sufficient condition for the first, the last is sufficient if the density is right, to get the first. The case of collapsing seems central (it also implies clause (c)) so we repeat the summary from Balcar, Simon [BaSi95] of what was known of the collapse of cardinals by $\mathbb{P}_\mu$, i.e., $\otimes_1(b)$. Let $\chi \to_\mu \theta$ denote the fact that χ is collapsed to θ by $\mathbb{P}_\mu$

$\boxtimes_1$ (i) for $\mu = \aleph_0, 2^{\aleph_0} \to_\mu \mathfrak{h}$, (but $\mathbb{P}_\mu$ adds no new sequence of length $< \mathfrak{h}$ so we are done), Balcar, Pelant, Simon [BPS]

(ii) for μ uncountable and regular, $\mathfrak{b}_\mu \to_\mu \aleph_0$, (hence $\mu^+ \to_\mu \aleph_0$), Balcar, Simon [BaSi88]

(iii) for μ singular with $\mathrm{cf}(\mu) = \aleph_0, 2^{\aleph_0} \to_\mu \aleph_1$, Balcar, Simon [BaSi95]

(iv) for μ singular with $\mathrm{cf}(\mu) \neq \aleph_0, \mathfrak{b}_{\mathrm{cf}(\mu)} \to_\mu \aleph_0$, Balcar, Simon [BaSi95];

under additional assumptions on cardinal arithmetic for singular cardinals more is known

(v) for μ singular with $\mathrm{cf}(\mu) = \aleph_0$ and $\mu^{\aleph_0} = 2^\mu, \mu^{\aleph_0} \to_\mu \aleph_1$,

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Balcar, Simon [BaSi88]

(*vi*) for μ singular with $\mathrm{cf}(\mu) \neq \aleph_0$ and $2^\mu = \mu^+, 2^\mu \to_\mu \aleph_0$, [BaSi88].

Now [BaSi95] finish with the following very reasonable conjecture.
<u>0.1 Conjecture</u>: (Balcar and Simon) in ZFC: for a singular cardinal μ with countable cofinality, $\mu^{\aleph_0} \to_\mu \aleph_1$ and for a singular cardinal μ with an uncountable cofinality $\mu^+ \to_\mu \aleph_0$ (here we concentrate on the case $\mathrm{cf}(\mu) > \aleph_0$, see below).

Concerning the other questions they prove

$\boxtimes_2$ (*i*) Balcar, Franek [BaFr87]:
if $\mu > \mathrm{cf}(\mu) > \aleph_0, 2^{\mathrm{cf}(\mu)} = \mathrm{cf}(\mu)^+$ then $\mathbb{P}_\mu$ is $(\aleph_0, \mu^+)$-nowhere distributive

(*ii*) Balcar, Simon [BaSi89, 5.20, pg.380]:
if $2^\mu = \mu^+$ and $2^{\mathrm{cf}(\mu)} = \mathrm{cf}(\mu)^+$ <u>then</u> $\mathbb{P}_\mu$ is equivalent:
to $\mathrm{Levy}(\aleph_0, \mu^+)$ if $\mathrm{cf}(\mu) > \aleph_0$ and
to $\mathrm{Levy}(\aleph_1, \mu^+)$ if $\mathrm{cf}(\mu) = \aleph_0$

(*iii*) Balcar, Franek [BaFr87]:
if $2^\mu = \mu^+, \mu = \mathrm{cf}(\mu) > \aleph_0, J$ a μ-complete ideal on μ and J is nowhere precipitous extending $[\mu]^{<\mu}$ then $\mathscr{P}(\mu)/J$ is equivalent to $\mathrm{Levy}(\aleph_0, \mu^+)$; also the parallel of (ii).

So under G.C.H. the picture was complete; getting clause (*ii*) of $\boxtimes_2$. Also under ZFC for regular cardinals $\mu > \aleph_0$ the picture is reasonable, particularly if we recall that by Baumgartner [Ba]

$\boxtimes_3$ if $\kappa = \mathrm{cf}(\mu) < \theta = \theta^{<\theta} < \mu < \chi$ and $\mathbf{V} \models$ G.C.H. for simplicity and $\mathbb{P}$ is forcing for adding χ Cohen subsets to θ then

(*a*) forcing with $\mathbb{P}$ collapses no cardinal, changes no cofinality, adds no new sets of $< \theta$ ordinals

(*b*) in $\mathbf{V}^{\mathbb{P}}, ([\mu]^\mu, \supseteq)$ satisfies the μ_1^+-c.c. where $\mu_1 = (2^\mu)^{\mathbf{V}}$; hence does not collapse any cardinal $\geq \mu_1^+$.

Lately, Kojman, Shelah [KjSh 720] prove the conjecture 0.1 for the case when $\mu > \mathrm{cf}(\mu) = \aleph_0$; morever

$\boxtimes_4$ (*i*) if $\mu > \mathrm{cf}(\mu) = \aleph_0$ then $\mathrm{Levy}(\aleph_1, \mu^{\aleph_0})$ can be completely embedded into the completion of $\mathbb{P}_\mu$. Moreover,

(*ii*) the embedding is "natural": $\mathrm{Levy}(\aleph_1, \mu^{\aleph_0})$ is equivalent to $\mathbb{Q}_\mu$ which is $\lessdot \mathbb{P}_\mu$ where
$\mathbb{Q}_\mu = (\{A \subseteq \mu : A$ a closed subset of μ of cardinality $\mu\}, \supseteq)$.

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Here we continue [KjSh 720] in §1, [BaSi89] in §2 but make it self contained. Both sections use results on pcf (in addition to guessing clubs) Naturally we may add to the questions (answered positively for the case $\mathrm{cf}(\mu) = \aleph_0$ by [KjSh 720])

$\otimes_2$ (a) can we strengthen "$\mathbb{P}_\mu$ collapse χ to $\aleph_{\ell(\mu)}$" to "Levy$(\aleph_{\ell(\mu)}, \chi)$ is completely embeddable into $\mathbb{P}_\mu$ (really $\hat{\mathbb{P}}_\mu$)"

(b) can we find natural such embeddings.

We may add that by [BaSi95] the Baire number of $\mathscr{U}[\mu]$, the space of all uniform ultrafilters over uncountable μ is $\aleph_1$, except when $\mu > \mathrm{cf}(\mu) = \aleph_0$ and in that case it is $\aleph_2$ under some reasonable assumptions. By [KjSh 720] the Baire number of $\mathscr{U}[\mu]$ is always $= \aleph_2$ when $\mu > \mathrm{cf}(\mu) = \aleph_0$.

Our original aim in this work has been to deal with $\mu > \mathrm{cf}(\mu) > \aleph_0$, proving the conjecture of Balcar and Simon above (i.e., that μ^+ is collapsed to $\aleph_0$), first of all when $2^{\mathrm{cf}(\mu)} < \mu$ answering $\otimes_2(a)+(b)$ using pcf (and replacing μ^+ by $\mathrm{pp}_{J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}}(\mu)$). In fact this seems, at least to me, the best we can reasonably expect. But a posteriori we have more to say.

For $\mu = \kappa = \mathrm{cf}(\mu) > \aleph_0$, though by the above we know that some cardinal $> \mu$ is collapsed (that is $\mathfrak{b}_\kappa$), we do not know what occurs up to 2^μ or when the c.c. fails. This leads to the following conjecture, (stronger than the Balcar, Simon one mentioned above). Of course, it naturally breaks to cases according to μ.

0.2 Conjecture. If $\mu > \aleph_0$ and $\mathbb{P}_\mu$ does not satisfy the χ-c.c., <u>then</u> forcing with $\mathbb{P}_\mu$ collapse χ to $\aleph_{\ell(\mu)}$, see Definition 0.6 below.

Note that

0.3 Observation. If conjecture 0.2 holds for $\mu > \aleph_0$ then $\mathbb{P}_\mu$ is equivalent to a Levy collapse iff it fails the $d(\mathbb{P}_\mu)$-c.c. where $d(\mathbb{P}_\mu)$ is the density of $\mathbb{P}_\mu$.

Lastly, we turn to the results; by 1.16(1):

0.4 Theorem. *If $\mu > \kappa = \mathrm{cf}(\mu) > \aleph_0$ and $\mu > 2^\kappa$ <u>then</u> $\mathbb{Q}_\mu$ (a natural complete subforcing of $\mathbb{P}_\mu$, forcing with closed sets) is equivalent to Levy$(\aleph_0, \mathbf{U}_{J^{\mathrm{bd}}_\kappa}(\mu))$.*

By 1.17, 1.18 and 2.7 we have

0.5 Theorem. *Conjecture 0.2 holds except possibly when $\aleph_0 < \mathrm{cf}(\mu) < \mu < 2^{\mathrm{cf}(\mu)}$.*

We shall in a subsequent paper prove the Balcar, Simon conjecture fully, i.e., in all cases.

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0.6 Definition. For $\mu > \aleph_0$ we define $\ell(\mu) \in \{0, 1\}$ by
$\ell(\mu) = 0$ if $\mathrm{cf}(\mu) > \aleph_0$
$\ell(\mu) = 1$ if $\mu > \mathrm{cf}(\mu) = \aleph_0$
and may add
$\ell(\mu) = \alpha$ when $\mu = \aleph_0, \mathfrak{h} = \aleph_\alpha$.

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§1 Forcing with closed set is equivalent to the Levy algebra

1.1 Definition. 1) For $f \in {}^{\kappa}(\mathrm{Ord}\backslash\{0\})$ and ideal I on κ let

$$\begin{aligned}\mathbf{U}_I(f) = \mathrm{Min}\{|\mathscr{P}| :& \mathscr{P} \subseteq [\sup \mathrm{Rang}(f)]^{\leq\kappa} \\ & \text{such that for every } g \leq f \text{ for some } u \in \mathscr{P} \\ & \text{we have } \{i < \kappa : g(i) \in u\} \in I^+\}.\end{aligned}$$

2) Let $\mathbf{U}_I(\lambda)$ means $\mathbf{U}_I(f)$ where f is the function with domain $\mathrm{Dom}(I)$ which is constantly λ

1.2 Hypothesis.

(a) μ is a singular cardinal
(b) $\kappa = \mathrm{cf}(\mu) > \aleph_0$.

1.3 Definition. 1) $\mathbb{P}_\mu$ is the following forcing notion

$$p \in \mathbb{P}_\mu \ \underline{\text{iff}}\ p \in [\mu]^\mu$$

$$\mathbb{P}_\mu \models p \leq q \text{ iff } p \supseteq q.$$

2) $\mathbb{P}'_\mu$ is the forcing notion with the same set of elements and with the partial order

$$\mathbb{P}'_\mu \models p \leq q \text{ iff } |q\backslash p| < \mu.$$

3) $\mathbb{Q}_\mu = \mathbb{Q}^0_\mu$ is $\mathbb{P}_\mu \restriction \{p \in \mathbb{P}_\mu : p \text{ is closed in the order topology of } \mu\}$.

1.4 Choice/Definition. 1) Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence of regular cardinals $> \kappa$ with limit μ.
2) Let $\lambda_i^- = \cup\{\lambda_j : j < i\}$.
3) For $p \in \mathbb{P}_\mu$ let $a(p) = \{i < \kappa : p \cap [\lambda_i^-, \lambda_i) \neq \emptyset\}$.
4) $\mathbb{Q}^1_\mu = \{p \in \mathbb{P}_\mu : i < \kappa \Rightarrow |p \cap \lambda_i| < \lambda_i$ and for each $i \in a(p)$ the set $p \cap \lambda_i \backslash \lambda_i^-$ has no last element, is closed in its supremum and has cardinality $> |p \cap \lambda_i^-|\}$.
5) For $p \in \mathbb{Q}^1_\mu$ let $\mathrm{ch}_p \in \prod\limits_{i \in a(p)} \lambda_i$ be $\mathrm{ch}_p(i) = \cup\{\alpha + 1 : \alpha \in p \cap [\lambda_i^-, \lambda_i)\}$ and $\mathrm{cf}_p \in \prod\limits_{i \in a(p)} \lambda_i$ be $\mathrm{cf}_p(i) = \mathrm{cf}(\mathrm{ch}_p(i))$.
6) $\mathbb{Q}^2_\mu = \{p \in \mathbb{Q}^1_\mu : \mathrm{cf}_p(i) > |p \cap \lambda_i^-| \text{ for } i \in a(p)\}$.

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1.5 Claim. *1)* $\mathbb{Q}^0_\mu, \mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$ *are complete sub-forcings of* $\mathbb{P}_\mu$.
2) For $\ell = 0,1,2$ *and* $p, q \in \mathbb{Q}^\ell_\mu$ *we have* $p \Vdash_{\mathbb{Q}^\ell_\mu}$ "$q \in \underset{\sim}{G}$" *iff* $|p \backslash q| < \mu$ *and similarly for* $\mathbb{P}_\mu$.
3) $\mathbb{Q}_\mu = \mathbb{Q}^0_\mu, \mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$ *are equivalent, in fact* $\mathbb{Q}^2_\mu$ *is a dense subset of* $\mathbb{Q}^1_\mu$ *and for* $\ell = 0,1, \{p/\approx : p \in \mathbb{Q}^\ell_\mu\}$ *does not depend on* ℓ *where* $\approx$ *is the equivalence relation of* $\mathbb{P}_\mu$, *defined by* $p_1 \approx p_2$ *iff* $(\forall q \in \mathbb{P}_\mu)(q \Vdash_{\mathbb{P}_\mu} p_1 \in \underset{\sim}{G} \Leftrightarrow q \Vdash_{\mathbb{P}_\mu} p_2 \in \underset{\sim}{G})$.

Proof. Easy.

Recall

1.6 Claim. *1)* $\mathbb{P}_\kappa$ *can be completely embedded into* $\mathbb{P}_\mu$ *(naturally).*
2) $\mathbb{Q}_\mu$ *can be completely embedded into* $\mathbb{P}_\mu$ *(naturally).*
3) $\mathbb{P}_\kappa$ *is completely embeddable into* $\mathbb{Q}_\mu$ *(naturally).*

Proof. 1) Known: just $a \in [\kappa]^\kappa$ can be mapped to $\cup\{[\lambda^-_i, \lambda_i) : i \in a\}$.
2) By [KjSh 720, 2.2].
3) Should be clear (map $A \in [\kappa]^\kappa$ to $\cup\{[\lambda^-_i, \lambda_i] : i \in A\}$). $\square_{1.6}$

1.7 Choice/Definition. $\lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$.

Recall

1.8 Claim. *Assume* $\mu > 2^\kappa$.
1) $\lambda_* = \sup\{{\rm pp}_{J^{\rm bd}_\kappa}(\mu') : \kappa < \mu' \le \mu,\ cf(\mu') = \kappa\} = \sup\{{\rm tcf}(\prod\limits_{i<\kappa} \lambda'_i/J^{\rm bd}_\kappa) : \lambda'_i \in {\rm Reg} \cap (\kappa, \mu)$ *and* $\prod\limits_{i<\kappa} \lambda'_i/J^{\rm bd}_\kappa$ *has true cofinality*$\}$.
2) For every regular cardinal $\theta \in [\mu, \lambda_*]$, *for some increasing sequence* $\langle \lambda^*_i : i < \kappa\rangle$ *of regulars* $\in (\kappa, \mu)$ *we have* $\theta = {\rm tcf}(\prod\limits_{i<\kappa} \lambda^*_i, <_{J^{\rm bd}_\kappa})$.
3) $\lambda_* = |\mathscr{P}|$ *where* $\mathscr{P} \subseteq [\mu]^\kappa$ *is any maximal almost disjoint family.*

Proof. 1) Note that $J^{\rm bd}_\kappa \restriction A \approx J^{\rm bd}_\kappa$ if $A \in (J^{\rm bd}_\kappa)^+$, we use this freely. By their definitions the second and third terms are equal. Also by the definition the second is smaller or equal to the first.

By [Sh 589, 1.1], the first, $\lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$ is $\le$ than the second number (well it speaks on $T^2_{J^{\rm bd}_\kappa}(\mu)$ instead of $\mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$ but as $2^\kappa < \mu$ they are the same).

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2) By [Sh 589, 1.1] we actually get the stronger conclusion.
3) It follows easily from the definitions 1.1 and 1.7, and from the inequalities $2^\kappa < \mu < \lambda_*$. $\square_{1.8}$

1.9 Claim/Definition. Fix a set $\mathscr{P} \subseteq [\mu]^\kappa$ exemplifying $\lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$.
1) There is $\bar{C}^* = \langle C^*_\alpha : \alpha < \mu \rangle$ such that:

(a) C^*_α is a subset of $[\lambda^-_i, \lambda_i)$ closed in its supremum when $\alpha \in (\lambda^-_i, \lambda_i]$
(b) if $i < \kappa, \gamma < \lambda_i$, γ is a regular cardinal and C is a closed subset of $[\lambda^-_i, \lambda_i)$ of order type γ^{++}, then for some $\alpha \in (\lambda^-_i, \lambda_i), C^*_\alpha \subseteq C$ and $\mathrm{otp}(C^*_\alpha) = \gamma$.

2) $\mathbb{Q}^3_\mu = \{p \in \mathbb{Q}^2_\mu :$ if $i \in a(p)$ then $p \cap [\lambda^-_i, \lambda_i) \in \{C^*_\alpha : \alpha \in (\lambda^-_i, \lambda_i)\}$ and for some $u \in \mathscr{P}$, $\{\alpha < \mu :$ for some $i < \kappa, p \cap [\lambda^-_i, \lambda_i) = C^*_\alpha\} \subseteq u\}$ is a dense subset of $\mathbb{Q}^1_\mu, \mathbb{Q}^2_\mu$, hence of $\mathbb{Q}_\mu$.
3) For $p \in \mathbb{Q}^3_\mu$ let $\mathrm{cd}_p \in \prod\limits_{i \in a(p)} \lambda_i$ be such that $\mathrm{cd}_p(i) \in (\lambda^-_i, \lambda_i)$ is the minimal $\alpha \in [\lambda^-_i, \lambda_i)$ such that $p \cap [\lambda^-_i, \lambda_i) = C^*_\alpha$. Notice that for every $p \in \mathbb{Q}^3_\mu$, there is some $u \in \mathscr{P}$ with $\mathrm{Rang}(\mathrm{cd}_p) \subseteq u$.

Proof. 1) It is enough, for any limit $\delta \in (\lambda^-_i, \lambda_i)$ and regular $\theta, \theta^+ < \mathrm{cf}(\delta)$, to find a family $\mathscr{P}_{\delta,\theta}$ of closed subsets of (λ^-_i, δ) of order type θ such that any club of δ contains (at least) one of them. This holds by guessing clubs, see [Sh:g, III,§2].
2), 3) By the definitions. $\square_{1.9}$

1.10 Claim. *1) If $\mu > 2^\kappa$ (or just $\lambda_* \geq 2^\kappa$) <u>then</u> $\mathbb{Q}^2_\mu$ (hence $\mathbb{Q}^1_\mu$) has a dense subset of cardinality λ_*.*
2) If $\mu > 2^\kappa$ (or just $\lambda_* \geq 2^\kappa$) <u>then</u> $\mathbb{Q}^3_\mu$ is a dense subset of $\mathbb{Q}^1_\mu$ and has cardinality λ_*.

Proof. 1) By part (2).
2) By 1.9(2) it suffices to deal with $\mathbb{Q}^3_\mu$. The cardinality of the set $\mathscr{P}$ from 1.9 is λ_*. Whenever $p \in \mathbb{Q}^3_\mu$, then the function cd_p is uniquely determined by its range, because $i \in \mathrm{Dom}(\mathrm{cd}_p)$ iff $\mathrm{Rang}(\mathrm{cd}_p) \cap [\lambda^-_i, \lambda_i) \neq \emptyset$ and the value $\mathrm{cd}_p(i) = \alpha$ iff $\alpha \in [\lambda^-_i, \lambda_i) \cap \mathrm{Rang}(\mathrm{cd}_p)$. Also, the function cd_p uniquely determines p by $p = \bigcup\{C^*_{\mathrm{cd}_p(i)} : i \in \mathrm{Dom}(p)\}$. Since $\mathrm{Rang}(\mathrm{cd}_p) \subseteq u$, $u \in \mathscr{P}$, we get $|\mathbb{Q}^3_\mu| \leq 2^\kappa \cdot \lambda_* = \lambda_*$. $\square_{1.10}$

From now on (till the end of this section)

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1.11 Hypothesis. $2^\kappa < \mu$ (in addition to 1.2).

Recall (Claim 1.13(1) is Balcar, Simon [BaSi89, 1.15] and 1.13(2) is a variant).

1.12 Definition. A forcing notion $\mathbb{P}$ is (θ,λ)-nowhere distributive <u>when</u> there are maximal antichains $\bar{p}^\varepsilon = \langle p^\varepsilon_\alpha : \alpha < \alpha_\varepsilon\rangle$ of $\mathbb{P}$ for $\varepsilon < \theta$ such that for every $p \in \mathbb{P}$ for some $\varepsilon < \theta$, we have $\lambda \le |\{\alpha < \alpha_\varepsilon : p, p^\varepsilon_\alpha$ are compatible$\}|$.

1.13 Claim. *1) If*

(a) *$\mathbb{P}$ is a forcing notion, (θ,λ)-nowhere distributive*

(b) *$\mathbb{P}$ has density λ*

(c) *$\theta > \aleph_0 \Rightarrow \mathbb{P}$ has a θ-complete dense subset*

<u>then</u> $\mathbb{P}$ is equivalent to Levy(θ,λ).
2) If $\mathbb{P}$ is a forcing notion of density λ collapsing λ to $\aleph_0$ <u>then</u> $\mathbb{P}$ is equivalent to Levy$(\aleph_0,\lambda)$.
3) If $\mathbb{P}$ is a forcing notion of density λ and is (θ,λ)-nowhere distributive <u>then</u> $\mathbb{P}$ collapses λ to θ (and may or may not collapse θ). $\square_{1.13}$

1.14 Claim. *Assume $\langle b_\varepsilon : \varepsilon < \kappa\rangle$ is a sequence of pairwise disjoint members of $[\kappa]^\kappa$ with union b. <u>Then</u> we can find an antichain $\mathscr{I}$ of $\mathbb{Q}^3_\mu$ such that:*

(∗) *if $q \in \mathbb{Q}^3_\mu$ and $(\forall\varepsilon < \kappa)(a(q) \cap b_\varepsilon \in [\kappa]^\kappa)$, <u>then</u> q is compatible with $\lambda_* =: \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$ of the members of $\mathscr{I}$.*

Proof. Let

$$\begin{aligned}\mathscr{I}^* = \{p \in \mathbb{Q}^3_\mu :\ & \text{we can find an increasing sequence } \langle i_\varepsilon : \varepsilon < \kappa\rangle \\ & \text{such that } i_\varepsilon \in b_\varepsilon \backslash \varepsilon, a(p) \subseteq \{i_\varepsilon : \varepsilon < \kappa\} \text{ and} \\ & i_\varepsilon \in a(p) \Rightarrow p \cap [\lambda^-_{i_\varepsilon}, \lambda_{i_\varepsilon}) \text{ has order type } \lambda_\varepsilon\}.\end{aligned}$$

Let $\mathscr{J}^* = \{p \in \mathbb{Q}^3_\mu$: for every $\varepsilon < \kappa$ we have $a(p) \cap b_\varepsilon \in [\kappa]^\kappa\}$.

Clearly

(a) $|\mathscr{I}^*| \le \lambda_* = \mathbf{U}_{J^{\rm bd}_\kappa}(\mu)$
[Why? As $\mathscr{I}^* \subseteq \mathbb{Q}^3_\mu$.]

(b) if $\mathscr{I} \subseteq \mathscr{I}^*, |\mathscr{I}| < \lambda_*$ and $q \in \mathscr{J}^*$ then there is r such that $q \le r \in \mathscr{I}^*$ and r is incompatible with every $p \in \mathscr{I}$.

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[Why? Let $\theta = |\mathscr{I}|+\mu$, it is $< \lambda_*$, hence we can find an increasing sequence $\langle \theta_\varepsilon : \varepsilon < \kappa \rangle$ of regular cardinals with limit μ such that $\prod\limits_{\varepsilon<\kappa} \theta_\varepsilon / J^{\rm bd}_\kappa$ has true cofinality θ^+, this by 1.8 + the no hole lemma [Sh:g, II,§3]. By renaming without loss of generality $\theta_\varepsilon > \lambda_\varepsilon$.

Let $u = \{\varepsilon < \kappa : a(q) \cap b_\varepsilon \in [\kappa]^\kappa\}$, so we know that u is κ. For each $\varepsilon \in u$ we know that $a(q) \cap b_\varepsilon \in [\kappa]^\kappa$, and so for some $\zeta_\varepsilon < \kappa$ we have $\theta_\varepsilon < \lambda_{{\rm otp}(a(q)\cap\zeta_\varepsilon)}$. Now choose $i(\varepsilon) \in b_\varepsilon$ such that $i(\varepsilon) > \varepsilon \wedge i(\varepsilon) > \zeta_\varepsilon \wedge (\forall \varepsilon_1 < \varepsilon)(i(\varepsilon_1) < i(\varepsilon))$. As $q \in \mathbb{Q}^3_\mu$ it follows that $(q \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}))$ has order type $\geq \lambda_{{\rm otp}(a(q)\cap\zeta_\varepsilon)} > \theta_\varepsilon$. Let $C_{q,\varepsilon} = \{\alpha : \alpha \in q, \alpha \in [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)})$ and ${\rm otp}(q \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}) \cap \alpha)$ is $< \theta_\varepsilon\}$. Now for every $p \in \mathscr{I}^*$ the set $p \cap [\lambda^-_{i(\varepsilon)}, \lambda_{i(\varepsilon)}) \subseteq \cup\{[\lambda^-_i, \lambda_i) : i \in b_\varepsilon\}$ if non-empty has cardinality $\leq \lambda_\varepsilon$ which is $< \theta_\varepsilon$ hence $p \cap C_{q,\varepsilon}$ is a bounded subset of $C_{q,\varepsilon}$, call the lub $\alpha_{p,\varepsilon}$. As $\theta = |\mathscr{I}| + \mu < {\rm tcf}(\prod\limits_{\varepsilon<\kappa} \theta_\varepsilon / J^{\rm bd}_\kappa)$ clearly there is $h \in \prod\limits_{\varepsilon\in u} C_{q,\varepsilon}$ such that $p \in \mathscr{I}^* \Rightarrow \langle \alpha_{p,\varepsilon} : \varepsilon < \kappa \rangle <_{J^{\rm bd}_\kappa} h$ and let

$$\begin{aligned} r = \{\alpha :& \text{for some } \varepsilon \in u \text{ we have } \alpha \in C_{q,\varepsilon} \backslash h(\varepsilon) \\ & \text{and } |C_{q,\varepsilon} \cap \alpha \backslash h(\varepsilon)| < \lambda_\varepsilon\}. \end{aligned}$$

So r is as required in clause (b). (We can assume that $r \in \mathbb{Q}^3_\mu$, since by the density propositions of 1.10 we can find $r \leq r' \in \mathbb{Q}^3_\mu$ as required.) So clause (b) holds.]
As by 1.10(2) in the conclusion of the claim it is enough to deal with $q \in \mathbb{Q}^3_\mu$, there are only λ_* such q's so we can finish easily by (clause (b) and) diagonalization. $\square_{1.14}$

1.15 Claim. *The forcing notion $\mathbb{Q}^3_\mu$ is $(\mathfrak{b}_\kappa, \lambda_*)$-nowhere distributive.*

Proof. Let $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ be such that: $\bar{A}_\alpha = \langle A_{\alpha,i} : i < \kappa \rangle, A_{\alpha,i} \in [\kappa]^\kappa, i < j \Rightarrow A_{\alpha,i} \cap A_{\alpha,j} = \emptyset$ and $(\forall B \in [\kappa]^\kappa)(\exists \alpha < \mathfrak{b}_\kappa)(\forall i < \kappa)[\kappa = |B \cap A_{\alpha,i}|]$, exists by 2.7(2) below. Hence for each $\alpha < \mathfrak{b}_\kappa, \mathscr{I}^*_\alpha \subseteq \mathbb{Q}^3_\mu$ as in 1.14 for the sequence $\bar{A}_\alpha$ exists. So $\langle \mathscr{I}^*_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ is a sequence of $\mathfrak{b}_\kappa$ antichains of $\mathbb{Q}^3_\mu$ and we shall show that it witnesses the conclusion. Now

- $\circledast$ if $q \in \mathbb{Q}^3_\mu$ then for some $\alpha < \mathfrak{b}_\kappa$ the set $\{p \in \mathscr{I}^*_\alpha : p$ compatible with $q \in \mathbb{Q}^3_\mu\}$ has cardinality λ_*.

Why? By the choice of $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ there is $\alpha < \mathfrak{b}_\kappa$ such that

- $(*)$ $a(q) \cap A_{\alpha,i} \in [\kappa]^\kappa$ for every $i < \kappa$.

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Hence q fits the demand in 1.14 with $\bar{A}_\alpha$ here standing for $\langle b_\varepsilon : \varepsilon < \kappa \rangle$. Hence it is compatible with λ_* members of $\mathscr{I}^*_\alpha$ which, of course, shows that we are done.

$\square_{1.15}$

1.16 Conclusion. 1) If $2^\kappa < \mu$ (and $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$, of course) <u>then</u> $\mathbb{Q}_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \lambda_*)$, i.e., they have isomorphic completions (recalling $\mathbb{Q}_\mu$ is naturally completely embeddable into the completion of $\mathbb{P}_\mu = ([\mu]^\mu, \supseteq)$).
2) If $(\forall \alpha < \mu)(|\alpha|^\kappa < \mu)$ <u>then</u> $\mathbb{Q}_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \mu^\kappa)$.
3) If μ is strong limit (singular of uncountable cofinality κ), <u>then</u> $\mathbb{P}_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \mu^\kappa) = \mathrm{Levy}(\aleph_0, 2^\mu)$.

Proof. 1) By 1.10(1), $\mathbb{Q}^3_\mu$ has density (even cardinality) λ_* and by 1.15 it is $(\mathfrak{b}_\kappa, \lambda_*)$-nowhere distributive hence by 1.13(3), we know that $\mathbb{Q}^3_\mu$ collapses λ_* to $\mathfrak{b}_\kappa$. But $\mathbb{P}_\kappa$ is completely embeddable into $\mathbb{Q}^2_\mu$ (see 1.6(3)) and $\mathbb{P}_\kappa$ collapses $\mathfrak{b}_\kappa$ to $\aleph_0$ (e.g. see §2) and $\mathbb{Q}^3_\mu$ is dense in $\mathbb{Q}^2_\mu$. Together forcing with $\mathbb{Q}^3_\mu$ collapses λ_* to $\aleph_0$. As $\mathbb{Q}^3_\mu$ has density λ_*, by 1.13(2) we get that $\mathbb{Q}^2_\mu$ is equivalent to $\mathrm{Levy}(\aleph_0, \lambda_*)$.

Lastly $\mathbb{Q}_\mu, \mathbb{Q}^3_\mu$ are equivalent by 1.5(3) + 1.9(2) so we are done.
2) Recalling 1.8, by [Sh:g, VIII] we have $\lambda_* = \mu^\kappa$ (alternatively directly as in [Sh 506, §3]). Now apply part (1).
3) By easy cardinal arithmetic $\mu^\kappa = 2^\mu$. Enough to check the demands in 1.13(2). Now as $\mathbb{Q}_\mu$ collapses λ_* to $\aleph_0$ by part (1) and $\mathbb{Q}_\mu$ can be completely embeddable into $\mathbb{P}_\mu$ (see 1.6(2)) clearly $\mathbb{P}_\mu$ collapses λ_* to $\aleph_0$. But $|\mathbb{P}_\mu| \le |[\mu]^\mu| = 2^\mu$, so $\mathbb{P}_\mu$ has density $\le 2^\mu$.

Lastly $\lambda_* = 2^\mu$ by [Sh:g, VIII]. So we are done. $\square_{1.16}$

1.17 Claim. *Assume that $\mathbb{P}_\mu$ does not satisfy the χ-c.c. <u>Then</u> forcing with $\mathbb{P}_\mu$ collapses χ to $\aleph_0$.*

Proof. By the nature of the conclusion without loss of generality χ is regular. Now we can find $\bar{X}$ such that

$(*)_1$ (a) $\bar{X} = \langle X_\xi : \xi < \chi \rangle$
(b) $X_\xi \in \mathbb{P}_\mu$
(c) $X_\zeta \cap X_\xi \in [\mu]^{<\mu}$ for $\zeta \ne \xi < \chi$.

As $\mathbb{Q}_\mu \lessdot \mathbb{P}_\mu$, by the earlier proof (e.g., 1.16(1)) it suffices to prove that $\mathbb{P}_\mu$ collapses χ to λ_*. There exists $\mathbf{P} \subseteq \mathbf{P}_* := \{\bar{A} : \bar{A} = \langle A_\alpha : \alpha < \mu \rangle$, the A_α's are pairwise disjoint and each A_α belongs to $[\mu]^\mu\}$ such that $|\mathbf{P}| = \lambda_*$ and

$(*)_2$ for every $p \in \mathbb{P}_\mu$ there is an $\bar{A} \in \mathbf{P}$ such that $(\forall \alpha < \mu)[|A_\alpha \cap p| = \mu]$.

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[Why? For each $i < \kappa$ fix some partition $\langle W_{i,\alpha} : \alpha < \lambda_i \rangle$ of λ_i into λ_i (pairwise disjoint) sets each of cardinality λ_i. Now for each $p \in \mathbb{P}_\mu$ we shall choose $\bar{A} = \bar{A}^p \in \mathbf{P}_\lambda$ as required in $(*)_2$ such that $\mathbf{P} := \{\bar{A}^p : p \in \mathbb{P}_\mu\}$ has cardinality $\leq \lambda_*$ this suffice; so fix $p \in \mathbb{P}_\mu$. By induction on $\varepsilon < \kappa$ we can find $\delta_\varepsilon < \mu$ of cofinality λ_ε^{++} such that $p \cap \delta_\varepsilon$ is unbounded in δ_ε and $\delta_\varepsilon > \cup\{\delta_\zeta : \zeta < \varepsilon\}$. There is a club C^1_ε of δ_ε of order type λ_ε^{++} with $\min(C^1_\varepsilon) > \cup\{\delta_\zeta : \zeta < \varepsilon\}$. Let $C^2_\varepsilon = \{\delta \in C^1_\varepsilon : \delta$ is a limit ordinal such that $C^1_\epsilon \cap p$ is unbounded in δ and has order type divisible by $\lambda^+_\epsilon\}$, it is a club of δ_ε. But by the club guessing (see 1.9) there is C^3_ε such that: $C^3_\varepsilon \subseteq C^2_\varepsilon (\subseteq C^1_\varepsilon)$ and $\mathrm{otp}(C^3_\varepsilon) = \lambda_\varepsilon$.

By the definition of $\mathbb{Q}^3_\mu$, there is some $a \in [\kappa]^\kappa$ such that $\bigcup\{C^3_\varepsilon : \varepsilon \in a\} \in \mathbb{Q}^3_\mu$. Lastly, let us define $\bar{A} = \langle A_\alpha : \alpha < \mu \rangle$ by

$$\begin{aligned} A_\alpha = \cup\{[\beta, \min(C^3_\varepsilon \backslash (\beta+1)) : & \varepsilon \in a \text{ satisfies} \\ & \alpha < \lambda_\varepsilon \text{ and } \beta \in C^3_\varepsilon \text{ and} \\ & \mathrm{otp}(C^3_\varepsilon \cap \beta) \in W_{\varepsilon,\alpha}\}. \end{aligned}$$

Easily $\langle A_\alpha : \alpha < \mu \rangle$ is as required in $(*)_2$, and since $\bar{A}$ is determined by an element of $\mathbb{Q}^3_\mu$ (and the constant $\langle W_{i,\alpha} : \alpha < \lambda_i : i < \kappa \rangle$), the cardinality $|\mathbf{P}| \leq |\mathbb{Q}^3_\mu| \leq \lambda_*$.] Now for $\bar{A} \in \mathbf{P}$ we define a $\mathbb{P}_\mu$-name $\underset{\sim}{\tau}_{\bar{A}}$ as follows: for $\mathbf{G} \subseteq \mathbb{P}_\mu$ generic over $\mathbf{V}$,

$(*)_3$ $\underset{\sim}{\tau}_{\bar{A}}[\mathbf{G}] = \xi$ iff ξ is minimal such that $\cup\{A_\alpha : \alpha \in X_\xi\} \in \mathbf{G}$

clearly

$(*)_4$ $\underset{\sim}{\tau}_{\bar{A}}[\mathbf{G}]$ is defined in at most one way;

$(*)_5$ for every $p \in \mathbb{P}_\mu$ for some $\bar{A} \in \mathbf{P}$ for every $\xi < \chi$ we have $p \nVdash$ "$\underset{\sim}{\tau}_{\bar{A}} \neq \xi$".

[Why? Let $\bar{A} \in \mathbf{P}$ be such that $(\forall \alpha < \mu)(\mu = |p \cap A_\alpha|)$, it exists by $(*)_2$. Now we can find q satisfying $p \leq q \in \mathbb{P}_\mu$ such that $(\forall \alpha < \mu)(q \cap A_\alpha$ is a singleton) and for each $\xi < \chi$ let $q_\xi = \cup\{A_\alpha \cap q : \alpha \in X_\xi\}$. Clearly $\zeta < \xi \Rightarrow |X_\zeta \cap X_\xi| < \mu \Rightarrow \cup\{A_\alpha : \alpha \in X_\zeta\} \cap q_\xi \subseteq \cup\{A_\alpha \cap q_\xi : \alpha \in X_\zeta\} = \cup\{A_\alpha \cap q : \alpha \in X_\zeta \cap X_\xi\} \in [\mu]^{<\mu}$, hence $q_\xi \Vdash$ "$\xi = \underset{\sim}{\tau}_{\bar{A}}$".]
So

$(*)_6$ $\Vdash_{\mathbb{P}_\mu}$ "$\chi = \{\underset{\sim}{\tau}_{\bar{A}}[\underset{\sim}{\mathbf{G}}] : \bar{A} \in \mathbf{P}\}$".

Together clearly $\mathbb{P}_\mu$ collapses χ to $\lambda_* + |\mathbf{P}|$ which is $\leq \lambda_*$, so as said above we are done. $\square_{1.17}$

Lastly, concerning the singular μ_* of cofinality $\aleph_0$ so we forget the hypothesis 1.2, 1.11.

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1.18 Claim. *If $\mu_* > \mathrm{cf}(\mu_*) = \aleph_0$ and $\mathbb{P}_{\mu_*}$ fails the χ-c.c., <u>then</u> $\mathbb{P}_{\mu_*}$ collapses χ to $\aleph_1$; note that in this case $\mathbb{Q}_{\mu_*}$ is equivalent to Levy$(\aleph_1, \mu_*^{\aleph_0})$ by [KjSh 720].*

Proof. Let $\lambda_* = \mu_*^{\aleph_0}$.

By Kojman, Shelah [KjSh 720], $\mathbb{P}_{\mu_*}$ collapses λ_* to $\aleph_1$ hence it suffices to prove that $\mathbb{P}_\mu$ collapse χ to λ_* assuming $\chi > \lambda_*$ (otherwise the conclusion is known). Let $\langle \lambda_n : n < \omega \rangle$ be a sequence of regular uncountable cardinals with limit μ_*. Now repeat the proof of 1.17 $\square_{1.18}$

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§2 The regular uncountable case

We prove that (for κ regular uncountable), $\mathbb{P}_\kappa$ collapse λ to $\aleph_0$ iff $\mathbb{P}_\kappa$ fail the λ-c.c. This continues Balcar, Simon [BaSi88, 2.8] so we first re-represent what they do; the proof of 2.6 is made to help later. In the present notation they let $\lambda = \mathfrak{b}_\kappa$ (rather that $\lambda \in \mathfrak{b}^{\rm spc}\kappa$ as below, let $\langle f_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ be a sequence exemplifying it; let $C_\alpha = \{\delta < \kappa : (\forall \beta < \delta)(f_\alpha(\beta) < \delta), \delta$ a limit ordinal$\}$ and let $B_\alpha = \kappa \backslash C_\alpha$, so $\langle B_\alpha : \alpha < \lambda \rangle$ is a (κ, λ)-sequence (see 2.5(1)), derive a good $(\kappa, {}^{\omega>}\lambda)$-sequence from it (see 2.5(2)), define $\alpha_n(A), \beta_n(A)$ and used the $A_{\eta,\delta,i}$'s to define the $\mathbb{P}_\kappa$-names $\underset{\sim}{\beta}_n$ and prove $\Vdash_{\mathbb{P}_\kappa}$ "$\{g^*(\underset{\sim}{\beta}_n) : n < \omega\} = \mathfrak{b}_\kappa$" (see 2.6). We then prove the new result: if $\mathbb{P}_\kappa$ fail the χ-c.c. then it collapses χ to $\aleph_0$.

2.1 Context. κ is a fixed regular uncountable cardinal.

2.2 Definition. 1) Let $\mathfrak{b}^{\rm spc}_\kappa$ be the set of regular $\lambda > \kappa$ such that there is a $<_{J^{\rm bd}_\kappa}$-increasing sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of members of ${}^\kappa\kappa$ with no $\leq_{J^{\rm bd}_\kappa}$-upper bound in ${}^\kappa\kappa$.
2) Let $\mathfrak{b}_\kappa = {\rm Min}(\mathfrak{b}^{\rm spc}_\kappa)$.

2.3 Definition. 1) We say $\bar{B}$ is a (κ, λ)-sequence when

(a) $\bar{B} = \langle B_\alpha : \alpha < \lambda \rangle$
(b) $B_\alpha \in [\kappa]^\kappa$ and $\kappa \backslash B_\alpha \in [\kappa]^\kappa$ and $B_{\alpha+1} \backslash B_\alpha \in [\kappa]^\kappa$
(c) for every $B \in [\kappa]^\kappa$ for some $\alpha, B \cap B_\alpha \in [\kappa]^\kappa$
(d) $B_\alpha \subseteq^* B_\beta$ when $\alpha < \beta < \lambda$, i.e., $B_\alpha \backslash B_\beta \in [\kappa]^{<\kappa}$.

2) We say that $\bar{B}$ is a $(\kappa, {}^{\omega>}\lambda)$-sequence when:

(a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda \rangle$
(b) $B_\eta \in [\kappa]^\kappa$
(c) if $\eta_1 \triangleleft \eta_2 \in {}^{\omega>}\lambda$ then $B_{\eta_2} \subseteq^* B_{\eta_1}$ which means $B_{\eta_2} \backslash B_{\eta_1} \in [\kappa]^{<\kappa}$
(d) $B_{<>} = \kappa$
(e) if $\eta \in {}^{\omega>}\lambda$ and $A \in [B_\eta]^\kappa$ then for some $\alpha < \lambda$ we have $A \cap B_{\eta^\frown <\alpha>} \in [\kappa]^\kappa$
(f) if $\eta \in {}^{\omega>}\lambda$ and $\alpha < \beta < \lambda$ then $B_{\eta^\frown <\alpha>} \subseteq^* B_{\eta^\frown <\beta>}$ and $B_\eta \backslash B_{\eta^\frown <\alpha>} \in [\kappa]^\kappa$ and $B_{\eta^\frown <\beta>} \backslash B_{\eta^\frown <\alpha>} \in [\kappa]^\kappa$.

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3) For a $(\kappa, {}^{\omega>}\lambda)$-sequence $\bar{B}$ and $A \in [\kappa]^\kappa$ we try to define an ordinal $\alpha_k(A, \bar{B})$ by induction on $k < \omega$. If $\eta = \langle \alpha_\ell(A, \bar{B}) : \ell < k \rangle$ is well defined (holds for $k = 0$) and there is an $\alpha < \lambda$ such that $A \subseteq^* B_{\eta^\frown <\alpha>} \wedge (\forall \beta < \alpha)(A \cap B_{\eta^\frown <\beta>} \in [\kappa]^{<\kappa})$ then we let $\alpha_k(A, \bar{B}) = \alpha$; note that α, if exists, is unique. Let $n(A, \bar{B})$ be the $n \leq \omega$ such that $\alpha_\ell(A, \bar{B})$ is well defined iff $\ell < n$.
4) We say that $(\bar{B}, \bar{\nu})$ is a $(\kappa, {}^{\omega>}\lambda)$-parameter when:

(a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda \rangle$ is a $(\kappa, {}^{\omega>}\lambda)$-sequence

(b) $\bar{\nu}$ is an S^λ_κ-ladder which means that $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa \rangle, \nu_\delta$ is an increasing sequence of ordinals of length κ with limit δ, where $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

5) We say $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter when (a)+(b) of part (4) holds and

(c) if $A \in [\kappa]^\kappa$ then for some $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ and $A' \in [A]^\kappa$ we have

  (α) $\alpha_\ell(A', \bar{B}) = \eta(\ell)$ for $\ell < n$

  (β) for κ many ordinals $\zeta < \kappa$ we have $(\forall \varepsilon < \zeta)(A' \cap B_{\eta^\frown <\nu_\delta(\zeta)>} \backslash B_{\eta^\frown <\nu_\delta(\varepsilon)>}$ belongs to $[\kappa]^\kappa)$.

6) $\bar{B}$ is a good $(\kappa, {}^{\omega>}\lambda)$-sequence if clause (a) of (4) and clause (c) of (5) holds for some S^λ_κ-ladder (see above). We say $\bar{B}$ is a weakly good sequence if clause (a) of (4) and clause $(c)^-$ of (5) which means that we ignore subclause (α) there. Similarly $(\bar{B}, \bar{\nu})$ is a weakly good $(\kappa, {}^{\omega>}\lambda)$-parameter.

2.4 Observation. 1) In 2.3(5)(c)(β), the "for κ many ordinal $\zeta < \kappa$" implies "for club many ordinals $\zeta < \kappa_0$.
2) In 2.3(6) it doesn't matter which S^λ_κ-ladder you choose.

Proof. If $\nu_1, \nu_2 \in {}^\kappa\delta$ are increasing and $\sup(\nu_1) = \sup(\nu_2) = \delta$, then $\{i < \kappa : \bigcup_{j<i} \nu_1(j) = \bigcup_{j<i} \nu_2(j)\}$ is a club of κ.

Note that for §1 we need no more than Claim 2.5 (actually the weakly good version is enough for §1 except presenting the proof that $\mathfrak{b}_\kappa$ is collapsed).

2.5 Claim. *1) Assume $\lambda = \mathfrak{b}_\kappa$ or just $\lambda \in \mathfrak{b}^{\mathrm{spc}}_\kappa$. Then λ is regular $> \kappa$ and there is a $\subseteq^*$-decreasing sequence $\langle C_\alpha : \alpha < \lambda \rangle$ of clubs of κ such that for no $A \in [\kappa]^\kappa$ do we have $\alpha < \lambda \Rightarrow A \subseteq^* C_\alpha$. Hence $\langle \kappa \backslash C_\alpha : \alpha < \lambda \rangle$ is a (κ, λ)-sequence.*
2) Assume $\bar{C} = \langle C_\alpha : \alpha < \lambda \rangle$ is as above and $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa \rangle$ is an S^λ_κ-ladder, see Definition 2.3(4), clause (b) (such $\bar{\nu}$ always exists). Then $\bar{B} = \bar{B}_{\bar{C}}, \bar{f} = \bar{f}_{\bar{C}}$ are well

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defined and the pair $(\bar{B}, \bar{\nu})$ *is a good* $(\kappa, {}^{\omega>}\lambda)$*-parameter where we define* $\bar{B}$ *and* $\bar{f}$ *as follows:*

$\circledast$ (a) $\bar{B} = \langle B_\eta : \eta \in {}^{\omega>}\lambda \rangle$

(b) $\bar{f} = \langle f_\eta : \eta \in {}^{\omega>}\lambda \rangle$

(c) $B_{<>} = \kappa, f_{<>} = \mathrm{id}_\kappa$

(d) $B_\eta \in [\kappa]^\kappa, f_\eta$ *is a function from* B_η *onto* κ*, non-decreasing, and not eventually constant*

(e) *if the pair* (B_ρ, f_ρ) *is defined and* $\alpha < \lambda$ *then we let*

$$B_{\rho^\frown<\alpha>} = \{\gamma \in B_\rho : f_\rho(\gamma) \in \kappa \backslash C_\alpha\}$$

(f) *if* $\eta = \rho^\frown\langle\alpha\rangle$ *and* B_ρ, f_ρ *and* B_η *are defined then we let* $f_\eta : B_\eta \to \kappa$ *be defined by* $f_\eta(i) = \mathrm{otp}(C_\alpha \cap f_\rho(i))$ *for each* $i < \kappa$*,*

hence

(g) *if* $\eta^\frown\langle\alpha\rangle \in {}^{\omega>}\lambda$ *then* $B_{\eta^\frown<\alpha>} \subseteq B_\eta$ *and* $i \in B_{\eta^\frown<\alpha>} \wedge f_\eta(i) > 0 \Rightarrow f_\eta(i) > f_{\eta^\frown<\alpha>}(i)$.

Proof. 1) Recall $S^\lambda_\kappa := \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$.

By the definition of $\mathfrak{b}^{\mathrm{spc}}_\kappa$ there is an $<_{J^{\mathrm{bd}}_\kappa}$-increasing sequence $\langle f^*_\alpha : \alpha < \lambda\rangle$ of members of ${}^\kappa\kappa$ with no $\leq_{J^{\mathrm{bd}}_\kappa}$-upper bound from ${}^\kappa\kappa$. Let $C_\alpha := \{\delta < \kappa : \delta$ is a limit ordinal such that $(\forall\gamma < \delta)(f^*_\alpha(\gamma) < \delta)\}$.

Clearly

$(*)_1$ C_α is a club of κ
[why? as κ is regular uncountable]

$(*)_2$ if $\alpha < \beta < \lambda$ then $C_\beta \subseteq^* C_\alpha$; i.e., $C_\beta \backslash C_\alpha \in [\kappa]^{<\kappa}$
[why? as if $\alpha < \beta$ then $f^*_\alpha <_{J^{\mathrm{bd}}_\kappa} f^*_\beta$, i.e., for some $\varepsilon < \kappa, (\forall\zeta)(\varepsilon \leq \zeta < \kappa \Rightarrow f^*_\alpha(\zeta) < f^*_\beta(\zeta))$ hence letting $\epsilon_1 = \sup(\mathrm{Rang} f^*_\alpha \restriction \alpha)$, we have $C_\beta \backslash (\varepsilon_1 + 1) \subseteq C_\alpha$ as required]

$(*)_3$ for every club C of κ for some $\zeta < \lambda$ we have $C \backslash C_\zeta \in [\kappa]^\kappa$
[why? as $\bar{f}$ has no $\leq_{J^{\mathrm{bd}}_\kappa}$-bound in ${}^\kappa\kappa$]

hence

$(*)_4$ for every unbounded subset A of κ for some $\zeta < \lambda$ we have $A \backslash C_\zeta \in [\kappa]^\kappa$.
[Why? Otherwise the closure of A contradicts $(*)_3$.]

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Clearly $\langle C_\alpha : \alpha < \lambda \rangle$ is as required.

Lastly, let $B_\alpha = \kappa \backslash C_\alpha$, it is easy to check that $\langle B_\alpha : \alpha < \lambda \rangle$ is a (κ, λ)-sequence.

2) Clearly $\bar{B}_{\bar{C}}, \bar{f}_{\bar{C}}$ are well defined and $(\bar{B}, \bar{\nu})$ is a $(\kappa, {}^{\omega>}\lambda)$-parameter and clauses (a)-(g) of $\circledast$ holds. Why is it good? Toward contradiction assume that it is not, so choose $A \in [\kappa]^\kappa$ which exemplify the failure of clause (c) of Definition 2.3(5) and define

$$\mathscr{T}_0 = \mathscr{T}_A^0 = \big\{\eta \in {}^{\omega>}\lambda : \text{there is } A' \in [A]^\kappa \text{ such that } \langle \alpha_\ell(A', \bar{B}) : \ell < \ell g(\eta) \rangle \text{ is well defined and equal to } \eta\big\}.$$

and define

$$\mathscr{T}_1 = \mathscr{T}_A^1 := \big\{\eta \in \mathscr{T}_A^0 : \text{for every } k < \ell g(\eta) \text{ there are } < \kappa \text{ ordinals } \alpha < \eta(k) \text{ such that } (\eta \restriction k)^\frown \langle \alpha \rangle \in \mathscr{T}_0\big\}.$$

Clearly

$(*)_1$ $\mathscr{T}_0 \supseteq \mathscr{T}_1$ are non-empty subsets of ${}^{\omega>}\lambda$ (in fact $<>\in \mathscr{T}_1 \subseteq \mathscr{T}_0$)

$(*)_2$ $\mathscr{T}_0, \mathscr{T}_1$ are closed under initial segments.

For $\eta \in \mathscr{T}_\ell$ let $\mathrm{Suc}_{\mathscr{T}_\ell}(\eta) = \{\rho \in \mathscr{T}_\ell : \ell g(\rho) = \ell g(\eta) + 1 \text{ and } \eta \lhd \rho\}$.

We define $A_\eta \in [B_\eta]^\kappa$ for $\eta \in \mathscr{T}_1$ by induction on $\ell g(\eta)$:

$(*)_3$ (a) $A_{<>} = A$

(b) if A_ν is defined and $\nu^\frown\langle\alpha\rangle \in \mathscr{T}_1$ then we let
$$A_{\nu^\frown<\alpha>} = A_\nu \cap B_{\nu^\frown<\alpha>} \backslash \bigcup\{B_{\nu^\frown<\beta>} : \beta < \alpha \text{ and } \nu^\frown\langle\beta\rangle \in \mathscr{T}_1\}.$$

Now

$(*)_4$ if $\nu \in \mathscr{T}_1$ then

(a) if $B \in [A]^\kappa$ and $\langle \alpha_\ell(B, \bar{B}) : \ell < \ell g(\nu) \rangle$ is well defined and equal to ν <u>then</u> $B \subseteq^* A_\nu$

(b) if $\mathrm{Suc}_{\mathscr{T}_j}(\nu)$ has cardinality $< \kappa$ then $A_\nu \backslash \cup \{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_j}(\nu)\}$ has cardinality $< \kappa$ for $j = 1$ (actually $j = 0$ is O.K., too).

(c) If $\mathrm{Suc}_{\mathscr{T}_1}(\nu)$ has cardinality $< \kappa$ then $\mathrm{Suc}_{\mathscr{T}_0}(\nu) = \mathrm{Suc}_{\mathscr{T}_1}(\nu)$

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[Why? First we can prove clause (a) by induction on $\ell g(\nu)$ using the definition of $\mathscr{T}_1$ and clause (c) of 2.3(2). Second, we can prove clause (b) from it. Third why clause (c) holds?
Otherwise, as $\mathscr{T}_1 \subseteq \mathscr{T}_0$, there is an α with $\nu_n{}^\frown\langle\alpha\rangle \in \mathrm{Suc}_{\mathscr{T}_0}(\nu_n)\backslash \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$. Hence by the definition of $\mathscr{T}_1$ the set $u := \{\beta < \alpha : \nu_n{}^\frown\langle\beta\rangle \in \mathscr{T}_0\}$ has cardinality $\geq \kappa$ but then $\beta \in u \wedge |\beta \cap u| < \kappa \Rightarrow \nu_n{}^\frown\langle\beta\rangle \in \mathscr{T}_1$ which implies that $|\mathrm{Suc}_{\mathscr{T}_1}(\nu_n)| \geq \kappa$, contradiction to the assumption of clause (c).]

$(*)_5$ $|\mathscr{T}_1| \geq \kappa$

[Why? Otherwise by $(*)_4$ the set $A' := \cup\{A_\nu \backslash \cup \{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_0}(\nu)\} : \nu \in \mathscr{T}_1\}$ is a subset of κ of cardinality $< \kappa$ and by clause (d) of $\circledast$ of the present claim also $A'' = \cup\{f_\nu^{-1}\{0\} : \nu \in \mathscr{T}_1\}$ is a subset of κ of cardinality $< \kappa$. So we can choose $j \in A\backslash(A' \cup A'')$. Now we try to choose $\nu_n \in \mathscr{T}_1$ by induction on n such that $\ell g(\nu_n) = n, \nu_{n+1} \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$ and $j \in A_{\nu_n}$.

So $\nu_0 =<>$ belongs to $\mathscr{T}_1$ by $(*)_1 + (*)_3$(a). Now assume ν_n is well defined, then $\mathrm{Suc}_{\mathscr{T}_0}(\nu_n) = \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)$ by $(*)_4(2)$ and our present assumption toward contradicting $|\mathscr{T}_1| < \kappa$.

Now $j \notin A', A' \supseteq A_{\nu_n} \backslash \cup \{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)\}$, but $j \in A_{\nu_n}$ hence clearly $j \in \cup\{A_\rho : \rho \in \mathrm{Suc}_{\mathscr{T}_1}(\nu_n)\}$, so we can choose ν_{n+1} as required. So we have carried the definition of $\langle \nu_n : n < \omega\rangle$.

As $j \in A_{\nu_n} \subseteq B_{\nu_n}$ by $(*)_3(b)$ above, clearly $f_{\nu_n}(j)$ is well defined (for each $n < \omega$). As $j \notin A''$ and $f_{\nu_n}^{-1}\{0\} \subseteq A''$, so $j \notin f_{\nu_n}^{-1}\{0\}$, necessarily $f_{\nu_n}(j) \neq 0$ and so $f_{\nu_n}(j) > f_{\nu_{n+1}}(j)$ by the choice of $f_{\nu_{n+1}}$ in clauses (g) of $\circledast$. Hence $\langle f_{\nu_n}(j) : n < \omega\rangle$ is decreasing (sequence of ordinals), contradiction. So $(*)_5$ holds.]

Let $n < \omega$ be maximal such that $|\mathscr{T}_1 \cap {}^{n\geq}\lambda| < \kappa$, it exists as $|\mathscr{T}_1| \geq \kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $n = 0 \Rightarrow |\mathscr{T}_1 \cap {}^{n\geq}\lambda| = 1 < \kappa$, and let $\eta \in \mathscr{T}_1 \cap {}^n\lambda$ be such that $\mathrm{Suc}_{\mathscr{T}_1}(\eta)$ has $\geq \kappa$ members; it exists as κ is regular. We can choose an increasing sequence $\langle \alpha_i : i < \kappa\rangle$ of ordinals such that α_i is the i-th member of the set $\{\alpha < \lambda : \eta^\frown\langle\alpha\rangle \in \mathscr{T}_1\}$ and let $A_i \in [A]^\kappa$ be such that $\langle \alpha_\ell(A_i, \bar{B}) : \ell \leq n\rangle = \eta^\frown\langle\alpha_i\rangle$ and let $\delta = \cup\{\alpha_i : i < \kappa\}$, so $\delta \in S^\lambda_\kappa$. Let

$$A_* = \cup\{A_i : i < \kappa\} \cap B_\eta \backslash \cup \{A_{(\eta\restriction\ell)^\frown<\gamma>} : \ell < \ell g(\eta) \text{ and } \gamma < \eta(\ell) \text{ and } (\eta \restriction \ell)^\frown\langle\gamma\rangle \in \mathscr{T}_1\}$$

(note that that number of pairs (ℓ, γ) as mentioned above is $< \kappa$).

Clearly $\alpha_\ell(A_*, \bar{B}) = \eta(\ell)$ for $\ell < \ell g(\eta)$ hence $\alpha_\ell(A_* \cap A_i, \bar{B}) = \eta(\ell)$ for $i < \kappa, \ell < n$ so clause (α) of (c) of Definition 2.3(5) holds, as well as clause (β) because $\alpha_n(A_* \cap A_i, \bar{B}) = \alpha_i$ for $i < \kappa$ and $\langle B_{\eta^\frown\langle\alpha\rangle} : \alpha < \lambda\rangle$ is $\subseteq^*$-increasing.

$\square_{2.5}$

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2.6 Claim. *If there is a good $(\kappa, {}^{\omega>}\lambda)$-parameter and $\lambda_1 \in \mathfrak{b}^{\rm spc}_\kappa$ <u>then</u> the forcing notion $\mathbb{P}_\kappa$ collapses λ_1 to $\aleph_0$.*

Proof. Let $(\bar{B}, \bar{\nu})$ be a good $(\kappa, {}^{\omega>}\lambda)$-parameter.
Note

$\circledast_1$ if $A_1 \subseteq A_2$ are from $[\kappa]^\kappa$ and $\alpha_\ell(A_2, \bar{B})$ is well defined then $\alpha_\ell(A_1, \bar{B})$ is well defined and equal to $\alpha_\ell(A_2, \bar{B})$, recalling Definition 2.3(3).

Let $\bar{h} = \langle h_\gamma : \gamma < \lambda_1 \rangle$ exemplify $\lambda_1 \in \mathfrak{b}^{\rm spc}_\kappa$, i.e., is as in Definition 2.2 and without loss of generality $[i < j < \kappa \Rightarrow i < h_\gamma(i) < h_\gamma(j)]$. For each $\delta \in S^\lambda_\kappa$ and $\eta \in {}^{\omega>}\lambda$ let $A_{\eta,\delta,i} = B_{\eta^\frown <\nu_\delta(i+1)>} \setminus \cup \{B_{\eta^\frown <\nu_\delta(j+1)>} : j < i\}$ for $i < \kappa$ so $\langle A_{\eta,\delta,i} : i < \kappa \rangle$ are pairwise disjoint subsets of κ (each of cardinality κ). For $n < \omega$ and $A \in [\kappa]^\kappa$ we try to define an ordinal $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h})$ as follows:

$\circledast_2$ $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$ <u>iff</u> for some $\eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ we have $\langle \alpha_\ell(A, \bar{B}) : \ell \leq n \rangle = \eta^\frown \langle \delta \rangle$ so in particular is well defined and $A \subseteq^* \cup \{A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa\}$ but for every $\beta < \gamma$ we have $A \cap \cup \{A_{\eta,\delta,i} \cap h_\beta(i) : i < \kappa\} \in [\kappa]^{<\kappa}$.

Next we define a $\mathbb{P}_\kappa$-name $\underset{\sim}{\beta}_n = \underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h})$ by:

$\circledast_3$ for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}$: $\underset{\sim}{\beta}_n[\mathbf{G}] = \gamma$ <u>iff</u> for some $A \in \mathbf{G}$ we have $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \gamma$ or there is no such A and $\gamma = 0$.

Now

$\circledast_4$ if $A \in [\kappa]^\kappa$ and $(\mathscr{T}^0_A, \mathscr{T}^1_A)$ n, η, δ are chosen as in the proof of 2.5(2), then $u := \{\beta < \lambda_1 : A \nVdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h}) \neq \beta$"$\}$ is a κ-closed unbounded subset of λ_1.

[Why? We know that $w := \{i < \kappa : A \cap A_{\eta,\delta,i} \in [\kappa]^\kappa\}$ has cardinality κ. Why is u "unbounded"? For any $\gamma_1 < \lambda_1$, we define a function $h \in {}^\kappa\kappa$ as follows, $h(i)$ is the minimal $i_1 < \kappa$ such that for some $i_0, i < i_0 < i_1$ the set $A \cap A_{\eta,\delta,i_0} \cap i_1 \backslash h_{\gamma_1}(i_0)$ is not empty, clearly h is well defined because $|w| = \kappa$. So for some $\gamma_2 \in (\gamma_1, \lambda_1)$ the set $v := \{i < \kappa : h(i) < h_{\gamma_2}(i)\}$ has cardinality κ. Let C be the club $\{\delta < \kappa : \delta$ is a limit ordinal and $i < \delta \Rightarrow h(i) < \delta \wedge h_{\gamma_2}(i) < \delta\}$ and let $\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ list $C \cup \{0\}$ increasing order κ and let $A' = \cup\{A \cap A_{\eta,\delta,i} \cap [\alpha_\varepsilon, \alpha_{\varepsilon+1}) : i < \kappa, \varepsilon < \kappa$ and $\alpha_\varepsilon \leq i < \alpha_{\varepsilon+1}\}$, now $A' \in [\kappa]^\kappa$ (really $i < j < \kappa \Rightarrow i < h_{\gamma_1}(i) < h_{\gamma_2}(j)$). So $\mathbb{P}_\kappa \models$ "$A \leq A'$" and $A' \Vdash$ "$\underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h}) \in (\gamma_1, \gamma_2]$", recalling that the h_γ's are $<_{J^{\rm bd}_\kappa}$ increasing. Why "the set u is κ-closed" (that is the limit of any increasing sequence of length κ of members belong to it)? Easy, too.]

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Let $\langle S_\varepsilon : \varepsilon < \lambda_1 \rangle$ be pairwise disjoint stationary subsets of $S^{\lambda_1}_\kappa$ and define $g^* : \lambda_1 \to \lambda_1$ by $g^*(\gamma) = \varepsilon$ if $\gamma \in S_\varepsilon \vee (\gamma \in \lambda_1 \setminus \bigcup\limits_{\zeta<\lambda_1} S_\zeta \wedge \varepsilon = 0)$. So

$\circledast_5$ for every $p \in \mathbb{P}_\kappa$ for some n, for every $\varepsilon < \lambda_1$, $p \nVdash$ "$g^*(\underset{\sim}{\beta}_n) \neq \varepsilon$"

so we are done. $\square_{2.6}$

Now we arrive to the main point.

2.7 Main Claim. *1) If $\mathbb{P}_\kappa$ does not satisfy the χ-c.c. <u>then</u> forcing with $\mathbb{P}_\kappa$ collapses χ to $\aleph_0$.*
2) There is $\langle \bar{A}_\alpha : \alpha < \mathfrak{b}_\kappa \rangle$ such that $\bar{A}_\alpha = \langle A_{\alpha,i} : i < \kappa \rangle$ is a sequence of pairwise disjoint subsets of κ each of cardinality κ (without loss of generality each is a partition of κ) such that for every $B \in [\kappa]^\kappa$ for some $\alpha < \mathfrak{b}_\kappa$ we have $i < \kappa \Rightarrow \kappa = |A_{\alpha,i} \cap B|$; i.e., for every $i < \kappa$ not just for κ many $i < \kappa$.

Remark. 1) In part (2) we can replace $\mathfrak{b}_\kappa$ by any $\lambda \in \mathfrak{b}^{\rm spc}_\kappa$, but this does not add information. The proof gives a little more for "many" $\alpha < \lambda$.
2) In case 1 we could have assumed $\mathfrak{b}_\kappa > \kappa^+$, this suffice
3) We could have seperated the different roles of λ in the proof of case 1. Say

(a) $(\bar{B}, \bar{\nu}^1)$ will be a good $(\kappa, {}^{\omega>}(\lambda_1))$-parameter,

(b) $\langle h_\alpha : \alpha < \lambda_2 \rangle$ exemplify $\lambda_2 \in \mathfrak{b}^{\rm spc}_\kappa$ and $\langle \nu^*_\delta : \delta \in S^{\lambda_2}_\kappa \rangle$ is an $S^{\lambda_2}_\kappa$-ladder system (so $\delta^* \in S^{\lambda_2}_\kappa$ in the proof)

4) Actually, we can revise case 2 to cover Case 1, too: for $\delta_* \in S^\lambda_{\kappa^+}$ choose C'_{δ_*} a club of δ_* of order type κ^+. Now for each δ we can repeat the construction of names from the proof of Case 2, for each $p \in \mathbb{P}_\kappa$ for some δ_* we succeed to show $\circledast$ below.

Proof. The proof is divided to two cases.

<u>Case 1</u>: $\lambda \in \mathfrak{b}^{\rm spc}_\kappa, \lambda > \kappa^+$, e.g. $\lambda = \mathfrak{b}_\kappa$.

So λ is regular $> \kappa^+$ and a good $(\kappa, {}^{\omega>}\lambda)$ sequence $\bar{B}$ exists (by 2.5).

Let $\bar{\nu} = \langle \nu_\delta : \delta \in S^\lambda_\kappa \rangle$ be such that $\nu_\delta \in {}^\kappa\delta$ is increasing continuous with limit δ and $\bar{\nu}$ guesses clubs (i.e. for every club C of λ, for stationarily many $\delta \in S^\lambda_\kappa$ we have ${\rm Rang}(\nu_\delta) \subseteq C$); exists by [Sh:g, III,§2] because $\lambda = {\rm cf}(\lambda) > \kappa^+$. As $\bar{B}$ is a good $(\kappa, {}^{\omega>}\lambda)$-sequence, $(\bar{B}, \bar{\nu})$ is a good $(\kappa, {}^{\omega>}\lambda)$-parameter by 2.4 (or use 2.5).

Let $\langle h_\alpha : \alpha < \lambda \rangle$ exemplify $\lambda \in \mathfrak{b}^{\rm spc}_\kappa$ without loss of generality $i < j < \kappa \Rightarrow i < h(i) < h(j)$.

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For $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$ and $i < \kappa$, recall that $A_{\eta,\delta,i} = B_{\eta^\frown<\nu_\delta(i+1)>} \backslash \cup\{B_{\eta^\frown<\nu_\delta(j+1)>} : j < i\}$ and let $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}), \underset{\sim}{\beta}_n = \underset{\sim}{\beta}_n(\bar{B}, \bar{\nu}, \bar{h})$ be defined as in the proof of 2.6. For $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$ and $\gamma < \lambda$ let $B^*_{\eta,\delta,\gamma} := \cup\{A_{\eta,\delta,i} \cap h_\gamma(i) : i < \kappa\}$. So clearly (for each $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$) the sequence $\langle B^*_{\eta,\delta,\gamma} : \gamma < \lambda\rangle$ is $\subseteq^*$-increasing. For $\delta^* \in S^\lambda_\kappa$ and $i < \kappa$ let $A^*_{\eta,\delta,\delta^*,i} := B^*_{\eta,\delta,\nu_{\delta^*}(i+1)} \backslash \cup\{B^*_{\eta,\delta,\nu_{\delta^*}(j+1)} : j < i\}$. So $\langle A^*_{\eta,\delta,\delta^*,i} : i < \kappa\rangle$ are pairwise disjoint subsets of κ. Note that (by the proof of 2.6 but not used) for each pair (η, δ) as above for some club $E_{\eta,\delta}$ of λ, for every $\delta^* \in S^\lambda_\kappa \cap E_{\eta,\delta}$ and $i < \kappa, A^*_{\eta,\delta,\delta^*,i}$ has cardinality κ. We shall show during the proof of (1) that $\{\langle A^*_{\eta,\delta,\delta^*,i} : i < \kappa\rangle : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \delta^* \in S^\lambda_\kappa\}$ is as required in part (2), so this will prove part (2) when $\mathfrak{b}_\kappa > \kappa^+$.

Let $\langle X^*_\xi : \xi < \chi\rangle$ be an antichain of $\mathbb{P}_\kappa$, it exists by the assumption. We now for η, δ, δ^* as above define $\mathbb{P}_\kappa$-names $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$: for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}$ we let:

$\circledast_0$ $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}[\mathbf{G}] = \xi$ <u>iff</u> for some $A \in \mathbf{G}, n < \omega$ and $\eta \in {}^n\lambda$ and $\delta, \delta^* \in S^\lambda_\kappa$ we have:

- (a) $\langle \alpha_\ell(A, \bar{B}) : \ell < n\rangle = \eta$ so in particular is well defined
- (b) $\alpha_n(A, \bar{B}) = \delta \in S^\lambda_\kappa$
- (c) $\beta_n(A, \bar{B}, \bar{\nu}, \bar{h}) = \delta^* \in S^\lambda_\kappa$
- (d) $A \cap A^*_{\eta,\delta,\delta^*,i}$ has at most one member for each $i < \kappa$
- (e) $A \subseteq \cup\{A^*_{\eta,\delta,\delta^*,i} : i \in X^*_\xi\}$

Note that demands (a),(b),(c) are natural but actually not being used; with them we could have defined the $\mathbb{P}_\kappa$-names $\underset{\sim}{\gamma}_n$ which is $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$ when defined. Now clearly

$\circledast_1$ $\underset{\sim}{\gamma}_{\eta,\delta,\delta^*}$ is a $\mathbb{P}_\kappa$-name of an ordinal $< \chi$ (may have no value)

$\circledast_2$ for every $p \in \mathbb{P}_\kappa$ for some $\eta \in {}^{\omega>}\lambda$ and $\delta, \delta^* \in S^\lambda_\kappa$, for every $\varepsilon < \chi$ there is q such that $p \le q \in \mathbb{P}_\kappa$ and $q \Vdash_{\mathbb{P}_\kappa}$ "$\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} = \varepsilon$".

[Why? We start as in the proof of 2.6. First there are $n < \omega, \eta \in {}^n\lambda$ and $\delta \in S^\lambda_\kappa$ such that $p \cap A_{\eta,\delta,i} \in [\kappa]^\kappa$ for κ many ordinals $i < \kappa$.
Second, there is a club C_p of λ such that:
if $\beta < \gamma < \lambda$ are from C_p then $p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa$. Indeed, $C_p = \{\gamma < \lambda$: for every $\beta < \gamma$ the set $p \cap B^*_{\eta,\delta,\gamma} \setminus B^*_{\eta,\delta,\beta}$ is from $[\kappa]^\kappa\}$ is as required.

Now by the choice of $\bar{\nu}$, i.e., club guessing, there is $\delta^* \in \mathrm{acc}(C_p) \cap S^\lambda_\kappa$ such that $(\forall i < \kappa)(\nu_{\delta^*}(i) \in C_p)$. So (as we have used $\nu_{\delta^*}(i+1), \nu_{\delta^*}(j+1)$ in the definition of $A^*_{\eta,\delta,\delta^*,i}$)

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$$i < \kappa \Rightarrow p \cap A^*_{\eta,\delta,\delta^*,i} \in [\kappa]^\kappa.$$

This fulfills the promise needed for proving part (2) in the present case 1. Choose $\zeta_i \in p \cap A^*_{\eta,\delta,\delta^*,i}$ for $i < \kappa$. Now for every $\xi < \chi$ let $q_\xi = \{\zeta_i : i \in X^*_\xi\}$. Recall that $\langle X^*_\zeta : \zeta < \chi\rangle$ is an antichain in $\mathbb{P}_\kappa$. Clearly for $\xi < \chi$ we have $\mathbb{P}_\kappa \models$ "$p \le q_\xi$" and $q_\xi \Vdash$ "$\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} = \xi$"; so we have finished proving $\circledast_2$.]

This is enough for proving

$\circledast_3$ forcing with $\mathbb{P}_\kappa$ collapse χ to $\aleph_0$.
[Why? By $\circledast_1 + \circledast_2$ we know that $\Vdash_{\mathbb{P}_\kappa}$ "$\chi = \{\underset{\sim}{\gamma}_{\eta,\delta,\delta^*} : \eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa$ and $\delta^* \in S^\lambda_\kappa\}$", so it is forced that $|\chi| \le |\lambda|$. As we already have by 2.6 that $\Vdash_{\mathbb{P}_\kappa}$ "$|\lambda| = \aleph_0$", we are done.]

<u>Case 2</u>: $\mathfrak{b}_\kappa = \kappa^+$.

Let $\lambda = \kappa^+$ and $\bar{B}$ be a good $(\kappa, {}^{\omega>}\lambda)$-sequence. Let $\langle S_\varepsilon : \varepsilon < \kappa\rangle$ be a partition of $S^{\kappa^+}_\kappa$ to (pairwise disjoint) stationary sets. For $\alpha < \kappa^+$ let $\langle u^\alpha_i : i < \kappa\rangle$ be an increasing continuous sequence of subsets of α each of cardinality $< \kappa$ with union α and without loss of generality $\alpha < \beta \Rightarrow (\forall^* i < \kappa)(u^\alpha_i = u^\beta_i \cap \alpha)$. Let $\bar{h} = \langle h_\beta : \beta < \kappa^+\rangle$ exemplifying $\kappa^+ \in \mathfrak{b}^{\rm spc}_\kappa$ be such that each h_β is strictly increasing, $(\forall i) h_\beta(i) > i$ and let $C_\beta = \{\delta < \kappa : \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta\}$ and let $(\bar{B}, \bar{\nu})$ be a good $(\kappa, {}^{\omega>}\lambda)$-parameter; exists by 2.5(2). Now for $\eta \in {}^{\omega>}\lambda$ and $\delta \in S^\lambda_\kappa$ we define $A_{\eta,\delta,i}(i < \kappa), B^*_{\eta,\delta,\gamma}(\gamma < \lambda)$ as in Case 1. Now for $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define the sequence $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha\rangle$ by

$$Y_{\eta,\delta,\alpha,\beta,\gamma} := \cup\{B^*_{\eta,\delta,\gamma} \cap [i, \mathrm{Min}(C_\beta \backslash (i+1)) \backslash \cup\{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma \cap u^\alpha_i\} : i \in C_\beta \text{ satisfy } \gamma \in u^\alpha_i\}.$$

So $\langle Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma < \alpha\rangle$ is a sequence of pairwise disjoint subsets of κ and for $\varepsilon < \kappa$ let

$$Z_{\eta,\delta,\alpha,\beta,\varepsilon} := \cup\{Y_{\eta,\delta,\alpha,\beta,\gamma} : \gamma \in S_\varepsilon \cap \alpha\}.$$

Clearly

$\boxdot_1$ $\bar{Z}_{\eta,\delta,\alpha,\beta} = \langle Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon < \kappa\rangle$ is a sequence of pairwise disjoint subsets of κ.

We shall show during the proof of (1) that

$$\big\langle \langle Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon < \kappa\rangle : \eta \in {}^{\omega>}\lambda,\ \delta \in S^\lambda_\kappa,\ \alpha < \lambda,\ \beta < \lambda \big\rangle$$

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exemplify part (2); you may wonder: possibly for some quadruple $(\eta, \delta, \beta, \zeta)$ we do not have $(\forall \epsilon < \kappa)[|Z_{\eta,\delta,\alpha,\beta,\epsilon}| = \kappa]$, so? However the quadruple $(\eta, \delta, \alpha, \beta)$ for which this fails, cannot satisfy the desired property in part (2), so we can just omit them.

Let $\langle X^*_\xi : \xi < \chi \rangle$ be a family of sets from $[\kappa]^\kappa$ such that the intersection of any two have cardinality $< \kappa$, it exists as $\mathbb{P}_\kappa$ fail the χ-c.c.. For each $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+$ and $\beta < \kappa^+$ we define a $\mathbb{P}_\kappa$-name $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}$ as follows:

$\boxdot_2$ for $\mathbf{G} \subseteq \mathbb{P}_\kappa$ generic over $\mathbf{V}$, $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}[\mathbf{G}] = \xi$ iff

- (α) for some $A \in \mathbf{G}$ we have
  - (a) $\varepsilon < \kappa \Rightarrow A \cap Z_{\eta,\delta,\alpha,\beta,\varepsilon}$ has at most one member
  - (b) $A \subseteq \cup\{Z_{\eta,\delta,\alpha,\beta,\varepsilon} : \varepsilon \in X^*_\xi\}$
- (β) if for no $A \in \mathbf{G}$ does (a)+(b) hold and $\xi = 0$.

Clearly

$\boxdot_3$ $\underset{\sim}{\tau}_{\eta,\gamma,\alpha,\beta}$ is a well defined ($\mathbb{P}_\kappa$-name) (by $\boxdot_2$).

Now

$\boxdot_4$ for every $p \in \mathbb{P}_\kappa$, for some $\eta \in {}^{\omega>}\lambda, \delta \in S^\lambda_\kappa, \alpha < \kappa^+, \beta < \kappa^+$ we have: for every $\xi < \chi$ for some $q \in \mathbb{P}_\kappa$ above p we have $q \Vdash$ "$\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta} = \xi$" and $\epsilon < \kappa \Rightarrow |Z_{\eta,\delta,\alpha,\beta} \cap p| = \kappa$.

As in Case 1, this is enough for proving that $\mathbb{P}_\kappa$ collapse χ to $\lambda = \kappa^+$. But by 2.6 we already know that forcing with $\mathbb{P}_\kappa$ collapses κ^+ to $\aleph_0$ and so we are done.

Note: we can eliminate η from the $\underset{\sim}{\tau}_{\eta,\delta,\alpha,\beta}$, but not worth it. So we are left with proving $\boxdot_4$.

Why does $\boxdot_4$ hold? First, as in the earlier cases, find $\eta \in {}^{\omega>}\lambda$ and $\delta \in S^\lambda_\kappa$ such that $p \cap A_{\eta,\delta,i} \in [\kappa]^\kappa$ for κ ordinals $i < \kappa$. Second, for some club C_p of λ we have $\beta < \gamma \wedge \gamma \in C_p \Rightarrow p \cap B^*_{\eta,\delta,\gamma} \backslash B^*_{\eta,\delta,\beta} \in [\kappa]^\kappa$. As S_ε (for $\varepsilon < \kappa$) is a stationary subset of λ and C_p a club of λ for each $\varepsilon < \kappa$ we can choose $\gamma^*_\varepsilon \in S_\varepsilon \cap C_p$. Hence there is $\alpha^* < \kappa^+$ large enough such that $\varepsilon < \kappa \Rightarrow \gamma^*_\varepsilon < \alpha^* \in C_p$. Now define a function $h : \kappa \to \kappa$ by induction on i, as follows:

$$\begin{aligned} h(i) = \mathrm{Min}\{j : & j \in (i, \kappa) \text{ and } i_1 < i \Rightarrow h(i_1) < j \text{ and} \\ & \text{if the pair } (\gamma, \epsilon) \text{ is such that } \gamma \in u^{\alpha^*}_i \cap S_\varepsilon \text{ then} \\ & p \cap (i, j) \cap B^*_{\eta,\delta,\gamma} \backslash \cup \{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma \cap u^{\alpha^*}_i\} \text{ is not empty}\}. \end{aligned}$$

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it is well defined as for a given $i < \kappa$ the number of pairs (γ, ε) such that $\gamma \in u_i^{\alpha^*} \cap S_\varepsilon$ is $\leq |u_i^{\alpha^*}| < \kappa$ and is increasing; next we define

$$C = \{j < \kappa : j \text{ is a limit ordinal such that } i < j \Rightarrow h(i) < j\}.$$

Clearly C is a club of κ and let $h' \in {}^\kappa\kappa$ be defined by $h'(i) = h(\mathrm{Min}(C \backslash (i+1))$. By the choice of $\langle h_\beta : \beta < \lambda \rangle$ there is $\beta < \lambda$ such that for κ many ordinals $i < \kappa, h'(i) < h_\beta(i)$. Recall that $C_\beta = \{\delta < \kappa : \delta$ is a limit ordinal and for every $i < \delta$ we have $h_\beta(i) < \delta\}$.

So $W_1 = \{i < \kappa : h'(i) < h_\beta(i)\}$, by the choice of β clearly $W_1 \in [\kappa]^\kappa$. Let $\langle i_j^0 : j < \kappa \rangle$ be an enumeration of the club $C \cap \mathrm{acc}(C_\beta)$ of κ in an increasing order, so clearly $\mathscr{U} := \{j < \kappa : W_1 \cap [i_j^0, i_{j+1}^0) \neq \emptyset\}$ is unbounded in κ. For each $j \in \mathscr{U}$ let i_j^2 be the first member of $W_1 \cap [i_j^0, i_{j+1}^0)$, then let $i_j^1 = \sup(C_\beta \cap (i_j^2 + 1))$, it is well defined as $i_j^0 \in C \cap \mathrm{acc}(C_\beta)$, and so $i_j^0 \leq i_j^1$ and let $i_j^3 = \min\,(C_\beta \setminus (i_j^2 + 1))$ so $i_j^2 < i_j^3$ and

$(*)$ $i_j^0 \leq i_j^1 \leq i_j^2 < h(i_j^2) < h'(i_j^2) < h_\beta(i_j^2) < i_j^3$.

[why? as said above by the choice of i_j^1 by the choice of h, by the choice of the pair (C, h'), by $i_j^2 \in W_1$, by the choice of i_j^3 and C_β respectively.]

(1) "$(*)'$" $i_j^1 < i_j^3$ are successive members of C_β
[Why? both are members of C_β by their choices hence it is enough to prove that $C_\beta \cap (i_j^1, i_j^3) = \emptyset$. But $C_\beta \cap (i_j^1, i_j^2] = \emptyset$ by the choice of i_j^1 and $\beta \cap (i_j^2, i_j^3) = \emptyset$ by the choice of i_j^3]

Now for each $\varepsilon < \kappa$ we know that $\gamma_\varepsilon^* \in \alpha^* \cap S_\varepsilon \cap C_p \subseteq \alpha^* = \cup\{u_i^{\alpha^*} : i < \kappa\}$ and $\langle u_i^{\alpha^*} : i < \kappa \rangle$ is $\subseteq$-increasing hence for some $j(\varepsilon) < \kappa$ if $j \in \mathscr{U} \setminus j(\varepsilon)$ then $\gamma_\varepsilon^* \in u_{i_j^0}^{\alpha^*}$ hence by the choice of $h(i_j^1)$ and $(*)$ we have $p \cap (i_j^1, i_j^3) \cap B^*_{\eta,\delta,\gamma_\varepsilon^*} \backslash \cup \{B^*_{\eta,\delta,\gamma_1} : \gamma_1 \in \gamma_\varepsilon^* \cap u_{i_j^0}^{\alpha^*}\}$ is not empty; but $i_j^1 < i_j^3$ are successive members of C_β by $(*)'$ so the definition of $Y_{\eta,\delta,\alpha^*,\beta,\gamma_\varepsilon^*}$ implies that $p \cap Y_{\eta,\delta,\alpha^*,\beta,\gamma_\varepsilon^*} \cap (i_j^1, i_j^3) \neq \emptyset$.

As this holds for every large enough $j \in \mathscr{U}$ i.e., for every $j \in \mathscr{U} \backslash j(\varepsilon)$ and $\mathscr{U} \in [\kappa]^\kappa$ it follows that $p \cap Y_{\eta,\delta,\alpha^*,\beta,\gamma_\varepsilon^*} \in [\kappa]^\kappa$. By the definition of $Z_{\eta,\delta,\alpha^*,\beta,\varepsilon}$ it follows that $p \cap Z_{\eta,\delta,\alpha^*,\beta,\varepsilon} \in [\kappa]^\kappa$.

We have proved this for every $\epsilon < \kappa$. Choose $\zeta_\varepsilon \in p \cap Z_{\eta,\delta,\alpha^*,\beta,\varepsilon}$ for every $\epsilon < \kappa$. Now for each $\xi < \chi$ let

$$q_\xi = \{\zeta_\varepsilon : \varepsilon \in X_\xi^*\}.$$

So clearly:

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$$\xi < \chi \Rightarrow \mathbb{P}_\kappa \models \text{“}p \leq q_\xi\text{” and } q_\xi \Vdash_{\mathbb{P}_\kappa} \text{“}\underset{\sim}{\tau}_{\eta,\delta,\alpha^*,\beta} = \xi\text{”}.$$

$\square_{2.7}$

2.8 Conclusion. If κ is regular uncountable and $\mathbb{P}_\kappa$ fail the 2^κ-c.c. <u>then</u> $\mathrm{comp}(\mathbb{P}_\kappa)$ is isomorphic to the completion of $\mathrm{Levy}(\aleph_0, 2^\kappa)$.

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REFERENCES.

[BaFr87] Bohuslav Balcar and František Franěk. Completion of factor algebras of ideals. *Proceedings of the American Mathematical Society*, **100**:205–212, 1987.

[BPS] Bohuslav Balcar, Jan Pelant, and Petr Simon. The space of ultrafilters on N covered by nowhere dense sets. *Fundamenta Mathematicae*, **CX**:11–24, 1980.

[BaSi88] Bohuslav Balcar and Petr Simon. On collections of almost disjoint families. *Commentationes Mathematicae Universitatis Carolinae*, **29**:631–646, 1988.

[BaSi89] Bohuslav Balcar and Petr Simon. Disjoint refinement. In *Handbook of Boolean Algebras*, volume 2, pages 333–388. North–Holland, 1989. Monk D., Bonnet R. eds.

[BaSi95] Bohuslav Balcar and Petr Simon. Baire number of the spaces of uniform ultrafilters. *Israel Journal of Mathematics*, **92**:263–272, 1995.

[Ba] James E. Baumgartner. Almost disjoint sets, the dense set problem and partition calculus. *Annals of Math Logic*, **9**:401–439, 1976.

[KjSh 720] Menachem Kojman and Saharon Shelah. Fallen Cardinals. *Annals of Pure and Applied Logic*, **109**:117–129, 2001. arxiv:math.LO/0009079.

[Sh:g] Saharon Shelah. *Cardinal Arithmetic*, volume 29 of *Oxford Logic Guides*. Oxford University Press, 1994.

[Sh 506] Saharon Shelah. The pcf-theorem revisited. In *The Mathematics of Paul Erdős, II*, volume 14 of *Algorithms and Combinatorics*, pages 420–459. Springer, 1997. Graham, Nešetřil, eds.. arxiv:math.LO/9502233.

[Sh 589] Saharon Shelah. Applications of PCF theory. *Journal of Symbolic Logic*, **65**:1624–1674, 2000. arxiv:math.LO/9804155.

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